CONTIGUITY RELATIONS FOR GENERALIZED HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. It is well known that the hypergeometric functions
\[ 2F_1(\alpha \pm 1, \beta, \gamma; t), \quad 2F_1(\alpha, \beta \pm 1, \gamma; t), \quad 2F_1(\alpha, \beta, \gamma \pm 1; t), \]
which are contiguous to \( 2F_1(\alpha, \beta, \gamma; t) \), can be expressed in terms of
\[ 2F_1(\alpha, \beta, \gamma; t) \quad \text{and} \quad 2F_1^{(\alpha, \beta, \gamma; t)}. \]
We explain how to derive analogous formulas for generalized hypergeometric functions. Our main point is that such relations can be deduced from the geometry of the cone associated in a recent paper by B. Dwork and F. Loeser to a generalized hypergeometric series.

1. Introduction

Let \( A = (A_{ij}) \) be an \((m \times n)\)-matrix with entries in \( \mathbb{Z} \). For \( i = 1, \ldots, m \), let \( \ell_i \) be the linear form defined by the \( i \)th row of \( A \):
\[ \ell_i(s_1, \ldots, s_n) = \sum_{j=1}^{n} A_{ij} s_j. \]
Let \( a = (a_1, \ldots, a_m) \in \mathbb{C}^m \). We suppose \( a \) satisfies the condition:
\[ (1.1) \quad \text{If } a_i \in \mathbb{N}^\times, \text{ then } A_{ij} \in \mathbb{N} \text{ for } j = 1, \ldots, n, \]
where \( \mathbb{N} \) denotes the nonnegative integers and \( \mathbb{N}^\times \) denotes the positive integers. We may then define the generalized hypergeometric series
\[ Y(a; t) = \sum_{s \in \mathbb{N}^n} t_1^{s_1} \cdots t_n^{s_n} \frac{(-1)^{s_1+\cdots+s_n}}{s_1! \cdots s_n!} \prod_{i=1}^{m} (a_i)_{\ell_i(s)}, \]
where as usual for \( \rho \in \mathbb{Z} \), \( (z)_\rho = \Gamma(z + \rho)/\Gamma(z) \).

Let \( \epsilon_i \) be the unit vector in the \( i \)th coordinate direction in \( \mathbb{C}^m \). It is easy to verify that if \( a, a + \epsilon_i \) satisfy (1.1), then
\[ (1.2) \quad a_i Y(a + \epsilon_i; t) = (a_i + \ell_i(\delta)) Y(a; t), \]
where \( \ell_i(\delta) = \sum_{j=1}^{n} A_{ij} \delta_j \) and \( \delta_j = t_j \partial/\partial t_j \). The purpose of this note is to invert this relation. We solve the following problem.

**Problem.** Find \( P_i \in \mathbb{Q}(a)[t, \partial/\partial t_1, \ldots, \partial/\partial t_n] \) such that for generic values of \( a \),

\[
Y(a - \epsilon_i; t) = P_i(a, t, \partial/\partial t_1, \ldots, \partial/\partial t_n)Y(a; t).
\]

We give an algorithm for constructing \( P_i \) and show that the coefficients in \( \mathbb{Q}(a) \) appearing in \( P_i \) have denominators which are products of linear factors involving the faces of codimension one of the cone associated with \( Y \) in earlier work [1, 2, 3]. We give estimates for the degree of \( P_i \) as a polynomial in \( t \) and we describe the set of \( a \in \mathbb{C}^m \) for which (1.3) is valid. Under an additional condition, which is satisfied by all the classically studied hypergeometric series, we bound the order of \( P_i \) as a partial differential operator.

This problem has a lengthy history. The function \( \gamma F_1 \) had been treated by Gauss and Appell's \( F_1 \) had been treated by Lavasseur in his 1893 Paris thesis. Professor Kita has brought to our attention the recent works [5, 7]. The methods and scope of [5, 7] are quite different.

2. **Exponential modules**

Let \( A^{(j)} \) be the \( j \)th column of the matrix \( A \). We recall that, in earlier work, the polynomial

\[
-x(t, x) = x_1 + \cdots + x_m + \sum_{j=1}^{n} t_j x^{A^{(j)}},
\]

where \( x^{A^{(j)}} = x_1^{A_{1j}} \cdots x_m^{A_{mj}} \), has been associated with \( Y(a; t) \). Let \( \Omega = \mathbb{Q}(a) \), \( R' = \Omega(t)[x_1, x_1^{-1}, \ldots, x_m, x_m^{-1}] \), \( E_i = x_i \partial/\partial x_i \) for \( i = 1, \ldots, m \), and \( g_i = E_i(g) \). Define operators on \( R' \): \( D_{a,i,t} = E_i + a_i + g_i \) for \( i = 1, \ldots, m \), and \( \sigma_j = \partial/\partial t_j - x^{A^{(j)})(= \partial/\partial t_j + \partial g/\partial t_j) \) for \( j = 1, \ldots, n \). The \( D_{a,i,t} \) and \( \sigma_j \) commute with one another for all \( i \) and \( j \). We define

\[
\mathcal{W}_{a,t} = R'/\sum_{i=1}^{m} D_{a,i,t} R',
\]

which is viewed as an \( \mathcal{R}_1 \)-module, where \( \mathcal{R}_1 \) is the noncommutative ring \( \Omega(t)[\sigma_1, \ldots, \sigma_n] \).

We make the hypothesis

\[
a_i \notin \mathbb{N}^\times \quad \text{for} \quad i = 1, \ldots, m.
\]

Let

\[
R* = \left\{ \sum_{u \in \mathbb{Z}^m} A_u(t)x^{-u} \mid A_u(t) \in \Omega[[t]] \right\}
\]

and let \( \xi_{a,t}^* \in R^* \) be defined by

\[
\xi_{a,t}^* = \exp\left( -\sum_{j=1}^{n} t_j x^{A^{(j)}} \right) \sum_{u \in \mathbb{Z}^m} \prod_{i=1}^{m}(a_i) u_i x^u.
\]
By a direct calculation, for \( u \in \mathbb{Z}^m \)

\[
(2.3) \quad \left( \prod_{i=1}^{m} (a_i)_{ui} \right) Y(a + u; t) = \langle \xi^*_{a,t}, x^u \rangle,
\]

where for \( \xi^* \in R^*, \xi \in R', \langle \xi^*, \xi \rangle \) is defined to be the coefficient of \( x^0 \) in the product \( \xi^* \xi \). In particular, taking \( u = 0 \) gives

\[
Y(a; t) = \langle \xi^*_{a,t}, 1 \rangle.
\]

For any \( \xi^* \in R^*, \xi \in R' \), one checks easily that

\[
\frac{\partial}{\partial t_j} \langle \xi^*, \xi \rangle = \langle \sigma^*_j(\xi^*), \xi \rangle + \langle \xi^*, \sigma_j(\xi) \rangle,
\]

where \( \sigma^*_j = \partial / \partial t_j + x^A(j) \). From (2.2) it follows that \( \sigma^*_j(\xi^*_{a,t}) = 0 \), hence

\[
(2.4) \quad \frac{\partial}{\partial t_j} \langle \xi^*_{a,t}, \xi \rangle = \langle \xi^*_{a,t}, \sigma_j(\xi) \rangle.
\]

Applying this with \( \xi = 1 \), we conclude that for \( P \in \mathbb{Q}(a)[t, Z_1, \ldots, Z_n] \),

\[
P(a, t, \partial / \partial t_1, \ldots, \partial / \partial t_n)Y(a; t) = P(a, t, \partial / \partial t_1, \ldots, \partial / \partial t_n)\langle \xi^*_{a,t}, 1 \rangle
\]

(2.5)

\[
= \langle \xi^*_{a,t}, P(a, t, \sigma_1, \ldots, \sigma_n) \rangle.
\]

Under the pairing \( \langle , \rangle \), the adjoint on \( R^* \) of the mapping \( D_{a,i,t} \) on \( R' \) is the mapping \( D^*_{a,i,t} = -E_i + a_i + g_i \). One checks that \( D^*_{a,i,t}(\xi^*_{a,t}) = 0 \) for \( i = 1, \ldots, m \). It follows that \( \xi^*_{a,t} \) annihilates \( \sum_{i=1}^{m} D_{a,i,t} R' \) under the pairing. Taking \( u = -e_i \) in (2.3) and comparing with (2.5) reduces the problem stated in the introduction to the problem of finding \( P_i \in \mathcal{R}_1 \) such that

\[
(2.6) \quad \frac{a_i - 1}{x_i} \equiv P_i(a, t, \sigma_1, \ldots, \sigma_n) \quad (\text{mod} \sum_{i=1}^{m} D_{a,i,t} R').
\]

One then has

\[
(2.7) \quad P_i(a, t, \partial / \partial t_1, \ldots, \partial / \partial t_n)Y(a; t) = Y(a - e_i; t).
\]

Let \( \ell_i(t \sigma) = \sum_{j=1}^{m} A_{ij} t_j \sigma_j \). One checks from the definitions that

\[
a_i + \ell_i(t \sigma) - \ell_i(\delta) = -E_i + x_i + D_{a,i,t},
\]

hence for \( v \in \mathbb{Z}^m \),

\[
(a_i + \ell_i(t \sigma) + v_i)x^v \equiv x^{v + e_i} \quad (\text{mod} \sum_{i=1}^{m} D_{a,i,t} R').
\]

One then proves by induction that for \( r \in \mathbb{N}^m \),

\[
(2.8) \quad x^r \equiv \prod_{i=1}^{m} (a_i + \ell_i(t \sigma) + v_i) \quad (\text{mod} \sum_{i=1}^{m} D_{a,i,t} R').
\]
Let $\mathcal{H}_0$ be the monoid generated by $\epsilon_1, \ldots, \epsilon_m, A^{(1)}, \ldots, A^{(n)}$. If $u \in \mathcal{H}_0$, then $u = r + \sum_{j=1}^n s_j A^{(j)}$, where $r \in \mathbb{N}^m$, $(s_1, \ldots, s_n) \in \mathbb{N}^n$. One has trivially

$$(-\sigma_1)^{s_1} \cdots (-\sigma_n)^{s_n}(x^r) = x^r + \sum_{j=1}^m s_j A^{(j)} = x^u.$$ 

Hence by (2.8),

$$x^u \equiv (-\sigma_1)^{s_1} \cdots (-\sigma_n)^{s_n} \prod_{i=1}^m (a_i + \ell_i(t\sigma)) r_i^1 \pmod{\sum_{i=1}^m D_{a_i, i, r}},$$

Thus to find $P_i$ satisfying (2.6), it suffices to find a formula of the type

$$x^u = (-1)^{s_1 + \cdots + s_n} \prod_{j=1}^m \left(\frac{\partial}{\partial t_j}\right)^{s_j} \prod_{i=1}^m (a_i + \ell_i(\delta)) r_i Y(a; t).$$

3. The contiguity algorithm

Let $\mathcal{C}$ be the cone in $\mathbb{R}^m$ determined by the monomials of $g$:

$$\mathcal{C} = \{ z \in \mathbb{R}^m \mid z = \sum_{i=1}^m r_i \epsilon_i + \sum_{j=1}^n s_j A^{(j)}, \text{ all } r_i, s_j \in [0, \infty) \}.$$ 

Let $\mathcal{H}_0 = \mathcal{C} \cap \mathbb{Z}^m$. We introduce $\mathcal{H}_0$ because it can be characterized by a system of linear inequalities.

**Lemma 3.1.** There exists $w \in \mathcal{H}_0$ such that $\mathcal{H}_0 + w \subseteq \mathcal{H}_0$. In particular, we may take $w = \sum_{i=1}^m T_i \epsilon_i$, where

$$T_i = \sup \left(0, -1 + \sum_{j=1}^n \sup(0, -A_{ij})\right).$$

**Remark.** For classical hypergeometric functions the matrices $A$ are made explicit in the appendix of [2] and it is not hard to check that in all these cases we have $\mathcal{H}_0 = \mathcal{H}_0$, i.e., one may take $w = 0$ in the classical examples.

**Proof.** If $v \in \mathcal{H}_0$ then $v = \sum_{i=1}^m r_i \epsilon_i + \sum_{j=1}^n s_j A^{(j)}$ where all $r_i$ and $s_j$ are nonnegative. Putting $r_i = \alpha_i + \alpha'_i$, $\alpha_i \in \mathbb{N}$, $\alpha'_i \in [0, 1)$, and $s_j = \beta_j + \beta'_j$, $\beta_j \in \mathbb{N}$, $\beta'_j \in [0, 1)$, we conclude that $v = u + \mu$, where $u \in \mathcal{H}_0$ and $\mu = \sum_{i=1}^m \alpha'_i \epsilon_i + \sum_{j=1}^n \beta'_j A^{(j)} \in \mathcal{H}_0$. Since $\mu$ lies in a bounded set, there are only a finite number of possibilities for $\mu$ and hence there exists $w \in \mathbb{N}^m$ such that $w + \mu \in \mathcal{H}_0$ for all $\mu$. This shows the existence of $w$. To check our particular choice for $w$ it is enough to check that for all $i, \alpha'_i + \sum_{j=1}^n \beta'_j A_{ij} \in \mathbb{Z}$ implies that $\alpha'_i + \sum_{j=1}^n \beta'_j A_{ij} + T_i \geq 0$. This follows from the fact that if
inf_{j=1,\ldots,n}\{A_{ij}\} < 0, \text{ then }
\sum_{j=1}^{n} \beta_j A_{ij} > \sum_{j=1}^{n} \inf(0, A_{ij}) = -1 - T_i. \quad \square

We now recall that the cone $\mathcal{C}$ may also be defined by linear inequalities. Let $\tau_1, \ldots, \tau_\rho$ be the hyperplanes through the faces of $\mathcal{C}$ of codimension one. Then for $k = 1, \ldots, \rho$, $\tau_k$ is defined by a linear form

$$f_k(u) = \sum_{i=1}^{m} B_{ki} u_i,$$

where the $B_{ki}$ are integers with greatest common divisor 1, $f_k(u) = 0$ is the equation of $\tau_k$, and $f_k(u) \geq 0$ for all $u \in \mathcal{C}$. Let us write

$$f_k(Da) = \sum_{i=1}^{m} B_{ki} Da, i, t,$$

$$f_k(g) = \sum_{i=1}^{m} B_{ki} g_i$$

$$= - \sum_{i=1}^{m} x_i f_k(e_i) - \sum_{j=1}^{n} t_j x^{A(j)} f_k(A^{(j)})$$

$$f_k(E) = \sum_{i=1}^{m} B_{ki} E_i.$$

The key point is that all monomials appearing in $f_k(g)$ have exponents lying in the region $f_k(u) \geq 1$.

**Lemma 3.2.** For each $v \in \mathbb{Z}^m$ there exists a representation

$$x^v \equiv \sum_{u \in \hat{H}_0} c_{v, u} x^u \pmod{\sum_{i=1}^{m} Da, i, i R'},$$

where the sum on the right-hand side is finite and each $c_{v, u} \in \mathbb{Q}(a)[t]$.

**Proof.** We use induction on $N_v = \sum_{k=1}^{\rho} \sup(0, f_k(w - v))$. If $N_v = 0$, then $f_k(v - w) \geq 0$ for all $k$ which shows that $v \in w + \hat{H}_0 \subseteq \hat{H}_0$ by Lemma 3.1. Thus we may assume $f_k(v - w) < 0$ for some $k$. We compute

$$f_k(Da)x^v = (f_k(a) + f_k(v))x^v + f_k(g)x^v,$$

and so

$$x^v \equiv -\frac{1}{f_k(a) + f_k(v)} f_k(g)x^v \pmod{\sum_{i=1}^{m} Da, i, i R'}.$$

We now apply the induction hypothesis to $f_k(g)x^v$, which is a $\mathbb{Z}[t]$-linear combination of terms $x^{v'}$ such that $N_{v'} < N_v$.
Note that equation (3.3) remains valid under specialization of $a$ for all $a$ such that $f_k(a) + f_k(v) \neq 0$. As an immediate consequence of the proof, we have:

**Corollary 3.4.**

\[
\text{deg}_{t} c_{v, u} \leq N_v,
\]

(3.5)

\[
\left( \prod_{k=1}^{\rho} (f_k(a) + f_k(v)) \sup(0, f_k(w-v)) \right) c_{v, u} \in \mathbb{Q}[a, t].
\]

(3.6)

Specializing $v$ to $-\epsilon_i$ and combining Lemma 3.2 with (2.11), (2.3) and Corollary 3.4 gives:

**Theorem 3.7.** There exists $P_i \in \mathbb{Q}(a)[t, \partial/\partial t_1, \ldots, \partial/\partial t_n, \delta_1, \ldots, \delta_n]$ such that

\[
Y(a - \epsilon_i; t) = P_i(a, t, \partial/\partial t_1, \ldots, \partial/\partial t_n, \delta_1, \ldots, \delta_n) Y(a; t).
\]

As polynomials in $t$, the coefficients of $P_i$ have degree bounded by

\[
N_{-\epsilon_i} = \sum_{k=1}^{\rho} \sup(0, f_k(w + \epsilon_i))
\]

and $H_i(a)P_i$ has coefficients in $\mathbb{Q}[a, t]$, where

\[
H_i(a) = \prod_{i=1}^{\rho} (f_k(a) + f_k(-\epsilon_i)) \sup(0, f_k(w+\epsilon_i)).
\]

Thus this contiguity relation is valid provided $a_j \notin \mathbb{N}^\times$ for $j = 1, \ldots, m$ and $H_i(a) \neq 0$.

Of course, if one expresses $P_i$ as a polynomial in the $\partial/\partial t_i$'s only (i.e., replace $\delta_i$ by $t_i \partial/\partial t_i$), then the degrees of its coefficients as polynomials in $t$ change. These new degrees can be bounded by the methods of the next section, under the additional assumption that $\hat{H}_0 = \hat{H}_0$.

We believe that this theorem gives the basic set of contiguity relations. We observe that other contiguity relations may be deduced from

\[
\langle \xi^*_a, t, D_{a, i, t}x^u \rangle = 0
\]

for all $v \in \mathbb{Z}^m$, $i = 1, \ldots, m$, together with either (2.11) for $v \in \hat{H}_0$ or (2.3) for arbitrary $v \in \mathbb{Z}^m$.

### 4. Bounding the Order of $P_i$

To bound the order of $P_i$ as a differential operator, we introduce some auxiliary functions. For $u \in \hat{H}_0$, put

\[
W(u) = \inf \left\{ \sum_{i=1}^{m} r_i + \sum_{j=1}^{n} s_j \mid u = \sum_{i=1}^{m} r_i \xi_i + \sum_{j=1}^{n} s_j A^{(j)}, \ r_i, s_j \in \mathbb{N} \text{ for all } i, j \right\}.
\]

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From (2.11) we see that the differential operator on $Y(a; t)$ that corresponds to $x^u$ (more precisely, that corresponds to a representation of $u$ minimizing $\sum_{i=1}^{m} r_i + \sum_{j=1}^{n} s_j$) has order $W(u)$. Thus the problem of bounding the order of the differential operator corresponding to $x^v$, $v \in \mathbb{Z}^m$, is reduced to the problem of bounding $W(u)$ as $u$ ranges over all terms with $c_{v, u} \neq 0$ on the right-hand side of Lemma 3.2. To accomplish this, we need to extend the definition of $W$ to all $u \in \mathbb{Z}^m$.

It is clear from (4.1) that if $u_1, u_2 \in \tilde{H}_0$, then

$$W(u_1 + u_2) \leq W(u_1) + W(u_2).$$

Any $u \in \mathbb{Z}^m$ can be written $u = u_1 - u_2$ with $u_1, u_2 \in \tilde{H}_0$. If in addition $u \in \tilde{H}_0$, then (4.2) implies $W(u_1) - W(u_2) \leq W(u)$. Thus we may extend (4.1) by defining for $u \in \mathbb{Z}^m$

$$W(u) = \sup\{W(u_1) - W(u_2) \mid u = u_1 - u_2, \ u_1, u_2 \in \tilde{H}_0\}.$$ 

**Remark.** We shall establish later that, under the hypothesis $\tilde{H}_0 = \tilde{H}_0$, $W(u) < \infty$ for all $u \in \mathbb{Z}^m$. This will show that our bound on the order of $P_i$ is nontrivial.

**Lemma 4.4.** If $u \in \mathbb{Z}^m$, $u' \in \tilde{H}_0$, then

$$W(u + u') \leq W(u) + W(u').$$

**Proof.** Pick $u_1, u_2 \in \tilde{H}_0$ such that $u + u' = u_1 - u_2$. By (4.2), $W(u' + u_2) \leq W(u') + W(u_2)$, hence

$$W(u_1) - W(u_2) \leq W(u_1) + W(u') - W(u' + u_2).$$

But $u_1 - (u' + u_2) = u$, so $W(u_1) - W(u' + u_2) \leq W(u)$. This implies the lemma. □

**Proposition 4.5.** If $v \in \mathbb{Z}^m$, the partial differential operator corresponding to $x^v$ under (2.11) and Lemma 3.2 has order $\leq W(v) + N_v$.

**Proof.** The proof is by induction on $N_v$. If $N_v = 0$, then as noted in the proof of Lemma 3.2, $v \in \tilde{H}_0$. Suppose $N_v > 0$. By Lemma 4.4, $f_k(g)x^v$ is a $\mathbb{Z}^m$-linear combination of terms $x^{v'}$ such that $W(v') \leq W(v) + 1$. Since $N_{v'} \leq N_v - 1$, we are done by (3.3) and the induction hypothesis. □

**Corollary 4.6.** The partial differential operator $P_i$ of Theorem 3.7 can be chosen to have order $\leq W(-\epsilon_i) + N_{-\epsilon_i}$.

We now show this bound is nontrivial when $\tilde{H}_0 = \tilde{H}_0$. We introduce a function $W'$ on $\tilde{H}_0$ defined by

$$W'(u) = \inf \left\{ \sum_{i=1}^{m} r_i + \sum_{j=1}^{n} s_j \mid u = \sum_{i=1}^{m} r_i \epsilon_i + \sum_{j=1}^{n} s_j A^{(j)} \right\},$$

where $r_i, s_j \in [0, \infty)$ for all $i, j$.

Trivially, $W'(u) \leq W(u)$ for all $u \in \tilde{H}_0$. The function $W'$ has a geometric interpretation: $W'(u)$ is the smallest nonnegative real number such that $u \in$
\[ W'(u) = \text{sup} \{ \lambda_i(u) \mid i = 1, \ldots, p \}. \]

**Lemma 4.9.** Fix \( u \in \hat{H}_0 \). Then the set \( \{ W'(u + u_1) - W'(u_1) \mid u_1 \in \hat{H}_0 \} \) is bounded above and below.

**Proof.** From the definition of \( W' \) it is clear that

\[ W'(u + u_1) \leq W'(u) + W'(u_1), \]

hence the given set is bounded above by \( W'(u) \). By (4.8) we may choose \( i_1, i_2 \in \{1, \ldots, p\} \) such that

\[ W'(u + u_1) = \lambda_{i_1}(u + u_1), \quad W'(u_1) = \lambda_{i_2}(u_1). \]

Then \( (\lambda_{i_1} - \lambda_{i_2})(u_1) \leq 0 \) but \( (\lambda_{i_1} - \lambda_{i_2})(u + u_1) \geq 0 \), hence there exists \( \alpha \in [0, 1] \) such that

\[ (\lambda_{i_1} - \lambda_{i_2})(\alpha u + u_1) = 0. \]

Then

\[ W'(u + u_1) - W'(u_1) = \lambda_{i_1}(u + u_1) - \lambda_{i_2}(u_1) = (1 - \alpha)\lambda_{i_1}(u) + \alpha\lambda_{i_2}(u) \]

by (4.10). This latter quantity is clearly bounded above and below independently of \( u_1 \). \( \square \)

**Lemma 4.11.** Suppose \( \hat{H}_0 = \hat{H}_0 \). There exists a positive constant \( \kappa \) such that for all \( u \in \hat{H}_0 \),

\[ W(u) \leq W'(u) + \kappa. \]

**Proof.** Let \( u \in \hat{H}_0 \). Choose \( r_i, s_j \in [0, \infty) \) such that

\[ u = \sum_{i=1}^{m} r_i \epsilon_i + \sum_{j=1}^{n} s_j A^{(j)} \quad \text{and} \quad W'(u) = \sum_{i=1}^{m} r_i + \sum_{j=1}^{n} s_j. \]

Put \( [u] = \sum_{i=1}^{m} [r_i] \epsilon_i + \sum_{j=1}^{n} [s_j] A^{(j)} \in \hat{H}_0 \). Then \( u - [u] = \mu \in \mathbb{Z}^m \cap \beta = \hat{H}_0 \). Furthermore, \( \mu \) lies in a bounded (hence finite) subset of \( \hat{H}_0 \). Let \( \kappa \) be the maximum value of \( W \) on this finite subset. Now \( u = [u] + \mu \), hence

\[ W(u) \leq \sum_{i=1}^{m} [r_i] + \sum_{j=1}^{n} [s_j] + \kappa \]

\[ \leq W'(u) + \kappa. \quad \square \]

**Proposition 4.12.** Suppose \( \hat{H}_0 = \hat{H}_0 \). Then \( W(u) < \infty \) for all \( u \in \mathbb{Z}^m \).
Proof. Write \( u = u_1 - u_2 \) with \( u_1, u_2 \in \tilde{H}_0 \). By Lemma 4.4, \( W(u) \leq W(u_1) + W(-u_2) \). Thus it suffices to show \( W(-u) < \infty \) for all \( u \in \tilde{H}_0 \). So suppose \(-u = u_1 - u_2 \) with \( u, u_1, u_2 \in \tilde{H}_0 \). Then \( u_2 = u_1 + u \), so

\[
W(u_1) - W(u_2) = W(u_1) - W(u_1 + u) \\
\leq W'(u_1) + \kappa - W'(u_1 + u)
\]

by Lemma 4.11. By Lemma 4.9, this quantity is bounded above independently of \( u_1 \). \( \square \)

5. Examples

Consider the classical Gaussian hypergeometric function

\[
_{2}F_1(\alpha, \beta, \gamma; t) = \sum_{s=0}^{\infty} \frac{(\alpha)_s (\beta)_s (\gamma)_s}{s!} t^s.
\]

Using the relation \((\gamma)_s (1 - \gamma)_s = (-1)^s\) we have

\[
_{2}F_1(\alpha, \beta, \gamma; t) = \sum_{s=0}^{\infty} \frac{(\alpha)_s (\beta)_s (1 - \gamma)_s (-t)^s}{s!} = Y(\alpha, \beta, 1 - \gamma; t),
\]

so \( a = (a_1, a_2, a_3) = (\alpha, \beta, 1 - \gamma) \) and \( \ell_1(s) = \ell_2(s) = s \), \( \ell_3(s) = -s \). This corresponds to

\[
-g(t, x) = x_1 + x_2 + x_3 + t\frac{x_1 x_2}{x_3}.
\]

The codimension-one faces of the corresponding cone \( \mathfrak{C} \) are given by the forms \( f_1(u) = u_2 + u_3 \), \( f_2(u) = u_1 + u_3 \), \( f_3(u) = u_1 \), \( f_4(u) = u_2 \) and one checks that \( \tilde{H}_0 = \tilde{H}_0 \), hence \( w = 0 \) in Lemma 3.1. One has \( f_1(-\epsilon_1), f_4(-\epsilon_1) \geq 0 \) but \( f_2(-\epsilon_1) = f_3(-\epsilon_1) = -1 \). Following the algorithm described in the proof of Lemma 3.2 by first applying \( f_2(Da) \) to \( 1/x_1 \) and then applying \( f_3(Da) \) to \( x_3/x_1 \) gives

\[
\frac{\alpha - 1}{x_1} \equiv \frac{\alpha - 1 + x_2 + tx_2}{\alpha - \gamma} \pmod{\sum_{i=1}^{3} D_{a,i,t} R'}.
\]

By (2.9),

\[
x_3 \equiv (1 - \gamma - t\tau)1 \pmod{\sum_{i=1}^{3} D_{a,i,t} R'},
\]

\[
x_2 \equiv (\beta + t\tau)1 \pmod{\sum_{i=1}^{3} D_{a,i,t} R'},
\]

so

\[
\frac{\alpha - 1}{x_1} \equiv \frac{(\alpha - \gamma - t\tau) + t(\beta + t\tau)}{\alpha - \gamma} \pmod{\sum_{i=1}^{3} D_{a,t} R'}.
\]
From (2.7) we get

$$(\gamma - \alpha)_{2F1}(\alpha - 1, \beta; \gamma; t) = (t(1-t) \frac{\partial}{\partial t} + (\gamma - \alpha - \beta t))_{2F1}(\alpha, \beta, \gamma; t),$$

a well-known classical formula (see [4, section 2.8, equation (23)] or [6, Chapter VI, section 24]).

We give some details for the calculation of the contiguity relations for the Lauricella series $F_A$, which we write in the form

$$Y(a; t) = \sum_{s \in \mathbb{N}^n} c(s) \frac{t^s}{s_1! \cdots s_n!},$$

where

$$c(s) = (a_{2n+1})_{s_1+\cdots+s_n} \frac{\prod_{i=1}^n (a_i+n)_{s_i}}{\prod_{i=1}^n (1-a_i)_{s_i}}.$$ 

The associated polynomial [2] is

$$-g = x_1 + \cdots + x_{2n+1} + \sum_{j=1}^{n} t_j x_{2n+1} \frac{x_{n+j}}{x_j}.$$ 

There are $2^n + 2n$ linear forms which define the associated cone:

$$f_j(u) = u_{n+j} \quad (j = 1, \ldots, n),$$

$$f_{n+j}(u) = u_{n+j} + u_j \quad (j = 1, \ldots, n),$$

and for each subset $S$ of $\{1, \ldots, n\}$ the form

$$f_S(u) = u_{2n+1} + \sum_{j \in S} u_j.$$ 

Clearly these forms are nonnegative on the cone of $g$. We omit the proof that these forms define the cone.

We give some of the calculations for $Y(a - \epsilon_1; t)$. By applying $f_{n+1}(D_a)$ to $1/x_1$ we obtain

$$(a_{n+1} + a_1 - 1) \frac{1}{x_1} \equiv (x_1 + x_{n+1}) \frac{1}{x_1} = 1 + \frac{x_{n+1}}{x_1}.$$ 

Letting $S = \{1\}$ and applying $f_S(D_a)$ to $x_{n+1}/x_1$ we obtain

$$(a_{2n+1} + a_1 - 1) \frac{x_{n+1}}{x_1} \equiv x_{n+1} (x_1 + x_{2n+1} + \sum_{j=2}^{n} t_j x_{2n+1} \frac{x_{n+j}}{x_j}).$$ 

Letting $y_l = x_{2n+1} x_{n+l}/x_l$, $l = 1, \ldots, n$, this becomes

$$(a_{2n+1} + a_1 - 1) \frac{x_{n+1}}{x_1} \equiv x_{n+1} + y_1 + \sum_{j=2}^{n} t_j y_1 \frac{x_{n+j}}{x_j}. $$
Thus we are reduced to the problem of reducing \( y_1 \cdots y_{l-1} x_{n+1}/x_l \) modulo \( \sum_{i=1}^{2n+1} D_{a,i,t} R' \). Applying \( f_S(D_a) \) with \( S = \{1, \ldots, l\} \) we obtain

\[
(a_{2n+1} + a_1 + \cdots + a_l - 1)y_1 \cdots y_{l-1} x_{n+1}/x_l = (x_1 + \cdots + x_l + x_{2n+1} + \sum_{j=l+1}^{n} t_j y_j) y_1 \cdots y_{l-1} x_{n+1}/x_l
\]

\[
= y_1 \cdots y_l + \sum_{i=1}^{l} y_1 \cdots \hat{y}_i \cdots y_l x_{n+i} + \sum_{j=l+1}^{n} t_j y_1 \cdots y_j x_{n+j}/x_j
\]

By iteration we arrive at a representation of \( 1/x_l \) as a polynomial in \( x_1, \ldots, x_{2n+1}, y_1, \ldots, y_n \) with coefficients in \( \mathbb{Q}(a)[t] \). The number of steps is quite large since \( f_{n+1}(-\epsilon_1) = -1 \) and \( f_{\emptyset}(-\epsilon_1) = -1 \) for every subset \( S \) of \( \{1, \ldots, n\} \) that contains \( 1 \) (thus \( N_{-\epsilon_1} = 1 + 2^{n-1} \)).

**REFERENCES**


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