CONTIGUITY RELATIONS FOR GENERALIZED HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. It is well known that the hypergeometric functions

\[ \begin{align*}
2F_1(a \pm 1, \beta, \gamma; t), \quad 2F_1(a, \beta \pm 1, \gamma; t), \quad 2F_1(a, \beta, \gamma \pm 1; t),
\end{align*} \]

which are contiguous to \( 2F_1(a, \beta, \gamma; t) \), can be expressed in terms of

\[ \begin{align*}
2F_1(a, \beta, \gamma; t) \quad \text{and} \quad 2F_1(a, \beta, \gamma; t). 
\end{align*} \]

We explain how to derive analogous formulas for generalized hypergeometric functions. Our main point is that such relations can be deduced from the geometry of the cone associated in a recent paper by B. Dwork and F. Loeser to a generalized hypergeometric series.

1. INTRODUCTION

Let \( A = (A_{ij}) \) be an \((m \times n)\)-matrix with entries in \( \mathbb{Z} \). For \( i = 1, \ldots, m \), let \( \ell_i \) be the linear form defined by the \( i \)th row of \( A \):

\[ \ell_i(s_1, \ldots, s_n) = \sum_{j=1}^{n} A_{ij} s_j. \]

Let \( a = (a_1, \ldots, a_m) \in \mathbb{C}^m \). We suppose \( a \) satisfies the condition:

\[ (1.1) \quad \text{If } a_i \in \mathbb{N}^\times, \text{ then } A_{ij} \in \mathbb{N} \text{ for } j = 1, \ldots, n, \]

where \( \mathbb{N} \) denotes the nonnegative integers and \( \mathbb{N}^\times \) denotes the positive integers. We may then define the generalized hypergeometric series

\[ Y(a; t) = \sum_{s \in \mathbb{N}^n} t_1^{s_1} \cdots t_n^{s_n} \frac{(-1)^{s_1+\cdots+s_n}}{s_1! \cdots s_n!} \prod_{i=1}^{m} (a_i)^{\ell_i(s)}, \]

where as usual for \( \rho \in \mathbb{Z}, \ (z)_\rho = \Gamma(z + \rho)/\Gamma(z) \).

Let \( e_i \) be the unit vector in the \( i \)th coordinate direction in \( \mathbb{C}^m \). It is easy to verify that if \( a, a + e_i \) satisfy (1.1), then

\[ (1.2) \quad a_i Y(a + e_i; t) = (a_i + \ell_i(\delta)) Y(a; t), \]

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where $\ell_i(\delta) = \sum_{j=1}^{n} A_{ij} \delta_j$ and $\delta_j = t_j \partial / \partial t_j$. The purpose of this note is to invert this relation. We solve the following problem.

**Problem.** Find $P_i \in \mathbb{Q}(a)[t, \partial / \partial t_1, \ldots, \partial / \partial t_n]$ such that for generic values of $a$,

$$Y(a - \epsilon_i; t) = P_i(a, t, \partial / \partial t_1, \ldots, \partial / \partial t_n)Y(a; t).$$

We give an algorithm for constructing $P_i$ and show that the coefficients in $\mathbb{Q}(a)$ appearing in $P_i$ have denominators which are products of linear factors involving the faces of codimension one of the cone associated with $Y$ in earlier work [1, 2, 3]. We give estimates for the degree of $P_i$ as a polynomial in $t$ and we describe the set of $a \in \mathbb{C}^m$ for which (1.3) is valid. Under an additional condition, which is satisfied by all the classically studied hypergeometric series, we bound the order of $P_i$ as a partial differential operator.

This problem has a lengthy history. The function $\gamma F_1$ had been treated by Gauss and Appell's $F_1$ had been treated by Lavasseur in his 1893 Paris thesis. Professor Kita has brought to our attention the recent works [5, 7]. The methods and scope of [5, 7] are quite different.

### 2. Exponential modules

Let $A^{(j)}$ be the $j$th column of the matrix $A$. We recall that, in earlier work, the polynomial

$$-g(t, x) = x_1 + \cdots + x_m + \sum_{j=1}^{n} t_j x^{A^{(j)}},$$

where $x^{A^{(j)}} = x_1^{A_{1j}} \cdots x_m^{A_{mj}}$, has been associated with $Y(a; t)$. Let $\Omega = \mathbb{Q}(a)$, $R' = \Omega(t)[x_1, x_1^{-1}, \ldots, x_m, x_m^{-1}]$, $E_{i} = x_i \partial / \partial x_i$ for $i = 1, \ldots, m$, and $g_i = E_{i}(g)$. Define operators on $R': D_{a,i,t} = E_{i} + a_{i} + g_{i}$ for $i = 1, \ldots, m$, and $\sigma_{j} = \partial / \partial t_j - x^{A^{(j)}} \partial / \partial t_j + \partial g / \partial t_j$ for $j = 1, \ldots, n$. The $D_{a,i,t}$ and $\sigma_{j}$ commute with one another for all $i$ and $j$. We define

$$\mathcal{W}_{a,t} = R' / \sum_{i=1}^{m} D_{a,i,t} R',$$

which is viewed as an $\mathcal{B}_1$-module, where $\mathcal{B}_1$ is the noncommutative ring $\Omega(t)[\sigma_1, \ldots, \sigma_n]$.

We make the hypothesis

$$a_i \notin \mathbb{N}^* \quad \text{for} \quad i = 1, \ldots, m.$$  

Let

$$R^* = \left\{ \sum_{u \in \mathbb{Z}^m} A_{u}(t)x^{-u} \mid A_{u}(t) \in \Omega[[t]] \right\}$$

and let $\xi_{a,t}^* \in R^*$ be defined by

$$\xi_{a,t}^* = \exp \left( -\sum_{j=1}^{n} t_j x^{A^{(j)}} \right) \cdot \sum_{u \in \mathbb{Z}^m} \prod_{i=1}^{m} (a_{i})_{u_{i}} x^{u}.$$
By a direct calculation, for \( u \in \mathbb{Z}^m \)

\[
(2.3) \quad \left( \prod_{i=1}^m (a_i u_i) \right) Y(a + u; t) = \langle \xi^*_{a,i,t}, x^u \rangle,
\]

where for \( \xi^* \in R^*, \xi \in R', \langle \xi^*, \xi \rangle \) is defined to be the coefficient of \( x^0 \) in the product \( \xi^* \xi \). In particular, taking \( u = 0 \) gives

\[
Y(a; t) = \langle \xi^*_{a,i,t}, 1 \rangle.
\]

For any \( \xi^* \in R^*, \xi \in R' \), one checks easily that

\[
\frac{\partial}{\partial t_j} \langle \xi^*_{a,i,t}, \xi \rangle = \langle \sigma_j^* (\xi^*_{a,i,t}), \xi \rangle + \langle \xi^*_{a,i,t}, \sigma_j(\xi) \rangle,
\]

where \( \sigma_j^* = \partial / \partial t_j + x A_i^j \). From (2.2) it follows that \( \sigma_j^* (\xi^*_{a,i,t}) = 0 \), hence

\[
(2.4) \quad \frac{\partial}{\partial t_j} \langle \xi^*_{a,i,t}, \xi \rangle = \langle \xi^*_{a,i,t}, \sigma_j(\xi) \rangle.
\]

Applying this with \( \xi = 1 \), we conclude that for \( P \in \mathbb{Q}(a)[t, Z_1, \ldots, Z_n] \)

\[
(2.5) \quad P(a, t, \partial / \partial t_1, \ldots, \partial / \partial t_n)Y(a; t) = P(a, t, \partial / \partial t_1, \ldots, \partial / \partial t_n) \langle \xi^*_{a,i,t}, 1 \rangle.
\]

Under the pairing \( \langle \ , \ \rangle \), the adjoint on \( R^* \) of the mapping \( D_{a,i,t} \) on \( R' \) is the mapping \( D^*_{a,i,t} = -E_i + a_i + g_i \). One checks that \( D^*_{a,i,t} (\xi^*_{a,i,t}) = 0 \) for \( i = 1, \ldots, m \). It follows that \( \xi^*_{a,i,t} \) annihilates \( \sum_{i=1}^m D_{a,i,t} R' \) under the pairing. Taking \( u = -\epsilon_i \) in (2.3) and comparing with (2.5) reduces the problem stated in the introduction to the problem of finding \( P_i \in \mathbb{R}_1 \) such that

\[
(2.6) \quad \frac{a_i - 1}{x_i} \equiv P_i(a, t, \sigma_1, \ldots, \sigma_n) \ (\text{mod} \ \sum_{i=1}^m D_{a,i,t} R').
\]

One then has

\[
(2.7) \quad P_i(a, t, \partial / \partial t_1, \ldots, \partial / \partial t_n)Y(a; t) = Y(a - \epsilon_i; t).
\]

Let \( \ell_i(t \sigma) = \sum_{j=1}^m A_{ij} t_j \sigma_j \). One checks from the definitions that

\[
a_i + \ell_i(t \sigma) - \ell_i(\delta) = -E_i + x_i + D_{a,i,t},
\]

hence for \( v \in \mathbb{Z}^m \)

\[
(a_i + \ell_i(t \sigma) + v_i)x^v \equiv x^{v + \epsilon_i} \ (\text{mod} \ \sum_{i=1}^m D_{a,i,t} R').
\]

One then proves by induction that for \( r \in \mathbb{N}^m \)

\[
(2.8) \quad x^r \equiv \prod_{i=1}^m (a_i + \ell_i(t \sigma))_r \ 1 \ (\text{mod} \ \sum_{i=1}^m D_{a,i,t} R').
\]
Let $\hat{H}_0$ be the monoid generated by $\epsilon_1, \ldots, \epsilon_m, A^{(1)}, \ldots, A^{(n)}$. If $u \in \hat{H}_0$, then $u = r + \sum_{j=1}^n s_j A^{(j)}$, where $r \in \mathbb{N}^m$, $(s_1, \ldots, s_n) \in \mathbb{N}^n$. One has trivially

$$(-\sigma_1)^{s_1} \cdots (-\sigma_n)^{s_n}(x^r) = x^r + \sum_{j=1}^n s_j A^{(j)} = x^u.$$ 

Hence by (2.8),

$$x^u \equiv (-\sigma_1)^{s_1} \cdots (-\sigma_n)^{s_n} \prod_{i=1}^m (a_i + \ell_i(t\sigma)) r_i 1 \pmod{\sum_{i=1}^m D_{a,i},, R'}.$$ 

Thus to find $P_i$ satisfying (2.6), it suffices to find a formula of the type

$$\prod_{j=1}^n \left( \frac{\partial}{\partial t_j} \right)^{s_j} \prod_{i=1}^m (a_i + \ell_i(\sigma)) r_i Y(a; t).$$

3. The contiguity algorithm

Let $\mathcal{C}$ be the cone in $\mathbb{R}^m$ determined by the monomials of $g$:

$$\mathcal{C} = \{ z \in \mathbb{R}^m \mid z = \sum_{i=1}^m r_i \epsilon_i + \sum_{j=1}^n s_j A^{(j)}, \text{ all } r_i, s_j \in [0, \infty) \}.$$ 

Let $\hat{H}_0 = \mathcal{C} \cap \mathbb{Z}^m$. We introduce $\hat{H}_0$ because it can be characterized by a system of linear inequalities.

**Lemma 3.1.** There exists $w \in \hat{H}_0$ such that $\hat{H}_0 + w \subseteq \hat{H}_0$. In particular, we may take $w = \sum_{i=1}^m T_i \epsilon_i$, where

$$T_i = \sup \left( 0, -1 + \sum_{j=1}^n \sup(0, -A_{ij}) \right).$$

**Remark.** For classical hypergeometric functions the matrices $A$ are made explicit in the appendix of [2] and it is not hard to check that in all these cases we have $\hat{H}_0 = \hat{H}_0$, i.e., one may take $w = 0$ in the classical examples.

**Proof.** If $v \in \hat{H}_0$ then $v = \sum_{i=1}^m r_i \epsilon_i + \sum_{j=1}^n s_j A^{(j)}$ where all $r_i$ and $s_j$ are nonnegative. Putting $r_i = \alpha_i + \alpha'_i$, $\alpha_i \in \mathbb{N}$, $\alpha'_i \in [0, 1)$, and $s_j = \beta_j + \beta'_j$, $\beta_j \in \mathbb{N}$, $\beta'_j \in [0, 1)$, we conclude that $v = u + \mu$, where $u \in \hat{H}_0$ and $\mu = \sum_{i=1}^m \alpha'_i \epsilon_i + \sum_{j=1}^n \beta'_j A^{(j)} \in \hat{H}_0$. Since $\mu$ lies in a bounded set, there are only a finite number of possibilities for $\mu$ and hence there exists $w \in \mathbb{N}^m$ such that $w + \mu \in \mathbb{N}^m \subseteq \hat{H}_0$ for all $\mu$. This shows the existence of $w$. To check our particular choice for $w$ it is enough to check that for all $i$, $\alpha'_i + \sum_{j=1}^n \beta'_j A_{ij} \in \mathbb{Z}$ implies that $\alpha'_i + \sum_{j=1}^n \beta'_j A_{ij} + T_i \geq 0$. This follows from the fact that if
\[ \sum_{j=1}^{n} \beta_j A_{ij} > \sum_{j=1}^{n} \inf(0, A_{ij}) = -1 - T_i. \]

We now recall that the cone \( \mathcal{C} \) may also be defined by linear inequalities. Let \( \tau_1, \ldots, \tau_p \) be the hyperplanes through the faces of \( \mathcal{C} \) of codimension one. Then for \( k = 1, \ldots, p \), \( \tau_k \) is defined by a linear form

\[ f_k(u) = \sum_{i=1}^{m} B_{ki} u_i, \]

where the \( B_{ki} \) are integers with greatest common divisor 1, \( f_k(u) = 0 \) is the equation of \( \tau_k \), and \( f_k(u) \geq 0 \) for all \( u \in \mathcal{C} \). Let us write

\[ f_k(D_a) = \sum_{i=1}^{m} B_{ki} D_{a,i}, \]
\[ f_k(g) = \sum_{i=1}^{m} B_{ki} g_i \]
\[ = -\sum_{i=1}^{m} x_i f_k(e_i) - \sum_{j=1}^{n} t_j x^{A^{(j)}} f_k(A^{(j)}) \]
\[ f_k(E) = \sum_{i=1}^{m} B_{ki} E_i. \]

The key point is that all monomials appearing in \( f_k(g) \) have exponents lying in the region \( f_k(u) \geq 1 \).

**Lemma 3.2.** For each \( v \in \mathbb{Z}^m \) there exists a representation

\[ x^v \equiv \sum_{u \in \hat{H}_0} c_{v,u} x^u \pmod{\sum_{i=1}^{m} D_{a,i,i} R'}, \]

where the sum on the right-hand side is finite and each \( c_{v,u} \in \mathbb{Q}(a)[t] \).

**Proof.** We use induction on \( N_v = \sum_{k=1}^{p} \sup(0, f_k(w - v)) \). If \( N_v = 0 \), then \( f_k(v - w) \geq 0 \) for all \( k \) which shows that \( v \in w + \hat{H}_0 \subseteq \hat{H}_0 \) by Lemma 3.1. Thus we may assume \( f_k(v - w) < 0 \) for some \( k \). We compute

\[ f_k(D_a)x^v = (f_k(a) + f_k(v))x^v + f_k(g)x^v, \]

and so

\[ x^v \equiv -\frac{1}{f_k(a) + f_k(v)} f_k(g)x^v \pmod{\sum_{i=1}^{m} D_{a,i,i} R'}. \]
Note that equation (3.3) remains valid under specialization of $a$ for all $a$ such that $f_k(a) + f_k(v) \neq 0$. As an immediate consequence of the proof, we have:

**Corollary 3.4.**

$$\deg c_{v,u} \leq N_v,$$

$$\left( \prod_{k=1}^{\rho} (f_k(a) + f_k(v))^{\sup(0,f_k(w-v))} \right) c_{v,u} \in \mathbb{Q}[a,t].$$

Specializing $v$ to $-\epsilon_i$ and combining Lemma 3.2 with (2.11), (2.3) and Corollary 3.4 gives:

**Theorem 3.7.** There exists $P_i \in \mathbb{Q}(a)[t, \partial/\partial t_1, \ldots, \partial/\partial t_n, \delta_1, \ldots, \delta_n]$ such that

$$Y(a-\epsilon_i; t) = P_i(a, t, \partial/\partial t_1, \ldots, \partial/\partial t_n, \delta_1, \ldots, \delta_n)Y(a; t).$$

As polynomials in $t$, the coefficients of $P_i$ have degree bounded by

$$N_{-\epsilon_i} = \sum_{k=1}^{\rho} \sup(0, f_k(w+\epsilon_i))$$

and $H_i(a)P_i$ has coefficients in $\mathbb{Q}[a,t]$, where

$$H_i(a) = \prod_{i=1}^{\rho} (f_k(a) + f_k(-\epsilon_i))^{\sup(0,f_k(w+\epsilon_i))}.$$ 

Thus this contiguity relation is valid provided $a_j \notin \mathbb{N}^\times$ for $j = 1, \ldots, m$ and $H_i(a) \neq 0$.

Of course, if one expresses $P_i$ as a polynomial in the $\partial/\partial t_i$'s only (i.e., replace $\delta_i$ by $t_i\partial/\partial t_i$), then the degrees of its coefficients as polynomials in $t$ change. These new degrees can be bounded by the methods of the next section, under the additional assumption that $\tilde{H}_0 = \tilde{H}_0$.

We believe that this theorem gives the basic set of contiguity relations. We observe that other contiguity relations may be deduced from

$$\langle \xi^*, a, D_{a,i}, tx^v \rangle = 0$$

for all $v \in \mathbb{Z}^m$, $i = 1, \ldots, m$, together with either (2.11) for $v \in \tilde{H}_0$ or (2.3) for arbitrary $v \in \mathbb{Z}^m$.

### 4. Bounding the Order of $P_i$

To bound the order of $P_i$ as a differential operator, we introduce some auxiliary functions. For $u \in \tilde{H}_0$, put

$$(4.1) \quad W(u) = \inf \left\{ \sum_{i=1}^{m} r_i + \sum_{j=1}^{n} s_j \mid u = \sum_{i=1}^{m} r_i \epsilon_i + \sum_{j=1}^{n} s_j A^{(j)}, \ r_i, s_j \in \mathbb{N} \text{ for all } i, j \right\}.$$
From (2.11) we see that the differential operator on \( Y(a; t) \) that corresponds to \( x^u \) (more precisely, that corresponds to a representation of \( u \) minimizing \( \sum_{i=1}^{m} r_i + \sum_{j=1}^{n} s_j \)) has order \( W(u) \). Thus the problem of bounding the order of the differential operator corresponding to \( x^v \), \( v \in \mathbb{Z}^m \), is reduced to the problem of bounding \( W(u) \) as \( u \) ranges over all terms with \( c_{v,u} \neq 0 \) on the right-hand side of Lemma 3.2. To accomplish this, we need to extend the definition of \( W \) to all \( u \in \mathbb{Z}^m \).

It is clear from (4.1) that if \( u_1, u_2 \in \hat{H}_0 \), then
\[
W(u_1 + u_2) \leq W(u_1) + W(u_2).
\]

Any \( u \in \mathbb{Z}^m \) can be written \( u = u_1 - u_2 \) with \( u_1, u_2 \in \hat{H}_0 \). If in addition \( u \in \hat{H}_0 \), then (4.2) implies \( W(u_1) - W(u_2) \leq W(u) \). Thus we may extend (4.1) by defining for \( u \in \mathbb{Z}^m \)
\[
W(u) = \sup \{ W(u_1) - W(u_2) \mid u = u_1 - u_2, \; u_1, u_2 \in \hat{H}_0 \}.
\]

Remark. We shall establish later that, under the hypothesis \( \hat{H}_0 = \hat{H}_0 \), \( W(u) < \infty \) for all \( u \in \mathbb{Z}^m \). This will show that our bound on the order of \( P_i \) is nontrivial.

Lemma 4.4. If \( u \in \mathbb{Z}^m \), \( u' \in \hat{H}_0 \), then
\[
W(u + u') \leq W(u) + W(u').
\]

Proof. Pick \( u_1, u_2 \in \hat{H}_0 \) such that \( u + u' = u_1 - u_2 \). By (4.2), \( W(u' + u_2) \leq W(u') + W(u_2) \), hence
\[
W(u_1) - W(u_2) \leq W(u_1) + W(u') - W(u' + u_2).
\]

But \( u_1 - (u' + u_2) = u \), so \( W(u_1) - W(u' + u_2) \leq W(u) \). This implies the lemma. \( \square \)

Proposition 4.5. For \( v \in \mathbb{Z}^m \), the partial differential operator corresponding to \( x^v \) under (2.11) and Lemma 3.2 has order \( \leq W(v) + N_v \).

Proof. The proof is by induction on \( N_v \). If \( N_v = 0 \), then as noted in the proof of Lemma 3.2, \( v \in \hat{H}_0 \). Suppose \( N_v > 0 \). By Lemma 4.4, \( f_k(g)x^v \) is a \( \mathbb{Z}[I] \)-linear combination of terms \( x^{v'} \) such that \( W(v') \leq W(v) + 1 \). Since \( N_{v'} \leq N_v - 1 \), we are done by (3.3) and the induction hypothesis. \( \square \)

Corollary 4.6. The partial differential operator \( P_i \) of Theorem 3.7 can be chosen to have order \( \leq W(-\epsilon_i) + N_{-\epsilon_i} \).

We now show this bound is nontrivial when \( \hat{H}_0 = \hat{H}_0 \). We introduce a function \( W' \) on \( \hat{H}_0 \) defined by
\[
W'(u) = \inf \left\{ \sum_{i=1}^{m} r_i + \sum_{j=1}^{n} s_j \mid u = \sum_{i=1}^{m} r_i \epsilon_i + \sum_{j=1}^{n} s_j A^{(j)}, \quad r_i, s_j \in [0, \infty) \text{ for all } i, j \right\}.
\]

Trivially, \( W'(u) \leq W(u) \) for all \( u \in \hat{H}_0 \). The function \( W' \) has a geometric interpretation: \( W'(u) \) is the smallest nonnegative real number such that \( u \in
$W'(u)\Delta$, where $\Delta$ is the convex hull of the points $\epsilon_1, \ldots, \epsilon_m, A^{(1)}, \ldots, A^{(n)}$ and the origin and $W'(u)\Delta$ is the dilation of $\Delta$ by the factor $W'(u)$. Let $\lambda_1, \ldots, \lambda_p$ be linear forms defining the codimension-one faces of $\Delta$ that do not contain the origin. We assume them to be normalized so that the corresponding codimension-one face lies in the hyperplane $\lambda_i(u) = 1$ for $i = 1, \ldots, p$. This determines the $\lambda_i$'s uniquely. Then for $u \in \tilde{H}_0$,

(4.8) \hspace{1cm} W'(u) = \sup\{\lambda_i(u) \mid i = 1, \ldots, p\}.

Lemma 4.9. Fix $u \in \tilde{H}_0$. Then the set $\{W'(u + u_1) - W'(u_1) \mid u_1 \in \tilde{H}_0\}$ is bounded above and below.

Proof. From the definition of $W'$ it is clear that

$$W'(u + u_1) \leq W'(u) + W'(u_1),$$

hence the given set is bounded above by $W'(u)$. By (4.8) we may choose $i_1, i_2 \in \{1, \ldots, p\}$ such that

$$W'(u + u_1) = \lambda_{i_1}(u + u_1), \hspace{0.5cm} W'(u_1) = \lambda_{i_2}(u_1).$$

Then $(\lambda_{i_1} - \lambda_{i_2})(u_1) \leq 0$ but $(\lambda_{i_1} - \lambda_{i_2})(u + u_1) \geq 0$, hence there exists $\alpha \in [0, 1]$ such that

(4.10) \hspace{1cm} (\lambda_{i_1} - \lambda_{i_2})(\alpha u + u_1) = 0.

Then

$$W'(u + u_1) - W'(u_1) = \lambda_{i_1}(u + u_1) - \lambda_{i_2}(u_1) = (1 - \alpha)\lambda_{i_1}(u) + \alpha \lambda_{i_2}(u)$$

by (4.10). This latter quantity is clearly bounded above and below independently of $u_1$. \Box

Lemma 4.11. Suppose $\tilde{H}_0 = \tilde{H}_0$. There exists a positive constant $\kappa$ such that for all $u \in \tilde{H}_0$,

$$W(u) \leq W'(u) + \kappa.$$

Proof. Let $u \in \tilde{H}_0$. Choose $r_i, s_j \in [0, \infty)$ such that

$$u = \sum_{i=1}^{m} r_i \epsilon_i + \sum_{j=1}^{n} s_j A^{(j)} \hspace{1cm} \text{and} \hspace{1cm} W'(u) = \sum_{i=1}^{m} r_i + \sum_{j=1}^{n} s_j.$$

Put $[u] = \sum_{i=1}^{m} r_i \epsilon_i + \sum_{j=1}^{n} s_j A^{(j)} \in \tilde{H}_0$. Then $u - [u] = \mu \in \mathbb{Z}^m \cap \mathbb{Z}^n = \tilde{H}_0$. Furthermore, $\mu$ lies in a bounded (hence finite) subset of $\tilde{H}_0$. Let $\kappa$ be the maximum value of $W$ on this finite subset. Now $u = [u] + \mu$, hence

$$W(u) \leq \sum_{i=1}^{m} [r_i] + \sum_{j=1}^{n} [s_j] + \kappa \leq W'(u) + \kappa. \Box$$

Proposition 4.12. Suppose $\tilde{H}_0 = \tilde{H}_0$. Then $W(u) < \infty$ for all $u \in \mathbb{Z}^m$. 

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Proof. Write \( u = u_1 - u_2 \) with \( u_1, u_2 \in \tilde{H}_0 \). By Lemma 4.4, \( W(u) \leq W(u_1) + W(-u_2) \). Thus it suffices to show \( W(-u) < \infty \) for all \( u \in \tilde{H}_0 \). So suppose \( -u = u_1 - u_2 \) with \( u, u_1, u_2 \in \tilde{H}_0 \). Then \( u_2 = u_1 + u \), so

\[
W(u_1) - W(u_2) = W(u_1) - W(u_1 + u) \\
\leq W'(u_1) + \kappa - W'(u_1 + u)
\]

by Lemma 4.11. By Lemma 4.9, this quantity is bounded above independently of \( u_1 \). \qed

5. Examples

Consider the classical Gaussian hypergeometric function

\[
_{2}F_{1}(\alpha, \beta, \gamma; t) = \sum_{s=0}^{\infty} \frac{(\alpha)_{s}(\beta)_{s}}{(\gamma)_{s}s!} t^{s}.
\]

Using the relation \((\gamma)_{s}(1 - \gamma)_{-s} = (-1)^{s}\) we have

\[
_{2}F_{1}(\alpha, \beta, \gamma; t) = \sum_{s=0}^{\infty} \frac{(\alpha)_{s}(\beta)_{s}(1 - \gamma)_{-s}}{s!} \frac{(-t)^{s}}{s!}
\]

so \( a = (a_1, a_2, a_3) = (\alpha, \beta, 1 - \gamma) \) and \( \ell_1(s) = \ell_2(s) = s, \ell_3(s) = -s \). This corresponds to

\[
-g(t, x) = x_1 + x_2 + x_3 + t \frac{x_1 x_2}{x_3}.
\]

The codimension-one faces of the corresponding cone \( C \) are given by the forms \( f_1(u) = u_2 + u_3, f_2(u) = u_1 + u_3, f_3(u) = u_1, f_4(u) = u_2 \) and one checks that \( \tilde{H}_0 = \tilde{H}_0 \), hence \( w = 0 \) in Lemma 3.1. One has \( f_1(-\varepsilon_1), f_4(-\varepsilon_1) \geq 0 \) but \( f_2(-\varepsilon_1) = f_3(-\varepsilon_1) = -1 \). Following the algorithm described in the proof of Lemma 3.2 by first applying \( f_2(D_a) \) to \( 1/x_1 \) and then applying \( f_3(D_a) \) to \( x_3/x_1 \) gives

\[
\frac{\alpha - 1}{x_1} \equiv \frac{\alpha - 1 + x_3 + tx_2}{\alpha - \gamma} \mod \sum_{i=1}^{3} D_{a, i, t} R',
\]

By (2.9),

\[
x_3 \equiv (1 - \gamma - t\sigma)1 \mod \sum_{i=1}^{3} D_{a, i, t} R',
\]

\[
x_2 \equiv (\beta + t\sigma)1 \mod \sum_{i=1}^{3} D_{a, i, t} R',
\]

so

\[
\frac{\alpha - 1}{x_1} \equiv \frac{\alpha - \gamma - t\sigma + t(\beta + t\sigma)1}{\alpha - \gamma} \mod \sum_{i=1}^{3} D_{a, i, t} R'.
\]
From (2.7) we get

\[(\gamma - \alpha) {}_2F_1(\alpha - 1, \beta, \gamma; t) = (t(1 - t) \frac{\partial}{\partial t} + (\gamma - \alpha - \beta t)) {}_2F_1(\alpha, \beta, \gamma; t),\]

a well-known classical formula (see [4, section 2.8, equation (23)] or [6, Chapter VI, section 24]).

We give some details for the calculation of the contiguity relations for the Lauricella series \( F_A \), which we write in the form

\[Y(a; t) = \sum_{s \in \mathbb{N}^n} c(s) \frac{t^s}{s_1! \cdots s_n!},\]

where

\[c(s) = (a_{2n+1})_{s_1+\cdots+s_n} \prod_{i=1}^n (a_i + n)_{s_i} \prod_{i=1}^n (1 - a_i)^{s_i}.\]

The associated polynomial [2] is

\[-g = x_1 + \cdots + x_{2n+1} + \sum_{j=1}^n t_j x_{2n+1} \frac{x_{n+j}}{x_j}.\]

There are \( 2^n + 2n \) linear forms which define the associated cone:

\[f_j(u) = u_{n+j} \quad (j = 1, \ldots, n),\]
\[f_{n+j}(u) = u_{n+j} + u_j \quad (j = 1, \ldots, n),\]

and for each subset \( S \) of \( \{1, \ldots, n\} \) the form

\[f_S(u) = u_{2n+1} + \sum_{j \in S} u_j.\]

Clearly these forms are nonnegative on the cone of \( g \). We omit the proof that these forms define the cone.

We give some of the calculations for \( Y(a - \epsilon_1; t) \). By applying \( f_{n+1}(D_a) \) to \( 1/x_1 \) we obtain

\[(a_{n+1} + a_1 - 1) \frac{1}{x_1} \equiv (x_1 + x_{n+1}) \frac{1}{x_1} = 1 + \frac{x_{n+1}}{x_1}.\]

Letting \( S = \{1\} \) and applying \( f_S(D_a) \) to \( x_{n+1}/x_1 \) we obtain

\[(a_{2n+1} + a_1 - 1) \frac{x_{n+1}}{x_1} \equiv x_{n+1} (x_1 + x_{2n+1} + \sum_{j=2}^n t_j x_{2n+1} \frac{x_{n+j}}{x_j}).\]

Letting \( y_l = x_{2n+1} x_{n+l}/x_l, \ l = 1, \ldots, n \), this becomes

\[(a_{2n+1} + a_1 - 1) \frac{x_{n+1}}{x_1} \equiv x_{n+1} + y_1 + \sum_{j=2}^n t_j y_1 \frac{x_{n+j}}{x_j}.\]
Thus we are reduced to the problem of reducing $y_1 \cdots y_{l-1} x_{n+l} / x_l$ modulo $\sum_{i=1}^{2n+1} D_{a,i} R'$. Applying $f_S(D_a)$ with $S = \{1, \ldots, l\}$ we obtain

\[(a_{2n+1} + a_1 + \cdots + a_l - 1)y_1 \cdots y_{l-1} \frac{x_{n+l}}{x_l} \equiv \bigg(x_1 + \cdots + x_l + x_{2n+1} + \sum_{j=l+1}^{n} t_j y_j\bigg)y_1 \cdots y_{l-1} \frac{x_{n+l}}{x_l} \]

\[= y_1 \cdots y_l + \sum_{i=1}^{l} y_1 \cdots \hat{y}_i \cdots y_l x_{n+i} + \sum_{j=l+1}^{n} t_j y_1 \cdots y_l x_{n+j} / x_j.\]

By iteration we arrive at a representation of $1/x_1$ as a polynomial in $x_1, \ldots, x_{2n+1}, y_1, \ldots, y_n$ with coefficients in $\mathbb{Q}(\alpha)[[t]]$. The number of steps is quite large since $f_{n+1}(-\epsilon_1) = -1$ and $f_S(-\epsilon_1) = -1$ for every subset $S$ of $\{1, \ldots, n\}$ that contains 1 (thus $N_{-\epsilon_1} = 1 + 2^{n-1}$).

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