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DISTINGUISHED KÄHLER METRICS ON HIRZEBRUCH SURFACES

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ABSTRACT. Let $\mathfrak{F}_n$ be a Hirzebruch surface, $n \geq 1$. Using the family of extremal metrics on these surfaces constructed by Calabi [1], we study a closely related scale-invariant variational problem, and show that only $\mathfrak{F}_1$ admits an extremal Kähler metric which is critical for this new functional. Applying a result of Derdzinski [3], we prove that this metric cannot be conformally equivalent to an Einstein metric on $\mathfrak{F}_1$. When $n = 2$, we show there is a critical orbifold metric on the space obtained from $\mathfrak{F}_2$ by blowing down the negative section.

1. INTRODUCTION

Let (M, J) be a *polarized compact complex manifold* of dimension n. That is to say, the complex manifold (M, J) is equipped with a positive class $\Omega \in H^{1,1}(M, \mathbb{R}) \subset H^2(M, \mathbb{R})$ which can be represented by the Kähler form of some Kähler metric. We denote by Ω^+ the set of all Kähler forms representing this cohomology class. In an attempt to represent the given class by a canonical metric, Calabi [1], [2] proposed that one should search for critical points of the functional

$$\begin{aligned} \Phi_\Omega \colon \Omega^+ &\to \mathbb{R} \\ \omega &\mapsto \int_M \mathbf{s}_\omega^2 d\mu_\omega, \end{aligned} \tag{1}$$

where the metric associated with the form ω has scalar curvature $\mathbf{s}_\omega$ and volume form $d\mu_\omega$. He called these critical points *extremal Kähler metrics*, and then observed that a Kähler metric is extremal iff the gradient of its scalar curvature $\mathbf{s}$ is a real-holomorphic vector field.

The functional Φ_Ω is scale invariant when $n = 2$. In this article, we study instead a closely related functional which has this property for any n. Indeed, let g be a Kähler metric on (M, J) with Kähler form ω. Recall that in terms of the Ricci tensor $\mathbf{r}$ of g, the *Ricci form* ρ is defined by $\rho(X, Y) = \mathbf{r}(JX, Y)$. This form is, up to a constant, the curvature of the anti-canonical

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line bundle. Let $\mathbf{c}_1$ be the *first Chern class* of M. Since

$$d\mu_\omega = \frac{\omega^n}{n!}, \quad \mathbf{s}\omega^n = n\rho \wedge \omega^{n-1},$$

the volume and total scalar curvature,

$$\begin{aligned} \mathbf{v} = \mathbf{v}_{[\omega]} &= \int_M d\mu_\omega = \frac{[\omega]^n}{n!}, \\ \bar{\mathbf{s}} = \bar{\mathbf{s}}_{[\omega],J} &= \int_M \mathbf{s}_\omega d\mu_\omega = \frac{2\pi}{(n-1)!}\mathbf{c}_1 \cup [\omega]^{\cup(n-1)}, \end{aligned} \tag{2}$$

only depend on the Kähler class $[\omega]$ and the complex structure J. Evidently, $\Phi_{[\omega]}$ is bounded below by $\bar{\mathbf{s}}^2/\mathbf{v}$, and this bound is attained precisely when $[\omega]$ is represented by a metric of constant scalar curvature. For Kähler metrics with nonzero total scalar curvature, consider the functional

$$\omega \mapsto \Psi(\omega) = \frac{\Phi_{[\omega]}(\omega)\mathbf{v}_{[\omega]}}{\mathbf{s}^2_{[\omega],J}} - 1. \tag{3}$$

The functional Ψ is clearly scale-invariant and bounded below by 0. This bound is achieved precisely when ω has constant scalar curvature. In some sense, the minimum value of Ψ on a fixed Kähler class measures the extent to which the class fails to contain a constant scalar curvature representative.

By (2) the critical points of Ψ, restricted to representatives of a fixed Kähler class Ω, are identical to the critical points of Φ_Ω. In other words, a critical metric for Ψ must be, *a fortiori*, an extremal Kähler metric. This shows part of the following.

Proposition 1. *The critical points of* (3) *are extremal Kähler metrics* g *whose Kähler form* ω *satisfies the relation*

$$\bar{\mathbf{s}}_{[\omega]} \int_M (\mathbf{s}\rho, \alpha) d\mu_\omega = \Phi_{[\omega]}(\omega) \int_M (\rho, \alpha) d\mu_\omega \tag{4}$$

for all real g*-harmonic* $(1,1)$*-forms* α. *Here,* $(\cdot,\cdot)$ *is the pairing of* 2*-forms induced by* g, *and so, by the harmonicity of* α, (ω, α) *is a constant.*

Remark. The relation indicated in the proposition always holds if $\mathbf{s}_\omega$ is constant. Also, it always holds when $\alpha = \omega$. Therefore, an extremal metric g is critical for (3) if it satisfies $h^{1,1}(M) - 1$ additional relations as in (4), $h^{1,1}(M)$ being the rank of $H^{1,1}(M, \mathbb{R})$.

Proof. Let g be a critical point with Kähler form ω. Then g is extremal. Since all deformations which change the Kähler class are parametrized by the harmonic $(1,1)$-forms, we take one such form α and consider the deformation of g given by $\omega(t) = \omega + t\alpha$. Then

$$\begin{aligned} d\mu(t) &= (1 + t(\omega, \alpha))d\mu + O(t^2), \\ \rho(t) &= \rho - i\partial\bar{\partial}(\omega, \alpha) + O(t^2), \\ \mathbf{s}(t) &= \mathbf{s} - t(\rho, \alpha) + O(t^2). \end{aligned}$$

Computing the t derivative of $\Psi(\omega(t))$ at $t = 0$, and equating it to zero, we see that

$$\bar{\mathbf{s}}_{[\omega]}(\dot{\Phi}_{[\omega]}\bar{\mathbf{v}}_{[\omega]} + \Phi_{[\omega]}\dot{\bar{\mathbf{v}}}_{[\omega]}) = 2\dot{\bar{\mathbf{s}}}_{[\omega]}\Phi_{[\omega]}\bar{\mathbf{v}}_{[\omega]}. \tag{5}$$

The result is obtained using the expressions above to conclude that

$$\dot{\bar{\mathbf{v}}}_{[\omega]} = (\omega, \alpha)\bar{\mathbf{v}}_{[\omega]},$$

$$\dot{\bar{\mathbf{s}}}_{[\omega]} = (\omega, \alpha)\bar{\mathbf{s}}_{[\omega]} - \int (\rho, \alpha) d\mu_\omega,$$

$$\dot{\Phi}_{[\omega]} = (\omega, \alpha)\Phi_{[\omega]} - 2\int (s\rho, \alpha) d\mu_\omega,$$

and substituting into (5). □

Before proceeding any further, we make an important observation, which we shall not exploit in this article. Let ω be a critical metric for Ψ, and $\Xi = \partial^\# \mathbf{s} = \partial^\#(\mathbf{s} - \bar{\mathbf{s}}_0)$, where $\partial^\# \mathbf{s} = (\bar{\partial}\mathbf{s})^\#$ is the type $(1, 0)$ piece of the gradient of $\mathbf{s}$ with respect to the metric, and $\bar{\mathbf{s}}_0$ is the projection of $\mathbf{s}$ onto the constants. Since ω is extremal, Ξ is a holomorphic vector field and one can compute the Futaki character [2] of Ξ for the class $[\omega]$. This number, $\mathscr{F}(\Xi, [\omega])$, changes [7] when the class $[\omega]$ varies. The expression

$$\int_M ((\mathbf{s} - \bar{\mathbf{s}}_0)\rho, \alpha) d\mu_\omega$$

is precisely the t-derivative of $\mathscr{F}(\Xi, [\omega + t\alpha])$ evaluated at $t = 0$. Thus, the linearization of the Futaki character appears in (4) in a way which is reminiscent of its *nondegeneracy*, a condition discovered in [7] to play a crucial rôle in the analysis of extremal metrics under deformations of the complex structure.

We study critical points of Ψ for the Hirzebruch surfaces. In this case, the complex dimension is 2, and the index n is used to parametrize the surfaces under consideration, instead of the dimension. In other words, for an integer $n \geq 0$, let $\mathfrak{L}$ be the total space of the line bundle $\mathscr{O}_{\mathbb{P}^1}(-n)$ over $\mathbb{P}^1$. Then Hirzebruch surface $\mathfrak{F}_n$ is the $\mathbb{P}^1$-bundle

$$\mathfrak{F}_n = \mathbb{P}(\mathfrak{L} \oplus \mathscr{O}).$$

The surface $\mathfrak{F}_0$ is $\mathbb{P}_1 \times \mathbb{P}_1$, while $\mathfrak{F}_1$ is $\mathbb{P}_2$ blown-up at one point, and for higher values of n, $\mathfrak{F}_n$ is diffeomorphic (but of course not biholomorphic) to $\mathfrak{F}_0$ or $\mathfrak{F}_1$ when n is even or odd, respectively. Thus, $H^2(\mathfrak{F}_n, \mathbb{R}) = \mathbb{R} \oplus \mathbb{R}$, and since they are all Kähler manifolds, we have $H^2(\mathfrak{F}_n, \mathbb{R}) = H^{1,1}(\mathfrak{F}_n, \mathbb{R}) = \mathbb{R} \oplus \mathbb{R}$. As proven by Calabi [1], for $n \geq 1$ each $\mathfrak{F}_n$ admits a family of extremal Kähler metrics, and up to the action of the connected group of automorphisms, there is exactly one in each Kähler class. None of these metrics has constant scalar curvature, and none of them has zero total scalar curvature. Since $h^{1,1}(\mathfrak{F}_n) = 2$, an extremal metric must satisfy one additional relation for it to be a critical point of (3). To describe this condition in further detail, we recall Calabi's metrics on $\mathfrak{F}_n$ in a way amenable to calculations.

2. Extremal metrics on $\mathfrak{F}_n$

The construction described here is due to Koiso and Sakane [6], see also [4]. We sketch the general construction, then apply it to the Hirzebruch surfaces.

Let (B, g_B) be a Kähler-Einstein manifold with positive curvature, and assume the Kähler form ω_B of g_B is integral. For each integer n, there is a Hermitian holomorphic line bundle $p\colon (\mathfrak{L}, h) \to (B, g_B)$ with $c_1(\mathfrak{L}, h) = n\omega_B$.

Let $\mathfrak{L}^0$ denote the complement of the zero section of $\mathfrak{L}$, regarded as the total space of a holomorphic $\mathbb{C}^\times$-bundle over B, and let $\widehat{\mathfrak{L}} = \mathbb{P}(\mathfrak{L} \oplus \mathscr{O})$ be the associated projective line bundle. There is a natural $\mathbb{C}^\times$-action on $\mathbb{P}(\mathfrak{L} \oplus \mathscr{O})$. Decompose $\mathbb{C}^\times = \mathbb{R}^+ \times \mathbb{S}^1$, and let H and $S = JH$ be the associated holomorphic vector fields.

Let $\varphi\colon [-1, 1] \to [0, \infty)$ be a smooth function vanishing exactly at ± 1, and satisfying $\varphi'(\pm 1) = \mp 2$. Define smooth functions $u\colon (0, \infty) \to (-1, 1)$ and $t\colon (0, \infty) \to (0, R)$, where $R = \int_{-1}^{1} dx/\sqrt{\varphi(x)}$, by

$$\log r = \int_0^{u(r)} \frac{dx}{\varphi(x)}, \quad t(r) = \int_{-1}^{u(r)} \frac{dx}{\sqrt{\varphi(x)}}.$$

Precomposing with the Hermitian norm function $s\colon \mathfrak{L}^0 \to (0, \infty)$, we transfer these functions to $\mathfrak{L}^0$, still denoting them by u and t.

Proposition 2. *The metric*

$$g = dt^2 + (dt \circ J)^2 + (a - nu)p^* g_B \tag{6}$$

is Kähler on $\mathfrak{L}^0$. If φ extends smoothly to an open neighborhood of $[-1, 1]$, then the metric g extends to a smooth Kähler metric, also denoted by g, on $\widehat{\mathfrak{L}}$.

Proof. See, for example, [6], [4]. Note that when $\varphi(x) = 1 - x^2$, the fibers of $\widehat{\mathfrak{L}}$ are round spheres. □

Remark. An entirely analogous construction works over a product of positive Einstein-Kähler manifolds. Under certain conditions, one can extend metrics on $\mathfrak{L}^0$ to compactifications other than the "natural" one.

We now specialize to the case where (B, g_B) is $\mathbb{P}^1$ with the integral Fubini-Study metric and $\mathfrak{L}$ is the total space of $\mathscr{O}_{\mathbb{P}^1}(-n)$ for a positive integer n, so that $\widehat{\mathfrak{L}} = \mathfrak{F}_n$. Fix $a > n \geq 1$. We discuss the geometry of the metric g on $\mathfrak{F}_n$ constructed above.

Note that $\mathrm{Pic}(\mathfrak{F}_n) = \mathbb{Z} \oplus \mathbb{Z}$ is generated by Poincaré duals of the negative section C (i.e., the divisor C with $C \cdot C = -n$) and the fiber F. If we let ω be the Kähler form of g, we have

$$\int_C \omega = a - n, \qquad \int_F \omega = 4\pi. \tag{7}$$

By varying $a > n$, it is clear that every Kähler class (up to scaling) is represented by such a metric.

The function t is the distance to the negative section of $\mathfrak{F}_n$, the function u is the moment map for the $\mathbb{S}^1$-action, and $\varphi(u) = g(H, H)$. Each regular level set of u is the total space of the principal $\mathbb{S}^1$-bundle over $\mathbb{P}^1$ inducing $\mathfrak{L}$ whose fiber has length $2\pi(\varphi(u))^{1/2}$, and the restriction of p is a Riemannian submersion to the metric $(a - nu)g_{\mathbb{P}^1}$ on $\mathbb{P}^1$. Topologically, each level set is a lens space $\mathbb{S}^3/\mathbb{Z}_n$. By integrating over level sets of u, one obtains the following.

Lemma 1. *If $f\colon [-1, 1] \to \mathbb{R}$ is any smooth function, then*

$$\int_{\mathfrak{F}_n} f(u)\, d\mu_g = 2\pi'' \int_{-1}^{1} f(x)(a - nx)\, dx. \tag{8}$$

Proposition 3. *Choose coordinates z^0, z^1 such that z^0 is a fiber coordinate with $\partial/\partial z^0 = H - \sqrt{-1}S$ and z^1 is a coordinate on $\mathbb{P}^1$. On a fiber where*

$\partial u/\partial z^1 = 0$, the components of the metric, the Hessian of a function $f(u)$, and the Ricci tensor, are given by

$$g_{0\bar{0}} = 2\varphi(u), \quad g_{0\bar{1}} = 0, \quad g_{1\bar{1}} = (a - nu)g_B, \tag{9}$$

$$f(u)_{0\bar{0}} = \varphi(\varphi f')'(u), \quad f(u)_{0\bar{1}} = 0, \quad f(u)_{1\bar{1}} = \frac{n}{2}(\varphi f')(u)g_B, \tag{10}$$

$$\begin{aligned} r_{0\bar{0}} &= -\varphi\left(\varphi' + \frac{Q'}{Q}\varphi\right)'(u), \quad r_{0\bar{1}} = 0, \\ r_{1\bar{1}} &= \left(2 + \frac{n}{2}\varphi(\log\varphi Q)'(u)\right)g_B, \end{aligned} \tag{11}$$

where $Q(x) = a - nx$. In particular, the scalar curvature is given by

$$\mathrm{s}(u) = \frac{1}{Q(u)}\left(2 - \frac{1}{2}(\varphi Q)''(u)\right). \tag{12}$$

Proof. See [6] for the first three assertions. To compute the scalar curvature, take the trace of r with respect to g:

$$\begin{aligned} \mathrm{s}(u) &= -\frac{1}{2}\left(\varphi' + \frac{Q'}{Q}\varphi\right)'(u) + \frac{1}{a - nu}\left(2 + \frac{Q'}{Q}\varphi(\log\varphi Q)'(u)\right) \\ &= \frac{2}{Q(u)} - \frac{1}{2}\left(\varphi' + \frac{Q'}{Q}\varphi\right)'(u) + \frac{1}{2}\frac{q'(\varphi Q)'}{Q^2}(u) \\ &= \frac{1}{Q(u)}\left(2 - \frac{1}{2}(\varphi Q)''(u)\right) \end{aligned}$$

as claimed. □

It follows that for a function f as in Lemma 1, we have

$$\operatorname{grad} f(u) = f'(u)H, \quad \Delta f(u) = \frac{(\varphi Q f')'}{2Q}(u). \tag{13}$$

The vector field $\operatorname{grad} f(u)$ is globally holomorphic precisely when $f(u) = \alpha + \beta u$, so the Euler-Lagrange equation for extremal metrics may be stated as follows.

Proposition 4. *The metric* (6) *is extremal if and only if the scalar curvature is of the form $\alpha + \beta u$.*

For any positive integer n and any real number $a > n \geq 1$, there exists a function φ as in Proposition 2 whose scalar curvature is an affine function of u. In fact, take

$$\varphi(x) = \frac{2}{Q(x)}\left(Q(-1)(x+1) - \int_{-1}^{x}(Q(y)(\alpha + \beta y) - 2)(x - y)\,dy\right), \tag{14}$$

where α and β are chosen so that $\varphi(1) = 0$ and $\varphi'(1) = -2$, the conditions $\varphi(-1) = 0$ and $\varphi'(-1) = 2$ being automatic.

Theorem 1. *For any integer $n \geq 1$ and real number $a > n$, consider the function* (14) *with*

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{3}{3a^2 - n^2} \begin{pmatrix} a^2 + 2a - n^2 \\ -2n(a-1) \end{pmatrix}. \tag{15}$$

Then the metric (6) *on $\mathfrak{F}_n$ is an extremal Kähler metric. In each Kähler class, there is exactly one such representative.*

Proof. A simple count shows that $(\varphi Q)''$ does not have enough zeroes for φ to become negative. It only remains to calculate α and β in terms of a and n. If we multiply equation (12) by $a - nx$, by Lemma 1 we obtain

$$\int_{-1}^{1} (\alpha + \beta x)(a - nx)\, dx = \int_{\mathfrak{F}_n} \mathbf{s}(u) d\mu_g = 2(a+2).$$

Similarly,

$$\int_{-1}^{1} x(\alpha + \beta x)(a - nx)\, dx = \int_{\mathfrak{F}_n} u\mathbf{s}(u) d\mu_g = -2n.$$

Carrying out the integrations and solving for α and β leads to (15). □

These metrics on $\mathfrak{F}_n$ were originally constructed by Calabi [1]. Although they can be reproduced in various ways, the method used here is very convenient for our purpose of analyzing critical points of the functional Ψ in (3). Before doing that, we quickly recall a result of Derdzinski [3], which we shall use. Let (M, J, g) be a Kähler manifold of complex dimension 2, and nonconstant scalar curvature $\mathbf{s}$. If for some positive function k the metric kg is an Einstein metric, then, up to a constant, one must have $k = \mathbf{s}^{-2}$. Thus, one can ask when is $\mathbf{s}^{-2}g$ an Einstein metric. This is so if, and only if, the metric g is extremal and

$$8\mathbf{s}^3 + 48\mathbf{s}\Delta\mathbf{s} - 48|d\mathbf{s}|^2 = \text{ constant}. \tag{16}$$

In this expression, $\mathbf{s}$ stands for the scalar curvature as used in this paper, which is half of the usual scalar curvature in Riemannian geometry, and Δ is the positive complex Laplacian. That is why the coefficients in (16) differ from those in the analogous equation in [3].

3. Critical metrics

For the metric on $\mathfrak{F}_n$ defined in Theorem 1, we have

Proposition 5. *Let ω be the Kähler form of the metric on $\mathfrak{F}_n$ of parameter a constructed above. Then*

$$\Psi(\omega) = \frac{4(n(a-1))^2}{(a+2)^2(3a^2 - n^2)}. \tag{17}$$

Such a metric is a critical point for Ψ iff a is a real root of the polynomial

$$p(a) = a^3 - 2a^2 - 2a + n^2, \tag{18}$$

satisfying the condition $a > n \geq 1$.

Proof. We use Proposition 4 and Lemma 1 to conclude that $\mathbf{v}_{[\omega]} = 2a$, $\mathbf{s}_{[\omega],J} = 2(a+2)$ and $\Phi_{[\omega]}(\omega) = 2\alpha^2 a - 4\alpha\beta n/3 + 2\beta^2 a/3$. The result follows substituting these into the definition (3) of $\Psi(\omega)$. □

We are now ready to prove our main result.

Theorem 2. *If $n > 2$, the Hirzebruch surfaces $\mathfrak{F}_n$ do not admit critical points for the functional* (3). *When $n = 2$, there exists a critical point which is an orbifold metric on the space obtained from $\mathfrak{F}_2$ by blowing down the negative section. Away from the negative section, this metric is conformal to a (non-Kähler) Einstein metric. When $n = 1$, there is exactly one critical metric, which cannot be conformally deformed to an Einstein metric on $\mathfrak{F}_1$.*

Proof. When $n \geq 3$, the polynomial (18) has only one real root, which is negative. For $n = 2$, the roots are $a = 2$ and $a = \pm\sqrt{2}$. The first of these roots corresponds to an orbifold metric on the space obtained from $\mathfrak{F}_2$ by blowing down the negative section, which evidently is a critical point for the functional Ψ. Finally, when $n = 1$ the roots are $a = -1$ and $a = (3 \pm \sqrt{5})/2$. In this case, the root $a = (3+\sqrt{5})/2$ is greater than 1, and it corresponds to an extremal metric on $\mathfrak{F}_1$, unique up to scaling and automorphisms.

In order to prove the remaining statements, we use Derdzinski characterization (16), performing an explicit calculation. So let g be the metric of parameter a on $\mathfrak{F}_n$, exhibited in Theorem 1. We have that $\mathbf{s} = \alpha + \beta u$, where α, β are given by (15). Using (13), we conclude that $\Delta\mathbf{s} = \beta(\varphi Q)'/2Q$, evaluated at u. Also, $|d\mathbf{s}|^2 = \beta^2\varphi(u)$. By direct calculation we find that

$$\begin{aligned}\varphi Q = {} & \frac{n\beta}{6}x^4 + \frac{n\alpha - a\beta}{3}x^3 + (2 - a\alpha)x^2 \\ & + \left(4 + 2a - 2a\alpha + 2n + a\beta - n\alpha + \frac{2n\beta}{3}\right)x \\ & + 2 + 2n + 2a + \frac{2a\beta}{3} - a\alpha + \frac{n\beta}{2} - \frac{2n\alpha}{3}.\end{aligned}$$

Hence, for the extremal metric on $\mathfrak{F}_n$ with parameter $a > n$, we have that

$$8\mathbf{s}^3 + 48\mathbf{s}\Delta\mathbf{s} - 48|d\mathbf{s}|^2 = 4\frac{c_1 x + c_0}{Q(x)}|_{x=u},$$

where the coefficients c_1 and c_0 are given by

$$\begin{aligned}c_1 &= 24\alpha\beta - 2n\alpha^3 - 6a\alpha^2\beta - 6\beta^2\left(4 + 2a - 2a\alpha + 2n + a\beta - n\alpha + \frac{2n\beta}{3}\right), \\ c_0 &= 2a\alpha^3 + 6\alpha\beta\left(4 + 2a - 2a\alpha + 2n + a\beta - n\alpha + \frac{2n\beta}{3}\right) \\ &\quad - 12\beta^2\left(2 + 2n + 2a + \frac{2a\beta}{3} - a\alpha + \frac{n\beta}{2} - \frac{2n\alpha}{3}\right).\end{aligned}$$

If we specialize to the case where $a = n = 2$, we find that $\alpha = 3/2 = -\beta$, and

$$\varphi(x) = (3 - x)(1 - x^2)/4, \quad \mathbf{s}^3 = (3/2)^3(1 - x)^3|_{x=u},$$
$$c_1 = c_0 = 0.$$

Obviously, the scalar curvature vanishes on the negative section, that is to say, $\mathbf{s} = 0$ when $u = 1$. Thus, the claim about the orbifold metric on the blow-down of $\mathfrak{F}_2$ along the negative section follows from the Derdzinski condition.

Consider now the case where $n = 1$. As functions of a, the coefficients c_1 and c_2 in the expression above are rational functions of the form

$$q_1(a)/(3a^2 - n^2)^3 \quad \text{and} \quad q_2(a)/(3a^2 - n^2)^3,$$

where

$$q_1(a) = 54(5a^6 - 12a^5 - 33a^4 + 24a^3 + 31a^2 - 28a + 5),$$

and

$$q_2(a) = 54(a^7 + 6a^6 - 9a^5 + 50a^4 - 37a^3 - 30a^2 + 37a - 10),$$

respectively. For the Derdzinski condition to hold, we must have $c_1 = -\lambda$ and $c_0 = \lambda a$ for some constant λ. Calculating the quotient c_0/c_1, and equating the result to $-a$, we get that

$$-a = \frac{-10 + 37a - 30a^2 - 37a^3 + 50a^4 - 9a^5 + 6a^6 + a^7}{5 - 28a + 31a^2 + 24a^3 - 33a^4 - 12a^5 + 5a^6},$$

which leads to the polynomial equation

$$\begin{aligned} q(a) &= 6a^7 - 6a^6 - 42a^5 + 74a^4 - 6a^3 - 58a^2 + 42a - 10 \\ &= 2(a-1)^2(a+1)(3a^4 - 18a^2 + 16a - 5) = 0. \end{aligned}$$

There is only one real root of $q(a)$ which is greater than one (this root lies in between 1.91 and 1.92). Hence, the Derdzinski condition isolates one and only one extremal metric on $\mathfrak{F}_1$ which can be conformally deformed to an Einstein metric. This metric is not the critical metric on $\mathfrak{F}_1$ corresponding to the parameter $a = (3+\sqrt{5})/2$, as $q((3+\sqrt{5})/2) \neq 0$. Thus, our critical metric cannot be conformally deformed to an Einstein metric. Observe, also, that the root $a = 1$ of the polynomial $q(a)$ corresponds to the well known Einstein metric on $\mathbb{P}^2$, metric obtained from the one on $\mathfrak{F}_1$ by blowing-down the -1 curve. □

Our main result implies the existence of Kähler manifolds with critical points of (3) which are not Kähler metrics of constant scalar curvature. Higher-dimensional examples could be constructed by analogous means.

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