A NOTE ON THE PROBLEM OF PRESCRIBING
GAUSSIAN CURVATURE ON SURFACES

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Abstract. The problem of existence of conformal metrics with Gaussian curvature equal to a given function $K$ on a compact Riemannian 2-manifold $M$ of negative Euler characteristic is studied. Let $K_0$ be any nonconstant function on $M$ with $\max K_0 = 0$, and let $K_\lambda = K_0 + \lambda$. It is proved that there exists a $\lambda^* > 0$ such that the problem has a solution for $K = K_\lambda$ iff $\lambda \in (-\infty, \lambda^*)$. Moreover, if $\lambda \in (0, \lambda^*)$, then the problem has at least 2 solutions.

Let $M$ be a closed 2-dimensional smooth manifold and $g$ be a Riemannian metric on $M$. Let $k$ denote the Gaussian curvature of $g$. If $g' = e^{2u}g$ is another Riemannian metric conformal to $g$, and has Gaussian curvature $k'$, then it is well known that

$$k' = e^{-2u}(k - \Delta u),$$

where $\Delta$ is the Laplacian of $g$. Given a function $K \in C^\infty(M)$, the problem of prescribing Gaussian curvature asks whether one can find $u \in C^\infty(M)$ such that the metric $g' = e^{2u}g$ has the given $K$ as its Gaussian curvature. Obviously, this is equivalent to the problem of solvability of the following elliptic equation

$$\Delta u - k + Ke^{2u} = 0, \quad \text{on} \, M.$$

If $u$ is a solution of (1), then we have by integrating (1)

$$\int_M Ke^{2u}dv = \int_M kdv,$$

where $dv$ is the area element with respect to the metric $g$. It follows from the Gauss-Bonnet formula that

$$\int_M ke^{2u}dv = 2\pi \chi(M),$$

where $\chi(M)$ is the Euler characteristic of $M$. Note that (2) poses restrictions on the given function $K$ for the solvability of (1), according to the sign of $\chi(M)$.

If $\chi(M) = 0$, the problem of the solvability of (1) has been completely resolved. (See [K-W].) If $\chi(M) > 0$, then $M$ is either $RP^2$ (the real projective...
plane) or $S^2$ (the 2-sphere). While the case where $M = RP^2$ has been well understood (see [M], [A]), the case where $M = S^2$ is much more complicated. Many authors have studied the problem on $S^2$ with its standard metric, known as Nirenberg problem (see e.g. [C-D], [C-Y1, 2], [C-L]).

In this note we consider only the case where $\chi(M) < 0$; in other words,

$$\int_M k d\nu < 0.$$ 

This case has been studied by Kazdan and Warner in [K-W] using the method of super- and sub-solutions for second order elliptic equations. The following are some facts derived by them.

**Fact (i).** One can always find an arbitrarily negative subsolution $\varphi$ for equation (1). Indeed, such a subsolution can be of the form $\varphi_c = f - c$, where $f$ is a solution to the equation $\Delta f = k - \bar{k}$ with $\bar{k}$ being the mean value of $k$, and $c$ is any sufficiently large number. Therefore, to solve (1) one needs only to find a supersolution $\psi$ for (1).

**Fact (ii).** Let $K_1 \geq K_2$ are two smooth functions on $M$. Suppose that (1) has a solution $u_1$ for $K = K_1$. Then, since $u_1$ is a supersolution for (1) with $K = K_2$ as can be easily checked, we see that (1) is solvable for $K = K_2$ by Fact (i).

**Fact (iii).** It is easy to see from (2) that a necessary condition for (1) to be solvable is that the function $K$ is negative somewhere on $M$. On the other hand, if $K \leq 0$, then one can find a supersolution for (1). It follows from Fact (i) that (1) has a solution provided $K \leq 0$. Moreover, in such a case, one can show that the solutions of (1) are unique.

In view of Fact (iii), we are only interested in the case where the function $K$ changes sign. From now on, we assume that $K_0 \in C^\infty(M)$ is a nonconstant function which satisfies

$$\text{Max}_{x \in M} K_0(x) = 0$$

and let $K_\lambda = K_0 + \lambda$, where $\lambda$ is a real number. Consider the family of equations

$$(1)_\lambda \quad \Delta u - k + K_\lambda e^{2u} = 0.$$ 

By Fact (iii), $(1)_\lambda$ has a unique solution $u_\lambda$ for $\lambda \leq 0$. On the other hand, for the solution $u_0$ of $(1)_0$, the variational equation

$$\Delta v + 2K_0 e^{2u_0} v = 0$$

has only a trivial solution $v \equiv 0$, since $K_0 \leq 0$ and $K_0 \not= 0$. It follows from the implicit function theorem that $(1)_\lambda$ has a solution for sufficiently small $\lambda > 0$. So we have

**Lemma 1.** There exists a $\lambda^* > 0$ such that $(1)_\lambda$ is solvable for all $\lambda < \lambda^*$, and it has no solutions for $\lambda > \lambda^*$.

**Proof.** Let $\lambda^*$ be the supremum of all $\lambda$ for which $(1)_\lambda$ has a solution. We have known that $\lambda^* < 0$, and $\lambda^* < - \inf_M K_0$ by (iii). It follows from Fact (ii) that $\lambda^*$ has the claimed property.

Our main result is as follows.
Theorem. Let $K_0 \in C^\infty(M)$ be any nonconstant function satisfying (3), and let $K_\lambda = K_0 + \lambda$. Then there exists a $\lambda^* > 0$ such that (a) $(1)_{\lambda}$ has a unique solution for $\lambda \leq 0$; (b) $(1)_{\lambda}$ has at least two solutions if $0 < \lambda < \lambda^*$; and (c) $(1)_{\lambda^*}$ has at least one solution.

Remark. If we set
$$S = \{ K \in C^\infty(M) : (1) \text{ is solvable} \},$$
then the Theorem implies that the set $S \cup \{0\}$ is closed in $C^0$ topology. Indeed, let $\{K_i\} \subset S$ be a sequence such that $K_i \rightarrow K \in C^\infty(M) \setminus \{0\}$. Then for any $\varepsilon > 0$ we can find $K_i$ such that $K - \varepsilon \leq K_i$, and this shows that $K - \varepsilon \in S$ for any $\varepsilon > 0$. It follows from (c) of the Theorem that $K \in S$.

Now we turn to the proof of the Theorem. It is clear that conclusion (a) follows from Fact (iii). Hence we need only prove (b) and (c).

Proof of (b) of the Theorem. Note that $(1)_{\lambda}$ is the Euler-Lagrange equation of the functional
$$I_{\lambda}(u) = \int_M (|\nabla u|^2 + 2k u - K_{\lambda} e^{2u}) \, dv.$$
We are to apply variational methods (see [C]) to obtain multiple critical points for $I_{\lambda}$, which correspond to solutions of $(1)_{\lambda}$, for $\lambda \in (0, \lambda^*)$. Fixing any $\lambda \in (0, \lambda^*)$, we choose a $\lambda_1 \in (\lambda, \lambda^*)$. Let $\psi$ be a solution of $(1)_{\lambda_1}$. Then $\psi$ is a super-solution for the equation $(1)_{\lambda}$. By Fact (i), we can find a sub-solution $\varphi$ for $(1)_{\lambda}$ such that $\varphi \leq \psi$ on $M$. Let $[\varphi, \psi]$ be the order interval defined by
$$[\varphi, \psi] = \{ v \in C^1(M) : \varphi \leq v \leq \psi \text{ on } M \}.$$ The ordinary super- and sub-solution method asserts that $(1)_{\lambda}$ has a solution $u_{\lambda} \in [\varphi, \psi]$. Further variational considerations as in [C] permits one to assume that $u_{\lambda}$ is $I_{\lambda}$-minimizing in the interval $[\varphi, \psi]$, i.e.,

(4) $I_{\lambda}(u_{\lambda}) = \inf \{ I_{\lambda}(v) : v \in [\varphi, \psi] \}.$

Next, we note that there exist functions $w \in C^1(M)$ such that $I_{\lambda}(w) < I_{\lambda}(u_{\lambda})$. Indeed, since $\lambda > 0$, the set $M_\varepsilon = \{ x \in M : K_{\lambda}(x) > \varepsilon \}$ for small $\varepsilon > 0$ is nonempty and open. Let $f \in X$ be any function which is positive in $M_\varepsilon$ and vanishes on $M \setminus M_\varepsilon$. Then

$$I_{\lambda}(tf) = t^2 \int_M |\nabla f|^2 \, dv + t \int_M K_{\lambda} d\psi - \int_{M_\varepsilon} K_{\lambda} e^{2tf} \, dv - \int_{M \setminus M_\varepsilon} K_{\lambda} d\psi$$
$$\leq At^2 + Bt + C - \varepsilon \int_{M_\varepsilon} e^{2tf} \, dv$$
$$\leq At^2 + Bt + C - a \varepsilon e^{2ta^{-1}} \int_{M_\varepsilon} f \, dv \rightarrow -\infty, \quad \text{as } t \rightarrow \infty,$$
where $a$ is the area of $M_\varepsilon$. Thus, we may take $w = tf$ with $t$ big enough. Now, if the functional satisfies the Palais-Smale condition, a result of K. C. Chang [C] asserts the existence of a mountain-pass critical point $v_{\lambda}$ other than $u_{\lambda}$. The fact that $I_{\lambda}$ does satisfy the Palais-Smale condition is proved in the next lemma. This completes the proof of (b).
Lemma 2. Assume that the set $M_- = \{x \in M: K_1(x) < 0\}$ is nonempty. Then the functional $I_x$ satisfies the Palais-Smale condition in the function space $X = W^{1,2}(M)$. That is to say, if $\{u_k\}$ is any sequence in $X$ such that $I_x(u_k) \to c$ for some $c \in \mathbb{R}$ and $I_x'(u_k) \to 0$ in $X^*$ (the dual space of $X$), then a subsequence of $\{u_k\}$ converges in $X$.

Proof. Let $\{u_k\}$ be the sequence in the lemma. Then we have

$$I_x(u_k) = \int_M (|\nabla u_k|^2 + 2k u_k - K_x e^{2u_k}) \, dv \to c,$$

and

$$I_x'(u_k)(\varphi) = \int_M (\nabla u_k \cdot \nabla \varphi + k \varphi - K_x e^{2u_k} \varphi) \, dv = o(\|\varphi\|), \quad \forall \varphi \in X,$$

where $\|\cdot\|$ is the norm of $X$. Let $u^+_k = \max\{u_k, 0\}$. We claim that $\{u^+_k\}$ is locally $W^{1,2}$-bounded in the open set $M_-$. More precisely, we will prove that for any domain $\Omega \subset M_-$ with $\text{dist}(\Omega, \partial M_-) = d(\Omega) > 0$, we have $\|u^+_k\|_{W^{1,2}(\Omega)} \leq C$, where the constant $C$ depends only on $d(\Omega)$. To see that our claim holds it suffices to show that for any $p \in M_-$ with $\text{dist}(p, \partial M_-) = d$, we have

$$\int_{B_d/4} (|\nabla u^+_k|^2 + (u^+_k)^2) \, dv \leq C,$$

where $B_r$ denote the geodesic ball centered at $p$ with radius $r > 0$, and the constant $C > 0$ depends only on the distance $d$. To prove (7), let $\eta$ be a smooth cut-off function supported in $B_{d/2} = B_{d/2}(p)$, such that $\eta(x) = 1$ for $x \in B_{d/4}$, $\eta(x) = 0$ for $x \in M \setminus B_{d/2}$ and $0 \leq \eta \leq 1$, $|\nabla \eta| \leq Ad^{-1}$ on $M$. Substituting $\eta u_k$ in (6) we get

$$\int_{B_{d/2}} (\nabla u^+_k \cdot \nabla (\eta^2 u^+_k) + k \eta^2 u^+_k - K_x e^{2u^+_k} \eta^2 u^+_k) \, dv \leq C \|\eta^2 u^+_k\| \leq C \|\eta u^+_k\|.$$

Here and in the sequel we use $C$ to denote various constants depending only on $d$. Using

$$\nabla u^+_k \cdot \nabla (\eta^2 u^+_k) = |\nabla (\eta u^+_k)|^2 + |\nabla \eta|^2 (u^+_k)^2,$$

$K_x \leq -\epsilon$ in $B_{d/2}$ for some $\epsilon > 0$, and $e^{2t} \geq t^3$ for $t \in \mathbb{R}$,

we derive from (8) that

$$\int_{B_{d/2}} (|\nabla (\eta u^+_k)|^2 + \epsilon \eta^2 (u^+_k)^4) \, dv \leq - \int_{B_{d/2}} k \eta^2 u^+_k \, dv + C \|\eta u^+_k\|.$$

Since $(u^+_k)^4 > (u^+_k)^2 - 1$, it is easy to see from the above inequality that

$$\epsilon \|\eta u^+_k\|^2 \leq C \|\eta u^+_k\| + C.$$

From this it follows that $\|\eta u^+_k\| \leq C$, and consequently (7) holds since $\eta \equiv 1$ in $B_{d/4}$. Next, letting $\varphi \equiv 1$ in (6) we have

$$\int_M K_x e^{2u_k} \, dv - \int_M k \, dv \to 0, \quad \text{as } k \to \infty.$$
Combining with (5), this gives that

\[
\int_M (|\nabla u|^2 + 2k u_k) \, dv = I_\lambda (u_k) - \int_M k \alpha e^{2u_k} \, dv \to c - 2\pi \chi(M),
\]

as \( k \to \infty \).

Now we claim that \( \{ u_k \} \) is bounded in \( L^2(M) \). If the claim is true, then (10) implies that \( \{ u_k \} \) is also bounded in \( X = W^{1,2}(M) \). By passing to a subsequence if necessary we may assume that \( u_k \) converge weakly in \( X \) to some \( u_0 \). Then it is standard to show that \( u_k \) actually converge strongly in \( X \) using (6) and the fact that \( e^{2u_k} \to e^{2u_0} \) in \( L^p(M) \) for any \( p \geq 1 \). (Note that \( \dim M = 2 \).) This will finish our proof of Lemma 2.

To prove our claim we assume that on the contrary, \( \| u_k \|_{L^2(M)} \to \infty \) and consider \( v_k = u_k/\| u_k \|_{L^2} \), which satisfy \( \| v_k \|_{L^2} = 1 \) for all \( k \). We see from (10) that

\[
\int_M |\nabla v_k|^2 \, dv = -2\int_M k \frac{v_k}{u_k} \, dv + o(1) \to 0.
\]

It follows that \( v_k \) converges in \( X \) to some constant function \( v \equiv \beta \). Since \( \| v \|_{L^2} = 1 \) we have \( \beta \neq 0 \). Note that (10) also implies that

\[
\int_M k v_k \, dv \leq C \| u_k \|_{L^2}^{-1}.
\]

Taking the limit we get that

\[
\int_M \beta k \, dv = 2\pi \chi(M) \beta \leq 0.
\]

Since \( \beta \) is nonzero and \( \chi(M) < 0 \), we must have \( \beta > 0 \). Now, consider \( v_k^+ = u_k/\| u_k \|_{L^2} \). The above discussion shows that \( v_k^+ \) converge to \( \beta > 0 \) almost everywhere in \( M \). However, as we have proved, \( u_k^+ \) is locally \( W^{1,2} \)-bounded in \( M_\ast \), which implies that \( v_k^+ \) converge to 0 almost everywhere in \( M_\ast \), a contradiction! This completes our proof of Lemma 2.

We now turn to

**Proof of (c) of the Theorem.** We are to prove that \( (1)_\lambda \) has a solution. This will be proven by showing that certain solutions of \( (1)_\lambda \) converge in \( X \) as \( \lambda \to \lambda^* \).

We have seen in the proof of (b) that for \( \lambda < \lambda^* \), \( (1)_\lambda \) has a solution \( u_\lambda \) which is \( I_\lambda \)-minimizing in an order interval \( [\varphi, \psi] \) in \( C^1(M) \) (see (4)). By the maximum principle, we must have \( \varphi < u_\lambda < \psi \). This implies that \( u_\lambda \) is a local minima for \( I_\lambda \) in \( C^1(M) \). It follows that the second variation of \( I_\lambda \) at \( u_\lambda \) is nonnegative, i.e.,

\[
\int_M (|\nabla \varphi|^2 - 2K_\lambda e^{2u_\lambda} \varphi^2) \, dv \geq 0,
\]

where \( \varphi \in C^1(M) \). We also note that there is a \( C > 0 \) such that for \( \lambda \in (0, \lambda^*) \)

\[
u_\lambda \geq -C, \quad \text{on} \ M.
\]

Actually, let \( \varphi_c = f - c \) be the family of functions in Fact (i). Then for \( c \geq \) some \( c_0 \), \( \varphi_c \) is a continuous family of subsolutions for \( (1)_0 \), hence it is also a continuous family of subsolutions for \( (1)_\lambda \), where \( \lambda \in (0, \lambda^*) \). We claim that
$u_\lambda \geq \varphi_{c_0}$, and consequently (12) holds. For otherwise, by varying $c \in [c_0, \infty)$, we find that for some $c$ we have

$$u_\lambda \geq \varphi_c \text{ on } M, \text{ and } u_\lambda(x_0) = \varphi_c(x_0) \text{ for some } x_0 \in M.$$ 

This, by the maximum principle, can occur only if $u_\lambda \equiv \varphi_c$, which is impossible. So we see that (12) holds.

The crucial point of this proof is to show that $u_\lambda$ is uniformly bounded in $X$ as $\lambda \to \lambda^*$. If this is true, then by elliptic $L^p$-estimate for the solutions of (1), we see that $u_\lambda$ is uniformly bounded in $W^{2,p}(M)$ for any $p > 1$. The Sobolev imbedding theorem together with Schauder estimates then imply that $u_\lambda$ is uniformly $C^{2,\alpha}$-bounded. It follows that some subsequence of $u_\lambda$ converges in $C^2$ to a solution of $\lambda^*$. This will complete our proof. We now proceed to prove the $W^{1,2}$-boundedness of $u_\lambda$. To this end we need to use the conformal invariance of equation (1). Note that $u_\lambda$ being a solution of (1), is equivalent to the Gaussian curvature of $g_\lambda = e^{2u_\lambda} g$ being $K_\lambda$. If $g' = e^{2v} g$ is any metric conformal to $g$, then we have $g_\lambda = e^{2(u_\lambda - v)} g'$. This means that the function $w_\lambda = u_\lambda - v$ solves

$$(13) \quad \Delta g' w - k_{g'} + K_\lambda e^{2w} = 0,$$

where $\Delta g'$ and $k_{g'}$ are respectively the Laplacian and Gaussian curvature of $g'$.

Claim. The set $M^- = \{ x \in M : K_\lambda(x) < 0 \}$ is nonempty. We choose $g'$ in (13) to be the uniqueness metric $g_0 = e^{2v_0} g$ which has constant curvature $k_0 = -1$, where $v_0$ is the unique solution of $\Delta v - k - e^{2v} = 0$. Then $w_\lambda = u_\lambda - v_0$ is a solution of

$$(14) \quad \Delta_0 w + 1 + K_\lambda e^{2w} = 0.$$ 

Here and in the sequel, by the subscript $0$ we mean that the corresponding geometric objects are for the metric $g_0$. Multiplying (14) by $e^{-2w_0}$ and integrating over $M$ we get

$$\int_M K_\lambda dv_0 = - \int_M (2|\nabla w_\lambda|^2_0 + 1)e^{-2w_0} dv_0.$$ 

Letting $\lambda \to \lambda^*$ we see that $\int K_{\lambda^*} \leq 0$. If the Claim is false then we must have $K_{\lambda^*} \geq 0$, and consequently $K_{\lambda^*} \equiv 0$. This contradicts our assumption that $K_\lambda$ are nonconstant for all $\lambda$, showing that the Claim is true.

Now, let $h$ be a smooth function which vanishes outside an open set $D$ such that $\overline{D} \subset M_+$ and $h < 0$ in $D$. As in the proof of (b) of the Theorem, one may derive that $u_\lambda^+ \geq 0$, and consequently $K_{\lambda^+} \equiv 0$. This contradicts our assumption that $K_\lambda$ are nonconstant for all $\lambda$, showing that the Claim is true.

Next, let $g_1 = e^{2v_1} g$ be the metric with Gaussian curvature $h$, where $v_1$ is the unique solution of the equation $\Delta v - k + h e^{2v} = 0$. Then the function $w_\lambda = u_\lambda - v_1$ satisfies the equation

$$\Delta_1 w_\lambda - h + K_\lambda e^{2w_\lambda} = 0.$$
Since $\Delta_1 = e^{-2w_1} \Delta$, we have

\begin{equation}
\Delta w_\lambda - h e^{2w_1} + K \lambda e^{2(w_\lambda + v_\lambda)} = 0.
\end{equation}

Multiplying (16) by $e^{2w_\lambda}$ and integrating over $M$ gives

\begin{equation}
2 \int_M \left| \nabla e^{w_\lambda} \right|^2 dv + \int_M h e^{2w_1} e^{2w_\lambda} dv - \int_M K \lambda e^{2w_1} e^{4w_\lambda} dv = 0.
\end{equation}

On the other hand, letting $\varphi = e^{w_\lambda}$ in (11) we have

\begin{equation}
\int_M \left| \nabla e^{w_\lambda} \right|^2 dv - 2 \int_M K \lambda e^{2w_1} e^{4w_\lambda} \geq 0.
\end{equation}

Together with (17) this gives

\begin{equation}
\int_M \left| \nabla e^{w_\lambda} \right|^2 dv \leq -\frac{2}{3} \int_M h e^{2(w_\lambda + v_\lambda)} dv = -\frac{2}{3} \int_D h e^{2w_1} dv.
\end{equation}

Thus, by (15), $\left| \nabla e^{w_\lambda} \right|$ is uniformly bounded in $L^2(M)$. We claim that $\|e^{w_\lambda}\|_{L^2(M)}$ is uniformly bounded too, consequently $e^{w_\lambda}$ is uniformly bounded in $X$. In fact, if this is not true, we may assume that $\|e^{w_\lambda}\|_{L^2} \to \infty$ as $\lambda \to \lambda^*$. Set

\begin{equation}
v_\lambda = e^{w_\lambda}/\|e^{w_\lambda}\|_{L^2}.
\end{equation}

Then we have

\begin{equation}
\|v_\lambda\|_{L^2} = 1, \quad \text{and} \quad \|\nabla v_\lambda\|_{L^2} \to 0.
\end{equation}

It follows that $v_\lambda$ converges in $X$ to a constant function $v$ with $\|v\|_{L^2} = 1$. However, (15) implies that $\|v_\lambda\|_{L^2(D)} \to 0$ as $\lambda \to \lambda^*$, and hence $v \equiv 0$ in $D$. But, $v$ is constant on $M$, so $v \equiv 0$ on $M$, contradicting $\|v\|_{L^2} = 1$. This proves that $e^{w_\lambda}$ and also $e^{u_\lambda}$ are uniformly bounded in $L^2$. Actually, $e^{u_\lambda}$ is uniformly $L^p$-bounded for any $p > 1$ since it is bounded in $X$.

Now we observe that since $u_\lambda$ is bounded below by (12), the $L^p$-boundedness of $e^{u_\lambda}$ implies the $L^p$-boundedness of $u_\lambda$. Therefore, the elliptic $L^p$ and Schauder estimates for the solutions of (1)$_\lambda$ lead to a uniform $C^{2,\alpha}$-bound for $u_\lambda$. It follows that some subsequence of $u_\lambda$ converges in $C^2$ to a solution of (1)$_{\lambda^*}$. This completes the proof of (c) of the Theorem.

References


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