FUNCTIONS WITH BOUNDED SPECTRUM

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Abstract. Let $0 < p < \infty$, $f(x) \in L^p(\mathbb{R}^n)$, and $\text{supp } Ff$ be bounded, where $F$ is the Fourier transform. We will prove in this paper that the sequence $\|D^\alpha f\|_p^{1/|\alpha|}$, $\alpha \geq 0$, has the same behavior as the sequence $\sup_{\xi \in \text{supp } Ff} |\xi^\alpha|^{1/|\alpha|}$, $\alpha \geq 0$. In other words, if we know all "far points" of $\text{supp } Ff$, we can wholly describe this behavior without any concrete calculation of $\|D^\alpha f\|_p$, $\alpha \geq 0$. A Paley-Wiener-Schwartz theorem for a nonconvex case, which is a consequence of the result, is given.

1. Introduction

The following result showing a relation between behavior of the sequence of norms of derivatives of a function and the support of its Fourier transform [2] has been proved: Let $1 < p < \infty$ and $D^m f(x) \in L^p(\mathbb{R}^1)$, $m = 0, 1, \ldots$. Then there always exists the limit

$$d_f = \lim_{m \to \infty} \|D^m f\|_p^{1/m}$$

and moreover

$$d_f = \sup \{ |\xi| : \xi \in \text{supp } \hat{f} \},$$

where $\hat{f}(\xi) = F f(\xi)$ is the Fourier transform of the function $f(x)$.

This result is of value in the theory of Sobolev spaces of infinite order, in particular, in studying imbedding theorems for Sobolev spaces of infinite order [3–5].

The question arises as to what happens for the $n$-dimensional case? In this paper we give a complete answer to this question. It should be noted that here we do not assume any restriction on geometrical properties of $\text{supp } \hat{f}$ (which is called the spectrum of $f$).

We will use the following standard notation: $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_+$; $D = (D_1, \ldots, D_n)$; $D_j = \frac{\partial}{\partial x_j}$, $j = 1, \ldots, n$; $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$; $\text{sp}(f) = \text{supp } \hat{f}$.

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And we presuppose that $0^0 = 0 = 1$, $\frac{1}{0} = \infty$ for $\lambda > 0$, $f(x) \in \mathcal{S}'$, and $f(x) \neq 0$.

2. Results

We will show the following

**Theorem 1.** Let $0 < p \leq \infty$, $f(x) \in L_p(\mathbb{R}^n)$ and $\text{sp}(f)$ be bounded. Then

\[
\lim_{|\alpha| \to \infty} \left( \frac{\|D^\alpha f\|_p}{\sup_{\text{sp}(f)} |\xi^\alpha|} \right)^{1/|\alpha|} = 1.
\]

To prove Theorem 1 we need the following results: Let $0 < p < q < \infty$ and $K \subset \mathbb{R}^n$ be compact. Denote by $M_{K_p}$ the class of all functions in $\mathcal{S}' \cap L_p(\mathbb{R}^n)$ such that $\text{sp}(f) \subset K$. The following Nikolsky inequality is well known ([(9), [10, p. 125]): There exists a constant $C(p, q, K)$ such that

\[
\|f\|_q \leq C(p, q, K)\|f\|_p
\]

for all $f \in M_{K_p}$.

It follows from the Nikolsky inequality that $M_{K_p} \subset M_{K_\infty}$, $0 < p \leq \infty$.

Further, let $G$ be a domain in $\mathbb{R}^n$ and $m \in \mathbb{Z}_+$. Denote by $W_{m, 2}(G)$ the classical Sobolev space, i.e., the completion $C^m(G)$ with respect to the norm

\[
\|f\|_{m, 2} = \left( \sum_{|\alpha| \leq m} \|D^\alpha f\|_{L^2(G)} \right)^{1/2}.
\]

And $W_{m, 2}^0(G)$ is the subspace of all functions $f(x) \in W_{m, 2}(G)$ such that the zero extension of $f(x)$ outside $G$ belongs to $W_{m, 2}(\mathbb{R}^n)$. For $s \in \mathbb{R}$, we put

\[
H(s) = \{ f \in \mathcal{S}' : \|f\|_{(s)} = \left( \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |F f(\xi)|^2 d\xi \right)^{1/2} < \infty \}.
\]

Then

\[
H(k) = W_{k, 2}(\mathbb{R}^n)
\]

(topological imbedding)

if $k \in \mathbb{Z}_+$ (see, for example, [1, p. 45; 6, p. 53; 7, 7.9.1]).

**Proof of Theorem 1.** We divide the proof into three cases.

Case 1 ($1 \leq p < \infty$). We first establish the following inequality

\[
\lim_{|\alpha| \to \infty} \left( \frac{\|D^\alpha f\|_p}{|\xi^\alpha|} \right)^{1/|\alpha|} \geq 1
\]

for any point $\xi \in \text{sp}(f)$.

Actually, let $\xi^0 \in \text{sp}(f)$, $\xi_j^0 \neq 0$, $j = 1, \ldots, n$. (It is easy to show later that there exist such points because of $p < \infty$.) For the sake of convenience, we assume that $\xi_j^0 > 0$, $j = 1, \ldots, n$. We fix a number $0 < \epsilon < \frac{1}{\min_{1 \leq j \leq n} \xi_j^0}$ and choose a domain $G$ with a smooth boundary such that $\xi^0 \in G$ and $G \subset \{ \xi : \xi_j^0 - \epsilon \leq \xi_j \leq \xi_j^0 + \epsilon, j = 1, \ldots, n \}$. Further we fix a function $\hat{\vartheta}(\xi) \in C^\infty_0(G)$ such that $\xi^0 \in \text{supp}(\hat{\vartheta} \hat{f})$. Then

\[
(\hat{\vartheta}(\xi) \hat{f}(\xi), \hat{\vartheta}(\xi)) = (f(x), \varphi(x)),
\]

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where \( \bar{w}(\xi) \in C_0^\infty(G) \) is an arbitrary function, \( \varphi(x) = \bar{v} * \bar{w}(x) \), and \( \bar{u}(x) = u(-x) \). The distribution \( \bar{v}(\xi)\bar{f}(\xi) \) has a compact support; therefore, it can be represented in the form

\[
\bar{v}(\xi)\bar{f}(\xi) = \sum_{|\alpha| \leq m} D^\alpha h_\alpha(\xi),
\]

where \( m \) is a nonnegative integer and \( h_\alpha(\xi) \) are ordinary functions in \( G \). Without loss of generality we may assume that \( m \geq 2n \).

It is well known that the Dirichlet problem for the elliptic differential equation

\[
L_2m \tilde{z}(\xi) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha (D_m^\alpha \tilde{z}(\xi)) = \bar{v}(\xi)\bar{f}(\xi)
\]

has a (unique) solution \( \tilde{z}(\xi) \in W^0_{m,2}(G) \) (see, for example, [6, p. 82]). Since (3), we obtain

(4) \( \langle \tilde{z}(\xi), L_2m \bar{w}(\xi) \rangle = \langle f(x), \varphi(x) \rangle \)

for all \( \bar{w}(\xi) \in C_0^\infty(G) \). The left side of (4) admits a closure up to an arbitrary function \( \bar{w}(\xi) \in W^0_{m,2}(G) \). Hence, replacing \( \bar{w}(\xi) \) by \( \xi^\alpha \bar{w}(\xi) \), we get

(5) \( \langle \tilde{z}(\xi), L_2m(\xi^\alpha \bar{w}(\xi)) \rangle = \langle -i|\alpha| \langle D^\alpha f(x), \varphi(x) \rangle \)

for all \( \bar{w}(\xi) \in W^0_{m,2}(G) \).

Now let \( \bar{w}_0(\xi) \in W^0_{m,2}(G) \) be the solution of the equation \( L_2m \bar{w}_0(\xi) = \bar{z}(\xi) \). Since \( 0 \notin G \), we get

\[
L_2m(\xi^\alpha \bar{w}_0(\xi)) = \prod_{j=1}^n (\xi_j^0 - 2\epsilon)^{\alpha_j} \bar{z}(\xi),
\]

where \( \bar{w}_\alpha(\xi) = \prod_{j=1}^n (\xi_j^0 - 2\epsilon)^{\alpha_j} \xi^{-\alpha} \bar{w}_0(\xi) \) and \( \alpha \geq 0 \). Therefore, it follows from (5) that

(6) \( \prod_{j=1}^n (\xi_j^0 - 2\epsilon)^{\alpha_j} \langle \tilde{z}(\xi), \bar{z}(\xi) \rangle \leq ||D^\alpha f||p ||v||1 ||w_\alpha||q \),

where \( 1/p + 1/q = 1 \).

On the other hand, there exists a constant \( C > 0 \) such that

(7) \( ||v||1 ||w_\alpha||q \leq C, \quad \alpha \geq 0 \).

Indeed, let \( |\beta| \leq 2n \). Using

\[
x^\beta w_\alpha(x) = (-i)^{|\beta|} \prod_{j=1}^n (\xi_j^0 - 2\epsilon)^{\alpha_j} \int_G e^{ix^\xi} D^\beta(\xi^{-\alpha} \bar{w}_0(\xi)) d\xi,
\]

the Leibniz formula, and the definition of \( G \), we get

\[
\sup \{ |x^\beta w_\alpha(x)| \} \leq C_1 \prod_{j=1}^n \left( \frac{\xi_j^0 - 2\epsilon}{\xi_j^0 - \epsilon} \right)^{\alpha_j} \sum_{\gamma \leq \beta} \left( \frac{\beta}{\gamma} \right) \alpha_k \cdots (\alpha_k + \gamma_k - 1),
\]

where

\[
C_1 = \max \left\{ \int_G |\xi^{-\gamma} D^{\beta-\gamma} \bar{w}_0(\xi)| d\xi : \gamma \leq \beta, |\beta| \leq 2n \right\}.
\]
On the other hand, since
\[
\prod_{k=1}^{n} (\alpha_k \cdots (\alpha_k + \gamma_k - 1) < (|\alpha| + 2n)^{2n}
\]
(because of $|\gamma| \leq |\beta| \leq 2n$),
\[
2|\beta| = \sum_{\gamma \leq \beta} \binom{\beta}{\gamma}
\]
and
\[
\lim_{|\alpha| \to \infty} (|\alpha| + 2n)^{2n} \prod_{j=1}^{n} \left( \frac{\xi_j^0 - 2\epsilon}{\xi_j^0 - \epsilon} \right)^{\alpha_j} = 0,
\]
we obtain
\[
\sup_{x \in \mathbb{R}^n} |x^\beta \omega_\alpha(x)| \leq C_2
\]
for all $|\beta| \leq 2n$ and $\alpha \geq 0$. Therefore, there is an absolute constant $C_3$ such that
\[
\sup_{x \in \mathbb{R}^n} (1 + x_1^2) \cdots (1 + x_n^2) |w_\alpha(x)| \leq C_3, \quad \alpha \geq 0.
\]
So we have proved (7) with $C = C_3 \pi^n ||v||_1$. Combining (6) and (7) we obtain
\[
1 \leq \lim_{|\alpha| \to \infty} \left( ||D^\alpha f||_p \prod_{j=1}^{n} (\xi_j^0 - 2\epsilon)^{-\alpha_j} \right)^{1/|\alpha|}.
\]
Therefore, since $\epsilon > 0$ is arbitrarily chosen and
\[
\left[ \prod_{j=1}^{n} \left( \frac{\xi_j^0 - 2\epsilon}{\xi_j^0} \right)^{-\alpha_j} \right]^{1/|\alpha|} \leq \max_{1 \leq j \leq n} \frac{\xi_j^0}{\xi_j^0 - 2\epsilon},
\]
we obtain (2) (with $\xi = \xi^0$) by letting $\epsilon \to 0$.

Now we prove (2) for “zero points”: Let $\xi^0 \in \text{sp}(f)$, $\xi^0 \neq 0$, and $\xi^0 \cdots \xi^0 = 0$. For the sake of convenience, we assume that $\xi_j^0 > 0$, $j = 1, \ldots, k$, and $\xi_{k+1} = \cdots = \xi_n = 0$ ($1 \leq k < n$). Then it is enough to show (2) only for indices $\alpha$ such that $\alpha_{k+1} = \cdots = \alpha_n = 0$. Then the proof is analogous to the above one after the following modification of choosing $\epsilon$: We fix a number $0 < \epsilon < \frac{1}{2} \min_{1 \leq j \leq k} \xi_j^0$.

Second we prove that
\[
\lim_{|\alpha| \to \infty} \left( ||D^\alpha f||_p / \sup_{\text{sp}(f)} |\xi^0| \right)^{1/|\alpha|} \geq 1.
\]
Assume the contrary, that there exists a subsequence $I_1$ such that
\[
(I_1) \lim_{|\alpha| \to \infty} \left( ||D^\alpha f||_p / \sup_{\text{sp}(f)} |\xi^0| \right)^{1/|\alpha|} < 1,
\]
where the symbol \((I_1)\) in (9) means that we take the limit only for \(\alpha \in I_1\). Then there exist a subsequence \(I_2 \subset I_1\) and numbers \(0 \leq \beta_j \leq 1\), \(j = 1, \ldots, n\) such that \(|\beta| = 1\) and

\[
(I_2) \quad \lim_{|\alpha| \to \infty} \frac{\alpha_j}{|\alpha|} = \beta_j, \quad j = 1, \ldots, n.
\]

Now we prove

\[
(I_2) \quad \lim \sup_{\gamma \to \beta} \frac{\xi^\gamma}{|\xi^\beta|} = \sup_{\gamma \to \beta} \frac{\xi^\gamma}{|\xi^\beta|}
\]

if \(\gamma \in \mathbb{R}_+^n\) and \(\gamma \to \beta\).

Indeed, given \(h > 1\), there exists \(\epsilon > 0\) such that \(h \gamma \geq \beta\) for \(\gamma \in \mathbb{R}_+^n\) and \(|\gamma - \beta| \leq \epsilon\). Further, let \(|\xi| \leq M\) for all \(\xi \in \text{sp}(f)\). Then for \(\xi \in \text{sp}(f)\) and \(\gamma \in \mathbb{R}_+^n\), \(|\gamma - \beta| \leq \epsilon\) we obtain

\[
|\xi^\gamma| = |\xi^{\gamma - \beta/h}| |\xi^\beta|^{1/h} \leq M |\gamma - \beta/h| \sup_{\xi \in \text{sp}(f)} |\xi^\beta|^{1/h}.
\]

Therefore

\[
\lim_{\gamma \to \beta} \sup_{\xi \in \text{sp}(f)} |\xi^\gamma| \leq M \sup_{\xi \in \text{sp}(f)} |\xi^\beta|^{1/h}.
\]

Letting \(h \to 1\), we get

\[
\lim_{\gamma \to \beta} \sup_{\xi \in \text{sp}(f)} |\xi^\gamma| \leq \sup_{\xi \in \text{sp}(f)} |\xi^\beta|.
\]

To prove (11) it remains to show that

\[
(I_2) \quad \lim \sup_{\gamma \to \beta} \frac{|\xi^\gamma|}{|\xi^\beta|} \geq \sup_{\xi \in \text{sp}(f)} |\xi^\beta|.
\]

Let \(\xi^* \in \text{sp}(f)\) such that \(|\xi^*\beta| = \sup_{\xi \in \text{sp}(f)} |\xi^\beta|\). Then it follows from \(p < \infty\) that the distribution \(\hat{f}(\xi)\) cannot concentrate on the hyperplanes \(\xi_j = 0\), \(j = 1, \ldots, n\) (this fact will be shown later). Therefore, \(|\xi^*\beta| > 0\). Furthermore, let \(\eta\) be an arbitrary point of \(\text{sp}(f)\). Then we will show later that the restriction of the distribution \(\hat{f}(\xi)\) on any neighborhood of \(\eta\) also does not concentrate on the hyperplanes \(\xi_j = 0\), \(j = 1, \ldots, n\). Therefore, there exists a sequence \(m_{\xi} \in \text{sp}(f), m \geq 1\), such that \(m_{\xi_j} \neq 0\), \(j = 1, \ldots, n\), for any \(m \geq 1\) and \(m_{\xi} \to m^*, m \to \infty\). Then

\[
\sup_{\xi \in \text{sp}(f)} |\xi^\gamma| \geq |m_{\xi}^\gamma|
\]

for any \(m \geq 1\). Hence,

\[
\lim_{\gamma \to \beta} \sup_{\xi \in \text{sp}(f)} |\xi^\gamma| \geq \lim_{\gamma \to \beta} |m_{\xi}^\gamma| = |m_{\xi^\beta}|.
\]

Letting \(m \to \infty\), we obtain (12) and then (11).

Further, given \(\lambda > 1\), there is a number \(k \geq 1\) such that \(\lambda |\xi^\beta| \geq |\xi^*\beta|\).

Therefore, it follows from (10)–(11) and (2) that

\[
(I_2) \quad \lim_{|\alpha| \to \infty} \frac{\langle |D^\alpha f|_p / \sup_{\xi \in \text{sp}(f)} |\xi^\alpha|^{1/|\alpha|} \rangle}{|\alpha|} = (I_2) \quad \lim_{|\alpha| \to \infty} \frac{\langle |D^\alpha f|_p / |\xi^\alpha|^{1/|\alpha|} \rangle}{|\alpha|} \\
\geq (I_2) \quad \frac{1}{\lambda} \lim_{|\alpha| \to \infty} \langle |D^\alpha f|_p / |k_{\xi}^\beta| \rangle \\
= (I_2) \quad \frac{1}{\lambda} \lim_{|\alpha| \to \infty} \langle |D^\alpha f|_p / |k_{\xi}^\alpha|^{1/|\alpha|} \rangle \geq \frac{1}{\lambda}.
\]

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This contradicts (9) by letting $\lambda \to 1$. Thus we have proved (8).

Finally, we will show
\begin{equation}
\lim_{|\alpha| \to \infty} \left( \frac{\|D^\alpha f\|_p \sup_{sp(f)} |\xi^\alpha|^{1/|\alpha|}}{\sup_{sp(f)}} \right) \leq 1.
\end{equation}

We fix a domain $G \supset sp(f)$ and a function $\psi \in C_0^\infty(G)$ such that $\psi(\xi)$ equals 1 in some neighborhood of $sp(f)$. Further, let $0 < q \leq 1$. We put $h_\alpha(\xi) = \psi(\xi)|\xi^\alpha|$, $\alpha \geq 0$. Then it follows from Hölder's inequality that for any $s > n(1/q - 1/2)$
\begin{align}
\|F^{-1}h_\alpha\|_q^q &= \int (|\tilde{h}_\alpha(\xi)|^2)^{q/2} d\xi \\
&\leq \left( \int |\tilde{h}_\alpha(\xi)|^2 (1 + |\xi|^2)^s d\xi \right)^{q/2} \times \left( \int (1 + |\xi|^{2s})^{-s(1/2 - q)} d\xi \right)^{1-q/2}.
\end{align}

Therefore,
\begin{equation}
\|F^{-1}h_\alpha\|_q \leq C'||h_\alpha||_{(s)},
\end{equation}
where $C' = C'(s, q)$ is independent of $h_\alpha$.

Combining (14), the topological equality $H_k = W_{k,2}(\mathbb{R}^n)$, and
\begin{equation}
\|D^\alpha f\|_p = \|F^{-1}(\psi(\xi)|\xi^\alpha|) \ast f\|_p \leq \|F^{-1}(\psi(\xi)|\xi^\alpha|)||f||_p,
\end{equation}
we get
\begin{equation}
\|D^\alpha f\|_p \leq C\|\psi(\xi)|\xi^\alpha|\|k,2\|f\|_p, \quad \alpha \geq 0,
\end{equation}
where $C$ is independent of $f$ and $\alpha$ and $k = \lceil \frac{n}{2} \rceil + 1$.

Given the Leibniz formula we get a constant $C_1 = C_1(\psi, k)$ such that
\begin{equation}
\|\psi(\xi)|\xi^\alpha|\|k,2 \leq C_1|\alpha|^k \sup_G \{\sup|\xi^{\alpha-\gamma}|: \gamma \leq \alpha, \gamma | \leq k\}, \quad \alpha \geq 0.
\end{equation}

On the other hand, by an argument analogous to the previous one, we get
\begin{equation}
\lim_{|\alpha| \to \infty} \left( \sup_G \{\sup|\xi^{\alpha-\gamma}|: \gamma \leq \alpha, \gamma | \leq k\} \right)^{1/|\alpha|} / \sup_G |\xi^\alpha|^{1/|\alpha|} = 1.
\end{equation}

Actually, assume the contrary, that there exist a subsequence $I_1$ and a number $\delta > 1$ such that
\begin{equation}
\sup_G \{\sup|\xi^{\alpha-\gamma}|^{1/|\alpha|}: \gamma \leq \alpha, \gamma | \leq k\} \geq \delta \sup_G |\xi^\alpha|^{1/|\alpha|}, \quad \alpha \in I_1.
\end{equation}

Therefore, there are a subsequence $I_2 \subset I_1$, numbers $0 \leq \beta_j \leq 1$, $j = 1, \ldots, n$, and an index $\gamma^0$, $|\gamma^0| \leq k$, such that $|\beta| = 1$ and
\begin{equation}
(I_2) \lim_{|\alpha| \to \infty} \frac{\alpha_j - \gamma_j^0}{|\alpha|} = \beta_j, \quad j = 1, \ldots, n,
\end{equation}
and
\begin{equation}
\sup_G \{\sup|\xi^{\alpha-\gamma}|^{1/|\alpha|}: \gamma \leq \alpha, \gamma | \leq k\} = \sup_G |\xi^{\alpha-\gamma^0}|^{1/|\alpha|}
\end{equation}
for all $\alpha \in I_2$. Therefore, by an argument analogous to that used for the proof of (11), we get
\begin{equation}
(I_2) \lim_{|\alpha| \to \infty} \sup_G |\xi^{\alpha-\gamma^0}|^{1/|\alpha|} = (I_2) \lim_{|\alpha| \to \infty} \sup_G |\xi^\alpha|^{1/|\alpha|} = \sup_G |\xi^\beta| > 0,
\end{equation}
which contradicts (18). Thus we have proved (17).
Combining (15)--(17) we obtain

\begin{equation}
(19) \quad \lim_{|\alpha| \to \infty} \|D^\alpha f\|_p^{1/|\alpha|}/\sup_G|\xi^\alpha|^{1/|\alpha|} \leq 1.
\end{equation}

Now we assume the contrary that (13) does not hold. Then there exist a subsequence \(J\) and numbers \(\lambda > 1, 0 \leq \beta_j \leq 1, j = 1, \ldots, n\), such that \(|\beta| = 1\) and

\begin{align*}
(J) \quad &\lim_{|\alpha| \to \infty} \|D^\alpha f\|_p^{1/|\alpha|}/\sup_{sp(f)}|\xi^\alpha|^{1/|\alpha|} = \lambda, \\
(J) \quad &\lim_{|\alpha| \to \infty} \frac{\alpha_j}{|\alpha|} = \beta_j, \quad j = 1, \ldots, n.
\end{align*}

Therefore, given the validity of (11) when \(sp(f)\) is replaced by \(G\) (which can be proved analogously because \(G\) is open) and (19) we get

\[\sup_G|\xi^\beta|/\sup_{sp(f)}|\xi^\beta| \geq \lambda\]

for any domain \(G \supset sp(f)\), which is impossible because of \(\sup_{sp(f)}|\xi^\beta| > 0\). The proof of Case 1 is complete.

Case 2 \((p = \infty)\). This is the most complicated case. It should be noted that many facts used in the proof of Case 1 are false in this case (for example, equality (11)). We first prove that if \(\sup_{sp(f)}|\xi^\alpha| = 0\), then \(D^\alpha f(x) \equiv 0\) (for the same \(\alpha\)). Indeed, without loss of generality we may assume that \(\alpha_j \neq 0, j = 1, \ldots, k\), and \(\alpha_{k+1} = \cdots = \alpha_n = 0\) \((1 \leq k \leq n)\). Then the distribution \(\tilde{f}(\xi)\) concentrates on the hyperplanes \(\xi_j = 0, j \in \{1, \ldots, k\}\).

For each \(j \in I\) we put

\[G_j = \{\xi \in \mathbb{R}^n : \xi_i \neq 0, i \in I\setminus\{j\}\}.
\]

Then \(G_j\) is open. And let \(\tilde{f}_j(\xi)\) be the restriction of \(\tilde{f}(\xi)\) on \(\cup_{j \in I} G_j\). Then using a partition of unity (see, for example, \([7, \text{Theorem 1.4.5}]\)), we get

\[\tilde{f}_j(\xi) = \sum_{j=1}^k \varphi_j(\xi)\tilde{f}(\xi),
\]

where \(\varphi_j(\xi) \in C_0^\infty(G_j), j \in I\).

The distribution \(\varphi_j(\xi)\tilde{f}(\xi)\) concentrates on the hyperplanes \(\xi_j = 0\). Therefore, taking account of a remark on Theorem 2.3.5 mentioned in Example 5.1.2 in \([7]\), we get

\begin{equation}
(20) \quad F^{-1}(\varphi_j\tilde{f})(x) = \sum_{\ell=0}^N g_\ell(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)(-ix_j)\ell,
\end{equation}

where \(N\) is the order of the distribution \(\tilde{f}(\xi)\) \((N < \infty\) because \(\text{supp} \tilde{f}\) is compact), \(g_\ell(\xi_1, \ldots, \xi_{j-1}, \xi_{j+1}, \ldots, \xi_n), \quad \ell = 0, 1, \ldots, N\) - distributions with compact support.

On the other hand, we have

\[||F^{-1}(\varphi_j\tilde{f})||_{L^\infty} = ||F^{-1}\varphi_j * f||_{L^\infty} \leq ||F^{-1}\varphi_j||_1 ||f||_{L^\infty} < \infty.
\]
Therefore, (20) is possible only if $\ell = 0$. Consequently, the function $F^{-1}(\varphi_j \tilde{f})(x)$ is independent of $x_j$. Hence, since $\alpha_j \neq 0$, we get

$$D^\alpha F^{-1}(\varphi_j \tilde{f})(x) \equiv 0, \quad j \in I,$$

and then $D^\alpha f_j(x) \equiv 0$. Therefore, to prove $D^\alpha f(x) \equiv 0$, it is enough to show that $D^\alpha(f - f_1)(x) \equiv 0$.

Further, let $\xi \in \mathbb{R}^n \setminus \bigcup_{j \in I} G_j$. Then there exist at least two indices $i, j \in I$ such that $\xi_i = \xi_j = 0$. Therefore, the distribution $\tilde{f}(\xi) - \tilde{f}_1(\xi)$ will concentrate on the hyperplanes $\xi_i = \xi_j = 0, i, j \in I, i \neq j$.

Let $i, j \in I, i \neq j$. We put

$$G_{ij} = \{\xi \in \mathbb{R}^n : \xi_\ell \neq 0, \ell \in I \setminus \{i, j\}\}.$$

Then $G_{ij}$ is open. And let $\tilde{f}_2(\xi)$ be the restriction of $\tilde{f}(\xi) - \tilde{f}_1(\xi)$ on the union of the sets $G_{ij}, i, j \in I, i \neq j$. Then using the partition of unity, we get

$$\tilde{f}_2(\xi) = \sum_{i, j \in I, i \neq j} \varphi_{ij}(\xi)(\tilde{f}(\xi) - \tilde{f}_1(\xi)),$$

where $\varphi_{ij}(\xi) \in C_0^\infty(G_{ij})$.

The distribution $\varphi_{ij}(\xi)(\tilde{f}(\xi) - \tilde{f}_1(\xi))$ concentrates on the hyperplane $\xi_i = \xi_j = 0$. Therefore, $D^n F^{-1}[\varphi_{ij}(\tilde{f} - \tilde{f}_1)](x) \equiv 0$ because, as shown above, $F^{-1}([\varphi_{ij}(\tilde{f} - \tilde{f}_1)])(x) \equiv 0$ is independent of variables $x_i, x_j$. Consequently, $D^n f_2(x) \equiv 0$. Therefore, to prove $D^n f(x) \equiv 0$, it is enough to show that $D^n(f - f_1 - f_2)(x) \equiv 0$.

Now let $\xi \in \mathbb{R}^n \setminus \bigcup\{G_{ij} : i, j \in I, i \neq j\}$. Then there are at least three indices $i_1, i_2, i_3 \in I$ such that $\xi_{i_1} = \xi_{i_2} = \xi_{i_3} = 0$. Therefore, the distribution $\tilde{f}(\xi) - \tilde{f}_1(\xi) - \tilde{f}_2(\xi)$ concentrates on the hyperplanes $\xi_{i_1} = \xi_{i_2} = \xi_{i_3} = 0, i_1, i_2, i_3 \in I, i_1 \neq i_2 \neq i_3$.

Again for $i_1, i_2, i_3 \in I, i_1 \neq i_2 \neq i_3$, we put

$$G_{i_1i_2i_3} = \{\xi \in \mathbb{R}^n : \xi_j \neq 0, j \in I \setminus \{i_1, i_2, i_3\}\}$$

and call $\tilde{f}_3(\xi)$ the restriction of $\tilde{f}(\xi) - \tilde{f}_1(\xi) - \tilde{f}_2(\xi)$ on the union of the sets $G_{i_1i_2i_3}$, $i_1, i_2, i_3 \in I, i_1 \neq i_2 \neq i_3$. Then we have again $D^n f_3(x) \equiv 0$. So it is enough to prove $D^n(f - f_1 - f_2 - f_3)(x) \equiv 0$, where $\tilde{f}(\xi) - \tilde{f}_1(\xi) - \tilde{f}_2(\xi) - \tilde{f}_3(\xi)$ concentrates on the hyperplanes $\xi_{i_1} = \xi_{i_2} = \xi_{i_3} = \xi_{i_4} = 0, i_1, i_2, i_3, i_4 \in I, i_1 \neq i_2 \neq i_3 \neq i_4$. Repeating the above arguments, we obtain the distribution $\tilde{f}(\xi) - \tilde{f}_1(\xi) - \tilde{f}_2(\xi) - \cdots - \tilde{f}_{k-1}(\xi)$ which concentrates on the hyperplane $\xi_1 = \cdots = \xi_k = 0$ and $D^n f(x) \equiv 0$ if $D^n(f - f_1 - \cdots - f_{k-1})(x) \equiv 0$. The last fact is clear because, as shown above, $(f - f_1 - \cdots - f_{k-1})(x)$ does not depend on variables $x_1, \ldots, x_k$.

Thus we have proved $D^n f(x) \equiv 0$.

By the result just obtained, it is enough to show (1) only for multi-indices $\alpha \geq 0$ such that $\sup |\xi^{\alpha}| > 0$ and denote by $P$ the set of all such as multi-indices.

Second we notice that inequalities (2) and (19) have been proved for $1 \leq p < \infty$.

Next we prove that

$$(P) \lim_{|\alpha| \to \infty} \left( |D^\alpha f|_\infty / \sup_{\text{sp}(f)} |\xi^{\alpha}| \right)^{|\alpha|} \geq 1.$$
Assume to the contrary, that there exist a subsequence \( I \subset P \), a number \( \lambda < 1 \), and a vector \( \beta \geq 0, |\beta| = 1 \) such that
\[
(I) \quad \lim_{|\alpha| \to \infty} \left( \frac{1}{|\alpha|} \sup_{\text{sp}(\mathcal{F})} |\xi^\alpha|^{1/|\alpha|} \right) < \lambda.
\]
(22)

Note that
\[
(I) \quad \lim_{|\alpha| \to \infty} |\alpha| = \beta.
\]
(23)

Indeed, assume to the contrary, that there exists a subsequence \( J \subset I \) such that
\[
(J) \quad \lim_{|\alpha| \to \infty} \sup_{\text{sp}(\mathcal{F})} |\xi^\alpha|^{1/|\alpha|} = 0.
\]
(24)

For any \( 1 \leq k \leq n \) and \( i_1, \ldots, i_k \in \{1, \ldots, n\} \) we put
\[ T_{i_1 \ldots i_k} = \{ \alpha \geq 0 : \alpha_{i_1} \neq 0, \ldots, \alpha_{i_k} \neq 0 \} \] and \( J_{i_1 \ldots i_k} = J \cap T_{i_1 \ldots i_k} \) is unbounded. Therefore, clearly, we get
\[
(J_{i_1 \ldots i_k}) \quad \lim_{|\alpha| \to \infty} \sup_{\text{sp}(\mathcal{F})} |\xi^\alpha|^{1/|\alpha|} \geq (J_{i_1 \ldots i_k}) \quad \lim_{|\alpha| \to \infty} |\eta|^{1/|\alpha|} > 0,
\]
where \( \eta \) is any point of \( \text{sp}(\mathcal{F}) \) such that \( \eta_{i_1} \neq 0, \ldots, \eta_{i_k} \neq 0 \). This contradicts (25). So we have proved (24).

Further, let \( \alpha \xi \in \text{sp}(\mathcal{F}) : |\alpha \xi^\alpha| = \sup_{\text{sp}(\mathcal{F})} |\xi^\alpha| \). Then \( \alpha \xi_{i_1} \neq 0, \ldots, \alpha \xi_{i_k} \neq 0 \) for any \( \alpha \in J_{i_1 \ldots i_k} \) and, by taking a subsequence, without loss of generality we may assume that for some \( \xi^* \in \text{sp}(\mathcal{F}) \)
\[
(J_{i_1 \ldots i_k}) \quad \lim_{|\alpha| \to \infty} \alpha \xi = \xi^*.
\]
(26)

Now we consider two cases of \( \xi^* \):
If \( \xi_{i_j} \neq 0, j = 1, \ldots, k \), then, obviously,
\[
(J_{i_1 \ldots i_k}) \quad \lim_{|\alpha| \to \infty} |\alpha \xi^\alpha|^{1/|\alpha|} = |\xi^*| = (J_{i_1 \ldots i_k}) \quad \lim_{|\alpha| \to \infty} |\xi^\alpha|^{1/|\alpha|}
\]
which together with \( \xi^* \in \text{sp}(\mathcal{F}) \), (2), and (22) implies
\[
1 \leq (J_{i_1 \ldots i_k}) \lim_{|\alpha| \to \infty} \left( \frac{1}{|\alpha|} \sup_{\text{sp}(\mathcal{F})} |\xi^\alpha|^{1/|\alpha|} \right) < \lambda < 1
\]
which is impossible.

Otherwise, without loss of generality we may assume that \( \xi_{i_1}^* = \ldots = \xi_{i_m}^* = 0 \) and \( \xi_{i_{m+1}}^* \neq 0, \ldots, \xi_{i_k}^* \neq 0 \) for some \( 1 \leq m \leq k \).

Since (24) and (26), it follows that \( \xi^* \neq 0 \). Therefore, \( m < k \). Further, by virtue of (23)–(24), (26), the definition of \( \alpha \xi \), and \( \xi_{i_1}^* = \ldots = \xi_{i_m}^* = 0 \) we obtain \( \beta_{i_1} = \ldots = \beta_{i_m} = 0 \). Since, clearly,
\[
(J_{i_1 \ldots i_k}) \quad \lim_{|\alpha| \to \infty} |\alpha \xi_{i_{m+1}} \ldots \alpha \xi_{i_k}|^{1/|\alpha|} = |\xi_{i_{m+1}}^* \ldots \xi_{i_k}^*|^{1/|\alpha|}
\]
(27)

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there exist \( \nu \in J_{i_1 \ldots i_k} \) and \( N > 0 \) such that
\[
|\alpha \xi_{i_\ell}| \leq \lambda^{-1} |\nu \xi_{i_\ell}|, \quad \ell = m + 1, \ldots, k,
\]
for all \( |\alpha| \geq N \), \( \alpha \in J_{i_1 \ldots i_k} \).

On the other hand, it follows from \( \nu \xi_{i_1} \neq 0, \ldots, \nu \xi_{i_k} \neq 0 \) and
\[
(J_{i_1 \ldots i_k}) \lim_{|\alpha| \to \infty} \alpha \xi_{i_j} = \xi_{i_j}^*, \quad j = 1, \ldots, m,
\]
that there exists \( M > 0 \) such that
\[
|\alpha \xi_{i_j}| \leq |\nu \xi_{i_j}|, \quad j = 1, \ldots, m,
\]
for all \( |\alpha| \geq M \), \( \alpha \in J_{i_1 \ldots i_k} \). This together with (27) implies
\[
|\alpha \xi_{i_j}| \leq \lambda^{-1} |\nu \xi_{i_j}|, \quad j = 1, \ldots, k,
\]
for all \( |\alpha| \geq \max\{M, N\} \), \( \alpha \in J_{i_1 \ldots i_k} \). Therefore,
\[
\sup_{sp(f)} |\xi_{\alpha}|^{1/|\alpha|} = |\alpha \xi_{\alpha}|^{1/|\alpha|} \leq \lambda^{-1} |\nu \xi_{\alpha}|^{1/|\alpha|}
\]
which together with (2) and (22) implies
\[
1 \leq (J_{i_1 \ldots i_k}) \lim_{|\alpha| \to \infty} (||D^\alpha f||_\infty / |\nu \xi_{\alpha}|)^{1/|\alpha|}
\]
\[
\leq (J_{i_1 \ldots i_k}) \lambda^{-1} \lim_{|\alpha| \to \infty} (||D^\alpha f||_\infty / \sup_{sp(f)} |\xi_{\alpha}|)^{1/|\alpha|} < 1.
\]
We thus arrive at a contradiction. So we have proved (21).

Finally, to complete the proof it remains to show that
\[
(P) \lim_{|\alpha| \to \infty} (||D^\alpha f||_\infty / \sup_{sp(f)} |\xi_{\alpha}|)^{1/|\alpha|} \leq 1.
\]
Assume to the contrary, that there exist a subsequence \( I \subset P \), a number \( h > 1 \), and a vector \( \beta \geq 0 \), \( |\beta| = 1 \) such that
\[
(I) \lim_{|\alpha| \to \infty} (||D^\alpha f||_\infty / \sup_{sp(f)} |\xi_{\alpha}|)^{1/|\alpha|} > h,
\]
(30)
\[
(I) \lim_{|\alpha| \to \infty} \alpha = \beta.
\]

Notation being as above, we have \( 1 \leq k \leq n \) and \( i_1, \ldots, i_k \in \{1, \ldots, n\} \)
such that \( I_{i_1 \ldots i_k} = I \cap T_{i_1 \ldots i_k} \) is unbounded.

We put
\[
Q = \{ \eta \in \mathbb{R}^n : \exists \{m\xi\} \subset sp(f), m \xi_j \neq 0, \quad j \in \{i_1, \ldots, i_k\}, \quad m \geq 1, \quad \lim_{m \to \infty} m \xi = \eta \},
\]
\[
Q_\delta = \{ x + y : x \in Q, |y| < \delta \}, \quad \delta > 0,
\]
and \( H = \mathbb{R}^n \setminus Q \). Then \( Q \) is close, \( H \) and \( Q_\delta \) are open.

Therefore, since \( sp(f) \subset Q_\delta \cup H (= \mathbb{R}^n) \), we obtain
\[
\hat{f}(\xi) = \varphi_\delta(\xi) \hat{f}(\xi) + \psi(\xi) \hat{f}(\xi), \quad \varphi_\delta \in C^\infty_0(Q_\delta), \quad \psi \in C^\infty_0(H).
\]
By an argument analogous to the previous one, we can prove that \( D^\alpha F^{-1}(\psi \hat{f})(x) \equiv 0 \) for all \( \alpha \in I_{i_1 \ldots i_k} \). Hence, it follows from (29) that
\[
(I_{i_1 \ldots i_k}) \lim_{|\alpha| \to \infty} (||D^\alpha F^{-1}(\varphi_\delta \hat{f})||_\infty / \sup_{sp(f)} |\xi_{\alpha}|)^{1/|\alpha|} > h
\]
for any \( \delta > 0 \).
On the other hand, by an argument used in the proof of equality (11), we get

\[(I_{i_1 \ldots i_k}) \lim_{|\alpha| \to \infty} \sup_{Q_\delta} |\xi_\alpha|^{1/|\alpha|} = \sup_{Q_\delta} |\xi_\delta|.
\]

Further, let \( m \in Q_{1/m} : |m \theta|^p = \sup_{Q_{1/m}} |\xi_\delta|^p, m \geq 1 \). Then there exist a subsequence \( \{m_k\} \) (for simplicity of notation we assume that \( m_k = k, k \geq 1 \)) and a point \( \theta^* \in Q \) such that \( m \theta \to \theta^*, m \to \infty \). Then

\[0 < \sup_{Q} |\xi_\delta| \leq \lim_{m \to \infty} |m \theta|^p = |\theta^*|^p.
\]

Arguing as in the proof of (12) and taking account of \( \theta^* \in Q \) and (30) we obtain

\[(33) |\theta^*|^p \leq (I_{i_1 \ldots i_k}) \lim_{|\alpha| \to \infty} \sup_{Q} |\xi_\alpha|^{1/|\alpha|}.
\]

Further, because inequality (19) was proved for \( 1 \leq p \leq \infty \), we have

\[(34) (I_{i_1 \ldots i_k}) \lim_{|\alpha| \to \infty} \left( ||D^\alpha F^{-1}(\varphi_{1/m}\tilde{f})||_{\infty} / \sup_{Q_{1/m}} |\xi_\alpha|^{1/|\alpha|} \right) \leq 1
\]

for any \( m \geq 1 \).

Now we fix a number \( m \geq 1 \) such that \( |m \theta|^p \leq h |\theta^*|^p \). Then combining (31)–(34), we obtain

\[1 \geq (I_{i_1 \ldots i_k}) \lim_{|\alpha| \to \infty} \left( ||D^\alpha F^{-1}(\varphi_{1/m}\tilde{f})||_{\infty} / \sup_{Q_{1/m}} |\xi_\alpha|^{1/|\alpha|} \right)
\]

which is impossible.

The proof of Case 2 is complete.

Let us now return to prove the fact mentioned in the proof of Case 1 that the distribution \( \tilde{f}(\xi) \) does not concentrate on the hyperplanes \( \xi_j = 0, j = 1, \ldots, n \) if \( 1 \leq p < \infty \). Actually, assume to the contrary, that \( \tilde{f} \) concentrates on the hyperplanes \( \xi_j = 0, j = 1, \ldots, n \). Then, notation being as above, we have

\[\tilde{f}_1(\xi) = \sum_{j=1}^{n} \varphi_j(\xi) \tilde{f}(\xi), \quad \varphi_j \in C_0^\infty(G_j),
\]

where \( \tilde{f}_1 \) is the restriction of \( \tilde{f} \) on \( \bigcup_{j=1}^{n} G_j \).

On the other hand, it follows from the Nikol’sky inequality that \( f(x) \in L_\infty \). Therefore, as shown above, \( F^{-1}(\varphi_j \tilde{f})(x) \) is independent of \( x_j \), which is possible only if \( F^{-1}(\varphi_j \tilde{f})(x) \equiv 0 \) because of

\[||F^{-1}(\varphi_j \tilde{f})||_p \leq ||F^{-1} \varphi_j||_1 ||f||_p < \infty
\]
and \( p < \infty \). Therefore, \( \tilde{f} \) must concentrate on the hyperplanes \( \xi_i = \xi_j = 0, i, j \in \{1, \ldots, n\}, i \neq j \). Repeating the above arguments, taking account of \( p < \infty \), we obtain \( \text{supp} \tilde{f} \subset \{0\} \), which is impossible because of \( p < \infty \) and \( f(x) \neq 0 \).

Further, let \( \eta \) be an arbitrary point of \( \text{sp}(f) \). Then analogously we can prove that the restriction of \( \tilde{f}(\xi) \) on any neighborhood of \( \eta \) also cannot concentrate on the hyperplanes \( \xi_j = 0, j = 1, \ldots, n \).

Case 3 \((0 < p < 1)\). The inequality

\[
\lim_{|\alpha| \to \infty} \frac{\langle |D^\alpha f|_p \sup_{\text{sp}(f)} |\xi^\alpha| \rangle_1}{|\alpha|} \geq 1
\]

follows from the Nikolsky inequality and Case 1. The inverse inequality

\[
\lim_{|\alpha| \to \infty} \frac{\langle |D^\alpha f|_p \sup_{\text{sp}(f)} |\xi^\alpha| \rangle_1}{|\alpha|} \leq 1
\]

can be proved in the same way as shown above with the following modification of (15):

\[
(D^\alpha f)' \leq C\|\psi(\xi)\xi^\alpha\|_{k,2}\|f\|_p, \quad \alpha \geq 0,
\]

where \( k = \lfloor n(\frac{1}{p} - \frac{1}{2}) \rfloor + 1 \).

Let us now prove (35). Given (14), we get

\[
\|F^{-1}h_\alpha\|_p \leq C\|h_\alpha\|_{(k)}, \quad \alpha \geq 0,
\]

where \( C' \) is independent of \( f \) and \( \alpha \) and notation is as above.

On the other hand, given

\[
\text{supp } F(F^{-1}h_\alpha(.))f(x - .) \subset \text{supp } h_\alpha + \text{supp } F(f(x - .)) = \text{supp } h_\alpha - \text{supp } f \subset G - \text{sp}(f),
\]

the Nikolsky inequality, and (36), we get \( F^{-1}h_\alpha \in L_{\infty}, F^{-1}h_\alpha(.)f(x - .) \in L_p \) for any \( x \in \mathbb{R}^n, \alpha \geq 0 \), and

\[
\|F^{-1}h_\alpha Ff\|_p \leq C_1\|F^{-1}h_\alpha(y)f(x - y)\|_p \leq C_2\|h_\alpha\|_{(k)}\|f\|_p \leq C\|h_\alpha\|_{k,2}\|f\|_p
\]

for all \( \alpha \geq 0 \). So we have proved (35).

The proof of Theorem 1 is complete.

Remark 1. By an easier way we can prove Theorem 1 for functions defined on torus \( \mathbb{T}^n \).

Remark 2. Theorem 1 still holds for the case of fractional derivatives. And it can be extended to the cases of other derivatives as Riesz' or Bessel's ones (see [8]).

Remark 3. Equality (1) is not true if \( \text{sp}(f) \) is unbounded. However, we have
Theorem 2. Let $0 < p \leq \infty$, $f(x) \in L_{p}(\mathbb{R}^{n})$, and $\text{sp}(f)$ with respect to $\xi_1, \ldots, \xi_k$ $(1 \leq k \leq n)$ be bounded. Then $D^{\nu}f(x) \in L_{p}(\mathbb{R}^{n})$ for all $\nu = (\nu_1, \ldots, \nu_k, 0, \ldots, 0) \in \mathbb{Z}^{n}_{+}$ and
\[
\lim_{|\nu| \to \infty} \frac{(||D^{\nu}f||_{p} / \sup_{\text{sp}(f)} |\xi^{\nu}|)^{1/|\nu|}}{1/|\nu| = 1}.
\]

3. An Application

Now let us apply Theorem 1 to obtain a nonconvex case of the Paley–Wiener–Schwartz theorem. For this purpose we have to introduce a notion on a set generated by a number sequence: Let $0 \leq \lambda_{0} < \infty$, $\alpha \geq 0$. Denote by $G\{\lambda_{\alpha}\}$ the set of all points $\xi \in \mathbb{R}^{n}$ such that $|\xi^{\alpha}| \leq \lambda_{\alpha}$ for all $\alpha \geq 0$. Then it is easy to see that $G\{\lambda_{\alpha}\}$ is compact and $G\{h^{m}|\lambda_{\alpha}|\} = hG\{\lambda_{\alpha}\}$, $h \geq 0$. Note that $G\{\lambda_{\alpha}\}$ can be nonconvex. Actually, let $n = 2$, $\lambda_{(i,j)} = 2^{i+j}$, $i, j \in \mathbb{Z}_{+}$. Then $G\{\lambda_{(i,j)}\} = \{(x, y) \in \mathbb{R}^{2}: |xy| < 1, |x| < 1, |y| < 1\}$—the parabola cross.

If there exist $m \geq 1$ and $\beta \geq 0$ such that $\lambda_{\alpha}^{m} < \lambda_{\beta}$, then $G\{\lambda_{\alpha}\}$ does not change when we replace $\lambda_{\beta}$ by $\lambda_{\alpha}^{m}$. So, to define $G\{\lambda_{\alpha}\}$ we can always assume that the sequence $\{\lambda_{\alpha}\}$ is right, i.e. $\lambda_{\alpha} \geq \lambda_{\alpha}^{1/m}$ for all $\alpha \geq 0$ and $m \geq 1$. Using Theorem 1, we can prove the following

Theorem 3. Let $0 < p \leq \infty$, $f(x) \in L_{p}(\mathbb{R}^{n})$ and $\{\lambda_{\alpha}\}$ be right. Then $\text{sp}(f) \subset G\{\lambda_{\alpha}\}$ if and only if
\[
||D^{\alpha}f||_{p} / \sup_{\text{sp}(f)} |\xi^{\alpha}|^{1/|\alpha|} \leq 1.
\]

Proof. Let $\text{sp}(f) \subset G\{\lambda_{\alpha}\}$. Then
\[
\sup_{\text{sp}(f)} |\xi^{\alpha}| \leq \lambda_{\alpha}, \quad \alpha \geq 0.
\]

Therefore, since Theorem 1, we get (37).

Conversely, if (37) holds, then given Theorem 1 we have
\[
\lim_{|\alpha| \to \infty} \left(\sup_{\text{sp}(f)} |\xi^{\alpha}|^{1/|\alpha|} \lambda_{\alpha}\right)^{1/|\alpha|} \leq 1.
\]

Therefore, for any $\epsilon > 0$ there exists a number $N < \infty$ such that
\[
\sup_{\text{sp}(f)} |\xi^{\alpha}| \leq (1 + \epsilon)^{|\alpha|} \lambda_{\alpha}, \quad |\alpha| \geq N.
\]

On the other hand, the sequence $\{\lambda_{\alpha}\}$ is right; therefore,
\[
\sup_{\text{sp}(f)} |\xi^{\alpha}| \leq (1 + \epsilon)^{|\alpha|} \lambda_{\alpha}
\]
for all $\alpha \geq 0$. Hence,
\[
\text{sp}(f) \subset (1 + \epsilon)G\{\lambda_{\alpha}\}.
\]

Letting $\epsilon \to 0$, we get $\text{sp}(f) \subset G\{\lambda_{\alpha}\}$. The proof is complete.

References


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