THE DE BRANGES-ROVNYAK MODEL WITH FINITE-DIMENSIONAL COEFFICIENTS

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Abstract. A characterization in terms of the canonical model spaces of L. de Branges and J. Rovnyak is obtained for Hilbert spaces of formal power series with vector coefficients which satisfy a difference-quotient inequality, thereby extending the closed ideal theorems of A. Beurling and P. D. Lax.

1. Introduction

This paper extends the well-known invariant subspace characterization of A. Beurling [3] and P. D. Lax [11] for the shift on the Hardy space of square summable power series with vector coefficients (cf. [10, 13–15]). The focus is instead on certain (not necessarily orthogonal) complements of contractively contained invariant manifolds of the shift. These are the spaces \( \mathcal{H}(B) \) of L. de Branges and J. Rovnyak [6–8]. In the Beurling-Lax theory, the key point is a dimension inequality. The inequality is trivial when the coefficient space has infinite dimension, so the essential content is in the finite-dimensional case. Previously only special cases of the more abstract problem have been treated [6, 9], but our methods generalize an argument from [7, Theorem 6]. The main difficulty again comes down to a dimension inequality in the finite-dimensional case. The purpose here is to derive new results on the structure of \( \mathcal{H}(B) \) spaces which reveal what is needed for the inequality to hold. As a consequence, we obtain a complete characterization of the spaces \( \mathcal{H}(B) \).

2. \( \mathcal{H}(B) \) Spaces

A basic concept in the de Branges-Rovnyak theory is complementation: A Hilbert space \( \mathcal{F} \) is contained contractively in a Hilbert space \( \mathcal{H} \) if \( \mathcal{F} \) is a submanifold of \( \mathcal{H} \) and if the inclusion map of \( \mathcal{F} \) into \( \mathcal{H} \) is a contraction. If \( \mathcal{F} \) is contained contractively in \( \mathcal{H} \), then the space complementary to \( \mathcal{F} \) in \( \mathcal{H} \) is the Hilbert space \( \mathcal{G} \) of elements \( \zeta \) of \( \mathcal{H} \) with the property that

\[
\| \zeta \|_{\mathcal{H}}^2 = \sup \{ \| \zeta + f \|_{\mathcal{H}}^2 - \| f \|_{\mathcal{F}}^2 : f \in \mathcal{F} \}
\]

is finite. The space \( \mathcal{G} \) is contained contractively in \( \mathcal{H} \). Moreover, \( \mathcal{G} \) is the unique Hilbert space such that the inequality \( \| k \|_{\mathcal{H}}^2 \leq \| f \|_{\mathcal{G}}^2 + \| g \|_{\mathcal{G}}^2 \) holds whenever \( k = f + g \) is a decomposition of \( k \) in \( \mathcal{H} \) into \( f \) in \( \mathcal{F} \) and \( g \) in...
and such that every element \( k \) in \( \mathcal{H} \) admits a decomposition for which equality holds.

Let \( \mathcal{C} \) be a finite-dimensional Hilbert space, and let \( \mathcal{H} \) be a Hilbert space of formal power series \( f(z) \) whose coefficients are in \( \mathcal{C} \) such that

\[
(1) \quad \| [f(z) - f(0)]/z \|_\mathcal{C}^2 \leq \| f(z) \|_\mathcal{H}^2 - \| f(0) \|_\mathcal{H}^2.
\]

Then \( \mathcal{H} \) is contained contractively in \( \mathcal{C}(z) \), the Hilbert space of square summable power series \( \sum a_n z^n \) with \( a_n \) in \( \mathcal{C} \) and norm given by \( \| \sum a_n z^n \|_{\mathcal{C}(z)} = \sum |a_n|^2 \).

Let \( B(z) \) be a power series whose coefficients are operators on \( \mathcal{C} \) such that

\[
\| B(z)f(z) \|_{\mathcal{L}(z)} \leq \| f(z) \|_{\mathcal{C}(z)} \text{ whenever } f(z) \text{ is in } \mathcal{C}(z).
\]

Cauchy multiplication by \( B(z) \) thus defines a contraction operator on \( \mathcal{C}(z) \) which will be denoted by \( T_B \). The range \( \mathcal{M}(B) \) of \( T_B \) becomes a Hilbert space in the unique norm with the property that

\[
\| T_B f \|_{\mathcal{M}(B)} = \| f \|_{\mathcal{C}(z)} \text{ whenever } f \text{ is orthogonal to the kernel of } T_B.
\]

Furthermore, \( \mathcal{M}(B) \) is contained contractively in \( \mathcal{C}(z) \), and multiplication by \( z \) is a contraction on \( \mathcal{M}(B) \).

The de Branges-Rovnyak space \( \mathcal{H}(B) \) is defined to be the complementary space to \( \mathcal{M}(B) \) in \( \mathcal{C}(z) \). The space \( \mathcal{H}(B) \) satisfies (1) and is an underlying space for canonical models of contractions on Hilbert space \( [1, 2, 12, 16, 17] \).

Multiplication by \( z \) is a contraction on the space \( \mathcal{M} \) complementary to \( \mathcal{H} \) in \( \mathcal{C}(z) \). In \( [6] \) (cf. \([5, \text{Theorem 6}]\)), de Branges extended the Beurling-Lax theorem by showing that if multiplication by \( z \) is isometric on \( \mathcal{M} \), then \( \mathcal{H} \) is isometrically equal to a space \( \mathcal{H}(B) \). It should be further noted that when \( \mathcal{C} \) is infinite dimensional, any space \( \mathcal{H} \) which satisfies (1) is isometrically equal to a space \( \mathcal{H}(B) \) \([4, \text{Theorem 11}]\).

Let \( \mathcal{H}(B) \) be a given space. Then \( \mathcal{H}(B) \) is also contained contractively in \( \mathcal{H}(zB) \). The space \( \mathcal{H}(zB) \) may be obtained as those elements \( h(z) \) of \( \mathcal{C}(z) \) such that \( [h(z) - h(0)]/z \) is in \( \mathcal{H}(B) \) and \( \| h(z) \|_{\mathcal{H}(zB)} = \|[h(z) - h(0)]/z\|_{\mathcal{H}(B)} + \| h(0) \|_{\mathcal{H}}^2 \).

The complementary space to \( \mathcal{H}(B) \) in \( \mathcal{C}(zB) \) is the space \( \mathcal{B}(z) \) with \( \| B(z)c \|_{\mathcal{C}(z)} = |c|_\mathcal{C} \) for every \( c \) orthogonal to \( \mathcal{C} \cap \ker T_B \). Let us define linear transformations \( J_\pm \) from \( \mathcal{H}(B) \) into \( \mathcal{C} \), with ranges denoted \( \mathcal{C}_\pm \), as follows: \( J_+ f = f(0) \) and \( J_- \) is the operator whose adjoint is given by \( J_- c = [B(z) - B(0)]c/z \). Let \( B(z) = \sum B_n z^n \), and let \( B_n \) be the adjoint of \( B_n \) on \( \mathcal{C} \). Then \( J_+ c = [1 - B(z)B(0)]c/z \); and since \( \mathcal{C} \) is finite dimensional, \( \mathcal{C}_+ = (1 - B_0 B_0) \mathcal{C} \) and \( \mathcal{C}_- = (\bigvee_{n \geq 1} B_n^* \mathcal{C}) \subseteq (1 - \overline{B}_0 B_0) \mathcal{C} \).

Let \( R(0) \) denote the difference-quotient transformation on \( \mathcal{H}(B) \), which maps \( f(z) \) into \( [f(z) - f(0)]/z \). Then \( R(0)*f(z) = z f(z) - B(z)J_- f \) so that \( [1 - R(0)*R(0)]f(z) = [B(z) - B(0)](J_- - f)/z \) and \( [1 - R(0)*R(0)]f(z) = (J_+ f) + B(z)J_- R(0)f \). Note that if \( [1 - R(0)*R(0)]f(z) = c + B(z)c_- \) with \( c \) in \( \mathcal{C} \) and \( c_- \) in \( \mathcal{C}_- \), then necessarily \( c = J_+ f \) and \( c_- = J_- R(0)f \). Therefore, since \( \dim \mathcal{C} \) is finite,

\[
\text{rank}[1 - R(0)*R(0)] = \dim \{ (J_+ f, J_- R(0)f) : f \in \mathcal{H}(B) \}
\]

(2) \[ \geq \dim \mathcal{C}_+ = \text{rank}(1 - \overline{B}_0 B_0) \]

(3) \[ \geq \dim \mathcal{C}_- = \text{rank}[1 - R(0)R(0)*]. \]

More precisely, the following will turn out to be a defining property of the spaces \( \mathcal{H}(B) \).
Theorem 1. Let $R(0)$ be the difference-quotient transformation on a given space $\mathcal{H}(B)$. Then

$$\text{rank}[1 - R(0)^* R(0)] = \dim \{ c \in \mathcal{C} : B(z)c \in \mathcal{H}(B) \} + \text{rank}[1 - R(0) R(0)^*].$$

Proof. Suppose that $B(z)c$ is in $\mathcal{H}(B)$. Then $c = (J- f) + d$ where $f$ is in $\mathcal{H}(B)$ and $[B(z) - B(0)]d/z = 0$. Moreover,

$$(4) \quad [1 - R(0)^* R(0)] \{[R(0)^* f] + B(z)c\} = (B_0 d) + B(z) J_0 f.$$

Let $J_0 f_1, \ldots, J_0 f_{s_0}$ be a basis for the subspace $\mathcal{C}_- = \{ c \in \mathcal{C} : B(z)c \in \mathcal{H}(B) \}$, and let $J_0 g_1, \ldots, J_0 g_t$ be a basis for $\mathcal{C}_+$ where $f_i$ and $g_j$ are in $\mathcal{H}(B)$ for all $i$ and $j$. Suppose that there are constants $\lambda_1, \ldots, \lambda_{s_0 + t}$ such that

$$0 = \sum_{i=1}^{s_0} \lambda_i [1 - R(0)^* R(0)] \{[R(0)^* f_i] + B(z) J_0 f_i\} + \sum_{j=1}^{t} \lambda_{s_0 + j} [1 - R(0)^* R(0)] g_j.$$

Equivalently by (4) we have

$$0 = \left( \sum_{i=1}^{s_0} \lambda_{s_0 + j} J_0 g_j \right) + B(z) J_0 \left[ \sum_{i=1}^{t} \lambda_{s_0 + j} R(0) g_j + \sum_{i=1}^{s_0} \lambda_i f_i \right]$$

so that $\sum_{i=1}^{s_0} \lambda_{s_0 + j} J_0 g_j = 0$ and hence $\lambda_{s_0 + j} = 0$ $(j = 1, \ldots, t)$. It follows that $\sum_i \lambda_i J_0 f_i = 0$ and thus $\lambda_i = 0$ for all $i$. Therefore,

$$\text{rank}[1 - R(0)^* R(0)] \geq s_0 + t. \quad (5)$$

Let $c_i = J_0 f_i$ $(i = 1, \ldots, s_0)$ and expand $\{c_i\}$ to a basis $c_1, \ldots, c_s$ of $\{ c \in \mathcal{C} : B(z)c \in \mathcal{H}(B) \}$. For every $j > s_0$ let us write $c_j = (J_0 f_j) + d_j$ as above where $f_j$ is in $\mathcal{H}(B)$ and $d_j$ is orthogonal to $\mathcal{C}_-$. By (4), $B_0 d_j$ is in $\mathcal{C}_+$, so it is in $(B_0 \mathcal{C}) \cap (1 - B_0 B_0) \mathcal{C}$. But since $\mathcal{C}$ is finite dimensional, it follows that this intersection coincides with $B_0 (1 - \overline{B_0 B_0}) \mathcal{C}$, and hence $B_0 d_j = B_0 e_j$ where $e_j$ is in $(1 - \overline{B_0 B_0}) \mathcal{C}$. Thus $d_j - e_j$ is in $\ker B_0$, which is also contained in $(1 - \overline{B_0 B_0}) \mathcal{C}$, and consequently $d_j$ is in $[(1 - \overline{B_0 B_0}) \mathcal{C}] \subset \mathcal{C}_-$.

Now $\{d_j : j > s_0\}$ is linearly independent: For suppose $\sum \alpha_j d_j = 0$. Then $\sum_{j > s_0} \alpha_j c_j = \sum_{j > s_0} \alpha_j J_0 f_j$ is in $\mathcal{C}_+$, so there exist $\beta_i$ such that $\sum_{j > s_0} \alpha_j c_j = \sum_{i \leq s_0} \beta_i c_i$. Since $\{c_i\}$ is linearly independent, $\alpha_j = 0$ for all $j$, and hence

$$t = \dim \mathcal{C}_+ = \text{rank}(1 - B_0 \overline{B_0}) = \text{rank}(1 - \overline{B_0 B_0}) \quad = \dim \{[(1 - \overline{B_0 B_0}) \mathcal{C}] \subset \mathcal{C}_- \} + \dim \mathcal{C}_- \quad \geq (s - s_0) + \text{rank}[1 - R(0) R(0)^*].$$

In conjunction with (5) we have

$$\text{rank}[1 - R(0)^* R(0)] \geq s + \text{rank}[1 - R(0) R(0)^*].$$

To verify the reverse inequality, it suffices to show that there exist $r = \text{rank}[1 - R(0)^* R(0)] - \text{rank}[1 - R(0) R(0)^*]$ linearly independent vectors $a_i$ in $\mathcal{C}$ such that $B(z)a_i$ is in $\mathcal{H}(B)$. By inequalities (2) and (3), it follows that $r = r_0 + r_1$ where $r_0 = \text{rank}[1 - R(0)^* R(0)] - \dim \mathcal{C}_+$ and $r_1 = \dim \{ \text{ran}(1 - \overline{B_0 B_0}) \mathcal{C} \}$.
Suppose that $r_0 > 0$ and recall the basis $\{J+g_j\}$ of $\mathcal{C}_+$. As above, $\{[1 - R(0)^* R(0)]g_j\}$ is linearly independent, so if $\mathcal{G}$ is its span, then there are $r_0$ vectors $[1 - R(0)^* R(0)]g_i$ ($i = 1, \ldots, r_0$), with $\mathcal{G}$ in $\mathcal{H}(B)$, which form a basis of $\text{ran}[1 - R(0)^* R(0)] \oplus \mathcal{G}$. Now there exist constants $\lambda_{ij}$ such that $J+g_i = \sum_{j=1}^{r_0} \lambda_{ij} J+g_j$ for each $i$. Let us define $a_i = J_- R(0)(g_i - \sum_j \lambda_{ij} g_j)$ for $i = 1, \ldots, r_0$. Then $B(z)a_i = [1 - R(0)^* R(0)](g_i - \sum_j \lambda_{ij} g_j)$ is in $\mathcal{H}(B)$, and $\{a_1, \ldots, a_{r_0}\}$ is linearly independent: Suppose that $\sum \mu_i a_i = 0$. Then

$$\sum \mu_i [1 - R(0)^* R(0)]g_i = \sum \mu_i [1 - R(0)^* R(0)] \left( \sum_j \lambda_{ij} g_j \right)$$

which must be zero since it is in both $\mathcal{G}$ and $\mathcal{G}^\perp$. Therefore $\mu_i = 0$ for every $i$.

Next, suppose that $r_1 > 0$ and let $\hat{d}_1, \ldots, \hat{d}_{r_1}$ be a basis of $\text{ran}(1 - \overline{B}_0 B_0) \oplus \mathcal{C}_-$. Then $B(z)\hat{d}_j = B_0 \hat{d}_j$ and $\hat{d}_j = (1 - \overline{B}_0 B_0) b_j$ for some $b_j$ in $\mathcal{C}$. Let $\hat{f}_j(z) = [1 - B(z) \overline{B}(0)] B_0 b_j$ and define $a_{r_0+j} = \hat{d}_j + J_- R(0) \hat{f}_j$ for $j = 1, \ldots, r_1$. Then $B(z)a_{r_0+j} = [1 - R(0)^* R(0)] \hat{f}_j$ is in $\mathcal{H}(B)$.

Finally, $\{a_i : i = 1, \ldots, r = r_0 + r_1\}$ is linearly independent: Suppose that there are constants $\nu_1, \ldots, \nu_r$ such that

$$\nu_0 = \sum_{i=1}^{r_0} \nu_i a_i = \sum_{i=1}^{r_1} \nu_{i} a_i + \sum_{j=1}^{r_1} \nu_{r_0+j} [\hat{d}_j + J_- R(0) \hat{f}_j].$$

It follows that $\sum_{j=1}^{r_1} \nu_{r_0+j} \hat{d}_j = 0$ since $a_i$ ($1 \leq i \leq r_0$) and $J_- R(0) \hat{f}_j$ ($1 \leq j \leq r_1$) are in $\mathcal{C}_-$, and $\hat{d}_j$ is orthogonal to $\mathcal{C}_-$ for every $j$. Therefore $\nu_{r_0+j} = 0$ ($j = 1, \ldots, r_1$), and consequently $\sum_{j=1}^{r_1} \nu_{i} a_i = 0$ so that $\nu_i = 0$ for all $i$. \quad \square

### 3. The characterization

Let $\mathcal{H}$ be a space which satisfies (1), and let $\mathcal{H}'$ be the Hilbert space of all power series $h(z)$ such that $[h(z) - h(0)]/z$ is in $\mathcal{H}$ with $\|h(z)\|_{\mathcal{H}'}^2 = \|[h(z) - h(0)]/z\|_{\mathcal{H}'}^2 + |h(0)|_{\mathcal{H}}^2$. Then $\mathcal{H}'$ satisfies (1), and $\mathcal{H}$ is contained contractively in $\mathcal{H}'$. Let $\mathcal{R}$ be the complementary space to $\mathcal{H}$ in $\mathcal{H}'$, and let $i_{\mathcal{R}}$ and $i_{\mathcal{H}}$ denote the respective inclusion maps of $\mathcal{H}$ and $\mathcal{R}$ into $\mathcal{H}'$. Then every $h$ in $\mathcal{H}'$ admits the unique decomposition $h = (i_{\mathcal{R}}^* h) + (i_{\mathcal{H}}^* h)$ where $\|h\|_{\mathcal{H}'}^2 = \|i_{\mathcal{R}}^* h\|_{\mathcal{R}}^2 + \|i_{\mathcal{H}}^* h\|_{\mathcal{H}}^2$.

A fundamental result from the theory of $\mathcal{H}(B)$ spaces is: $\mathcal{H}$ is isometrically equal to a space $\mathcal{H}(B)$ if and only if the dimension of $\mathcal{R}$ does not exceed the dimension of $\mathcal{C}$ [6]. More generally, if $\mathcal{C} \subset \mathcal{H}$ and $\dim \mathcal{R} \leq \dim \mathcal{C}$, then $\mathcal{H}$ is a space $\mathcal{H}(B)$ where the coefficients of $B(z)$ act on $\mathcal{C}$.

**Lemma.** Let $\mathcal{F}$ be the subspace of elements of $\mathcal{H}$ for which equality holds in (1). Then $\mathcal{R}$ and $\mathcal{H} \cap \mathcal{R}$ are contained in $\mathcal{H}' \oplus \mathcal{F}$ and $\mathcal{H} \oplus \mathcal{F}$ respectively. Moreover, $\dim \mathcal{R} = \dim \mathcal{H}' \oplus \mathcal{F}$ and $\dim \mathcal{H} \cap \mathcal{R} = \dim \mathcal{H} \oplus \mathcal{F}$.

**Proof.** As in [9], $\mathcal{F}$ is a (closed) subspace of $\mathcal{H}$ and is contained isometrically in $\mathcal{H}'$. Therefore for any $f$ in $\mathcal{F}$ and $g$ in $\mathcal{R}$, we have

$$\langle f, g \rangle_{\mathcal{H}'} = \langle f, i_{\mathcal{R}} g \rangle_{\mathcal{H}} = \langle i_{\mathcal{R}}^* f, g \rangle_{\mathcal{R}} = \langle 0, g \rangle_{\mathcal{R}} = 0.$$

Hence $\mathcal{F}$ is a subset of $\mathcal{H}' \oplus \mathcal{R}$. \quad \square
The restriction of \( i^*_\mathbb{R} \) to \( \mathcal{H}' \oplus \mathcal{I} \) is linear and continuous and has trivial kernel: if \( i^*_\mathbb{R} h = 0 \) for some \( h \in \mathcal{H}' \oplus \mathcal{I} \), then \( i^*_\mathbb{R} h = h \), so \( h \) is also in \( \mathcal{I} \), and thus \( h = 0 \). It follows that \( \dim \mathcal{H}' \oplus \mathcal{I} = \dim i^*_\mathbb{R}(\mathcal{H}' \oplus \mathcal{I}) \leq \dim \mathbb{R} \), and hence \( \dim \mathbb{R} = \dim \mathcal{H}' \oplus \mathcal{I} \).

Next, let \( g \) be in \( \mathcal{H} \cap \mathbb{R} \). Then \( g \) is in \( \mathcal{H}' \oplus \mathcal{I} \) but also in \( \mathcal{H} \oplus \mathcal{I} \) since for any \( f \) in \( \mathcal{I} \)

\[
(f, g)_{\mathcal{H}} = \langle i^*_\mathbb{R} f, g \rangle_{\mathcal{I}} = \langle f, i^*_\mathbb{R} g \rangle_{\mathcal{H}} = \langle f, g \rangle_{\mathcal{I}} = 0.
\]

Therefore \( (\mathcal{H} \cap \mathbb{R}) \subseteq (\mathcal{H} \oplus \mathcal{I}) \). Finally \( \dim \mathcal{H} \cap \mathbb{R} = \dim \mathcal{H} \oplus \mathcal{I} \) as above since \( i^*_\mathbb{R}(\mathcal{H} \oplus \mathcal{I}) \) is contained in \( \mathcal{H} \cap \mathbb{R} \). □

The following will distinguish the spaces \( \mathcal{H}(B) \).

**Corollary 1.** Let \( \mathcal{F}(B) \) be the subspace of elements of a given space \( \mathcal{H}(B) \) for which equality holds in (1). Then

\[
\dim J_+ \mathcal{F}(B) = \dim (\mathcal{C} \cap \ker T_B) + \text{rank}[1 - R(0)R(0)^*].
\]

**Proof.** Since \( B(z)\mathbb{C} \) is finite dimensional, the lemma implies that \( \mathcal{H}(B) \oplus \mathcal{F}(B) \) coincides with \( \mathcal{H}(B) \cap B(z)\mathbb{C} \). By (1), the kernel of \( 1 - R(0)^*R(0) \) is contained in \( \mathcal{F}(B) \) and is exactly the kernel of the restriction of \( J_+ \) to \( \mathcal{F}(B) \). Thus since \( 1 - R(0)^*R(0) \) has finite rank and

\[
J_+ \mathcal{F}(B) = J_+ \{\text{ran}[1 - R(0)^*R(0)] \cap \mathcal{F}(B)\},
\]

it follows that

\[
\text{rank}[1 - R(0)^*R(0)] = \dim \{\text{ran}[1 - R(0)^*R(0)] \cap \mathcal{F}(B)\} + \dim[\mathcal{H}(B) \cap \mathcal{F}(B)] = \dim J_+ \mathcal{F}(B) + \dim[\mathcal{H}(B) \cap B(z)\mathbb{C}].
\]

The corollary now follows from Theorem 1 since we also have

\[
\text{rank}[1 - R(0)^*R(0)] = \dim (\mathcal{C} \cap \ker T_B) + \dim[\mathcal{H}(B) \cap B(z)\mathbb{C}] + \text{rank}[1 - R(0)R(0)^*].
\]

By [7, Lemma 4], equality holds in (1) for a given space \( \mathcal{H}(B) \) if and only if \( \mathcal{H}(B) \) contains no nonzero element of the form \( B(z)c \) with \( c \) in \( \mathbb{C} \). An immediate consequence of the above results is

**Corollary 2.** Let \( \mathcal{H}(B) \) be a given space. Then \( \text{rank}[1 - R(0)^*R(0)] = \text{rank}[1 - R(0)R(0)^*] \) if and only if equality holds in (1) for every \( f(z) \) and there is no nonzero vector \( c \) such that \( B(z)c = 0 \).

We now have the proposed characterization.

**Theorem 2.** Let \( \mathcal{H} \) be a Hilbert space of formal power series which satisfies (1), and let \( \mathcal{F} \) be the subspace of those series for which equality holds in (1). Then \( \mathcal{H} \) is isometrically equal to a space \( \mathcal{H}(B) \) if and only if the dimension of the space of constant coefficients of elements of \( \mathcal{F} \) is at least the rank of \( 1 - TT^* \) where \( T \) is the difference-quotient transformation on \( \mathcal{H} \).

**Proof.** Any space \( \mathcal{H}(B) \) has the stated property by Corollary 1.

Conversely, suppose that \( \mathcal{H} \) is a space which satisfies (1) and the dimension hypothesis. Let \( \mathcal{H}', \mathcal{R}, i_\mathcal{H} \) and \( i_\mathbb{R} \) be defined as above, and let \( f(z) \) and \( g(z) \) be in \( \mathcal{H} \). Since
\[ (i^\mathcal{H}_g^* f(z), g(z))_{\mathcal{H}} = (zf(z), i^\mathcal{H}_g g(z))_{\mathcal{H}} = (f(z), Tg(z))_{\mathcal{H}}, \]

it follows that \( T^* f(z) = i^\mathcal{H}_g z f(z) \).

Let \( S \) denote the difference-quotient transformation on \( \mathcal{H}' \). Then

\[ (1 - TT^*)f(z) = f(z) - Ti^\mathcal{H}_g zf(z) = f(z) - S[zf(z) - i^\mathcal{H}_g zf(z)] = Si^\mathcal{H}_g zf(z). \]

More generally, \( S\mathcal{H} \) is contained in the range of \( 1 - TT^* \): Let \( g(z) \) be in \( \mathcal{H} \) such that \( g(z) \) is orthogonal to \( i^\mathcal{H}_g z f(z) \) for every \( f(z) \) in \( \mathcal{H} \). Then

\[ 0 = (g(z), i^\mathcal{H}_g z f(z))_{\mathcal{H}} = (g(z), zf(z))_{\mathcal{H}} = (Sg(z), f(z))_{\mathcal{H}} \]

for every \( f(z) \) in \( \mathcal{H} \). Letting \( f(z) = Sg(z) \), we conclude that \( g(z) \) is constant. Hence \( S\mathcal{H} = S \oplus \{i^\mathcal{H}_g z f(z) : f(z) \in \mathcal{H}\} \), which is contained in \( (1 - TT^*)\mathcal{H} \) since the rank of \( 1 - TT^* \) is finite by the hypothesis.

It follows that \( \mathcal{H} \) is finite dimensional since

\[ \dim \mathcal{H} \leq \dim S\mathcal{H} + \dim \ker S \leq \text{rank}(1 - TT^*) + \dim \mathcal{E}. \]

Thus by the lemma \( \mathcal{H} = \mathcal{H}' \oplus \mathcal{F} \).

Furthermore, since \( \mathcal{H}' \) contains \( \mathcal{E} \), the kernel of the restriction of \( S \) to \( \mathcal{H}' \oplus \mathcal{F} \) is \( \mathcal{E} \oplus \{f(0) : f(z) \in \mathcal{F}\} \). Hence, we have that

\[ \dim \mathcal{H} = \dim [\mathcal{E} \oplus \{f(0) : f(z) \in \mathcal{F}\}] + \dim S\mathcal{H} \]

\[ \leq \dim \mathcal{E} - \dim \{f(0) : f(z) \in \mathcal{F}\} + \text{rank}(1 - TT^*) \]

\[ \leq \dim \mathcal{E} \]

by the hypothesis. Therefore, \( \mathcal{H} \) is isometrically equal to a space \( \mathcal{H}(B) \). \( \square \)

Finally, any space which satisfies (1) is at least a reducing subspace of \( R(0) \) on some space \( \mathcal{H}(B) \).

**Corollary 3.** Let \( \mathcal{H} \), \( \mathcal{F} \) and \( T \) be defined as in Theorem 2, but assume on the other hand that

\[ \delta = \text{rank}(1 - TT^*) - \dim \{f(0) : f(z) \in \mathcal{F}\} \]

is finite and positive. If \( \mathcal{E} \) is any Hilbert space with dimension at least \( \delta \), then \( \mathcal{H} \oplus \mathcal{E}(z) \) is isometrically equal to a space \( \mathcal{H}(B) \).

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