STOCHASTIC CONTROL PROBLEMS
AND SPHERICAL FUNCTIONS ON SYMMETRIC SPACES

T. E. DUNCAN AND H. UPMEIER

ABSTRACT. A family of explicitly solvable stochastic control problems is formulated and solved in noncompact symmetric spaces. The symmetric spaces include all of the classical spaces and four of the exceptional spaces. The stochastic control problems are the control of Brownian motion in these symmetric spaces by a drift vector field. For each symmetric space a family of stochastic control problems is formulated by using spherical functions in the cost functionals. These spherical functions are explicitly described and are polynomials in suitable coordinates. A generalization to abstract root systems is given.

0. INTRODUCTION

In the investigation of stochastic optimal control for nonlinear stochastic systems it is natural to study the optimal control of diffusion processes in a Riemannian manifold. Some sufficient, as well as necessary and sufficient, conditions have been obtained for the optimal controls for such controlled diffusions. The first results for these problems are given in [D2, D3]. A natural controlled diffusion in a manifold can be formed from a stochastic differential equation that is the sum of the formal vectors of a manifold-valued Brownian motion and a drift vector field that depends on the control. For many reasons it is important to have examples of such controlled diffusions where the optimal control is expressed explicitly as a function of the state of the controlled diffusion. However even in Euclidean space there are only a relatively few examples of controlled diffusions where the optimal control is expressed explicitly as a function of the state.

In the more general setting of Riemannian geometry a natural class of (curved) manifolds are the symmetric spaces which are closely related to the semisimple and reductive Lie groups and arise in many classical situations (e.g. matrix groups and hyperbolic spaces). By using the additional structure available on symmetric spaces we are able to construct large families of explicitly solvable stochastic control problems in symmetric spaces of arbitrary rank.

The first example of an explicitly solvable stochastic control problem in a noncompact manifold with nonzero curvature is a controlled diffusion in real hyperbolic three space [D4]. This example in real hyperbolic three space is generalized in [D5] to all the hyperbolic spaces, that is, irreducible noncompact symmetric spaces of rank one, and a countable family of nontrivially distinct cost functionals for each of these spaces. In all of these examples some spherical functions, that are polynomials in appropriate variables, are used to define the cost functions. For rank one symmetric spaces these spherical functions are hypergeometric functions. The stochastic control problem is to control Brownian motion in a hyperbolic space by a drift vector field so that it remains close to a fixed point in the space called the origin. The cost functionals are positive, monotone increasing functions of the distance of the state from the origin and are quadratic functions of the control.

In this paper a family of stochastic control problems are formulated and explicitly solved in noncompact symmetric spaces of arbitrary rank that are formed from semisimple and reductive Lie groups. Specifically all symmetric cones and (complex) symmetric balls are used. The stochastic control problem is the natural analogue of the stochastic control problems in the hyperbolic spaces solved in [D4, D5]. The problem is to control Brownian motion in a noncompact symmetric space by a drift so that it remains close to the origin. Spherical functions are used to construct a family of cost functionals. For the symmetric cones spherical polynomials have been constructed in the statistics literature (e.g. [M2]). In this paper these spherical polynomials are constructed more easily from the representation theory of the symmetric cones [U2]. For symmetric balls not much is known about a general construction of spherical polynomials [D1, H1]. In this paper an explicit procedure is given for the construction of all spherical polynomials in an arbitrary symmetric ball. This procedure uses the spherical polynomials in symmetric cones. These spherical polynomials are monotone increasing along rays in a positive Weyl chamber. An admissible control is required to induce a stochastic differential equation for the controlled diffusion that has a unique strong solution. The solution of each of these stochastic control problems is obtained by finding a smooth solution to the Hamilton-Jacobi or dynamic programming equation for the control problem.

The organization of this paper is as follows. In Sections 2, 3, 4, 7 and 8, we develop the relevant geometric features for the symmetric cones and symmetric balls, respectively. We start from one basic fact, namely the Jordan-theoretic description of these two classes of symmetric spaces and their infinitesimal isometries [L3, U1, FK], and develop the root decomposition, the structure of the Laplace operator in radial coordinates, and the basic properties of spherical functions in a coherent way. This makes the paper more self-contained and underlines the close analogy between (real) symmetric cones and (complex) symmetric balls. (A more comprehensive account of the Jordan-theoretic description of symmetric spaces, which so far is not readily available in the literature, can be found in a forthcoming book [FK].) In Sections 5, 6, and 9, these geometric results are then used in the study of the corresponding stochastic control problem. Finally, Section 10 indicates how these results can be generalized to the so-called abstract root systems [D1], replacing geometric arguments by combinatorial properties of the spherical polynomials.

1. Review of stochastic calculus

Stochastic optimal control of a diffusion process in a Riemannian manifold is based on the stochastic calculus of Brownian motion. This stochastic calculus is an interplay between the properties of a Brownian motion in a Euclidean space and the theory of smooth curves in a Riemannian manifold. To make this paper accessible to an audience including those without a background in probability, some of the basic concepts and approaches of stochastic calculus and control in manifolds are briefly reviewed. A Brownian motion in a Riemannian manifold can be defined in various ways. Assume first that X is a measurable space. Let $B(X)$ be the Banach algebra of all bounded measurable functions $f: X \to \mathbb{C}$, endowed with the norm

$$\|f\| := \sup_{x \in X} |f(x)|.$$

Let $B'(X)$ be the space of all complex measures μ on X such that

$$\|\mu\| := \int_X |\mu|(dx) < \infty.$$

Via the natural pairing

$$\langle \mu, f \rangle := \int_X \mu(dx)f(x),$$

$B'(X)$ becomes a (proper) subspace of the topological dual space of $B(X)$.

Let Y be another measurable space. A mapping $K: X \to B'(Y)$, denoted by $x \mapsto K(x, dy)$, is called a *measure kernel* if, for every measurable subset $B \subset Y$, the mapping

$$X \ni x \mapsto \int_B K(x, dy)$$

is measurable, and we have

$$\|K\| := \sup_{x \in X} \int_Y |K(x, dy)| < \infty.$$

K can be identified with a continuous linear operator $B(X) \xleftarrow{K} B(Y)$, defined by

$$(Kf)(x) := \int_Y K(x, dy)f(y)$$

for all $f \in B(Y)$ and $x \in X$. If $L: Y \to B'(Z)$ is another measure kernel, we may define

$$(KL)(x, dz) := \int_Y K(x, dy)L(y, dz)$$

and obtain a measure kernel $KL: X \to B'(Z)$ since we have, by Fubini's Theorem,

$$\int_Z |(KL)(x, dz)| \le \int_Y |K(x, dy)| \int_Z |L(y, dz)|$$

for all $x \in X$, showing that $\|KL\| \le \|K\| \|L\|$. As an operator, KL is the composition of K and L.

Let $B'_1(X)$ be the subset of $B'(X)$ consisting of all (positive) probability measures on X; i.e., $|\mu| = \mu$ and $\|\mu\| = 1$ for $\mu \in B'_1(X)$.

Put $T := \{t \in \mathbb{R} : t \geq 0\}$, and consider a family

$$(1.1) \qquad\qquad K_t \colon X \to B_1'(X) \qquad (t \in T)$$

of measure kernels satisfying $K_s K_t = K_{s+t}$ for all $s, t \in T$. For any subset $I \subset T$, consider the direct product measurable space

$$X_I := \prod_{i \in I} X$$

endowed with the σ-algebra of cylinder sets. For $I \subset \mathscr{I} \subset T$, we have a canonical projection $p_I^{\mathscr{I}} \colon X_{\mathscr{I}} \to X_I$. For any finite subset $I = \{t_1 < \cdots < t_n\}$ of T, there exists a measure kernel $K_I \colon X \to B_1'(X_I)$ uniquely determined by the property

$$K_I\left(x_0, \prod_{i \in I} B_i\right) := \int_{B_{t_1}} K_{t_1 - t_0}(x_0, dx_1)$$

$$\cdot \int_{B_{t_2}} K_{t_2 - t_1}(x_1, dx_2) \cdots \int_{B_{t_n}} K_{t_n - t_{n-1}}(x_{n-1}, dx_n)$$

for all $x_0 \in X$ and all measurable subsets $B_i \subset X$ $(i \in I)$. Here we put $t_0 := 0$. If $\mathscr{I} \supset I$ is another finite subset of T, there is a commuting diagram

$$
\begin{array}{ccc}
 & X & \\
K_{\mathscr{I}} \swarrow & & \searrow K_I \\
B_1'(X_{\mathscr{I}}) & \xrightarrow{\;\;p_I^{\mathscr{I}}\;\;} & B_1'(X_I)
\end{array}
$$

where the horizontal arrow is the direct image map under $p_I^{\mathscr{I}}$. Thus (K_I) is a projective system of measure kernels and there exists a measure kernel

$$(1.2) \qquad\qquad K_T \colon X \to B_1'(X_T)$$

such that for every finite subset $I \subset T$ the diagram

$$
\begin{array}{ccc}
 & X & \\
K_T \swarrow & & \searrow K_I \\
B_1'(X_T) & \xrightarrow{\;\;p_I^{T}\;\;} & B_1'(X_I)
\end{array}
$$

commutes. K_T is called the *projective limit* associated with the semigroup (1.1).

An important class of semigroups of measure kernels comes from elliptic differential operators. Let X be a locally compact smooth manifold and let $\mathscr{E}(X)$ be the Fréchet algebra of all smooth functions $f \colon X \to \mathbb{C}$, endowed with the seminorms

$$|f|_{D,K} := \sup_{x \in K} |(Df)(x)|$$

where $K \subset X$ is compact and D is a differential operator with smooth coefficients on X.

Let $\mathscr{E}'(X)$ be the topological dual space of $\mathscr{E}(X)$, consisting of all distributions δ on X with compact support. The natural pairing between $\mathscr{E}(X)$ and $\mathscr{E}'(X)$ is denoted by

$$\langle \delta, f \rangle =: \int_X \delta(dx) f(x).$$

If Y is another smooth manifold, a mapping $D: X \to \mathscr{E}'(Y)$, denoted by $x \mapsto D(x, dy)$, is called a *distribution kernel* if for every $f \in \mathscr{E}(Y)$ the function

$$(Df)(x) := \int_Y D(x, dy)f(y)$$

is smooth on X, and defines a continuous operator $\mathscr{E}(X) \overset{D}{\longleftarrow} \mathscr{E}(y)$. If $E: Y \to \mathscr{E}'(Z)$ is another distribution kernel, the continuity assumption implies that

$$(DE)(x, dz) := \int_Y D(x, dy)E(y, dz)$$

defines a distribution kernel $DE: X \to \mathscr{E}'(Z)$ corresponding to the composition of operators. Any *differential operator* $\mathscr{E}(X) \overset{D}{\longleftarrow} \mathscr{E}(X)$ is continuous and can be identified with a distribution kernel

$$\int_X D(x, dy)f(y) := (Df)(x)$$

from X to $\mathscr{E}'(X)$. For each x, $D(x, dy)$ has compact support $\subset \{x\}$.

If D is a second-order differential operator which is strictly elliptic in the sense of [Y2, p. 419 and p. 426], there exists a *fundamental solution* which is a semigroup of measure kernels

$$\exp_t D: X \to B_1'(X) \qquad (t \in T)$$

(or of operators on $B(X)$). According to (1.2), we obtain a measure kernel

(1.3) $\exp_T D: X \to B_1'(X_T).$

We now consider the important special case $X = \mathbb{R}^n$ and the algebra $B'(\mathbb{R}^n)$ of all bounded complex Borel measures, with convolution

$$\int_{\mathbb{R}^n} (\mu^* \nu)(dz)f(z) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mu(dx)\nu(dy)f(x+y).$$

Every $\mu \in B'(\mathbb{R}^n)$ defines a measure kernel $\mu^*: \mathbb{R}^n \to B'(\mathbb{R}^n)$, denoted by $x \mapsto \mu^*(x, dy)$, via the formula $\mu^*(x, B) := \mu(B - x)$ for all Borel subsets $B \subset \mathbb{R}^n$. The associated convolution operator

$$(\mu^* f)(x) = \int_{\mathbb{R}^n} \mu^*(x, dy)f(y) = \int_{\mathbb{R}^n} \mu(dy)f(x+y)$$

commutes with translations, and we have

(1.4) $\mu^* \nu^* = (\mu^* \nu)^*$

for the product of measure kernels. Now consider a family $\mu_t \in B_1'(\mathbb{R}^n)$ of probability measures, for $t \in T := \mathbb{R}_+$, satisfying

(1.5) $\mu_s^* \mu_t = \mu_{s+t}$

for all $s, t \in T$. For each finite subset $I = \{t_1 < t_2 < \cdots < t_n\}$ of T define a measure $\mu_I \in B_1'(\mathbb{R}_I^n)$ by putting

$$\mu_I\left(\prod_{i \in I} B_i\right) := \int_{B_{t_1}} \mu_{t_1 - t_0}(dx_1) \int_{B_{t_2} - x_1} \mu_{t_2 - t_1}(dx_2)$$

$$\cdots \int_{B_{t_n} - x_1 - \cdots - x_{n-1}} \mu_{t_n - t_{n-1}}(dx_n)$$

for all Borel measurable subsets $B_i \subset \mathbb{R}^n$ $(i \in I)$. Here we put $t_0 := 0$.

If $\mathscr{I} \supset I$ is another finite subset of T, the canonical projection gives rise to the direct image $\mu_{\mathscr{I}} \rtimes p_I^{\mathscr{I}} = \mu_I$. Thus we obtain a projective system (μ_I) of measures and there exists a measure $\mu_T \in B_1'(\mathbb{R}_T^n)$ such that

$$\mu_T \rtimes p_I^T = \mu_I \quad (I \subset T \text{ finite}).$$

The measure μ_T is called is called the *projective limit* associated with the semigroup (1.5). According to (1.4), the associated measure kernels

$$K_t = \mu_t^{\#}: \mathbb{R}^n \to B_1'(\mathbb{R}^n)$$

form a semigroup as well, and we have

$$K_I = \mu_I^{\#} \quad (I \subset T \text{ finite}) \quad \text{and} \quad K_T = \mu_T^{\#}.$$

An important class of semigroups of measures comes from *constant-coefficient differential operators*. Consider the algebra $\mathscr{E}_0'(\mathbb{R}^n)$ of all distributions δ on $\mathbb{R}^n$ supported at $\{0\}$, with convolution

$$\int_{\mathbb{R}^n} (\delta^{\#}\varepsilon)(dz)f(z) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \delta(dx)\varepsilon(dy)f(x+y)$$

for all $f \in \mathscr{E}(\mathbb{R}^n)$. Every $\delta \in \mathscr{E}_0'(\mathbb{R}^n)$ defines a distribution kernel $\delta^{\#}: \mathbb{R}^n \to \mathscr{E}'(\mathbb{R}^n)$, denoted by $x \mapsto \delta^{\#}(x, dy)$, via the formula

$$(1.6) \qquad (\delta^{\#}f)(x) = \int_{\mathbb{R}^n} \delta^{\#}(x, dy)f(y) := \int_{\mathbb{R}^n} \delta(dy)f(x+y)$$

for all $f \in \mathscr{E}(\mathbb{R}^n)$. We have $\delta^{\#}\varepsilon^{\#} = (\delta^{\#}\varepsilon)^{\#}$ for the product of distribution kernels. Any *constant-coefficient differential operator*

$$\delta = \sum_\alpha c_\alpha \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

can be identified with the distribution

$$\int_{\mathbb{R}^n} \delta(dx)f(x) = (\delta f)(0)$$

in $\mathscr{E}_0'(\mathbb{R}^n)$. By translation invariance, δ induces a differential operator

$$(1.7) \qquad\qquad \mathscr{E}(\mathbb{R}^n) \xleftarrow{\delta^{\#}} \mathscr{E}(\mathbb{R}^n)$$

with distribution kernel defined by (1.6).

In case δ is strictly elliptic of second order [H3] there exists a *fundamental solution* which is a semigroup of measures

$$\exp_t \delta \in B_1'(\mathbb{R}^n) \qquad (t \in T)$$

whose associated measure kernels

$$(\exp_t \delta)^{\#} = \exp_t \delta^{\#}: \mathbb{R}^n \to B_1'(\mathbb{R}^n)$$

form the fundamental solution of the elliptic operator (1.7). According to (1.3), we obtain a measure

$$(1.8) \qquad\qquad \exp_T \delta \in B_1'(\mathbb{R}_T^n).$$

Every *Riemannian manifold* X has a canonical elliptic second-order differential operator Δ called the *Laplace-Beltrami operator*. The associated measure kernel

$$(1.9) \qquad \exp_T \Delta \colon X \to B_1'(X_T)$$

(cf. (1.3)) is the transition probability of the Markov process called the *Brownian motion* on X. One can show [IW] that, for every $x \in X$, the probability measure $(\exp_T \Delta)(x, dy)$ has support on the *continuous* paths $\mathscr{C}(T, X) \subset X_T$. Thus we have

$$(1.10) \qquad \exp_T \Delta \colon X \to B_1'(\mathscr{C}(T, X)).$$

If $X = \mathbb{R}^n$ is Euclidean space and

$$\delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

is the (strictly elliptic) *Casimir operator* at 0, the Laplace-Beltrami operator is $\Delta = \delta^{\#}$ and we have

$$(\exp_t \delta)(dx) = (4\pi t)^{-n/2} e^{-|x|^2/4t} \, dx$$

for all $t \in T$. The associated measure

$$(1.11) \qquad \exp_T \delta \in B_1'(\mathbb{R}_T^n)$$

(cf. (1.8)) is called the transition probability of the *Brownian motion* at $0 \in \mathbb{R}^n$. One can show [IW] that the probability measure (1.11) has support on the continuous paths $\mathscr{C}(T, \mathbb{R}^n) \subset \mathbb{R}_T^n$, so that

$$\exp_T \delta \in B_1'(\mathscr{C}(T, \mathbb{R}^n)).$$

The two concepts (1.10) and (1.11) are related via the *geodesic coordinates*

$$(1.12) \qquad \Omega_x \xrightarrow{\exp_x} X$$

at any "base point" $x \in X$, assigning to a tangent vector v in a 0-neighborhood $\Omega_x \subset T_x X$ the geodesic $t \mapsto \exp_x(tv)$ passing through x in direction v. More precisely, $(\exp_T \Delta)(x, dy)$ can locally be described as the image measure of $(\exp_T \delta)(dy)$ under the local diffeomorphism (1.12). Here δ is the Casimir operator on $T_x X = \mathbb{R}^n$.

A sample path approach to the construction of a Brownian motion in a Riemannian manifold M can be made using the inverse of the development of É. Cartan that acts on the family of piecewise smooth curves in the manifold with the same initial point p. More generally or abstractly the development (of É. Cartan) and its inverse define a diffeomorphism between the Sobolev space of absolutely continuous functions in the tangent space $T_p M$ starting at the origin whose derivatives are square integrable and the Sobolev manifold of absolutely continuous M-valued functions whose derivatives are square integrable with respect to the Riemannian metric. Actually this construction of Brownian motion cannot be extended to all noncompact, complete Riemannian manifolds. It is necessary to put some conditions on the curvature. The curvature conditions ensure that the Brownian motion does not have a finite escape time. With this construction of Brownian motion, many of the results for a Brownian

motion in Euclidean space can be directly translated to the manifold M using the horizontal lift of Brownian motion to the bundle of orthonormal frames, $O(M)$.

In analogy with Euclidean space a stochastic optimal control problem in a Riemannian manifold M is described by the local description of a controlled diffusion process (that is, a stochastic differential equation) and a cost functional. The problem is to choose a control from some family of admissible controls that depends on the current positions (or past history) of the process so that the cost functional is minimized. An important method for the minimization problem is to find a smooth solution of the associated Hamilton-Jacobi equation and verify that the control obtained by the procedure is admissible. In the problems considered here the Riemannian manifold is a noncompact symmetric space. The controlled diffusion is Brownian motion with a control as the drift vector. The cost functional is the sum of a quadratic function of the control and a spherical function so it is desired to keep the diffusion close to the "origin" of the symmetric space. Since the spherical functions and the controlled diffusions have some group invariance properties the solution of the control problem is determined from the behavior of the controlled diffusion in a subspace of the symmetric space.

2. Symmetric spaces and Jordan triples

We now introduce the "phase spaces" for our stochastic control problems, the *symmetric spaces* of arbitrary rank (for rank 1, cf. [D4]). The non-Euclidean symmetric spaces fall into two dual classes, the compact type and the noncompact type which have a simpler topology (no antipodal points, no closed geodesics, etc.) but on the other hand exhibit a highly nontrivial (stratified) *boundary structure*. It is known [L1] that a Riemannian symmetric space is algebraically characterized by a ternary product structure (Lie triple system) $[xyz]$ on its tangent space Z at an arbitrary base point. This triple system is given by the curvature tensor. More important to us is another ternary product (Jordan triple system) $\{xy^*z\}$ on Z which is related to the Lie triple via

$$[xyz] = \{xy^*z\} - \{yx^*z\}.$$

Thus the Jordan triple allows for the decomposition of $[xyz]$ into "positive" and "negative" part. The symmetric spaces admitting such a Jordan triple structure are called *symmetric R-spaces* [L2]; they include all classical spaces and 4 exceptional types.

We will now indicate how Jordan triple systems can be used to give an explicit and uniform construction for

(i) the noncompact model of the symmetric R-space as an unbounded domain in Z;

(ii) the semisimple Lie group G (or rather its Lie algebra $\mathfrak{g}$) of isometries;

(iii) conical and spherical eigenfunctions of the Laplace-Beltrami operator Δ.

A *Jordan triple product* on Z is a trilinear map

$$Z \times Z \times Z \ni (x, y, z) \mapsto \{xy^*z\} \in Z$$

satisfying $\{xy^*z\} = \{zy^*x\}$ (commutativity) and the commutator identity

(2.1) $$[u\square v^*, x\square y^*] = \{uv^*x\}\square y^* - x\square\{yu^*v\}^*$$

for the linear endomorphisms $(x\square y^*)z := \{xy^*z\}$ on Z. We assume in addition that the bilinear form $(x, y) \mapsto$ trace $x\square y^*$ is symmetric and positive definite.

We denote vector fields on Z by the symbol $f(z)\frac{\partial}{\partial z}$ where $z \in Z$ is a "variable" and $f: Z \to Z$ is smooth. The commutator is

$$(2.2) \qquad \left[f_1(z)\frac{\partial}{\partial z}, f_2(z)\frac{\partial}{\partial z} \right] = (df_1(z)f_2(z) - df_2(z)f_1(z))\frac{\partial}{\partial z}$$

where df is the derivative.

For example, every linear endomorphism $M: Z \to Z$ gives rise to a vector field $Mz\frac{\partial}{\partial z}$ and we have

$$\left[M_1 z\frac{\partial}{\partial z}, M_2 z\frac{\partial}{\partial z} \right] = [M_1, M_2]z\frac{\partial}{\partial z}.$$

In particular, we have the linear vector fields $\{xy^*z\}\frac{\partial}{\partial z}$ $(x, y \in Z$ fixed) which, by (2.1) and (2.2), form a Lie algebra $\mathfrak{l}^0$. Moreover, the "Jordan triple identity" (2.2) implies that the "constant" vector fields

$$\mathfrak{l}^1 = \left\{ a\frac{\partial}{\partial z} : a \in Z \right\}$$

and the "quadratic" vector fields

$$\mathfrak{l}^{-1} = \left\{ \{za^*z\}\frac{\partial}{\partial z} : a \in Z \right\}$$

give rise to a Lie algebra

$$\mathfrak{l} = \mathfrak{l}^1 \oplus \mathfrak{l}^0 \oplus \mathfrak{l}^{-1}$$

which, by [L2, Theorem 1], is semisimple. Let L be the corresponding center-free connected lie group. Then the normalizer P of $\mathfrak{l}^1$ is a parabolic subgroup and L/P is a compact symmetric R-space.

We are interested in the *noncompact dual space* of L/P which can always be be realized as a symmetric domain (either bounded or unbounded) in Z. Realized as a bounded domain Ω, a transitive (reductive) Lie group of isometries has a Lie algebra $\mathfrak{g}$ with Cartan decomposition $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}$ where

$$(2.3) \qquad \mathfrak{l} = \left\langle (\{ab^*z\} - \{ba^*z\})\frac{\partial}{\partial z} : a, b \in Z \right\rangle$$

and

$$(2.4) \qquad \mathfrak{p} = \left\{ (a - \{za^*z\})\frac{\partial}{\partial z} : a \in Z \right\}.$$

More appropriate for our purposes is the *unbounded realization* Π as a "Siegel domain". In order to construct Π, we need the notion of the *Cayley transformation* [KW, L3] associated with an element $e \in Z$ satisfying the "tripotent" condition $\{ee^*e\} = e$. The Cayley transform is the analytic automorphism

$$g_e = \exp\left(\frac{\pi}{4}(e + \{ze^*z\})\frac{\partial}{\partial z} \right)$$

of L/P. Note that its defining vector field belongs to $\mathfrak{l}$ but not to $\mathfrak{g}$. Although the open dense subset $Z \subset L/P$ is not invariant under g_e, one can show that $\Pi := g_e(\Omega)$ is still a domain in Z.

Assume in the following that the tripotent e is *maximal*; i.e., $e \square e^*$ is invertible. Then there is a Peirce decomposition

$$(2.5) \qquad\qquad Z = X \oplus U \oplus V$$

where

$$(2.6) \qquad\qquad X := \{x \in Z : \{ee^*x\} = x = \{ex^*e\}\}$$

is a "formally real" Jordan algebra with unit element e and product $x \circ y := \{xe^*y\}$,

$$(2.7) \qquad\qquad U := \{\xi \in Z : \{ee^*\xi\} = \xi = -\{e\xi^*e\}\}$$

and

$$(2.8) \qquad\qquad V := \left\{z \in Z : \{ee^*z\} = \frac{z}{2}\right\}.$$

The interior of the set of squares in X is a *cone* Λ (cf. [BK]) and the unbounded model of the symmetric R-space is the Siegel domain

$$(2.9) \qquad\qquad \Pi := \{x + \xi + v \in Z : 2x - \Phi(v, v) \in \Lambda\}.$$

Here we put $\Phi(v_1, v_2) := 2\{v_1 v_2^* e\}$.

In the following sections we will present a detailed (stochastic) analysis on Π, separating the three cases (i) $Z = X$, (ii) $Z = X \oplus U$, and (iii) $Z = X \oplus U \oplus V$. In case (i), the domain (2.9) is the *cone* Λ itself. In case (ii), we obtain a symmetric *tube domain* $\Lambda \times U$. In case (iii), we obtain the (proper) *Siegel domains*.

In the irreducible case, the resulting symmetric spaces correspond to the root systems (cf. [L1]):

(i) A_{r-1},
(ii) C_r or D_r,
(iii) B_r or BC_r.

It follows that our analysis covers all classical spaces and 4 exceptional spaces (of dimensions 16, 26+1, 32, and 54). The remaining 13 exceptional spaces with root systems G_2; F_4; E_6; E_7; E_8 and dimensions 8, 14; 28, 40, 52, 64, 112; 42, 78; 70, 133; 128, 248 are not accessible to a purely Jordan algebraic treatment.

3. SYMMETRIC CONES

Every (formally real) Jordan algebra X with product $x \circ y$ defines a (positive-definite) Jordan triple product

$$(3.1) \qquad\qquad \{xy^*x\} = 2(x \circ y) \circ x - y \circ (x \circ x)$$

for which the unit element e is a tripotent. The corresponding Cayley transform

$$g_e(z) = \exp\left(\frac{\pi}{4}(e + z^2)\frac{\partial}{\partial z}\right)(z) = (e + z) \circ (e - z)^{-1}$$

(cf. [L3, U1]) maps Ω onto the *symmetric cone*

$$(3.2) \qquad\qquad \Lambda = \{x^2 : x \in X \text{ invertible}\}$$

associated with X. This cone is homogeneous under its (reductive) linear automorphism group

$$(3.3) \qquad\qquad G^0 = GL(\Lambda) := \{P \in GL(X) : P\Lambda = \Lambda\}$$

and

(3.4) $$K^0 = O(\Lambda) := \{P \in GL(\Lambda) : Pe = e\}$$

is a maximal compact subgroup, whose elements are the Jordan algebra automorphisms of X.

3.5. Example. For $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ (quaternions), the selfadjoint $r \times r$-matrices

$$X_{\mathbb{K}} = \{x \in \mathbb{K}^{r \times r} : x = x^*\}$$

form a Jordan algebra with unit $e = 1$ (identity matrix) under the commutator product

$$x \circ y := \frac{xy + yx}{2}.$$

This corresponding Jordan triple product (3.1) is

$$\{xy^* z\} = \frac{1}{2}(xy^* z + zy^* x)$$

and (3.2) is the *matrix cone*

$$\Lambda_{\mathbb{K}} = \{x = x^* \in \mathbb{K}^{r \times r} : x \text{ positive definite}\}.$$

Putting $P_g x := gxg^*$, we have

$$GL(\Lambda_{\mathbb{K}}) = \{P_g : g \in GL(r, \mathbb{K}) \text{ (invertible)}\}$$

and

$$O(\Lambda_{\mathbb{K}}) = \{P_g : g \in U(r, \mathbb{K}) \text{ (unitary)}\}.$$

For $r = 3$ and $\mathbb{K} = \mathbb{O}$ (octonions), one obtains the "exceptional" cone $\Lambda_{\mathbb{O}}$ in a similar way.

3.6. Example. For $1 + n \geq 3$, the Euclidean space $X = \mathbb{R}^{1,n} = \mathbb{R} \times \mathbb{R}^n$ of all vectors $(x_0, x_1, \ldots, x_n) = (x_0, x')$ has the Jordan product

$$(x_0, x') \circ (y_0, y') := (x_0 y_0 + x' \cdot y', x_0 y' + y_0 x')$$

with the unit element $e = (1, 0, \ldots, 0)$. The associated symmetric cone is the *forward light cone*

$$\Lambda_{1,n} := \{(x_0, x_1, \ldots, x_n) : x_0 > (x_1^2 + \cdots + x_n^2)^{1/2}\}.$$

In this case $GL(\Lambda_{1,n}) = O(1, n)$ is the (two-component) "orthochronous" Lorentz group, and

$$O(\Lambda_{1,n}) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix} : P \in O(n) \right\}.$$

In order to determine the (reductive) Lie algebra $\mathfrak{g}^0$ associated with Λ, we apply the (adjoint) Cayley transformation and obtain for all $x \in X$ and $M \in \mathfrak{l}^0$ (cf. (2.4))

$$\text{Ad}(g_e) \left((x - \{zx^* z\}) \frac{\partial}{\partial x} \right) = 2\{xe^* z\} \frac{\partial}{\partial z}$$

and $\text{Ad}(g_e) M = M$. Identifying linear vector fields with the corresponding endomorphisms, it follows that $\mathfrak{g}^0$ has a Cartan decomposition

(3.7) $$\mathfrak{g}^0 = \mathfrak{l}^0 \oplus \mathfrak{p}^0,$$

where $\mathfrak{l}^0$ is the Lie algebra of (3.4) and $\mathfrak{p}^0$ consists of all *multiplication operators*

(3.8) $M_x z := x \circ z = \{xe^* z\}$

with $x \in X$. One can show that

$$\mathfrak{g}^0 = \mathfrak{gl}(\Lambda) := \{M \in \mathfrak{gl}(X) : \exp(tm)\Lambda = \Lambda \; \forall t \in \mathbb{R}\}$$

is the Lie algebra of $GL(\Lambda)$. From now on we consider only *irreducible* cones. Consider a complete system $e_1, \ldots, e_r$ of minimal disjoint idempotents of X. The cardinality r of such a "frame" is an important invariant of X called its *rank*. By [L3] there is a Peirce *decomposition*

(3.9) $$X = \sum_{1 \le i \le j \le r} X_{ij}$$

into mutually orthogonal "eigenspaces"

$$X_{ij} := \left\{ x \in X : e_k \circ x = \frac{\delta_{ik} + \delta_{jk}}{2} x, \; \forall 1 \le k \le r \right\}.$$

We have $X_{ii} = \mathbb{R}e_i$ $(1 \le i \le r)$ and

(3.10) $a := \dim_{\mathbb{R}} X_{ij}$ $(1 \le i < j \le r)$

is an invariant of X called the *characteristic multiplicity* of X. We put $X_{ji} := X_{ij}$. One can show that

(3.11) $$\mathfrak{a} := \left\{ \sum_{i=1}^{r} t_i M_{e_i} : t_1, \ldots, t_r \in \mathbb{R} \right\}$$

is a maximal abelian subspace of $\mathfrak{p}^0$. Let

$$M_{e_i}^{\#}(M_{e_j}) := \delta_{ij} \quad \text{(Kronecker symbol)}$$

denote the dual basis of linear forms on $\mathfrak{a}$.

3.12. **Example.** The matrix cones have the canonical frame

$$e_i := \begin{pmatrix} 0 & & & & & & \\ & \ddots & & & & \mathbf{0} & \\ & & 0 & & & & \\ & & & 1 & & & \\ & & & & 0 & & \\ & \mathbf{0} & & & & \ddots & \\ & & & & & & 0 \end{pmatrix} \leftarrow i$$

for $1 \le i \le r$, and for $i < j$ the Peirce space X_{ij} consist of all selfadjoint matrices with zero coefficients except in the $(1, j)$th place and (j, i)th place. In particular $a = \dim X_{ij} = \dim_{\mathbb{R}} \mathbb{K}$.

3.13. **Example.** For the forward light cone $\Lambda_{1,n}$ we have $r = 2$ and any pair of orthogonal vectors e_1, e_2 of norm $1/\sqrt{2}$ constitutes a frame. The Peirce space X_{12} is the orthogonal complement of e_1 and e_2, and $a = n - 1$.

3.14. **Proposition.** *There is a real root decomposition*

(3.15) $$\mathfrak{g}^0 = \sum_{1 \le i, j \le r} \mathfrak{g}_{ij}^0 \oplus \mathfrak{m}$$

with respect to $\mathfrak{a}$, *where*

$$\mathfrak{g}_{ij}^0 = \left\{ \{xe_i^*z\}\frac{\partial}{\partial z} : x \in X_{ij} \right\}$$

is the root space of

(3.16)
$$\frac{1}{2}(M_{e_j}^{\#} - M_{e_i}^{\#})$$

and

$$\mathfrak{m} := \{M \in \mathfrak{l}^0 : Me_1 = \cdots = Me_r = 0\}.$$

Proof. As a consequence of (2.1), we have

$$x\square y^* = M_{x\circ y} + [M_x, M_y] \in \mathfrak{g}^0$$

for all $x, y \in X$. In particular, $\mathfrak{g}_{ij}^0 \subset \mathfrak{g}^0$. Since

$$[e_i\square e^*, x\square e_i^*] = \{e_ie^*x\}\square e_i^* - x\square\{e_i^*e\}^* = -\frac{1}{2}x\square e_i^*$$

and

$$[e_j\square e^*, x\square e_i^*] = \{e_je^*x\}\square e_i^* - x\square\{e_j^*e\}^* = -\frac{1}{2}x\square e_i^*$$

and $[e_k\square e^8, x\square e_i^*] = 0$ whenever $k \neq i, j$, $\mathfrak{g}_{ij}^0$ is a root space for (3.16).

Now let $M \in \mathfrak{g}^0$ be arbitrary. Then we have $Me_j = \sum_{k=1}^r x_j^k$ with $x_j^k \in X_{jk}$, as follows from the Peirce composition rules [L3, Theorem 3.14] applied to derivations and multiplication operators, respectively. Put

$$B := \sum_i \left(x_i^i + 2\sum_{k \neq i} x_i^k \right) \square e_i^*.$$

Then

$$Be_j = x_j^j + 2\sum_{k \neq j} x_j^k/2 = Me_j \qquad .$$

for all j, showing that $M - B \in \mathfrak{m}$.   Q.E.D.

4. CONICAL AND SPHERICAL FUNCTIONS ON Λ

Endow Λ with the unique G^0-invariant Riemannian structure such that the basis (3.11) of $\mathfrak{a} \subset \mathfrak{p}^0 \approx T_e(\Lambda)$ (tangent space) becomes orthonormal. Consider the mapping $\varphi: \mathfrak{a} \to \Lambda$ defined by

(4.1)
$$\varphi(t_1, \ldots, t_r) := \exp\left(\sum_i t_i M_{e_i}\right)(e) = \sum_i e^{t_i}e_i.$$

Let N^0 be the connected Lie subgroup with Lie algebra

$$\mathfrak{n}^0 := \sum_{i<j} \mathfrak{g}_{ij}^0.$$

Then we have the (direct) Iwasawa decomposition [H1, FK]

(4.2)
$$\Lambda = N^0 \cdot \varphi(\mathfrak{a}).$$

Let $\mathscr{D}(\Lambda)$ denote the algebra of all (scalar) differential operators D on Λ which are invariant under $GL(\Lambda)$; that is, they satisfy

(4.3) $(g_*D)f := D(f \circ g) \circ g^{-1} = Df$

for all $f \in \mathscr{C}^\infty(\Lambda)$ and $g \in GL(\Lambda)$. For example the *Laplace-Beltrami operator* Δ^0 of Λ belongs to $\mathscr{D}(\Lambda)$.

By [H1, Theorem II.3.6], every $D \in \mathscr{D}(\Lambda)$ has an N^0-*radial part* D_N which is a differential operator on $\mathfrak{a}$ satisfying

(4.4) $(Df) \circ \varphi = D_N(f \circ \varphi)$

for all N^0-invariant functions $f \in \mathscr{C}^\infty(\Lambda)$.

4.5. Proposition. Δ^0 *has the* N^0-*radial part*

$$\Delta_N^0 = \sum_i \left(\frac{\partial^2}{\partial t_i^2} - \frac{a}{2}(2i - r - 1)\frac{\partial}{\partial t_i} \right).$$

Proof. For any $D \in \mathscr{D}(\Lambda)$, D_N is a constant coefficient differential operator of the form

(4.6) $D_N = q \left(\frac{\partial}{\partial t_1} - \rho_1^0, \dots, \frac{\partial}{\partial t_r} - \rho_r^0 \right)$

where q is any symmetric (i.e., Weyl group invariant) polynomial, and

$$\rho_i^0 = \frac{a}{4}(2i - r - 1)$$

are the coefficients of the half-sum of positive roots

(4.7) $\rho^0 = \frac{a}{2} \sum_{i<j} \frac{1}{2}(M_{e_j}^\# - M_{e_i}^\#) = \frac{a}{4}\sum_i (2i - r - 1)M_{e_i}^\#$

(cf. Proposition 3.14). By [H1, Proposition II.3.8], Δ^0 corresponds to the polynomial

(4.8) $q(s_1, \dots, s_r) := \sum_i (s_i^2 - (\rho_i^0)^2)$

so that

$$\Delta_N^0 = \sum_i \left(\frac{\partial^2}{\partial t_i^2} - 2\rho_i^0 \frac{\partial}{\partial t_i} \right). \quad \text{Q.E.D.}$$

The Peirce decomposition (3.9) enables us to construct N^0-invariant "conical" eigenfunctions Δ^0 in an explicit way.

We first introduce the *minors* of an irreducible Jordan algebra X with respect to a frame $\{e_1, \dots, e_r\}$ of idempotents. The *determinant* of X is the unique irreducible polynomial N_r of degree r on X which satisfies the invariance property

$$N_r(Px) = N_r(x)N_r(Pe)$$

for all $P \in GL(\Lambda)$ and $x \in X$ and is normalized by putting $N_r(e) := 1$. For example, we have $N_r(x) = \text{Det}(x)$ for selfadjoint matrices x over $\mathbb{R}$ or $\mathbb{C}$, and

$$N_2(x_0, \dots, x_n) = x_0^2 - x_1^2 - \cdots - x_n^2$$

in case of the light cone.

For each $k \geq 1$, the Jordan subalgebra

$$(4.9) \qquad X_k := \sum_{1 \leq i \leq j \leq k} \bigoplus X_{ij}$$

with unit element $e_1 + \cdots + e_k$ has its own determinant $N_k : X_k \to \mathbb{R}$ of degree k. Via the orthogonal projection $X \to X_k$ we may define N_k on X. The polynomials $N_1, \ldots, N_r$ are called the *minors* of X. They are strictly positive on Λ.

For real or complex matrices, the kth minor is given by

$$N_k \begin{pmatrix} x_{11} & \cdots & x_{1r} \\ & & \\ x_{r1} & & x_{rr} \end{pmatrix} := \operatorname{Det} \begin{pmatrix} x_{11} & \cdots & x_{1k} \\ & & \\ x_{k1} & & x_{kk} \end{pmatrix}$$

whereas the light cone has

$$N_1(x_0, x_1, \ldots, x_n) = \frac{x_0 + x_1}{2}$$

for $e_1 := (\frac{1}{2}, \frac{1}{2}, 0, \ldots, 0)$ and $e_2 = (\frac{1}{2}, -\frac{1}{2}, 0, \ldots, 0)$.

4.10. Proposition. *For each* $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{C}^r$, *the smooth function*

$$(4.11) \qquad N_\alpha(x) := N_1(x)^{\alpha_1 - \alpha_2} N_2(x)^{\alpha_2 - \alpha_3} \cdots N_r(x)^{\alpha_r}$$

on Λ *is a* N^0-*invariant eigenfunction of* Δ^0, *with eigenvalue*

$$(4.12) \qquad \chi_\alpha^0 = \sum_{i=1}^r \alpha_i \left(\alpha_i + \frac{a}{2}(r + 1 - 2i) \right).$$

Proof. It was shown in [U2] that the minors $N_1, \ldots, N_r$ are N^0-invariant. Since

$$(N_\alpha \circ \varphi)(t_1, \ldots, t_r) = (e^{t_1})^{\alpha_1 - \alpha_2} (e^{t_1 + t_2})^{\alpha_2 - \alpha_3} \cdots (e^{t_1 + \cdots + t_r})^{\alpha_r}$$
$$= e^{t_1 \alpha_1 + t_2 \alpha_2 + \cdots + t_r \alpha_r},$$

every $D \in \mathscr{D}(\Lambda)$ with radial part (4.6) satisfies

$$DN_\alpha = q(\alpha_1 - \rho_1^0, \ldots, \alpha_r - \rho_r^0) N_\alpha.$$

For the polynomial (4.7), we obtain

$$\Delta^0 N_\alpha = \sum_i (\alpha_i^2 - 2\alpha_i \rho_i^0) N_\alpha. \quad \text{Q.E.D.}$$

The functions (4.9) are called *conical functions* on Λ. Put

$$\vec{\mathbb{R}}^r := \{\alpha \in \mathbb{R}^r : \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_r\}$$

and

$$\vec{\mathbb{N}}^r := \vec{\mathbb{R}}^r \cap \mathbb{N}^r.$$

The elements $\alpha = (\alpha_1, \ldots, \alpha_r) \in \vec{\mathbb{N}}^r$ are called *partitions* of $|\alpha| := \alpha_1 + \cdots + \alpha_r$. For each $\alpha \in \vec{\mathbb{N}}^r$, N_α is a polynomial of degree $|\alpha|$ on X called the *conical polynomial* of type α.

We now pass to the maximal compact subgroup $K^0 = O(\Lambda)$ of $GL(\Lambda)$. Let

$$(4.13) \qquad \mathfrak{a}_+^0 := \left\{ \sum_{i=1}^r t_i M_{e_i} : t_1 < \cdots < t_r \right\}$$

denote the "Weyl chamber". Then there is a *Cartan decomposition* [H1]

(4.14) $\Lambda = K^0 \cdot \varphi(\mathfrak{a}^0_+)$.

By [H1, Theorem II.3.6], every $D \in \mathscr{D}(\Lambda)$ has a K^0-*radial part* D_K which is a differential operator on $\mathfrak{a}^0_+$ satisfying

(4.15) $(Df) \circ \varphi = D_K(f \circ \varphi)$

for all K^0-invariant functions $f \in \mathscr{C}^\infty(\Lambda)$.

4.16. Proposition. Δ^0 *has the* K^0-*radial part*

(4.17) $\displaystyle \Delta^0_K = \sum_{i=1}^r \left(\frac{\partial^2}{\partial t_i^2} - \frac{a}{2}(r-1)\frac{\partial}{\partial t_i} \right) + a \sum_{i \neq j} \frac{1}{\exp(t_j - t_i) - 1} \frac{\partial}{\partial t_i}.$

Proof. The positive roots (3.16) induce the "density function"

$\displaystyle \delta(t_1, \ldots, t_r) := \prod_{i<j} \sinh^a \left(\frac{t_j - t_i}{2} \right) = \prod_{i<j} \left(\frac{1}{2}(e^{t_j} - e^{t_i})e^{-(t_i+t_j)/2} \right)^a = \text{const} \cdot \omega^a \pi$

where

(4.18) $\displaystyle \omega = \prod_{i<j}(e^{t_j} - e^{t_i}), \qquad \pi = \left(\prod_i e^{t_i} \right)^{-a(r-1)/2}.$

By [H1, p. 335] we have

$\displaystyle \Delta^0_K = \frac{1}{\delta} \sum_i \frac{\partial}{\partial t_i} \delta \frac{\partial}{\partial t_i} = \frac{1}{\omega^a \pi} \sum_i \frac{\partial}{\partial t_i} \omega^a \pi \frac{\partial}{\partial t_i}$

$\displaystyle = \sum_i \left(\frac{\partial}{\partial t_i} + \frac{1}{\pi} \frac{\partial \pi}{\partial t_i} + \frac{a}{\omega} \frac{\partial}{\partial t_i} \right) \frac{\partial}{\partial t_i}$

$\displaystyle = \sum_i \left(\frac{\partial}{\partial t_i} - \frac{a}{2}(r-1) + a \sum_{j \neq i} \frac{1}{\exp(t_j - t_i) - 1} \right) \frac{\partial}{\partial t_i}. \quad \text{Q.E.D.}$

4.19. Corollary.

$\displaystyle \varphi_* \Delta^0_K = \sum_i \left(x_i^2 \frac{\partial^2}{\partial x_i^2} + \left(1 - \frac{a}{2}(r-1) \right) x_i \frac{\partial}{\partial x_i} \right)$

$\displaystyle \qquad\qquad + a \sum_{i \neq j} \frac{x_i^2}{x_j - x_i} \frac{\partial}{\partial x_i}.$

Proof. Putting $x_i = e^{t_i}$ we have by (4.18)

$\displaystyle \omega = \prod_{i<j}(x_j - x_i), \qquad \pi = (x_1 \cdots x_r)^{-a(r-1)/2}$

and (4.17) implies

$\displaystyle \varphi_* \Delta^0_K = \sum_i \left(\frac{d^2 x_i}{dt_i^2} + \left(\frac{dx_i}{dt_i} \right)^2 \left(\frac{\partial}{\partial x_i} + \frac{1}{\pi} \frac{\partial \pi}{\partial x_i} + \frac{a}{\omega} \frac{\partial \omega}{\partial x_i} \right) \right) \frac{\partial}{\partial x_i}$

$\displaystyle = \sum_i \left(x_i + x_i^2 \left(\frac{\partial}{\partial x_i} - \frac{a}{2}(r-1)\frac{1}{x_i} + a \sum_{j \neq i} \frac{1}{x_j - x_i} \right) \right) \frac{\partial}{\partial x_i}. \quad \text{Q.E.D.}$

By integrating over the compact Lie group K^0 we obtain a smooth K^0-invariant function

$$(4.20) \qquad K_\alpha^0(x) := \int_{K^0} N_\alpha(sx)\,ds$$

on Λ, called the *spherical function* of type $\alpha \in \mathbb{C}^r$. In case $\alpha \in \vec{\mathbb{N}}^r$, K_α^0 is a polynomial on X called the *spherical polynomial* of type α. Applying Proposition 4.9, we obtain

$$\Delta^0 K_\alpha^0 = \chi_\alpha^0 K_\alpha^0$$

for every $\alpha \in \mathbb{C}^r$.

4.21. Proposition. *For $\alpha \in \vec{\mathbb{N}}^r$, the spherical polynomial K_α^0 has a finite expansion*

$$K_\alpha^0 \left(\sum_{i=1}^r x_i e_i \right) = \sum_{\beta \in \vec{\mathbb{N}}^r} c_\beta q_\beta (x_1, \dots, x_r)$$

where $c_\beta \geq 0$ and q_β is the monomial symmetric function.

Proof. Given $s \in K^0$ and $l \leq r$, consider the expansion

$$(4.22) \qquad N_l \left(s \cdot \sum_{i=1}^r x_i e_i \right) = \sum_\mu b_\mu(s) x_1^{\mu_1} \cdots x_r^{\mu_r}$$

where $\mu \in \mathbb{N}^r$ (not necessarily nonincreasing). Define

$$|\mu| := \mu_1 + \cdots + \mu_r$$

and

$$l(\mu) := \operatorname{card}\{i : \mu_i \neq 0\}.$$

If $|\mu| \neq l$, then $b_\mu(s) = 0$ since N_l is homogeneous of degree l. Thus we may assume $|\mu| = l$ and therefore $l(\mu) \leq l$. Now assume $l(\mu) < l$. Put $I := \{i : \mu_i > 0\}$. Then for every choice of x_i $(i \in I)$ we have

$$N_l \left(s \cdot \sum_{i \in I} x_i e_i \right) = N_{X_l} \left(P_l \left(s \cdot \sum_{i \in I} x_i e_i \right) \right) = 0$$

since

$$\operatorname{rank} P_l \left(s \cdot \sum_{i \in I} x_i e_i \right) \leq \operatorname{rank} s \cdot \sum_{i \in I} x_i e_i$$

$$= \operatorname{rank} \sum_{i \in I} x_i e_i = \operatorname{card}(I) < l.$$

Here $P_l : X \to X_l$ is the orthogonal projection (which may decrease the rank) and N_{X_l} is the determinant of the Jordan algebra X_l of rank l, defined in (4.9). It follows that $b_\mu(s) = 0$ whenever $l(\mu) < l$. Thus (4.22) simplifies to

$$(4.23) \qquad N_l \left(s \cdot \sum_{i=1}^r x_i e_i \right) = \sum_I b_I(s) \prod_{i \in I} x_i$$

where I runs over all subsets of $\{1, \ldots, r\}$ having l elements. Since $N_l \geq 0$ on the closure $\overline{\Lambda}$, we have

$$b_I(s) = N_l \left(s \cdot \sum_{i \in I} e_i \right) \geq 0$$

for every $s \in K^0$. Taking powers and products, (4.23) implies

$$N_\alpha \left(s \cdot \sum_{i=1}^r x_i e_i \right) = \sum_{\substack{\mu \in \mathbb{N}^r \\ |\mu| = |\alpha|}} b_\mu(s) x_1^{\mu_1} \cdots x_r^{\mu_r}$$

where $b_\mu(s) \geq 0$ for all $s \in K^0$. Applying (4.20), this implies

(4.24)
$$K_\alpha^0 \left(\sum_{i=1}^r x_i e_i \right) = \sum_{\substack{\mu \in \mathbb{N}^r \\ |\mu| = |\alpha|}} b_\mu(s) x_1^{\mu_1} \cdots x_r^{\mu_r}$$

where

$$b_\mu = \int_{K^0} b_\mu(s) \, ds \geq 0.$$

For every permutation $\sigma \in \mathfrak{S}_r$ there exists an element $s \in K^0$ such that

$$s e_i = e_{\sigma(i)} \qquad (1 \leq i \leq r).$$

Putting $\sigma \cdot \mu := (\mu_{\sigma(1)}, \ldots, \mu_{\sigma(r)})$, (4.24) implies $b_{\sigma \cdot \mu} = b_\mu$ since K_α^0 is K^0-invariant. Since

$$q_\beta(x_1, \ldots, x_r) = \sum_\mu x_1^{\mu_1} \cdots x_r^{\mu_r}$$

where μ runs over all distinct permutations of β, the assertion follows.   Q.E.D.

5. Stochastic control theory on symmetric cones: Conical functions

To commence the study of stochastic control problems in noncompact symmetric spaces of arbitrary rank, a symmetric cone is considered where the cost functional is N^0-invariant. Since the N^0-radial part of the Laplace-Beltrami operator is a constant coefficient differential operator and the controlled diffusion can be projected to a Euclidean space, the analysis of the control problem is quite straightforward. However this problem serves to introduce some of the notions in an elementary context.

A stochastic control problem is formulated in a symmetric cone using the N^0-invariance and a conical polynomial N_α. The stochastic control problem is the control of a Brownian motion in a symmetric cone Λ so that the controlled diffusion remains close to the base point of Λ. Using the N^0-invariance of the cost functional to be introduced and the Iwasawa decomposition of the Lie group we can use the coordinates $t = (t_1, \ldots, t_r)$ for the Cartan subalgebra, $\mathfrak{a}$.

The controlled diffusion in the symmetric cone has the infinitesimal generator

$$\frac{1}{2} \Delta^0 + \langle u(s, t), \nabla_t \rangle.$$

Since the N^0-invariance of the control problem is exploited, it suffices to consider the controlled diffusion projected to the abelian subgroup or equivalently the Cartan subalgebra $\mathfrak{a}$. The infinitesimal generator for the controlled diffusion projected to $\mathfrak{a}$ is

$$(5.1) \qquad \frac{1}{2}\Delta_N^0 + \langle u(s, t), \nabla_t \rangle$$

where Δ_N^0 is the N^0-radial part of the Laplace-Beltrami operator Δ^0, $t = (t_1, \ldots, t_r)$, $\nabla_t = (\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_r})$ is the gradient, and $s \in \mathbb{R}_+$. Suppose that $(X(s), s \in [0, T])$ is the controlled diffusion in $\mathfrak{a}$ with infinitesimal generator (5.1) where the dependence of X on u has been suppressed. The cost functional for the stochastic control problem is

$$(5.2) \qquad J_N^0(U, \alpha) = E_{X(0)} \int_0^T (N_\alpha \circ \varphi)(X(s)) + \langle U(s), U(s) \rangle \, ds$$

where N_α is the N^0-invariant polynomial eigenfunction of Δ^0 for the partition α.

Since the infinitesimal generator of $(X(s), s \in [0, T])$ is (5.1), the family of stochastic differential equations that $(X(s), s \in [0, T])$ satisfies is

$$(5.3) \qquad dX_k(s) = \left[-\frac{a}{4}(2k - r - 1) + U_k(s, X(s)) \right] ds + dB_k(s)$$

for $k = 1, 2, \ldots, r$, $X(0) \in \mathfrak{a}$, and $(B_1(s), \ldots, B_r(s), s \geq 0)$ is a standard r-dimensional Brownian motion.

To describe the family of admissible controls it suffices to give the requirements in terms of $(X(s), s \in [0, T])$ by the N^0-invariance of (5.1). An admissible control at time s is a Borel measurable function of $X(s)$ so that the family of stochastic differential equations (5.3) has one and only one strong solution.

It is elementary to verify that the stochastic control problem (5.1, 5.2) is well posed; that is, there is at least one control law (e.g. $U \equiv 0$) that gives a finite value to (5.2).

The solution of the stochastic control problem is given now.

5.4. Theorem. *The stochastic control problem described by* (5.1, 5.2) *has an optimal control given by*

$$(5.5) \qquad U^*(s, t) = -\frac{g(s)}{2} \nabla_t (N_\alpha \circ \varphi)(t)$$

where $s \in [0, T]$, $t = (t_1, \ldots, t_r)$, ∇_t *is the* t-*gradient, and* g *is the unique positive solution of the scalar Riccati differential equation*

$$(5.6) \qquad \frac{dg}{ds} + \frac{1}{2}\chi_\alpha^0 g - \frac{|\alpha|}{4}g^2 + 1 = 0, \quad g(T) = 0,$$

where

$$\chi_\alpha^0 = \sum_{i=1}^r (\alpha_i^2 - 2\alpha_i \rho_i^0) \quad \text{and} \quad \rho_i^0 = \frac{a}{4}(2i - r - 1).$$

Proof. A verification theorem [FR, p. 134] is applied to the smooth solution of the Hamilton-Jacobi equation

$$(5.7) \qquad 0 = \frac{\partial W}{\partial s} + \min_{v \in \mathbb{R}^r} \left[\frac{1}{2}\Delta_N^0 W + \sum_{j=1}^r v_j \frac{\partial W}{\partial t_j} + (N_\alpha \circ \varphi)(t) + \sum_{j=1}^r v_j^2 \right].$$

The family of controls obtained by the minimization in (5.6) is

$$(5.8) \qquad U_j^*(s, t) = -\frac{1}{2}\frac{\partial W(s, t)}{\partial t_j}$$

for $j = 1, 2, \ldots, r$.

Assume a solution W of (5.7) as

$$(5.9) \qquad W(s, t) = g(s)(N_\alpha \circ \varphi)(t).$$

Substituting (5.9) into (5.7) it follows that

$$(5.10) \qquad \begin{aligned} 0 &= g' N_\alpha \circ \varphi + \frac{1}{2}\chi_\alpha^0 N_\alpha \circ \varphi - \frac{|\alpha|}{4} g^2 N_\alpha \circ \varphi + N_\alpha \circ \varphi \\ &= N_\alpha \circ \varphi \left(g' + \frac{1}{2}\chi_\alpha^0 g - \frac{|\alpha|}{4} g^2 + 1 \right). \end{aligned}$$

If g in (5.10) satisfies (5.6) then (5.9) is a smooth solution of (5.7) that satisfies the boundary condition $W(s, t) = 0$ for $(s, t) \in \{T\} \times \mathfrak{a}$. The family of controls (5.5) is an admissible (optimal) control if the family of stochastic differential equations (5.3) with these controls has one and only one strong solution on $[0, T]$.

Since the coefficients of the family of stochastic differential equations (5.3) with the controls (5.8) are locally smooth in $\mathfrak{a}$, a unique strong solution can be constructed at least locally. Thus it suffices to verify that there is no escape of the process in $[0, T]$. Define the random times

$$S_i = \begin{cases} \inf\{s \in [0, T] : X_i(s) = +\infty\}, \\ +\infty \quad \text{if the above set is empty} \end{cases}$$

for $i = 1, 2, \ldots, r$. It is easy to verify by contradiction that

$$P(S_i < \infty \text{ for some } i = 1, 2, \ldots, r) = 0$$

by comparing (5.3) to Brownian motion with no drift term. Thus (5.5) is an admissible optimal control.   Q.E.D.

6. STOCHASTIC CONTROL THEORY ON SYMMETRIC CONES: SPHERICAL FUNCTIONS

A stochastic control problem is described in a symmetric cone using a K^0-invariance and a spherical polynomial K_α^0. The stochastic control problem is the control of a Brownian motion in a symmetric cone Λ by a drift vector field so that this controlled diffusion remains close to the base point of Λ.

The controlled diffusion in the symmetric cone Λ has the infinitesimal generator

$$(6.1) \qquad \frac{1}{2}\Delta^0 + \langle u(s, t), \nabla_t \rangle$$

where Δ^0 is the Laplace-Beltrami operator on Λ, $t = (t_1, \ldots, t_r)$ is the coordinator in $\mathfrak{a}_+^0$, and $s \in \mathbb{R}_+$. Since the K^0-invariance of the control problem is exploited, it suffices to consider the controlled diffusion projected to the positive Weyl chamber $\mathfrak{a}_+^0$ with the infinitesimal generator

$$(6.2) \qquad \frac{1}{2}\Delta_K^0 + \langle u(s, t), \nabla_t \rangle.$$

Let $(X(s), s \geq 0)$ be the controlled diffusion in the positive Weyl chamber whose infinitesimal generator is (6.2). The dependence on u has been suppressed. This controlled diffusion is the projection to the positive Weyl chamber by a suitable K^0-invariance. The cost functional for the stochastic optimal control problem is

(6.3)

$$J_K^0(U, \alpha) = E_{X(0)} \int_0^T (K_\alpha^0 \circ \varphi)(X(s)) + \langle A^0(X(s))U(s, X(s)), U(s, X(s)) \rangle \, ds$$

where K_α^0 is the spherical polynomial (4.24) for the partition α, $A^0(t) := \text{diag}(f_1(t), \dots, f_r(t))$,

$$f_i(t) = \frac{1}{{}_iK_\alpha^0(\varphi(t))} \left[\frac{\partial}{\partial t_i}(K_\alpha^0 \circ \varphi)(t) \right]^2,$$

and

$$_iK_\alpha^0(x) = \frac{x_i}{|\alpha|} \frac{\partial}{\partial x_i} K_\alpha^0(x)$$

for $i = 1, 2, \dots, r$. It is clear that

$$\sum_{i=1}^r {}_iK_\alpha^0(x) = K_\alpha^0(x).$$

Since the infinitesimal generator of $(X(s), s \in [0, T])$ is (6.2), the family of stochastic differential equations that $(X(s), s \in [0, T])$ satisfies is

(6.4)
$$dX_k(s) = \left[-\frac{a}{4}(r-1) + \frac{a}{2} \sum_{j \neq k} \frac{1}{1 - \exp(X_j(s) - X_k(s))} \right.$$
$$\left. + U_k(s, X(s)) \right] ds + dB_k(s)$$

for $k = 1, 2, \dots, r$, $X(0) = (X_1(0), \dots, X_r(0))$, $X_1(0) > X_2(0) > \cdots > X_r(0)$, $X(s) \in \mathfrak{a}_+^0$, and $(B_1(s), \dots, B_r(s); s \geq 0)$ is a standard r-dimensional Brownian motion.

Let $(Z(s) \in \Lambda, s \in [0, T])$ be the controlled diffusion with infinitesimal generator (6.1) where the dependence on the control law $U(\cdot)$ has been suppressed for notational simplicity. An admissible control at time s is a Borel measurable function of $Z(s)$ such that the stochastic differential equation whose generator is (6.1) has one and only one strong solution. By the K^0-invariance of the cost functional it suffices to consider controls at time s that are Borel measurable functions of $X(s)$ that satisfy (6.4) so that this family of stochastic differential equations has one and only one strong solution.

Initially it is necessary to verify that the stochastic control problem (6.2, 6.3) is well posed, specifically that there is at least one control law that gives a finite value to $J_\alpha^0(U)$ in (6.3).

6.5. Lemma. *Let $(\widetilde{X}(s), s \in [0, T])$ be the diffusion process in the positive Weyl chamber with the infinitesimal generator $\frac{1}{2}\Delta_K^0$, that is, the unique strong solution of (6.4) with $U(t) \equiv 0$. Then*

(6.6)
$$E_{X(0)} \int_0^T (K_\alpha^0 \circ \varphi)(\widetilde{X}(s)) \, dx < \infty.$$

Proof. The existence and the uniqueness of a strong solution of (6.4) with $U(s) \equiv 0$ follows immediately when it is verified that $(\widetilde{X}(s), s \in [0, T])$ does not hit the walls of the positive Weyl chamber. Since this verification follows by using a comparison theorem for stochastic differential equations [IW, p. 352] in a simpler manner than in the proof of Theorem 9.7, it is omitted here.

Let R be the Riemann curvature tensor of Λ. The Ricci tensor field S is defined as

$$S(X, Y) = \sum_{i=1}^{n} R(V_i, Y, V_i, X)$$

where $X, Y \in T_X\Lambda$, $\dim \Lambda = n$, and $(V_1, \ldots, V_n)$ is an orthonormal basis of $T_x M$. $S(\cdot, \cdot)$ is symmetric. Let ρ be defined as $\rho(X) = S(X, X)$ where $|X| = 1$. Since Λ is a Riemannian homogeneous space, it suffices to compute the Riemannian curvature and the Ricci curvature at one point. It follows that there is a $b < 0$ such that $(n - 1)b \le \rho(X)$ for a unit vector $X \in T_x\Lambda$ and $x \in \Lambda$. Define the real-valued process $(R(s), s \in [0, T])$ as

$$R(s) = d(e, \widetilde{X}(s))$$

where e is the base point of Λ, $d(\cdot, \cdot)$ is the distance function on Λ, and $(\widetilde{X}(s), s \in [0, T])$ is the process given in the statement of the lemma. Let $(Y_b(s), s \in [0, T])$ be the standard Brownian motion in the real hyperbolic space $\mathbb{H}^n(\mathbb{R})$ with the negative sectional curvature b. It follows [IW, p. 380] that

$$P(Y_b(s) \ge R(s); s \in [0, T]) = 1.$$

Thus is suffices to verify

$$E_{X(0)} \int_0^T (K_\alpha^0 \circ \varphi)(Y_b(s)) \, ds < \infty.$$

This verification is easily made [D5, Lemma]. Q.E.D.

The solution of the stochastic control problem (6.1, 6.3) is given now.

6.7. Theorem. *The stochastic control problem described by (6.1, 6.3) has an optimal control U^* given by*

(6.8) $$U^*(s, t) = -\frac{g(s)}{2} A^{-1}(t) \nabla_t (K_\alpha^0 \circ \varphi)(t)$$

where $s \in [0, T]$, $t = (t_1, \ldots, t_r)$, ∇_t is the t-gradient, and g is the unique positive solution of the scalar Riccati differential equation

(6.9) $$\frac{dg}{ds} + \frac{1}{2}\chi_\alpha^0 g - \frac{1}{4}g^2 + 1 = 0, \quad g(T) = 0,$$

where χ_α^0 is the eigenvalue (4.12).

Proof. It is well known (e.g. [FR]) that the Hamilton-Jacobi or dynamic programming equation for a stochastic optimal control problem of diffusion type is

(6.10) $$0 = \frac{\partial W}{\partial s} + \min_{u \in U}[A^v(s)W + L(s, x, v)]$$

where A^v is the infinitesimal generator of the diffusion using the control v and L is the integrand of the cost functional. To apply a verification theorem [FR,

p. 139] to the problem here it is required that the solution W of (6.10) with the boundary condition $W(s, y)$ for $(s, y) \in \{T\} \times \Lambda$ is $\mathscr{C}^{1,2}[(0, T) \times \Lambda]$ and continuous on $[0, T] \times \Lambda$.

To apply the Hamilton-Jacobi equation to (6.1, 6.3) it is important to use the fact that the integrand of the cost functional as a function of the state is K^0-invariant so that the Hamilton-Jacobi equation (6.10) can be reduced to the radial part of (6.10) as

$$(6.11) \qquad 0 = \frac{\partial W}{\partial s} + \min_{v \in \mathbb{R}^r} \left[\frac{1}{2} \Delta_K^0 W + \sum_{j=1}^{r} v_j \frac{\partial W}{\partial t_j} + (K_\alpha^0 \circ \varphi)(t) + \sum_{j=1}^{r} f_j v_j^2 \right].$$

It is clear that the family of controls obtained by the minimization in (6.11) is

$$(6.12) \qquad U_j^*(s, t) = -\frac{1}{2 f_j(t)} \frac{\partial W(s, t)}{\partial t_j}$$

for $j = 1, 2, \ldots, r$.

Assume a solution W of (6.11) as

$$(6.13) \qquad W(s, t) = (K_\alpha^0 \circ \varphi)(t) g(s).$$

Substituting (6.13) into (6.11) it follows that

$$
\begin{aligned}
(6.14) \qquad 0 &= g' K_\alpha^0 \circ \varphi + \frac{1}{2} \chi_\alpha^0 g K_\alpha^0 \circ \varphi - \frac{1}{4} g^2 K_\alpha^0 \circ \varphi + K_\alpha^0 \circ \varphi \\
&= K_\alpha^0 \circ \varphi' \left(g' + \frac{1}{2} \chi_\alpha^0 g - \frac{1}{4} g^2 + 1 \right).
\end{aligned}
$$

If g in (6.14) satisfies (6.9) then (6.13) is a smooth solution of (6.11) that satisfies the boundary condition $W(s, t) = 0$ for $(s, t) \in \{T\} \times \mathfrak{a}_+^0$. The family of controls (6.12) is an admissible (optimal) control if the family of stochastic differential equations (6.4) with these controls has one and only one strong solution in $[0, T]$.

Since the coefficients of the family of stochastic differential equations (6.4) with the controls (6.12) are locally in $\mathfrak{a}_+^0$, a unique strong solution can be constructed at least locally.

Let $(X(s), s \in [0, T])$ be the solution of the family of stochastic differential equations (6.4) with the family of controls (6.12). There is no finite escape time by the methods in the proof of Lemma 6.5. If it is assumed that $(X(s), s \in [0, T])$ does not hit the walls of the positive Weyl chamber then the local smoothness of the coefficients of the stochastic differential equations for $(X(s), s \in [0, T])$ can be used to construct a unique strong global solution (e.g. [IW, p. 164]).

Now it is verified that $(X(s), s \in [0, T])$ does not hit the walls of the Weyl chamber. It suffices to consider one wall of the Weyl chamber because the intersection of the walls of the Weyl chamber can be treated in a similar fashion.

Choose two positive constants $K \gg c > 0$ and define $\Gamma_k(c, K) \subset \mathfrak{a}_+^0$ for $k = \{1, \ldots, r-1\}$ as

$$
\begin{aligned}
(6.15) \qquad \Gamma_k(c, K) = \{ t : K > t_1 - rc > \cdots > t_{k+2} - (r - k + 1)c > t_{k+1} \\
\geq t_k > t_{k-1} - kc > \cdots > t_{r-1} - 2c > t_r > -K \}.
\end{aligned}
$$

To investigate the process $(X(s), s \in [0, T])$ in a neighborhood of $\{t: t_k = t_{k+1}\} \cap \Gamma_k(c, K)$ it suffices to consider the stochastic differential equation for

(6.16) $$Y_k(s) = X_k(s) - X_{k+1}(s)$$

for $s \in [0, T]$ and show that $(Y_k(s), s \in [0, T])$ does not hit zero. The stochastic differential equation for $(Y_k(s), s \in [0, T])$ is

(6.17)
$$
dY_k(s) = \left[\frac{a}{2} \sum_{j \neq k} \frac{1}{1 - \exp(X_j(s) - X_k(s))} \right.
$$
$$
- \frac{a}{2} \sum_{j \neq k+1} \frac{1}{1 - \exp(X_j(s) - X_{k+1}(s))}
$$
$$
\left. + U_k(s, X(s)) - U_{k+1}(s, X(s)) \right] ds + dB_k(s) - dB_{k+1}(s).
$$

It is assumed that $(X(s), s \in [0, T])$ is localized to $\Gamma_k(c, K)$.

Let $(Z(s), s \in [0, T])$ be the so-called two-dimensional Bessel process that is the unique strong solution of the stochastic differential equation

(6.18) $$dZ(s) = \frac{1}{Z(s)} ds + \sqrt{2} dB(s)$$

where $(B(s), s \geq 0)$ is a real-valued standard Brownian motion.

It is convenient to regroup the drift terms in (6.17) as

(6.19)
$$
dY_k(s) = \left[\frac{a/2}{1 - \exp(X_{k+1}(s) - X_k(s))} \right.
$$
$$
\left. - \frac{a/2}{1 - \exp(X_k(s) - X_{k+1}(s))} + h(s, X(s)) \right] ds
$$
$$
+ \sqrt{2} dB(s)
$$

where
$$
\sup_{(s,x) \in [0,T] \times \Gamma_k(c,K)} |h(s, x)| < M < \infty.
$$

By the expansion of the exponential function there is a $\delta > 0$ such that for $x \in (0, \delta]$

(6.20) $$-M + \frac{1/2}{1 - e^{-x}} - \frac{1/2}{1 - e^x} > \frac{1}{x}.$$

Localizing $\{X(s), s \in [0, T]\}$ to $\{t: t_k - t_{k+1} < \delta\} \cap \Gamma_k(c, K)$ the stochastic differential equations (6.4) and (6.17) can be compared [IW, p. 352].

Choosing sequences $c_n \downarrow 0$ and $K_n \uparrow +\infty$ and applying the above construction to $\Gamma_n(c_n, K_n)$ it follows that

(6.21)
$$
P(X_1(s) > X_2(s) > \cdots > X_k(s) = X_{k+1}(s) > \cdots > X_r(s)
$$
$$
\text{for some } s \in [0, T]) = 0.
$$

This completes the proof of the theorem.   Q.E.D.

7. Symmetric tube and Siegel domains

Every Jordan *-algebra Z with product $z \circ w$, unit element e, and involution z^* defines a Jordan triple product

$$(7.1) \qquad \{zw^*z\} = 2(z \circ w^*) \circ z - w^* \circ (z \circ z)$$

for which e is a tripotent. The corresponding Cayley transform

$$g_e(z) = \exp\left(\frac{\pi}{4}(e + z^2)\frac{\partial}{\partial z}\right)(z) = (e + z) \circ (e - z)^{-1}$$

maps Ω onto the *symmetric tube domain*

$$(7.2) \qquad \Pi = \{z \in Z : z + z^* \in \Lambda\} = \Lambda \times U,$$

where Λ is the symmetric cone of the formally real Jordan algebra

$$X := \{x \in Z : x^* = x\},$$

and

$$U := \{\xi \in Z : \xi^* = -\xi\}.$$

Of particular importance is the case $U = iX$; i.e., Z is the *complexification* of a formally real Jordan algebra X endowed with the complexified Jordan product and the involution

$$(x + iy)^* := x - iy \qquad (x, y \in X).$$

In this case $\Pi = \Lambda \times iX$ is a complex hermitian symmetric space. A basic example is Siegel's (right) half-space, corresponding to the cone $\Lambda_{\mathbf{R}}$ of positive definite symmetric matrices (cf. Example 3.5). In order to determine the (semisimple) Lie algebra $\mathfrak{g}$ associated with Π, we apply the (adjoint) Cayley transformation and obtain for all $x \in X$, $\xi \in U$, and $M \in \mathfrak{l}^0$ (cf. (3.7)).

$$\mathrm{Ad}(g_e)\left((x - \{zx^*z\})\frac{\partial}{\partial z}\right) = 2\{xe^*z\}\frac{\partial}{\partial z},$$

$$\mathrm{Ad}(g_e)\left((\xi - \{z\xi^*z\} - 2\{\xi e^*z\})\frac{\partial}{\partial z}\right) = 2\xi\frac{\partial}{\partial z},$$

$$\mathrm{Ad}(g_e)\left((\xi - \{z\xi^*z\} + 2\{\xi e^*z\})\frac{\partial}{\partial z}\right) = 2\{z\xi^*z\}\frac{\partial}{\partial z}$$

and

$$\mathrm{Ad}(g_e)M = M.$$

It follows that $\mathfrak{g}$ has a Cartan decomposition

$$(7.3) \qquad \mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p},$$

where

$$\mathfrak{l} = \mathfrak{l}^0 \oplus \left\{(\xi + \{z\xi^*z\})\frac{\partial}{\partial z} : \xi \in U\right\}$$

consists of all vector fields vanishing at e, and

$$\mathfrak{p} = \mathfrak{p}^0 \oplus \left\{(\xi - \{z\xi^*z\})\frac{\partial}{\partial z} : \xi \in U\right\}$$

is the tangent space at e.

Denoting by G the connected Lie subgroup of L with Lie algebra $\mathfrak{g}$, $\mathfrak{k}$ becomes the Lie algebra of the maximal compact subgroup

(7.4) $$K = \{g \in G\colon g(e) = e\}$$

(which contains nonlinear transformations if $U \neq \{0\}$). In the complex case, G is the identity component of the biholomorphic automorphism group of Π [L3, U1]. The Lie algebra $\mathfrak{gl}(\Lambda)$ has a representation

$$\mathfrak{gl}(\Lambda) \ni M \mapsto \widetilde{M} \in \mathfrak{gl}(Z)$$

such that

$$\widetilde{M}z_1 + (\widetilde{M}z_1)^* = M(z_1 + z_1^*)$$

for all $M \in \mathfrak{gl}(\Lambda)$. It follows that $\widetilde{M}$ leaves Π invariant. The domain Π is also invariant under the translations $\tau_\xi(z) := z + \xi$ parametrized by $\xi \in U$.

The vector field

$$z\frac{\partial}{\partial z} = \{ee^*z\}\frac{\partial}{\partial z} \in \mathfrak{g}^0 \subset \mathfrak{g}$$

induces a gradation

(7.5) $$\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^{-1},$$

where

$$\mathfrak{g}^\lambda := \left\{A \in \mathfrak{g}\colon \left[z\frac{\partial}{\partial z}, A\right] = \lambda A\right\}.$$

We have

$$\mathfrak{g}^1 = \left\{\xi\frac{\partial}{\partial z}\colon \xi \in U\right\}, \qquad \mathfrak{g}^0 = \left\{Mz\frac{\partial}{\partial z}\colon M \in \mathfrak{gl}(\lambda)\right\},$$

and

$$\mathfrak{g}^{-1} = \left\{\{z\xi z\}\frac{\partial}{\partial z}\colon \xi \in U\right\}.$$

Now fix a frame $e_1, \ldots, e_r$ of the Jordan algebra X, and consider the Peirce decomposition (3.9) of X. Similarly, we have

(7.6) $$U = \sum_{1 \leq i \leq j \leq r} \bigoplus U_{ij}$$

where

$$U_{ij} = \left\{\xi \in U;\ \{e_k e_k^* \xi\} = \frac{\delta_{ik} + \delta_{jk}}{2}\xi,\ \forall 1 \leq k \leq r\right\}.$$

The invariants

(7.7) $$c := \dim_{\mathbb{R}} U_{ii} \qquad (1 \leq i \leq r)$$

and

(7.8) $$a := \dim_{\mathbb{R}} U_{ij} \qquad (1 \leq i < j \leq r)$$

are independent of the frame and are called the *characteristic multiplicities* of U. One can show [N1] that, for irreducible symmetric spaces, (7.8) agrees with (3.10). The subspace $\mathfrak{a} \subset \mathfrak{p}^0 \subset \mathfrak{p}$ defined in (3.11) is still maximal abelian in $\mathfrak{p}$.

7.9. Proposition. *There is a real root decomposition*

$$(7.10) \qquad \mathfrak{g} = \sum_{i \le j} \mathfrak{g}_{ij}^1 \oplus \sum_{i,j} \mathfrak{g}_{ij}^0 \oplus \sum_{i \le j} \mathfrak{g}_{ij}^{-1} \oplus \mathfrak{m},$$

with respect to $\mathfrak{a}$, *where* $\mathfrak{g}_{ij}^0$ *and* $\mathfrak{m}$ *are as in Proposition 3.14, and*

$$\mathfrak{g}_{ij}^1 := \left\{ \xi \frac{\partial}{\partial z} : \xi \in U_{ij} \right\}$$

is the root space of

$$(7.11) \qquad \frac{1}{2}(M_{e_i}^\# + M_{e_j}^\#),$$

whereas

$$\mathfrak{g}_{ij}^{-1} := \left\{ \{z\xi^* z\} \frac{\partial}{\partial z} : \xi \in U_{ij} \right\}$$

is the root space of

$$(7.12) \qquad -\frac{1}{2}(M_{e_i}^\# + M_{e_j}^\#).$$

Proof. For $\xi \in U_{ij}$, (2.1), (2.2), and (7.6) imply

$$\left[\{e_k e^* z\} \frac{\partial}{\partial z}, \xi \frac{\partial}{\partial z} \right] = \{e_k e^* \xi\} \frac{\partial}{\partial z} = \frac{\delta_{ik} + \delta_{jk}}{2} \xi \frac{\partial}{\partial z}$$

and

$$\left[\{e_k e^* z\} \frac{\partial}{\partial z}, \{z\xi^* z\} \frac{\partial}{\partial z} \right]$$

$$= (\{e_k e^* \{z\xi^* z\}\} - 2\{\{e_k e^* z\}\xi^* z\}) \frac{\partial}{\partial z}$$

$$= -\{z\{e e_k^* \xi^*\}^* z\} \frac{\partial}{\partial z} = -\frac{\delta_{ik} + \delta_{jk}}{2} \{z\xi^* z\} \frac{\partial}{\partial z}.$$

This proves (7.11) and (7.12), and shows that $\mathfrak{g}^{\pm 1}$ is the direct sum of $\mathfrak{g}_{ij}^{\pm 1}$. Applying Proposition 3.14, the direct decomposition (7.10) follows.   Q.E.D.

We now treat the general case of a (positive definite) Jordan triple Z with a Peirce decomposition $Z = X \oplus U \oplus V$ with respect to a maximal tripotent e (the unit element of X). The corresponding Cayley transform

$$g_e(z_1, z_2) = \exp\left(\frac{\pi}{4}(e + \{ze^* z\}) \frac{\partial}{\partial z} \right)(z_1, z_2)$$

$$= ((e + z_1) \circ (e - z_1)^{-1}, \sqrt{2}z_2 \circ (e - z_1)^{-1})$$

(cf. [L3, U1]) maps Ω onto the *symmetric Siegel domain*

$$(7.13) \qquad \Pi = \{x + \xi + v : 2x - \Phi(v, v) \in \Lambda\}$$

associated with the cone Λ of X and the hermitian form $\Phi(v_1, v_2) := 2 \{v_1 v_2^* e\}$. In case $U = iX$ and V is complex, Π becomes an hermitian symmetric space.

7.14. **Example.** Let $r < R$ be integers and consider the matrix space

$$Z = {}_r\mathbb{K}^R = {}_r\mathbb{K}^r \oplus {}_r\mathbb{K}^{(R-r)}$$

over $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (quaternions), endowed with the Jordan triple product

$$\{zw^*z\} := zw^*z.$$

Writing the elements of Z as $z = (z_1, z_2)$ with z_1 of size $r \times r$ and z_2 of size $r \times (R-r)$, we obtain a tripotent $e = (1, 0)$, and the associated symmetric Siegel domain

$$\Pi = \{(z_1, z_2) \in Z : z_1 + z_1^* - z_2 z_2^* \in \Lambda_\mathbb{K}\}$$

belongs to the symmetric matrix cone $\Lambda_\mathbb{K}$ (cf. (3.5)) and the hermitian form $\Phi(z_2, w_2) := z_2 w_2^*$. For example if $r = 1$, we obtain

(7.15) $$\Pi = \{(z_0, z_1, \ldots, z_n) \in \mathbb{K}^{1+n} : z_0 + \bar{z}_0 > z_1\bar{z}_1 + \cdots + z_n\bar{z}_n\}.$$

For $\mathbb{K} = \mathbb{C}$, we obtain complex hermitian spaces in this way. For example, (7.15) is holomorphically equivalent to the unit ball in $\mathbb{C}^{1+n}$. In order to determine the (semisimple) Lie algebra $\mathfrak{g}$ associated with Π we apply the (adjoint) Cayley transformation and obtain for all $x \in X$, $\xi \in U$, $\eta \in V$, and $M \in \mathfrak{l}^0$

$$\mathrm{Ad}(g_e)\left((x - \{zx^*z\})\frac{\partial}{\partial z}\right) = 2\{xe^*z\}\frac{\partial}{\partial z},$$

$$\mathrm{Ad}(g_e)\left((\xi - \{z\xi^*z\} - 2\{\xi e^*z\})\frac{\partial}{\partial z}\right) = 2\xi\frac{\partial}{\partial z},$$

$$\mathrm{Ad}(g_e)\left((\xi - \{z\xi^*z\} + 2\{\xi e^*z\})\frac{\partial}{\partial z}\right) = 2\{z\xi^*z\}\frac{\partial}{\partial z}$$

$$\mathrm{Ad}(g_e)\left((\eta - \{z\eta^*z\} - 2\{\eta e^*z\} + 2\{e\eta^*z\})\frac{\partial}{\partial z}\right)$$
$$= \sqrt{2}(\eta + 2\{e\eta^*z\})\frac{\partial}{\partial z},$$

$$\mathrm{Ad}(g_e)\left((\eta - \{z\eta^*z\} + 2\{\eta e^*z\} - 2\{e\eta^*z\})\frac{\partial}{\partial z}\right)$$
$$= \sqrt{2}(\{z\eta^*z\} + 2\{\eta e^*z\})\frac{\partial}{\partial z}$$

and

$$\mathrm{Ad}(g_e)M = M.$$

It follows that $\mathfrak{g}$ has a Cartan decomposition

(7.16) $$\mathfrak{g} = \mathfrak{l} + \mathfrak{p},$$

where

$$\mathfrak{l} = \mathfrak{l}^0 \oplus \left\{(\xi + \{z\xi^*z\})\frac{\partial}{\partial z} : \xi \in U\right\}$$

$$\oplus \left\{(\eta + 2\{e\eta^*z\} - \{z\eta^*z\} - 2\{\eta e^*z\})\frac{\partial}{\partial z} : \eta \in V\right\}$$

consists of all vector fields vanishing at $(e, 0)$, and

$$p = p^0 \oplus \left\{ (\xi - \{z\xi^* z\}) \frac{\partial}{\partial z} : \xi \in U \right\}$$

$$\oplus \left\{ (\eta + 2\{e\eta^* z\} + \{z\eta^* z\} + 2\{\eta e^* z\}) \frac{\partial}{\partial z} : \eta \in V \right\}$$

is the tangent space at $(e, 0)$. Denoting by G the connected Lie subgroup of L with Lie algebra $\mathfrak{g}$, $\mathfrak{l}$ becomes the Lie algebra of the maximal compact subgroup

$$(7.17) \qquad\qquad K = \{g \in G \, g(e, 0) = (e, 0)\}.$$

In the complex case, G is the identity component of the biholomorphic automorphism group of Π. The Lie algebra $\mathfrak{gl}(\Lambda)$ has a representation

$$\mathfrak{gl}(\Lambda) \ni P \mapsto \widetilde{P} \in \mathfrak{gl}(Z)$$

such that

$$(7.18) \quad \widetilde{M}z_1 + (\widetilde{M}z_1)^* - \Phi(\widetilde{M}z_2, z_2) - \Phi(z_2, \widetilde{M}z_2) = M(z_1 + z_1^* - \Phi(z_2, z_2))$$

for all $M \in \mathfrak{gl}(\Lambda)$. The domain Π is invariant under the "pseudotranslations"

$$(7.19) \qquad \tau_{\xi, \eta}(z_1, z_2) := \left(z_1 + \xi + \Phi(z_2, \eta) + \frac{\Phi(\eta, \eta)}{2}, z_2 + \eta \right)$$

parametrized by $(\xi, \eta) \in U \oplus V$. The vector field

$$\{ee^* z\} \frac{\partial}{\partial z} = z_1 \frac{\partial}{\partial z_1} + \frac{z_2}{2} \frac{\partial}{\partial z_2} \in \mathfrak{g}$$

induces a gradation

$$(7.20) \qquad\qquad \mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^{1/2} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^{-1/2} \oplus \mathfrak{g}^{-1}$$

into the λ-eigenspaces $\mathfrak{g}^\lambda$ of $\mathrm{ad}(\{ee^* z\} \frac{\partial}{\partial z})$. We have

$$\mathfrak{g}^1 = \left\{ \xi \frac{\partial}{\partial z} : \xi \in U \right\},$$

$$\mathfrak{g}^{1/2} = \left\{ (\eta + 2\{z\eta^* e\}) \frac{\partial}{\partial z} : \eta \in V \right\}, \qquad \mathfrak{g}^0 = \left\{ \widetilde{M}z \frac{\partial}{\partial z} : M \in \mathfrak{gl}(\Lambda) \right\},$$

$$\mathfrak{g}^{-1/2} = \left\{ (\{z\eta^* z\} + 2\{ze^* \eta\}) \frac{\partial}{\partial z} : \eta \in V \right\},$$

and

$$g^{-1} = \left\{ \{z\xi^* z\} \frac{\partial}{\partial z} : \xi \in U \right\}.$$

Now fix a frame $e_1, \ldots, e_r$ of the Jordan algebra X and consider the Peirce decompositions (3.9) and (7.6) of X and U, respectively. Similarly, we have

$$(7.21) \qquad\qquad V = \sum_{j=1}^{r} \bigoplus V_j$$

where

$$V_j = \left\{ \eta \in V : \{e_k e_k^* \eta\} = \frac{\delta_{jk}}{2} \eta, \; \forall \, 1 \le k \le r \right\}.$$

The invariant

$$(7.22) \qquad b := \dim_{\mathbb{R}} V_j \qquad (1 \le j \le r)$$

is independent of the frame and is called the *characteristic multiplicity* of V. For example, in the matrix space $Z = {}_r \mathbb{K}^R$, V_j consists of all $r \times (R - r)$ matrices v with nonzero entries only in the j th row. The subspace $\mathfrak{a} \subset \mathfrak{p}^0 \subset \mathfrak{p}$ defined in (3.11) is still maximal abelian in $\mathfrak{p}$.

7.23. Proposition. *There is a real root decomposition*

$$(7.24) \qquad \begin{aligned} \mathfrak{g} &= \sum_{i \le j} \mathfrak{g}_{ij}^1 \oplus \sum_j \mathfrak{g}_j^{1/2} \oplus \sum_{i,j} \mathfrak{g}_{ij}^0 \\ &\oplus \sum_j g_j^{-1/2} \oplus \sum_{i \le j} g_{ij}^{-1} \oplus m, \end{aligned}$$

with respect to $\mathfrak{a}$, *where* $\mathfrak{g}_{ij}^{\pm 1}$, $\mathfrak{g}_{ij}^0$ *and* $\mathfrak{m}$ *are as in Propositions 7.9 and 3.14, and*

$$\mathfrak{g}_j^{1/2} := \left\{ (\eta + 2\{e\eta^* z\}) \frac{\partial}{\partial z} : \eta \in V_j \right\}$$

is the root space of

$$(7.25) \qquad \frac{1}{2} M_{e_j}^\#,$$

whereas

$$\mathfrak{g}_j^{-1/2} := \left\{ (\{z\eta^* z\} + 2\{\eta e^* z\}) \frac{\partial}{\partial z} : \eta \in V_j \right\}$$

is the root space of

$$(7.26) \qquad -\frac{1}{2} M_{e_j}^\#.$$

Proof. For $\eta \in V_j$, (2.1), (2.2), and (7.21) imply

$$\left[\{e_k e^* z\} \frac{\partial}{\partial z}, \ (\eta + 2\{e\eta^* z\}) \frac{\partial}{\partial z} \right]$$

$$= (\{e_k e^* \eta\} + 2(\{e_k e^* \{e\eta^* z\}\} - \{e\eta^* \{e_k e^* z\}\})) \frac{\partial}{\partial z}$$

$$= (\{e_k e^* \eta\} + 2\{e\{e_k e^* \eta\}^* z\}) \frac{\partial}{\partial z}$$

$$= \frac{\delta_{jk}}{2} (\eta + 2e\eta^* z) \frac{\partial}{\partial z}$$

since $\{e_k \eta^* e\} = 0$. Similarly,

$$\left[\{e_k e^* z\} \frac{\partial}{\partial z}, \ (\{z\eta^* z\} + 2\{\eta e^* z\}) \frac{\partial}{\partial z} \right]$$

$$= (\{e_k e^* \{z\eta^* z\}\} + 2\{e_k e^* \{\eta e^* z\}\}) \frac{\partial}{\partial z}$$

$$= (\{\{e_k e^* z\} \eta^* z\} - \{\eta e^* \{e_k e^* z\}\}) \frac{\partial}{\partial z}$$

$$= -(\{z \{e_k e^* \eta\}^* z\} + 2\{\{e_k e^* \eta\} e^* z\}) \frac{\partial}{\partial z}$$

$$= -\frac{\delta_{ij}}{2} (\{z\eta^* z\} + 2\{\eta e^* z\}) \frac{\partial}{\partial z}.$$

This proves (7.25) and (7.26), and shows that $\mathfrak{g}^{\pm 1/2}$ is the direct sum of $\mathfrak{g}_j^{\pm 1/2}$. Applying Propositions 7.9 and 3.14, the direct decomposition (7.24) follows. Q.E.D.

8. Conical and spherical functions on Π

We will now construct conical and spherical Laplace eigenfunctions for the symmetric tube and Siegel domains Π discussed is Section 7. Endow Π with the unique G-invariant Riemannian structure such that the basis (3.11) of $\mathfrak{a} \subset \mathfrak{p}^0 \subset \mathfrak{p} \approx T_e(\Pi)$ (tangent space) becomes orthonormal.

Let N be the connected Lie subgroup with Lie algebra

$$\mathfrak{n} = \mathfrak{g}^1 \oplus \mathfrak{g}^{1/2} \oplus \mathfrak{n}^0 = \sum_{i \le j} \mathfrak{g}_{ij}^1 \oplus \sum_j \mathfrak{g}_j^{1/2} \oplus \sum_{i < j} \mathfrak{g}_{ij}^0 .$$

(For tube domains, the $\mathfrak{g}^{1/2}$-part is missing.) Then there is a (direct) *Iwasawa decomposition* [H1]

$$\Pi = N \cdot \varphi(\mathfrak{a}) \tag{8.1}$$

where φ is defined in (4.1).

Let $\mathscr{D}(\Pi)$ denote the algebra of all (scalar) differential operators $\mathscr{D}$ on Π which are invariant under G, i.e., satisfy (4.3) for all $f \in \mathscr{C}^\infty(\Pi)$ and $g \in G$. For example, the *Laplace-Beltrami operator* Δ of Π belongs to $\mathscr{D}(\Pi)$. By [H1, Theorem II.3.6], every $D \in \mathscr{D}(\Pi)$ has an *N-radial part* D_N, which is a differential operator on $\mathfrak{a}$ satisfying (4.4) for all N-invariant functions $f \in \mathscr{C}^\infty(\Pi)$.

8.2. Proposition. Δ *has the N-radial part*

$$\Delta_N = \sum_i \left(\frac{\partial^2}{\partial t_1^2} - \left(a(i-1) + c + \frac{b}{2} \right) \frac{\partial}{\partial t_i} \right) .$$

Proof. For any $D \in \mathscr{D}(\Pi)$, D_N is a constant-coefficient differential operator of the form

$$D_N = q \left(\frac{\partial}{\partial t_1} - \rho_1 , \ldots , \frac{\partial}{\partial t_r} - \rho_r \right) , \tag{8.3}$$

where q is any even symmetric (i.e., Weyl group invariant) polynomial, and

$$\rho_i = \frac{a}{2}(i-1) + \frac{c}{2} + \frac{b}{4}$$

are the coefficients of the half-sum of positive roots

$$\begin{aligned}
\rho &= \frac{a}{2} \sum_{i < j} \left[\frac{1}{2}(M_{e_j}^\# - M_{e_i}^\#) + \frac{1}{2}(M_{e_j}^\# + M_{e_i}^\#) \right] + \frac{c}{2} \sum_i M_{e_i}^\# + \frac{b}{2} \sum_i \frac{1}{2} M_{e_i}^\# \\
&= \frac{1}{2} \sum_i \left(a(i-1) + c + \frac{b}{2} \right) M_{e_i}^\#
\end{aligned} \tag{8.4}$$

(cf. Propositions 7.9 and 7.23). By [H1, Proposition II.3.8], Δ corresponds to the polynomial

$$q(s_1, \ldots, s_r) = \sum_i (s_i^2 - \rho_i^2) \tag{8.5}$$

so that

$$\Delta_N = \sum_i \left(\frac{\partial^2}{\partial t_i^2} - 2\rho_i \frac{\partial}{\partial t_i} \right). \quad \text{Q.E.D.}$$

8.6. Proposition. *For each $\alpha \in \mathbb{C}^r$, the smooth function*

$$(8.7) \qquad N_\alpha(x + \xi + v) = N_\alpha \left(x - \frac{\Phi(v, v)}{2} \right)$$

on Π is an N-invariant eigenfunction of Δ, with eigenvalue

$$(8.8) \qquad \chi_\alpha = \sum_i \alpha_i \left(\alpha_i - a(i-1) - c - \frac{b}{2} \right).$$

Proof. The group N is generated by N^0 and the pseudo-translations (7.19). Since the expression $2x - \Phi(v, v)$ is invariant under pseudo-translations and satisfies (7.18) for all $P \in N^0$ it follows that (8.7) is N-invariant. Since

$$(N_\alpha \circ \varphi)(t_1, \ldots, t_r) = e^{t_1\alpha_1 + \cdots + t_r\alpha_r},$$

every $D \in \mathscr{D}(\Pi)$ given by (8.3) satisfies

$$DN_\alpha = q(\alpha_1 - \rho_1, \ldots, \alpha_r - \rho_r)N_\alpha.$$

For the polynomial (8.5), we obtain

$$\Delta N_\alpha = \sum_i (\alpha_i^2 - 2\alpha_i\rho_i)N_\alpha. \quad \text{Q.E.D.}$$

We now pass to the maximal compact subgroup K of G. Let

$$(8.9) \qquad \mathfrak{a}_+ := \left\{ \sum_{i=1}^r t_i M_{e_i}, \, 0 < t_1 < \cdots < t_r \right\}$$

denote the "Weyl chamber". Then there is a *Cartan decomposition*

$$(8.10) \qquad \Pi = K\varphi(\mathfrak{a}_+),$$

where φ is given by (4.1). By [H1, Theorem II.3.6], every $D \in \mathscr{D}(\Pi)$ has a *K-radial part* D_K which is a differential operator on $\mathfrak{a}_+$ satisfying (4.15) for all K-invariant functions $f \in \mathscr{C}^\infty(\pi)$.

8.11. Proposition. Δ *has the K-radial part*

$$(8.12) \qquad \begin{aligned} \Delta_K = &\sum_{i=1}^r \left(\frac{\partial^2}{\partial t_i^2} + \left(c \coth(t_i) + b \coth\left(\frac{t_i}{2} \right) \right) \frac{\partial}{\partial t_i} \right) \\ &+ a \sum_{i \neq j} \frac{\sinh(t_i)}{\cosh(t_j) - \cosh(t_i)} \frac{\partial}{\partial t_i}. \end{aligned}$$

Proof. The positive roots (3.16), (7.11), and (7.25) induce the density function

$$\begin{aligned} \delta(t_1, \ldots, t_r) :&= \prod_j \sinh^c(t_j) \sinh^b\left(\frac{t_j}{2} \right) \prod_{i<j} \sinh^a\left(\frac{t_j - t_i}{2} \right) \sinh^a\left(\frac{t_j + t_i}{2} \right) \\ &= \prod_j \sinh^c(t_j) \sinh^b\left(\frac{t_j}{2} \right) \prod_{i<j} \left[\sinh^2\left(\frac{t_j}{2} \right) - \sinh^2\left(\frac{t_i}{2} \right) \right]^a \\ &= \text{const} \cdot \omega^a \pi, \end{aligned}$$

where

$$\omega = \prod_{i<j} \left(\sinh^2 \left(\frac{t_j}{2} \right) - \sinh^2 \left(\frac{t_i}{2} \right) \right),$$

(8.13)

$$\pi = \prod_{j} \sinh^c(t_j) \sinh^b \left(\frac{t_j}{2} \right).$$

By [H1, p. 335] we have

$$\Delta_K = \frac{1}{\delta} \sum_i \frac{\partial}{\partial t_i} \delta \frac{\partial}{\partial t_i} = \frac{1}{\omega^a \pi} \sum_i \frac{\partial}{\partial t_i} \omega^a \pi \frac{\partial}{\partial t_i}$$

$$= \sum_i \left(\frac{\partial}{\partial t_i} + \frac{1}{\pi} \frac{\partial \pi}{\partial t_i} + \frac{a}{\omega} \frac{\partial \omega}{\partial t_i} \right) \frac{\partial}{\partial t_i}$$

$$= \sum_i \left(\frac{\partial}{\partial t_i} + c \coth(t_i) + b \coth \left(\frac{t_i}{2} \right) + a \sum_{j \neq i} \frac{\sinh(t_i)}{\cosh(t_j) - \cosh(t_i)} \right) \frac{\partial}{\partial t_i}.$$

Q.E.D.

Now consider the mapping $\psi : \mathfrak{a}_+ \to \Lambda \subset \Pi$ defined by

(8.14)

$$\psi(t_1, \ldots, t_r) = \sum_{i=1}^r \sinh^2 \left(\frac{t_i}{2} \right) e_i.$$

8.15. Corollary.

$$\psi_* \Delta_K = \sum_i \left(x_i(x_i + 1) \frac{\partial^2}{\partial x_i^2} + \left(\frac{1+b+c}{2} + \left(1 + c + \frac{b}{2} \right) x_i \right) \frac{\partial}{\partial x_i} \right)$$

$$+ a \sum_{i \neq j} \frac{x_i(x_i + 1)}{x_j - x_i} \frac{\partial}{\partial x_i}.$$

Proof. Putting $x_i = \sinh^2(\frac{t_i}{2})$ we have by (8.13)

$$\omega = \prod_{i<j} (x_j - x_i), \qquad \pi = \prod_i (x_i(x_i + 1))^{c/2} x_i^{b/2}$$

and (8.12) implies

$$\psi_* \Delta_K = \sum_i \left(\frac{d^2 x_i}{dt_i^2} + \left(\frac{dx_i}{dt_i} \right)^2 \left(\frac{\partial}{\partial x_i} + \frac{1}{\pi} \frac{\partial \pi}{\partial x_i} + \frac{a}{\omega} \frac{\partial \omega}{\partial x_i} \right) \right) \frac{\partial}{\partial x_i}$$

$$= \sum_i \left(x_i + \frac{1}{2} + x_i(1 + x_i) \left(\frac{\partial}{\partial x_i} + \frac{c}{2} \frac{1 + 2x_i}{x_i(1 + x_i)} \right. \right.$$

$$\left. \left. + \frac{b}{2x_i} + a \sum_{j \neq i} \frac{1}{x_j - x_i} \right) \right) \frac{\partial}{\partial x_i}. \qquad \text{Q.E.D.}$$

8.16. Theorem. *For each $\alpha \in \bar{\mathbb{N}}^r$, there exists a spherical function K_α on Π with eigenvalue*

(8.17)

$$\chi_\alpha = \sum_i \alpha_i \left(\alpha_i + a(r - i) + c + \frac{b}{2} \right)$$

which satisfies

(8.18)
$$K_\alpha \circ \varphi \circ \psi^{-1} = \sum_{\beta \le \alpha} e_\alpha^\beta K_\beta^0.$$

Here β runs over all partitions dominated by α and the coefficients c_α^β are nonnegative.

Proof. By Corollaries 4.19 and 8.15 we have

$$\varphi_* \Delta_K^0 = D_2 + \left(1 - \frac{a}{2}(r-1)\right) E$$

and

$$\psi_* \Delta_K = D_2 + D_1 + \frac{1+b+c}{2} \partial_e + \left(1 + c + \frac{b}{2}\right) E,$$

where

$$D_m := \sum_{i=1}^r x_i^m \frac{\partial^2}{\partial x_i^2} + a \sum_{i \ne j} \frac{x_i^m}{x_j - x_i} \frac{\partial}{\partial x_i},$$

$$\partial_e := \sum_{i=1}^r \frac{\partial}{\partial x_i} \quad \text{and} \quad E := \sum_{i=1}^r x_i \frac{\partial}{\partial x_i}.$$

The commutator identities $[\partial_e, D_m] = m D_{m-1}$ and $[\partial_e, E] = \partial_e$ imply

(8.19)
$$\psi_* \Delta_K = T + \frac{1}{2}[\partial_e, T] + \frac{b}{4} \partial_e,$$

where

$$T := \varphi_* \Delta_K^0 + \left(\frac{a}{2}(r-1) + c + \frac{b}{2}\right) E.$$

Since K_α^0 is homogeneous of degree $|\alpha| := \alpha_1 + \cdots + \alpha_r$, we have $T K_\alpha^0 = \chi_\alpha K_\alpha^0$, where

(8.20)
$$\chi_\alpha = \chi_\alpha^0 + \left(\frac{a}{2}(r-1) + c + \frac{b}{2}\right) |\alpha|$$
$$= \sum_i \alpha_i \left(\alpha_i + a(r-i) + c + \frac{b}{2}\right).$$

On the other hand, [Y1] implies

$$\partial_e K_\alpha^0 = \sum_i \binom{\alpha}{\alpha - \varepsilon_i} K_{\alpha - \varepsilon_i}^0,$$

with $\alpha - \varepsilon_i := (\alpha_1, \ldots, \alpha_{i-1}, \alpha_i - 1, \alpha_{i+1}, \ldots, \alpha_r)$ and "binomial" coefficients

(8.21)
$$\binom{\alpha}{\alpha - \varepsilon_i} \ge 0.$$

Assuming that χ_α is the Δ-eigenvalue of K_α, we obtain from (8.19)

$$\chi_\alpha \sum_\beta c_\alpha^\beta K_\beta^0 = \chi_\alpha (K_\alpha \circ \varphi \circ \psi^{-1})$$

$$= (\Delta K_\alpha) \circ \varphi \circ \psi^{-1} = \Delta_K (K_\alpha \circ \varphi \circ \psi^{-1})$$

$$= (\psi_* \Delta_K)(K_\alpha \circ \varphi \circ \psi^{-1}) = \sum_\beta c_\alpha^\beta (\psi_* \Delta_K) K_\beta^0$$

$$= \sum_\beta c_\alpha^\beta \left(T + \frac{1}{2} \partial_e T - \frac{1}{2} T \partial_e + \frac{b}{4} \partial_e \right) K_\beta^0$$

$$= \sum_\beta c_\alpha^\beta \left(\chi_\beta K_\beta^0 + \frac{1}{2} \sum_i \binom{\beta}{\beta - \varepsilon_i} \left(\chi_\beta - \chi_{\beta - \varepsilon_i} + \frac{b}{2} \right) K_{\beta - \varepsilon_i}^0 \right).$$

Equating coefficients of K_β^0, and $\beta < \alpha$ fixed, we obtain

$$(\chi_\alpha - \chi_\beta) c_\alpha^\beta = \frac{1}{2} \sum_i \binom{\beta + \varepsilon_i}{\beta} \left(\chi_{\beta + \varepsilon_i} - \chi_\beta + \frac{b}{2} \right) c_\alpha^{\beta + \varepsilon_i}.$$

Since (8.20) implies $\chi_\alpha > \chi_\beta$ and $\chi_{\beta + \varepsilon_i} > \chi_\beta$, we may solve for c_α^β inductively, starting with $c_\alpha^\alpha := 1$, and obtain nonnegative coefficients.   Q.E.D.

8.22. Corollary. *The spherical function K_α on Π has a finite expansion*

$$(8.23) \qquad (K_\alpha \circ \varphi \circ \psi^{-1})(x) = \sum_{\beta \le \alpha} a_\alpha^\beta q_\beta(x)$$

in terms of the monomial symmetric functions, with nonnegative coefficients $a_\alpha^\beta \ge 0$.

Proof. Apply Theorem 8.16 and Proposition 4.21.   Q.E.D.

9. STOCHASTIC CONTROL THEORY ON SYMMETRIC TUBE OR SIEGEL DOMAINS: SPHERICAL FUNCTIONS

A stochastic control problem is formulated in a symmetric tube or Siegel domain Π using a K-invariance and a spherical polynomial K_α. The stochastic control problem is the control of a Brownian motion in Π by a drift vector field so that this controlled diffusion remains close to the base point e of Π. Using the K-invariance we may use the coordinates $t = (t_1, \ldots, t_r)$ for the positive Weyl chamber (8.9).

The controlled diffusion in Π has the infinitesimal generator

$$(9.1) \qquad \frac{1}{2} \Delta + \langle u(s, t), \nabla_t \rangle$$

where Δ is the Laplace-Beltrami operator, $t = (t_1, \ldots, t_r)$, and $s \in \mathbb{R}_+$. Since the K-invariance of the control problem is exploited, it suffices to consider the controlled diffusion projected to the positive Weyl chamber $\mathfrak{a}_+$ with the infinitesimal generator

$$(9.2) \qquad \frac{1}{2} \Delta_K + \langle u(s, t), \nabla_t \rangle.$$

Let $(X(s), s \geq 0)$ be the controlled diffusion with the infinitesimal generator (9.2) where the dependence on the control u has been suppressed. The cost functional for this stochastic control problem is

$$(9.3) \quad J_\alpha(u) = E_{X(0)} \int_0^T (K_\alpha \circ \psi)(X(s)) + \langle A(X(s))U(s, X(s)), U(s, X(s)) \rangle \, ds$$

where K_α is the spherical polynomial (8.23) for the partition α, $A(t) := \text{diag}(f_1(t), \ldots, f_r(t))$,

$$f_i(t) = \frac{1}{{}_iK_\alpha(\psi(t))} \left[\frac{\partial}{\partial t_i}(K_\alpha \circ \psi)(t) \right]^2,$$

and

$$_iK_\alpha \left(\sum_{j=1}^r x_j e_j \right) = x_i \frac{\partial}{\partial x} \sum_\mu \frac{1}{|\mu|} c_\alpha^\mu q_\mu(x_1, \ldots, x_r)$$

for $i = 1, 2, \ldots, r$. It is clear that

$$K_\alpha \circ \psi = \sum_{i=1}^r {}_iK_\alpha \circ \psi +_0 K_\alpha$$

where ${}_0K_\alpha$ is a constant.

Since the infinitesimal generator of $(X(s), s \in [0, T])$ is (9.2), the family of stochastic differential equations that $(X(s), s \in [0, T])$ satisfies is
(9.4)

$$dX_k(s) = \left[\frac{c}{2} \coth(X_k(s)) + \frac{b}{2} \coth\left(\frac{X_k(s)}{2} \right) \right.$$

$$\left. + \frac{a}{2} \sum_{j \neq k} \frac{\sinh(X_k(s))}{\cosh(X_k(s)) - \cosh(X_j(s))} + U(s, X(s)) \right] ds = dB_k(s)$$

for $k = 1, 2, \ldots, r$, $X(0) = (X_1(0), \ldots, X_r(0))$, $X_1(0) > X_2(0) > \cdots > X_r(0) > 0$, $X(t) \in \mathfrak{a}_+$, and $(B_1(s), \ldots, B_r(s); s \geq 0)$ is a standard r-dimensional Brownian motion.

Let $(Z(s) \in \Omega, s \in [0, T])$ be the controlled diffusion with the infinitesimal generator (9.1) where the dependence on the control law $U(\cdot)$ has been suppressed for notational simplicity. An admissible control at time s is a Borel measurable function of $Z(s)$ such that the stochastic differential equation whose generator is (9.1) has one and only one strong solution. By the K-invariance of the cost functional it suffices to consider controls at time s that are Borel measureable functions of $X(s)$ that satisfy (9.4) so that this family of stochastic differential equations has one and only one strong solution.

Initially it is necessary to verify that the stochastic control problem (9.1, 9.3) is well posed, specifically that there is at least one control that gives a finite value to $J_\alpha(U)$ in (9.3). The verifcation of the following lemma is the same as the verification of Lemma 6.5.

9.5. Lemma. *Let $(\widetilde{X}(s), s \in [0, T])$ be the diffusion process in $\mathfrak{a}_+$ with the infinitesimal generator $\frac{1}{2}\Delta_K$, that is, the unique strong solution of (9.4) with*

$U(t) \equiv 0$. *Then*

(9.6) $$E_{X(0)} \int_0^T (K_\alpha \circ \psi)(\tilde{X}(s)) \, ds < \infty.$$

The solution of the stochastic control problem (9.1, 9.3) is given now.

9.7. **Theorem.** *The stochastic control problem described by* (9.1, 9.3) *has an optimal control* U^* *given by*

(9.8) $$U^*(s, t) = -\frac{g(s)}{2} A^{-1}(t) \nabla_t (K_\alpha \circ \psi)(t)$$

where $s \in [0, T]$, $t = (t_1, \ldots, t_r)$, ∇_t *is the* t-*gradient, and* g *is the unique positive solution of the scalar Riccati differential equation*

(9.9) $$\frac{dg}{ds} + \frac{1}{2}\chi_\alpha g - \frac{1}{4}g^2 + 1 = 0, \quad g(T) = 0,$$

and χ_α *is the eigenvalue* (8.17).

Proof. In analogy with the proof of Theorem 6.7, a smooth solution is found for the Hamilton-Jacobi equation for the stochastic optimal control problem of diffusion type. Since the integrand of the cost functional as a function of the state is K-invariant, the Hamilton-Jacobi equation can be reduced to the radial part of (9.1) as

(9.10) $$0 = \frac{\partial W}{\partial s} + \min_{v \in \mathbb{R}^r} \left[\frac{1}{2}\Delta_K + \sum_{j=1}^r v_j \frac{\partial W}{\partial t_j} + (K_\alpha \circ \psi)(t) + \sum_{j=1}^r f_j v_j^2 \right].$$

It is clear that the family of controls obtained by the minimization in (9.1) is

(9.11) $$U_j^*(s, t) = -\frac{1}{2f_j(t)} \frac{\partial W(s, t)}{\partial t_j}$$

for $j = 1, 2, \ldots, r$.

Assume a solution W of (9.10) as

(9.12) $$W(s, t) = g(s) \sum_i ({}_iK_\alpha \circ \psi)(t) + h(s).$$

Substituting (9.12) into (9.10) it follows that

$$0 = g' \sum {}_iK_\alpha \circ \psi + h' + \frac{1}{2}\chi_\alpha g \left(\sum {}_iK_\alpha \circ \psi + {}_0K_\alpha \right)$$

(9.13) $$+ \sum {}_iK_\alpha \circ \psi + {}_0K_\alpha - \frac{1}{4}g^2 \sum {}_iK_\alpha \circ \psi$$

$$= \sum {}_iK_\alpha \circ \psi \left(g' + \frac{1}{2}\chi_\alpha g - \frac{1}{4}g^2 + 1 \right) + h' + \frac{1}{2}\chi_\alpha g \, {}_0K_\alpha + {}_0K_\alpha.$$

If g satisfies (9.9) and h satisfies

(9.14) $$h' + \frac{1}{2}\chi_\alpha g \, {}_0K_\alpha + {}_0K_\alpha = 0, \quad h(T) = 0,$$

then (9.12) is a smooth solution of (9.10) that satisfies the boundary condition $W(s, t) = 0$ for $(s, t) \in \{T\} \times \mathfrak{a}_+$. The family of controls (9.11) is an admissible (optimal) control if the family of stochastic differential equations (9.4) with the controls has one and only one strong solution in $[0, T]$.

Let $(X(s), s \in [0, T])$ be the solution of the family of stochastic differential equations (9.4) with the family of controls (9.11). There is no finite escape time by the methods in the proof of Lemma 6.5. If it is assumed that $(X(s), s \in [0, T])$ does not hit the walls of the positive Weyl chamber then the local smoothness of the coefficients of the stochastic differential equations for $(X(s), s \in [0, T])$ can be used to construct a unique strong global solution (e.g. [IW, p. 164]).

Now it is necessary to verify that $(X(s), s \in [0, T])$ does not hit the walls of the positive Weyl chamber. Initially a subset of a wall away from the origin is investigated. In fact it suffices to consider the process $(X_{r-1}(s) - X_r(s), s \in [0, T])$ away from the origin because a similar verification is made for the other walls of the Weyl chamber. Choose two positive constants $K \gg c > 0$ and define $\Gamma_{r-1}(c, K)$ as

$$(9.15) \qquad \Gamma_{r-1}(c, K) = \{t \colon K > t_1 - rc > \cdots > t_{r-2} - 3c > t_{r-1} \geq t_r > c\}.$$

By localization by a stopping time it is assumed that $(X(s), s \in [0, T])$ is restricted to $\Gamma_{r-1}(c, K)$.

Let $Y_{r-1}(s)$ be defined as

$$Y_{r-1}(s) = X_{r-1}(s) - X_r(s)$$

for $s \in [0, T]$. The stochastic differential equation for $(Y_{r-1}(s), s \in [0, T])$ is

$$
\begin{aligned}
dY_{r-1}(s) = \Bigg[& \frac{a}{2} \frac{\sinh(X_{r-1}(s))}{\cosh(X_{r-1}(s)) - \cosh(X_r(s))} \\
& - \frac{a}{2} \frac{\sinh(X_r(s))}{\cosh(X_r(s)) - \cosh(X_{r-1}(s))} + h(s, X(s)) \Bigg] ds \\
& + dB_{r-1}(s) - dB_r(s)
\end{aligned}
$$

where

$$\sup_{(s, x) \in [0, T] \times \Gamma_{r-1}(c, K)} |h(s, x)| < M < \infty.$$

It easily follows from Taylor's formula that there is a $\delta > 0$ such that if $0 < t_{r-1} - t_r < \delta$ then

$$(9.17) \qquad \frac{a}{2} \frac{\sinh(t_{r-1}) + \sinh(t_r)}{\cosh(t_{r-1}) - \cosh(t_r)} - M \geq \frac{1}{t_{r-1} - t_r}.$$

Now localize $(X(s), s \in [0, T])$ to $\Gamma_{r-1}(c, K) \cap \{t \colon t_{r-1} - t_r < \delta\}$. Apply a comparison theorem [IW, p. 352] for the solutions of scalar stochastic differential equations to the above localization of (9.16) and the so-called two-dimensional Bessel stochastic differential equation

$$dZ(s) = \frac{1}{Z(s)} ds + \sqrt{2} dB(s).$$

Choosing sequences $c_n \downarrow 0$ and $K_n \uparrow +\infty$ and applying the above construction to $\Gamma_{r-1}(c_n, K_n)$ it follows that

$$(9.18) \quad P(X_1(s) > X_2(s) > \cdots > X_{r-1}(s) = X_r(s) > 0 \text{ for some } s \in [0, T]) = 0.$$

Now the behavior of $(X(s), s \in [0, T])$ is investigated near the origin. For this analysis it is more convenient to the coordinates $x = (x_1, \ldots, x_r)$ instead

of $t = (t_1, \ldots, t_r)$ where $x_i = \sinh^2(t_i/2)$. Since the radial part of the Laplace-Beltrami operator has been expressed in these coordinates by the chain rule, the family of stochastic differential equations for

$$(9.19) \qquad Y_k(s) = \sinh^2 \frac{X_k(s)}{2}$$

for $s \in [0, T]$ is

$$
\begin{aligned}
(9.20) \qquad d Y_k(s) = &\left[\frac{1}{2} Y_k(s)(1 + Y_k(s)) \vee 0\right]^{1/2} d B_k(s) \\
&+ \left[\frac{1}{4}(i + b + c) + \frac{1}{2}\left(1 + c + \frac{b}{2}\right) Y_k(s)\right. \\
&\left.+ \frac{a}{2} \sum_{j \neq k} \frac{Y_k(s)(1 + Y_k(s))}{Y_k(s) - Y_j(s)} - U_k^*(s, \psi^{-1}(Y(s)))\right] ds
\end{aligned}
$$

for $k = 1, 2, \ldots, r$.

Choose two positive numbers $K \gg c > 0$ and define $\Lambda_r(c, K)$ as

$$(9.21) \qquad \Lambda_r(c, K) = \{x : K > x_1 - rc > \cdots > x_{r-1} - 2c > x_r\}.$$

Assume that $(Y(s), s \in [0, T])$ and $(Z(s), s \in [0, T])$ are two pathwise solutions of the family of stochastic differential equations (9.20) that are localized to $\Lambda_r(c, K)$. It is elementary to verify that
(9.22)
$$\left|\left(\frac{1}{2} y(1 + y) \vee 0\right)^{1/2} - \left(\frac{1}{2} z(1 + z) \vee 0\right)^{1/2}\right| \leq 2|y - z|^{1/2} + 2|y - z| := \rho(|y - z|).$$

For $x = (x_1, \ldots, x_r)$ and $k \in \{1, \ldots, r\}$ define σ and b_k as

$$(9.23) \qquad \sigma(x_k) = \left(\frac{1}{2} x_k(1 + x_k) \vee 0\right)^{1/2},$$

(9.24)
$$b_k(s, x) = \frac{1}{4}(1 + b + c) + \frac{1}{2}\left(1 + c + \frac{b}{2}\right) x_k + \frac{a}{2} \sum_{j \neq k} \frac{x_k(1 + x_k)}{x_k - x_j} - U_k^*(s, \psi^{-1}(x)).$$

Following a construction for a scalar stochastic differential equation [IW, p. 168] let $1 > a_1 > a_2 > \cdots > a_n > \cdots > 0$ be defined by

$$\int_{a_1}^1 \rho^{-2}(u) \, du = 1, \qquad \int_{a_2}^1 \rho^{-2}(u) \, du = 2, \ldots, \qquad \int_{a_n}^1 \rho^{-2}(u) \, du = n, \ldots,$$

where ρ is given by (9.22). Clearly $a_n \downarrow 0$. Let $(\psi_n, n \in \mathbb{N})$ be a sequence of continuous functions such that the support of ψ_n is contained in (a_n, a_{n-1}),

$$0 \leq \psi_n(u) \leq \frac{2\rho^{-1}(u)}{n} \quad \text{and} \quad \int_{a_n}^{a_{n-1}} \psi_n(u) \, du = 1.$$

Define

$$\varphi_n(x) \int_0^{|x|} dy \int_0^y \psi_n(u) \, du, \qquad x \in \mathbb{R}^1.$$

Then $\varphi_n \in C^2(\mathbb{R}^1)$, $|\varphi_n'| \leq 1$, and $\varphi_n(x) \uparrow |x|$ as $n \to \infty$.

Fix $n \in \mathbb{N}$ and apply Itô's formula to $\varphi_n(Y_k(s) - Z_k(s))$ for $k = 1, 2, \ldots, r$ and add these r equations to obtain

$$\sum_{k=1}^{r} \varphi_n(Y_k(s) - Z_k(s))$$

(9.25)
$$= \sum_{k=1}^{r} \int_0^s \varphi_n'(Y_k(u) - Z_k(u))(\sigma(Y_k(u)) - \sigma(Z_k(u)) \, dB_k(u)$$

$$+ \frac{1}{2} \sum_{k=1}^{r} \int_0^s \varphi_n'(Y_k(u) - Z_k(u))(b_k(u, Y(u)) - b_k(u, Z(u))) \, du.$$

$$+ \frac{1}{2} \sum_{k=1}^{r} \int_0^s \varphi_n''(Y_k(u) - Z_k(u))(\sigma(Y_k(u)) - \sigma(Z_k(u)))^2 \, du.$$

Since the expectation of each of the stochastic integrals in (9.25) is zero, it follows that

$$E \left[\sum_{k=1}^{r} \varphi_n(Y_k(s) - Z_k(s)) \right]$$

(9.26)
$$= \sum_{k=1}^{r} E \int_0^s \varphi_n'(Y_k(u) - Z_k(u))(b_k(u, Y(u)) - b_k(u, Z(u))) \, du$$

$$+ \frac{1}{2} \sum_{k=1}^{r} E \int_0^s \varphi_n''(Y_k(u) - Z_k(u))(\sigma(Y_k(u)) - \sigma(Z_k(u)))^2 \, du$$

$$= I_1(n) + I_2(n).$$

On $\Lambda_r(c, K)$ the drift terms $(b_1, \ldots, b_r)$ are smooth so that

(9.27)
$$|I_1(n)| \le \sum_{k=1}^{r} E \int_0^s |b_k(u, Y(u)) - b_k(u, Z(u))| \, du$$

$$\le \sum_{k=1}^{r} E \int_0^s \widetilde{K} |Y(u) - Z(u)| \, du$$

and by the construction of φ_n

(9.28)
$$|I_2(n)| \le \frac{1}{2} \sum_{k=1}^{r} \int_0^s E \frac{2}{n} \rho^{-2}(|Y_k(u) - Z_k(u)|) \rho^2(|Y_k(u) - Z_k(u)|) \, du$$

$$\le r \frac{s}{n}.$$

By passing to the limit we have

(9.29)
$$E|Y(s) - Z(s)| \le \int_0^s r \widetilde{K} |Y(u) - Z(u)| \, du$$

so $Y(s) = Z(s)$ a.s. Choose $\varepsilon > 0$ such that $b_r(s, x) > 0$ for $x \in \Lambda_r(c, I) \cap \{x : |x_r| < \varepsilon\}$ and let

$$T_{-\varepsilon} = \inf\{s : Y_r(s) = -\varepsilon\}.$$

Let $S < T_{-\varepsilon}$ be such that $Y_r(S) < 0$ with positive probability. The stochastic differential equation for $(Y_r(s), s \in [0, T])$ implies that the map $s \mapsto Y_r(s)$ on $(S, T_{-\varepsilon})1_{\{Y_r(S)<0\}}$ is increasing which is impossible. Thus

$$P(T_{-\varepsilon} \leq T) = 0$$

and

$$P(Y_1(s) > \cdots > Y_{r-1}(s) > Y_r(s) \geq 0, \qquad s \in [0, T]) = 1.$$

Now it is shown that $(Y_{r-1}(s), s \in [0, T])$ does not hit the origin. From this verification it easily follows from the previous results that $(Y_k(s), s \in [0, T])$ does not hit the origin for $k \in \{1, \ldots, r-2\}$. Consider the following stochastic differential equation:

(9.30)
$$d\widetilde{Y}_{r-1}(s) = \left(\frac{1}{2}\widetilde{Y}_{r-1}(s)(1 + \widetilde{Y}_{r-1}(s)) \vee 0\right)^{1/2} dB_{r-1}(s)$$
$$+ \left[\frac{1}{4}(1 + b + c) + \frac{1}{2}\left(1 + c + \frac{b}{2}\right)\widetilde{Y}_{r-1}(s)\right] ds.$$

By a test for explosions of a one-dimensional diffusion process (e.g., [IW, p. 362]) it follows that

(9.31)
$$P_x\left(\inf_{s<e} \widetilde{Y}_{r-1}(s) > 0\right) = 1$$

where e is the explostion time. Choose two positive constants $K \gg c > 0$ and define $\Lambda_{r-1}(c, K)$ as

(9.32) $\quad \Lambda_{r-1}(c, K) = \{x: K > x_1 - rc > \cdots > x_{r-2} - 3c > x_{r-1} \geq x_r \geq 0\}.$

There is an $\varepsilon > 0$ such that on $\Lambda_{r-1}(c, K) \cap \{x: 0 \leq x \leq \varepsilon\}$ the sum of the last two terms in the drift of the stochastic differential equation for $(Y_{r-1}(s), s \in [0, T])$ is positive. This solution can be localized to $\Lambda_{r-1}(c, K) \cap \{x: 0 \leq x \leq \varepsilon\}$ and the stochastic differential equation for $(Y_{r-1}(s), s \in [0, T])$ can be compared with (9.20). Let $c_n \downarrow 0$ and $K_n \uparrow +\infty$ and apply the above techniques to $\Lambda_{r-1}(c_n, K_n)$ to verify that

(9.33) $\qquad P(Y_1(s) > Y_2(s) > \cdots > Y_{r-1}(s) > Y_r(s), s \in [0, T]) = 1.$

This completes the proof. Q.E.D.

10. Generalization to Abstract Root Systems

In this section we indicate how the previous results can be generalized, in a somewhat less geometric way, to root systems with "arbitrary" multiplicities (cf. [D1]). Consider Euclidean space $\mathbb{R}^r$ with coordinates $t = (t_1, \ldots, t_r)$ and consider the "abstract" Weyl chambers (compare with (4.13) and (8.9))

(10.1)
$$\mathfrak{a}_+^0 := \{t \in \mathbb{R}^r : t_1 > \cdots > t_r\}$$

and

(10.2)
$$\mathfrak{a}_+ := \{t \in \mathbb{R}^r : t_1 > \cdots > t_r > 0\},$$

respectively. Consider a density function $W(t)$, the associated volume element

$$dV(t) := W(t) dt_1 \cdots dt_r$$

and the "Laplace-Beltrami operator"

$$(10.3) \qquad \Delta = \frac{1}{w} \sum_i \frac{\partial}{\partial t_i} W \frac{\partial}{\partial t_i}$$

The density function is specified by

$$(10.4) \qquad W^0 = \prod_{1 \le i < j \le r} \sinh^a \left(\frac{t_j - t_i}{2} \right)$$

and

$$(10.5) \quad W = \prod_{i=1}^{r} \sinh^c(t_i) \sinh^b \left(\frac{t_i}{2} \right) \prod_{1 \le i < j \le r} \sinh^a \left(\frac{t_j - t_i}{2} \right) \sinh^a \left(\frac{t_j + t_i}{2} \right),$$

respectively. Here a, b, c are arbitrary positive constants. Note that for certain integer values of a, b, and c, we obtain the density functions associated with a symmetric cone or a symmetric domain, respectively (cf. Sections 4 and 8). We now pass to "algebraic" coordinates $x_1, \ldots, x_r$ by putting

$$(10.6) \qquad x_i := \exp(t_i), \qquad (x_1, \ldots, x_r) = \varphi(t_1, \ldots, t_r)$$

for (10.1) and

$$(10.7) \qquad x_i := \sinh^2 \left(\frac{t_i}{2} \right), \qquad (x_1, \ldots, x_r) = \psi(t_1, \ldots, t_r)$$

for (10.2). Then in both cases the Weyl chamber corresponds to $x_1 > x_2 > \cdots > x_r > 0$. Modifying the proofs of Corollaries 4.19 and 8.15 we obtain

10.8. Proposition. *In terms of the algebraic coordinates, the operators* (10.3) *have the form*

$$(10.9) \qquad \Delta^0 = \sum_{i=1}^{r} x_i^2 \frac{\partial^2}{\partial x_i^2} + a \sum_{i \ne j} \frac{x_i^2}{x_j - x_i} \frac{\partial}{\partial x_i} + \sum_{i=1}^{r} \left(1 - \frac{a}{2}(r-1) \right) x_i \frac{\partial}{\partial x_i}$$

and

$$\Delta = \sum_{i=1}^{r} x_i(x_i + 1) \frac{\partial^2}{\partial x_i^2} + a \sum_{i \ne j} \frac{x_i(x_i + 1)}{x_j - x_i} \frac{\partial}{\partial x_i}$$

$$(10.10)$$

$$+ \sum_{i=1}^{r} \left(\frac{1 + b + c}{2} + \left(1 + c + \frac{b}{2} \right) x_i \right) \frac{\partial}{\partial x_i}.$$

For every partition $\alpha = (\alpha_1, \ldots, \alpha_r)$ of integers $\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_r \ge 0$, let

$$(10.11) \qquad K_\alpha^0(x) = \mathscr{J}_\alpha \left(x, \frac{2}{a} \right) \Big/ \mathscr{J}_\alpha \left(1^r, \frac{2}{a} \right)$$

denote the "Jack polynomial" (cf. [S1]) for the parameter $2/a$, normalized by

$$K_\alpha^0(1, \ldots, 1) = 1.$$

It is known [S1, Theorem 3.1] that $\Delta^0 K_\alpha^0 = \chi_\alpha^0 K_\alpha^0$ where

$$(10.12) \qquad \chi_\alpha^0 = \sum_{i=1}^{r} \alpha_i \left(\alpha_i + \frac{a}{2}(r + 1 - 2i) \right).$$

If a is an integer coming from a symmetric cone Λ (cf. Section 3), K_α^0 is (the restriction of) a spherical function on Λ.

10.13. Lemma. *There is an expansion*

$$K_\alpha^0(x) = \sum_{\beta \leq \alpha} c_\alpha^\beta q_\beta(x)$$

where β runs over all partitions $\leq \alpha$ in the (reverse) lexicographic order, the q_β are the symmetric monomial functions [M1] and the coefficients c_α^β (depending on a) are nonnegative.

Proof. Apply [M2, Theorem 6.3] and observe that the coefficients $w(T)$ computed in [M2, (45)] involve only products and quotients of "generalized hook-lengths", which are certainly positive.   Q.E.D.

Fix a partition α and let $K_\alpha^0(x)$ be the polynomial (10.11) associated with α which is homogeneous of degree $|\alpha|$. Using the definition of the monomial symmetric functions we have

$$K_\alpha^0(x) = \sum a_\alpha(\underline{i}) x_1^{i_1} \cdots x_r^{i_r}$$

where $\underline{i} = (i_1, \ldots, i_r)$ and $a_\alpha(\underline{i}) = a_\alpha(\underline{j})$ from the definition of the monomial symmetric functions if $\underline{j}$ is a distinct permutation of $\underline{i}$. For $j \in \{1, \ldots, r\}$ define $a_\alpha^j(\underline{i})$ as

$$(10.14) \qquad\qquad a_\alpha^j(\underline{i}) = \frac{i_j}{|\alpha|} a_\alpha(\underline{i})$$

and

$$(10.15) \qquad\qquad {}_jK_\alpha^0(x) = \sum a_\alpha^j(\underline{i}) x_1^{i_1} \cdots x_r^{i_r}$$

where the sum is over distinct r-tuples such that $a_\alpha^j(\underline{i}) \neq 0$. Clearly

$$(10.16) \qquad\qquad K_\alpha^0(x) = \sum_{j=1}^r {}_jK_\alpha^0(x).$$

The cost functional for the stochastic control problem is

$$(10.17) \qquad J_\alpha^0(U) = E_{X(0)} \int_0^T (K_\alpha^0 \circ \varphi)(X(s)) + \sum_{j=1}^r f_j(X(s)) U_j^2(s) \, ds$$

where

$$(10.18) \qquad\qquad f_j(t) = \frac{1}{{}_jK_\alpha^0(\varphi(t))} \left[\frac{\partial}{\partial t_j} (K_\alpha^0 \circ \varphi)(t) \right]^2.$$

The process $(X(s), s \in [0, T])$ is the controlled diffusion in the "Weyl chamber" $\{(t_1, \ldots, t_r): t_1 > t_2 > \cdots > t_r\}$. The infinitesimal generator of $(X(s), s \in [0, T])$ is

$$\frac{1}{2}\Delta^0 + \sum_{j=1}^r u_j(s, t) \frac{\partial}{\partial t_j}$$

$$(10.19) \qquad = \frac{1}{2} \sum_{k=1}^r \left(\frac{\partial^2}{\partial t_k^2} - \frac{a}{2}(r-1)\frac{\partial}{\partial t_k} + a \sum_{j \neq k} \frac{1}{1 - \exp(t_j - t_k)} \frac{\partial}{\partial t_k} \right)$$

$$+ \sum_{k=1}^r u_k(s, t) \frac{\partial}{\partial t_k}.$$

The associated family of stochastic differential equations is

$$dX_k(s) = \left[-\frac{a}{4}(r-1) + \frac{a}{2}\sum_{j \neq k}\frac{1}{1 - \exp(X_j(s) - X_k(s))} \right.$$

$$\left. + U_k(s, X(s)) \right] ds + dB_k(s)$$

for $k = 1, 2, \ldots, r$, $X(0) = (X_1(0), \ldots, X_r(0))$, $X_1(0) > X_2(0) > \cdots > X_r(0)$, and $(B_1(s), \ldots, B_r(s))$; $s \geq 0$ is a standard r-dimensional Brownian motion.

An admissible control at time s is a Borel measurable function of $X(s)$ so that (10.20) with this control has one and only one strong solution.

It follows in analogy with Lemma 6.5 that $J_\alpha^0(0) < \infty$.

The solution of the stochastic optimal control problem (10.17, 10.19) is given now.

10.21.   Theorem. *Assume that* $a \geq 1$. *The stochastic control problem described by* (10.17, 10.19) *has an optimal control that is*

(10.22) $$U_j^*(s, t) = \left[\frac{-jK_\alpha^0(\varphi(t))}{2D_j(K_\alpha^0 \circ \varphi)(t)} \right] g(s)$$

where $j \in \{1, \ldots, r\}$, $s \in [0, T]$, $t = (t_1, \ldots, t_r)$, $D_j = \frac{\partial}{\partial t_j}$, *and* g *is the unique positive solution of*

$$\frac{dg}{ds} + \frac{1}{2}\chi_\alpha^0 g - \frac{1}{4}g^2 + 1 = 0, \quad g(T) = 0,$$

where χ_α^0 *is the eigenvalue of* K_α^0 *for* Δ^0 *given by* (10.12).

The proof of this result is not given because it follows closely the proof of Theorem 6.7. Basically it is required to verify that the optimally controlled diffusion does not hit the walls of the "Weyl chamber". Since it is assumed that $a \geq 1$, the inequality (6.20) is satisfied.

In order to study the operator Δ, note that Proposition 10.8 implies

(10.23)
$$\Delta = \Delta^0 + \left(c + \frac{b}{2} + \frac{a}{2}(r-1) \right) E$$
$$+ \frac{1}{2}\left[\partial_e, \Delta^0 + \left(c + \frac{b}{2} + \frac{a}{2}(r-1) \right) E \right] + \frac{b}{4}\partial_e,$$

where

$$E = \sum_{i=1}^r x_i \frac{\partial}{\partial x_i}$$

is the Euler operator,

$$\partial_e := \sum_{i=1}^r \frac{\partial}{\partial x_i},$$

and $[A, B] := AB - BA$ is the commutator. This follows as in the proof of Theorem 8.16. As shown in [D1], the operator Δ has eigenfunctions of the

form

(10.24)
$$K_\alpha(x) = \sum_{\beta \leq \alpha} a_\alpha^\beta K_\beta^0(x)$$

(called "generalized Jacobi polynomials"). Here $\alpha = (\alpha_1, \ldots, \alpha_r)$ is a partition and the coefficients a_α^β depend on a, b, and c. By [Y1], we have

(10.25)
$$\partial_e K_\alpha^0 = \sum_{i=1}^r \binom{\alpha}{\alpha - \varepsilon_i} K_{\alpha-\varepsilon_i}^0,$$

where $\alpha - \varepsilon_i := (\alpha_1, \ldots, \alpha_{i-1}, \alpha_i - 1, \alpha_{i+1}, \ldots, \alpha_r)$ and

$$\binom{\alpha}{\alpha - \varepsilon_i} \geq 0$$

are the generalized binomial coefficients. Since (10.25) implies that ∂_e decreases the "symmetric degree" of a partition, it follows from (10.23) that

$$\Delta K_\alpha = \chi_\alpha K_\alpha,$$

where

(10.26) $\quad \chi_\alpha = \chi_\alpha^0 + \left(\dfrac{a}{2}(r-1) + c + \dfrac{b}{2} \right) |\alpha| = \sum_i \alpha_i \left(\alpha_i + a(r-i) + c + \dfrac{b}{2} \right).$

10.27. **Proposition.** *There is an expansion*

(10.28)
$$K_\alpha = \sum_{\beta \leq \alpha} c_\alpha^\beta q_\beta(x)$$

where the coefficients c_α^β are nonnegative.

Proof. As in the proof of Theorem 8.16, we obtain a recursion formula

$$2c_\alpha^\beta \cdot (\chi_\alpha - \chi_\beta) = \sum_i c_\alpha^{\beta+\varepsilon_i} \binom{\beta + \varepsilon_i}{\beta} \cdot \left(\chi_{\beta+\varepsilon_i} - \chi_\beta + \dfrac{b}{2} \right)$$

for the coefficients c_α^β of (10.24), starting with $c_\alpha^\alpha := 1$. Clearly, (10.26) implies

$$\chi_\alpha - \chi_\beta > 0 \quad \text{and} \quad \chi_{\beta+\varepsilon_i} - \chi_\beta + \dfrac{b}{2} \geq 0.$$

Therefore $c_\alpha^\beta \geq 0$ for all $\beta \leq \alpha$. Now apply Lemma 10.13.   Q.E.D.

Fix a partition and let $K_\alpha(x)$ be the polynomial (10.28). Using the definition of the monomial symmetric functions we have

$$K_\alpha(x) = \sum a_\alpha(\underline{i}) x_1^{i_1} \cdots x_r^{i_r}$$

where $\underline{i} = (i_1, \ldots, i_r)$ and the sum is over distinct r-tuples such that $a_\alpha^j(\underline{i}) \neq 0$. For $j \in \{1, \ldots, r\}$ define $a_\alpha^j(\underline{i})$ as

(10.29)
$$a_\alpha^j(\underline{i}) = \frac{i_j}{\sum_{k=1}^r i_k} a_\alpha(\underline{i})$$

and

(10.30)
$$_j K_\alpha(x) = \sum a_\alpha^j(\underline{i}) x_1^{i_1} \cdots x_r^{i_r},$$

where the sum is over distinct r-tuples such that $a_\alpha^j(\underline{i}) \neq 0$. Clearly

$$(10.31) \qquad K_\alpha(x) = \sum_{j=1}^r {}_jK_\alpha(x) + {}_0K_\alpha,$$

where ${}_0K_\alpha$ is the constant term in K_α.

The cost functional for the stochastic control problem is

$$(10.32) \qquad J_\alpha(U) = E_{X(0)} \int_0^T (K_\alpha \circ \psi)(X(s)) + \sum_{j=1}^r f_j(X(s))U_j^2(s)\, ds$$

where

$$(10.33) \qquad f_j(t) = \frac{1}{{}_jK_\alpha(\psi(t))} \left[\frac{\partial}{\partial t_j}(K_\alpha \circ \psi)(t) \right]^2.$$

The process $(X(s), s \in [0, T])$ is the controlled diffusion in the "Weyl chamber" $\{(t_1, \ldots, t_r): t_1 > t_2 > \cdots > t_r > 0\}$. The infinitesimal generator of $(X(s), s \in [0, T])$ is

$$\frac{1}{2}\Delta + \sum_{j=1}^r u_j(s, t)\frac{\partial}{\partial t_j}$$

$$(10.34) \qquad \begin{aligned} &= \sum_{k=1}^r \frac{1}{2}\frac{\partial^2}{\partial t_k^2} + \left[\frac{c}{2}\coth t_k + \frac{b}{2}\coth \frac{t_k}{2} \right] \frac{\partial}{\partial t_k} \\ &\quad + \frac{a}{2}\sum_{j \neq k} \frac{\sinh(t_k)}{\cosh(t_j) - \cosh(t_k)}\frac{\partial}{\partial t_k} \\ &\quad + \sum_{k=1}^r u_k(s, t)\frac{\partial}{\partial t_k}. \end{aligned}$$

The associated family of stochastic differential equations is
(10.35)

$$dX_k(s) = \left[\frac{b}{2}\coth \frac{X_k(s)}{2} + \frac{c}{2}\coth X_k(s) \right.$$

$$\left. + \frac{a}{2}\sum_{j \neq k} \frac{\sinh X_k(s)}{\cosh(X_j(s)) - \cosh(X_k(s))} = U_k(s, X(s)) \right] ds + dB_k(s)$$

for $k = 1, 2, \ldots, r$, $X(0) = (X_1(0), \ldots, X_r(0))$, $X_1(0) > X_2(0) > \cdots > X_r(0) > 0$, and $(B_1(s), \ldots, B_r(s); s \geq 0)$ is a standard r-dimensional Brownian motion.

An admissible control at time s is a Borel measurable function of $X(s)$ so that (10.35) with this control has one and only one strong solution.

It follows in analogy with Lemma 9.5 that $J_\alpha(0) < \infty$.

The solution of the stochastic optimal control problem (10.32, 10.34) is given now.

10.36. Theorem. *Assume that $a \geq 1$. The stochastic control problem described by (10.32, 10.34) has an optimal control that is*

$$(10.37) \qquad U_j^*(s, t) = \left[\frac{-{}_jK_\alpha(\psi(t))}{2D_j(K_\alpha \circ \psi)(t)} \right] g(s)$$

where $j \in \{1, \ldots, r\}$, $s \in [0, T]$, $t = (t_1, \ldots, t_r)$, $D_j = \frac{\partial}{\partial t_j}$, and g is the unique positive solution of

$$\frac{dg}{ds} + \frac{1}{2}\chi_\alpha g - \frac{1}{4}g^2 + 1 = 0, \quad g(T) = 0,$$

where χ_α is the eigenvalue of K_α for Δ given by (10.26).

The proof of this result is not given because it follows closely the proof of Theorem 9.7. Basically it is required to verify that the optimally controlled diffusion does not hit the walls $\{t_i = t_{i-1}\}$ for $i = 1, \ldots, r$ of the "Weyl chamber". Since it is assumed that $a \geq 1$, the inequality (9.17) is satisfied. The behavior of this diffusion near the origin follows as in the proof of Theorem 9.7.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KANSAS, LAWRENCE, KANSAS 66045
E-mail address: duncan@kuhub.cc.ukans.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARBURG, HANS MEERWEIN STRASSE, 35032 MARBURG, GERMANY