TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 347, Number 4, April 1995

THE DERIVATIVES OF HOMOTOPY THEORY

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ABSTRACT. We construct a functor of spaces, M_n, and show that its multilinearization is equivalent to the nth layer of the Taylor tower of the identity functor of spaces. This allows us to identify the derivatives of the identity functor and determine their homotopy type.

The calculus of homotopy functors, developed by Goodwillie ([G1], [G2], [G3]), establishes that a homotopy functor, F, satisfying certain connectivity conditions, has associated to it a tower of functors, $\dots \to P_nF \to P_{n-1}F \to \dots$. These functors act like a Taylor series approximation to F in the sense that for a space, X, there is a map, $p_nF(X) : F(X) \to P_nF(X)$, for each n, and the connectivity of this map increases with n. This theory has been applied to the study of the functor A, Waldhausen's algebraic K-theory of spaces.

In this paper, we turn our attention to the Taylor tower of the identity functor of spaces, I. The ultimate goal is to identify the Taylor tower of I and use it to study homotopy theory. In this paper, we construct a collection of symmetric functors, $\{M_n\}$, and show that the multilinearization of M_n is equivalent to the nth layer, fiber $(P_nI \to P_{n-1}I)$, of the Taylor tower of I. This construction also allows us to identify the nth derivative of I. This is a spectrum with Σ_n-action which is equivalent to the functor fiber $(P_nI \to P_{n-1}I)$.

The paper is organized as follows. In section 1 we summarize the basic results and terminology of calculus that will be used throughout the paper. In section 2 we describe the problem of finding the Taylor tower of I in more detail and state the main results. In section 3 we outline the method by which the nth derivative of a functor is determined. In section 4 we construct the functor M_n and a natural transformation, T_n, used to establish the equivalence between the multilinearization of M_n and fiber$(P_nF \to P_{n-1}F)$. In section 5 we determine the homotopy type of the derivatives, and in section 6 we show that T_n is sufficiently connected to induce an equivalence between the multilinearization of M_n and fiber$(P_nF \to P_{n-1}F)$.

The results in this paper come from the author's thesis, written under the direction of Tom Goodwillie at Brown University. The author wishes to thank him for his guidance and for many insightful discussions. The author also wishes to thank Randy McCarthy for his helpful suggestions during the writing of this paper.

Received by the editors February 10, 1994.
1991 *Mathematics Subject Classification.* Primary 55P65.

1. The Taylor tower of a functor

To start, we need to describe the context in which we will be working. We need both to establish some notation and terminology and to describe (in brief) the language and main results of Goodwillie's calculus of homotopy functors which will be used throughout this paper. We will not give a complete exposition of the theory of calculus of homotopy functors. Instead we will outline the terms and results needed in this work. For further detail, explanation and examples the reader is referred to Goodwillie's papers ([G1], [G2], and [G3]). Specifically, [G2] contains results about n-cubes of spaces, excision, and analytic functors, while [G3] contains results about the Taylor tower of a functor.

We start with the conventions. By a space (generally denoted X) we will mean an object in one of two categories. For the general results from Goodwillie's works, a space X will be an object in the category of compactly generated topological spaces. For the specific results concerning the derivatives of the identity functor, a space will be a topological space having the homotopy type of a finite CW-complex. Moreover, we will assume that all the spaces considered have non-degenerate basepoints. When we say that two spaces are equivalent, we will mean that they are weakly homotopy equivalent. By the suspension of a space, ΣX, we will mean the reduced suspension, $S^1 \wedge X$.

By an *n-cube of spaces* we will mean the following. Let $\underline{n}$ denote the set $\{1, 2, \ldots, n\}$. Let N be the category whose objects are the subsets of $\underline{n}$ and whose morphisms are the inclusion maps among the subsets. An n-cube of spaces is a covariant functor from the category N to the category of spaces.

Goodwillie defines and uses particular n-cubes of spaces, namely Cartesian and co-Cartesian diagrams. Cartesian can be defined in several ways. One way is the following. Let $\mathbf{X}$ be an n-cube of spaces. Let

$$h_0(\mathbf{X}) = \operatorname*{holim}_{K \subseteq \underline{n},\, K \neq \varnothing} (\mathbf{X}(K)),$$

where holim means the homotopy inverse limit (as in [B-K]). There is a map $a(\mathbf{X})$ defined as the composition $a(\mathbf{X}) : \mathbf{X}(\varnothing) = \lim \mathbf{X} \xrightarrow{\simeq} \operatorname{holim}\mathbf{X} \to h_0(\mathbf{X})$. If $a(\mathbf{X})$ is an equivalence then we say that $\mathbf{X}$ is Cartesian. If $a(\mathbf{X})$ is k-connected, we say that $\mathbf{X}$ is k-Cartesian.

Dually, we define co-Cartesian. Let $\mathbf{Y}$ be an n-cube. Let

$$h_1(\mathbf{Y}) = \operatorname*{hocolim}_{K \subset \underline{n},\, K \neq \{1, \ldots, n\}} \mathbf{Y}(K),$$

where hocolim denotes the homotopy colimit (as in [B-K]). There is again a map $b(\mathbf{Y}) : h_1(\mathbf{Y}) \to \mathbf{Y}(\{1, \ldots, n\})$. If $b(\mathbf{Y})$ is an equivalence, then we say that $\mathbf{Y}$ is a co-Cartesian n-cube. If $b(\mathbf{Y}) : h_1(\mathbf{Y}) \to \mathbf{Y}(\{1, \ldots, n\})$ is k-connected then we say that $b(\mathbf{Y})$ is k-co-Cartesian. An n-cube is **strongly co-Cartesian** if each of its 2-faces is co-Cartesian.

Cartesianness and co-Cartesianness are related to a certain extent. A classical result, the Blakers-Massey theorem, estimates the degree to which a co-Cartesian square is Cartesian as a function of the connectivity of the maps $\mathbf{X}(\varnothing) \to \mathbf{X}(\{1\})$ and $\mathbf{X}(\varnothing) \to \mathbf{X}(\{2\})$. The Blakers-Massey theorem has been generalized in various forms to n-cubes by Barratt and Whitehead ([B-W]), Ellis and Steiner ([E-S]), and Goodwillie ([G2]).

By a homotopy functor we will mean a functor from spaces to spaces which preserves weak homotopy equivalences. As a generalization to homotopy functors of the connectivity estimate provided by the Blakers-Massey theorem, Goodwillie defines what it means for a homotopy functor to satisfy stable nth-order excision. Let $\mathbf{X}$ be an n-cube of spaces, and let F be a homotopy functor. By $F(\mathbf{X})$ we mean the n-cube for which $(F(\mathbf{X}))(S) = F(\mathbf{X}(S))$, for $S \subseteq \underline{n}$.

Definition 1.1. F is *stably n-excisive* with constants c and κ, if, for every strongly co-Cartesian $(n+1)$-cube of spaces such that the connectivity, k_s, of $\mathbf{X}(\varnothing) \to \mathbf{X}(s)$ is at least κ for every $s \in \{1, \dots, n+1\}$, then $F(\mathbf{X})$ is $(-c + \sum_{s=1}^{n+1} k_s)$-Cartesian. We will say that F satisfies $E_n(c, \kappa)$.

For example, the generalized form of the Blakers-Massey theorem ([G2], Theorem 2.3) states that a strongly co-Cartesian $(n+1)$-cube $\mathbf{X}$ where $\mathbf{X}(\varnothing) \to \mathbf{X}(\{s\})$ is k_s-connected for each $s \in \{1, \dots, n+1\}$ is k-Cartesian with $k = -n + \sum_{s=1}^{n+1} k_s$. In other words, applying the identity functor to $\mathbf{X}$ yields a $(-n + \sum_{s=1}^{n+1} k_s)$-Cartesian $(n+1)$-cube. It follows that the identity functor satisfies $E_n(n, \kappa)$ for any κ, and hence is stably n-excisive. If a functor satisfies $E_n(c, \kappa)$ for all c and κ, we say that it is n-excisive. That is,

Definition 1.2. F is *n-excisive* if $F(\mathbf{X})$ is Cartesian for every strongly co-Cartesian $(n+1)$-cube $\mathbf{X}$.

We also have the notion of analytic functors, which are stably n-excisive for all n.

Definition 1.3. A homotopy functor F is *ρ-analytic* if there is some number q such that F satisfies $E_n(n\rho - q, \rho + 1)$ for all n.

By the above, the identity functor satisfies $E_n(n\rho - q, \rho + 1)$ when $\rho = 1$ and $q = 0$. Thus, the identity functor is 1-analytic. Other examples of analytic functors include the stable homotopy functor, Q, Waldhausen's functor, A, and the functor $X \mapsto Q(\mathrm{Map}(K, X))$ where K is a fixed finite CW-complex and $\mathrm{Map}(K, X)$ is the space of all continuous maps from K to X. For details, see section 4 of [G2]. An example of a functor which is not analytic is the functor $X \mapsto \Omega^\infty(\mathbf{E} \wedge X^{[n]})$ where $\mathbf{E}$ is a spectrum that is not bounded below and $X^{[n]}$ denotes the n-fold smash product of X with itself (see [G3], Remark 1.16).

Finally, there is a sense in which functors can be approximated by other functors.

Definition 1.4. Let F and G be homotopy functors. We say that F and G agree to order n via a map $u : F \to G$ if there exist constants c and κ such that for every κ-connected space X, the map $u_X : F(X) \to G(X)$ is $(-c + (n+1)k)$-connected. We say that $u : F \to G$ satisfies $O_n(c, \kappa)$.

In [G3], Goodwillie provides a method by which an analytic functor can be approximated by a tower of excisive functors. The idea is that a stably n-excisive functor F can be approximated by an n-excisive functor P_nF. P_nF can be constructed along with a transformation $p_nF : F \to P_nF$ such that P_nF is n-excisive and F agrees with P_nF to order n via p_nF. P_nF is regarded as an "nth degree Taylor polynomial" approximation to F. If F is

ρ-analytic, then it follows that there is an entire sequence of functors $P_0 F = F(*)$, $P_1 F, \ldots, P_{n-1}F$, $P_n F, \ldots$ such that each $P_k F$ is k-excisive and agrees with F to order k. Furthermore, there are maps $q_n F : P_n F \to P_{n-1} F$ such that these functors fit together into a tower of functors. This tower is called the Taylor tower of F. More formally, we have the following results.

Theorem 1.5 (Goodwillie). *Let F be a ρ-analytic functor. To any basepointed space X there are naturally associated objects $P_n F(X)$ and maps $p_n F$ and $q_n F$ which fit together in a tower:*

$$\begin{array}{ccc}
 & & \vdots \\
 & & \downarrow \\
F(X) & \xrightarrow{p_n F} & P_n F(X) \\
 & & \Big\downarrow{\scriptstyle q_{n-1}F} \\
 & & \vdots \\
 & & \Big\downarrow{\scriptstyle q_1 F} \\
 & & P_0 F(X) \quad \xrightarrow{\simeq} \quad F(*)
\end{array}$$

which satisfies $(q_n F) \circ (p_n F) = p_{n-1} F$. $P_n F$ is an n-excisive functor. If X is $(\rho+1)$-connected then the connectivity of the map $p_n F : F(X) \to P_n F(X)$ tends to $+\infty$ with n. $F(X)$ is equivalent to the homotopy limit, $P_\infty F(X)$, of the tower.

For the construction of $P_n F$, $p_n F$, and $q_n F$ see section 1 of [G3].

The nth layer of the Taylor tower of an analytic functor F is

$$D_n F = \text{fiber}(P_n F \to P_{n-1} F).$$

Here and elsewhere, fiber will mean the homotopy fiber. Each layer of the Taylor tower is a functor of a special form, resembling a monomial of degree n. Specifically, it is a homogeneous functor of degree n, as we state below.

Definition 1.6. A homotopy functor F is *homogeneous of degree n* if it is n-excisive and $P_{n-1} F \simeq *$.

Goodwillie provides a classification of all such functors in [G3]. The result is stated below. By homotopy orbit spectrum we mean the spectrum obtained from a spectrum with G-action (for some finite group G) by taking the homotopy orbit space $(X_n \wedge_G EG_+)$ of each space, X_n, in the spectrum.

Theorem 1.7. *If F is homogeneous of degree n and X is a space, then*

$$F(X) \simeq \Omega^\infty (\mathbf{C} \wedge X^{[n]})_{h\Sigma_n}$$

where $\mathbf{C}$ is a spectrum with Σ_n-action, Σ_n is the symmetric group on n letters, $X^{[n]}$ is the n-fold smash product of X with itself, and $h\Sigma_n$ denotes the homotopy orbit spectrum.

As claimed, we have the following result.

Proposition 1.8. *If F is an analytic functor, then $D_n F$ is homogeneous of degree n.*

It follows that $D_nF(X)$ has the form $\Omega^\infty(\mathbf{C} \wedge X^{[n]})$ for some spectrum, $\mathbf{C}$, with Σ_n-action. $\mathbf{C}$ is regarded as the "coefficient" of D_nF. As we will see later, it is also the nth derivative of F at a point.

2. The identity functor

In this section, we describe the problem motivating this paper, and outline the results obtained thus far. Our object of study is the identity functor of topological spaces: the functor from the category of topological spaces to itself which takes a space to itself. This functor will be denoted by I. The goal is to determine the Taylor tower of I. As we saw in section 1, the generalized Blakers-Massey theorem tells us that I is a 1-analytic functor. I does not have finite degree, i.e., it is not homogeneous of degree n, nor does its Taylor tower split as a product of functors .

The Taylor tower of I is of interest because of the information it will provide about homotopy theory. The fact that I is 1-analytic means that its Taylor tower converges on 2-connected spaces. That is, each finite stage of the tower yields, for a k-connected space X, another space $P_nI(X)$ whose first $(n+1)k$ homotopy groups are the same as those of X. Hence, as n increases, $P_nI(X)$ approximates the homotopy of X in a greater and greater range. Goodwillie has also pointed out that there is a spectral sequence that converges to the homotopy groups of X in which the E^2 terms are given by the homotopy groups of the $D_nI(X)$'s. Before this spectral sequence can be utilized, the maps between the D_nI's must be determined, i.e., we need to know how the individual layers of the Taylor tower fit together within the tower.

If we look at convergence on the level of functors rather than spaces then we see that the Taylor tower is a sequence of functors which link stable and unstable homotopy theory. Specifically, the first layer of the tower is the stable homotopy functor Q. Considering the tower as converging to the "unstable homotopy functor" I, we see that each stage of the tower, P_nI, recovers increasingly more information about unstable homotopy theory, information which was lost when Q was applied. This tower should yield new ideas about the relationship between stable and unstable homotopy theory.

At this time, the Taylor tower of I has not been determined. In this paper, we complete a first step in the problem, that of determining the derivatives of I at a point. Goodwillie has previously determined the first two derivatives of I and the homogeneous degree n functors in [G1] and [G3]. Specifically, he has shown the following

Proposition 2.1.

(a) *The first derivative of I is the sphere spectrum $\mathbf{S}^0$.*

(b) *The second derivative of I is the (-1)-sphere spectrum, $\mathbf{S}^{-1}$, with trivial Σ_2-action.*

(c) $D_1I(X) \simeq Q(X)$ *for any 2-connected space X.*

(d) $D_2I(X) \simeq \Omega Q((X \wedge X)_{h\Sigma_2})$ *for any 2-connected space X.*

The homotopy type of the derivatives have also been known for a while. The homotopy types can be determined by the Hilton-Milnor theorem (see section 5). The elusive part of the problem has been to determine the Σ_n-action on the spectrum. One solution is given by John Rognes in his dissertation [R]. In it he computes the first nontrivial homotopy group of the cross effect functor of $\Omega^l\Sigma^l$. From this the homotopy of the derivatives of I with Σ_n-action can be recovered. He accomplishes this by identifying the homotopy group as the kernel of a map from $\pi_*(\Omega^l\Sigma^l(X_1 \vee X_2 \vee \cdots \vee X_n))$ to

$$\prod_{k=1}^{n} \pi_*(\Omega^l\Sigma^l(X_1 \vee X_2 \vee \cdots \vee \widehat{X_k} \vee \cdots \vee X_n))$$

where each component of the map is a collapsing map $\bigvee_{i=1}^n X_i \to (X_1 \vee X_2 \vee \cdots \vee \widehat{X_k} \vee \cdots \vee X_n)$. He then compares the kernel of the map of homotopy groups with the kernel of the same map in homology, and uses the Snaith splitting to identify the kernel in homology as the homology of a configuration space smashed with $X_1 \wedge X_2 \wedge \cdots \wedge X_n$. The final step is to show that the Hurewicz map between the kernel in homology and the kernel in homotopy is a Σ_n-isomorphism. This is done by means of the Browder operations on homology. The homology of the configuration space in the kernel is calculated by finding the homology of a related quotient complex of the standard $\binom{n}{2}$-simplex.

This paper computes the derivatives in a more direct fashion, working with spaces rather than homotopy and homology groups, and avoids the need for the Snaith splitting or the Browder operations. We construct a new symmetric functor on n spaces defined by

$$M_n(X_1, X_2, \ldots, X_n) = \mathrm{Map}_*(\Delta_n, X_1 \wedge X_2 \wedge \cdots \wedge X_n)$$

where Map_* denotes basepointed maps and Δ_n is a quotient space of the product of n copies of the $(n-1)$-cube. Δ_n has the same homotopy type as the wedge of $(n-1)!$ copies of the $(n-1)$-sphere. M_n will be related by Theorem 2.2 below to the nth cross effect of I, $\chi_n I$, a symmetric functor of n variables defined in section 3 (Definition 3.4). As will be shown in Proposition 4.1 and its corollary, Theorem 2.2 guarantees that M_n satisfies conditions necessary for its multilinearization to be equivalent to that of $\chi_n I$. In turn, Proposition 3.13 will establish the relationship between the multilinearization of $\chi_n I$ and the nth derivative of I. The main result is the following.

Theorem 2.2. *There is a natural transformation of symmetric functors:*

$$T : \chi_n I(X_1, X_2, \ldots, X_n) \to \mathrm{Map}_*(\Delta_n, X_1 \wedge X_2 \wedge \cdots \wedge X_n)$$

which satisfies the following properties:

(a) *T is Σ_n-equivariant, that is, it preserves the Σ_n-symmetry of $\chi_n I$ and M_n which permutes the spaces $X_1, X_2, \ldots, X_n$ and images of the $(n-1)$-cubes in Δ_n.*

(b) *If $X_1, X_2, \ldots, X_n$ are k-connected then*

$$\Omega T_n : \Omega\chi_n I(\Sigma X_1, \Sigma X_2, \ldots, \Sigma X_n) \to \Omega\mathrm{Map}_*(\Delta_n, \Sigma X_1 \wedge \Sigma X_2 \wedge \cdots \wedge \Sigma X_n)$$

is $(n+1)(k+1)-1$-connected.

From Theorem 2.2 it easily follows that:

Corollary 2.3.

(a) *The nth derivative of I, denoted $I^{(n)}$, is the spectrum whose kth term is $\mathrm{Map}_*(\Delta_n, \Sigma^k)$. This spectrum has the obvious Σ_n-action given by permuting the $(n-1)$-cubes of Δ_n.*

(b) *The homotopy type of* $I^{(n)}$ *is the same as that of the wedge of* $(n-1)!$ *copies of the* $(1-n)$*-sphere spectrum.*

The proofs of Theorem 2.2 and Corollary 2.3 will be given in the subsequent sections. In section 3 we will describe the general method for calculating derivatives of homotopy functors. In section 4 we will construct the quotient space Δ_n and the transformation T_n and establish the Σ_n-equivariance of T. In section 5 we will determine the homotopy type of the Δ_n (and consequently, of $I^{(n)}$), and section 6 will be devoted to proving Theorem 2.2b.

3. The nth derivative of a functor

In this section we will define the nth derivative of a homotopy functor and show how it can be calculated in general. In the traditional calculus of real-valued functions one first defines the derivatives of a function and then uses these to construct the Taylor series of the function. In the case of homotopy functors, the opposite is true. It is more natural to define the Taylor tower of a functor first and then define the derivatives of the functor as the coefficients of its Taylor tower. Recall, the nth layer of the Taylor tower of a functor F is $D_nF = \text{fiber}(P_nF \to P_{n-1}F)$. By Proposition 1.8, if F is analytic, D_nF is a homogeneous functor of degree n. We saw in Theorem 1.7 that such a functor is naturally equivalent to a functor of the form

$$G(X) = \Omega^\infty(\mathbf{C} \wedge X^{[n]})_{h\Sigma_n}$$

where $\mathbf{C}$ is a spectrum with Σ_n-action, $X^{[n]}$ is the n-fold smash product of X and $h\Sigma_n$ denotes the homotopy orbit spectrum. It is the spectrum that is the coefficient of the homogeneous degree n functor that Goodwillie defines to be the nth derivative of X.

One does not need to know P_nI and $P_{n-1}I$ to determine the nth derivative of I. We will work with another category of functors which are equivalent to homogeneous functors of degree n, namely symmetric multilinear functors. These functors will be defined and discussed in this section. Symmetric multilinear functors are functors of several variables and, as we will see in Proposition 3.11, are also equivalent to functors of the form

$$G(X_1, \dots, X_n) = \Omega^\infty(\mathbf{C} \wedge X_1 \wedge \cdots \wedge X_n)$$

where $\mathbf{C}$ is a spectrum with Σ_n-action. In defining the nth derivative we will identify, without needing to know P_nI or $P_{n-1}I$, a symmetric multilinear functor which is naturally equivalent to $D_n I$ and determine the nth derivative from this functor. This symmetric multilinear functor is the multilinearization of the nth cross effect functor of I and is called the nth differential of I. This section will be devoted to explicitly defining the derivative as outlined above. The treatment will consist of three parts: defining the cross effect functor, discussing multilinearization and symmetric multilinear functors, and defining the derivative. This process is the way in which the derivative is originally defined by Goodwillie. Most of the material can be found in section 1 of [G1] and sections 3 and 4 of [G3].

To begin, we will let F be a homotopy functor from the category of based spaces to itself. We will construct its nth cross effect. The nth cross effect is the total fiber of a particular n-cube of spaces. The first step in defining the nth cross effect of F is to define the total fiber of an n-cube of spaces.

Definition 3.1. The *total fiber* of an n-cube of spaces $\mathbf{X}$, denoted $\widetilde{f}\mathbf{X}$, is the homotopy fiber of the map $\gamma : \mathbf{X}(\varnothing) \to \operatorname{holim}_{U\in N, U\neq\varnothing}(\mathbf{X}(U))$.

We may also define the total fiber of an n-cube of spaces inductively.

Remark 3.2. For a 1-cube (i.e., a map of spaces)

$$f : X \to Y$$

the total fiber is just the homotopy fiber of f.

For an n-cube of spaces, $\mathbf{X}$,we can consider $\mathbf{X}$ as a map of $(n-1)$-cubes:

$$\mathbf{Y} \to \mathbf{Z}.$$

We can then define the total fiber of $\mathbf{X}$ inductively as

$$\widetilde{f}\mathbf{X} = \text{fiber}(\widetilde{f}\mathbf{Y} \to \widetilde{f}\mathbf{Z}).$$

That this inductive definition of the iterated homotopy fiber is equivalent to Definition 3.1 follows from properties of the homotopy inverse limit. (See [G2].)

We will need the following n-cube of spaces associated to F in order to define the nth cross effect of F.

Definition 3.3. Given a collection of n spaces $\mathbf{X} = \{X_1, \dots, X_n\}$, $F_{\mathbf{X}}$ is the n-cube of spaces defined by $F_{\mathbf{X}}(\{1, \dots, n\}) = F(*)$ and $F_{\mathbf{X}}(U) = F(\bigvee_{i\notin U} X_i)$ when $U \neq \{1, \dots, n\}$. If $U \subset V$ then the morphism $F_{\mathbf{X}}(U \to V)$ is $F(g_{U,V})$ where $g_{U,V}$ is the retraction of $\bigvee_{i\notin U} X_i$ to $\bigvee_{j\notin V} X_j$, collapsing any X_i such that $i \in V$ and $i \notin U$ to the basepoint.

For example, if we have $\mathbf{X} = \{X_1, X_2\}$ then $F_{\mathbf{X}}$ is the square below.

$$\begin{array}{ccc} F(X_1 \vee X_2) & \longrightarrow & F(X_1) \\ \downarrow & & \downarrow \\ F(X_2) & \longrightarrow & F(*). \end{array}$$

Now we can define the cross effect.

Definition 3.4. The *nth cross effect* of a functor F evaluated at the spaces $\{X_1, \dots, X_n\}$ is a functor of n variables defined as the total fiber of the n-cube $F_{\mathbf{X}}$ with $\mathbf{X} = \{X_1, \dots, X_n\}$. The cross effect will be denoted $\chi_n F$.

$\chi_n F$ is a symmetric functor, where a symmetric functor is defined as follows.

Definition 3.5. A homotopy functor F is called *symmetric* if for each $\pi \in \Sigma_n$ and spaces $X_1, \dots, X_n$ there is an isomorphism

$$F_\pi : F(X_1, \dots, X_n) \to F(X_{\pi(1)}, \dots, X_{\pi(n)})$$

and for every $\pi, \sigma \in \Sigma_n$

$$F_{\sigma\circ\pi} = F_\pi \circ F_\sigma.$$

That $\chi_n F$ is symmetric is clear from the symmetry of the n-cube with respect to the spaces $X_1, \dots, X_n$.

Now that we have defined the cross effect of F, we must explain what it means to multilinearize the cross effect, or to linearize a functor in general. The multilinearization of $\chi_n F$ will be the symmetric multilinear functor equivalent to $D_n F$ mentioned at the beginning of the section. First, we review the concept of linear functors. The process of linearization of a functor F is basically that of finding the linear functor which most closely agrees with F.

Definition 3.6. A homotopy functor F is called *linear* if it satisfies the properties below.

(a) F is reduced, i.e., $F(*)$ is contractible.
(b) F is 1-excisive.

For a discussion of linear functors see section 1 of [G1] or chapter 4 of [J].

To any functor F satisfying stable first order excision (recall Definition 1.1) we may also associate a linear functor. As stated in section 1, $P_1 F$ is the degree one (or excisive) functor which most closely approximates F. We make $P_1 F$ a linear functor by reducing it, i.e., by taking the homotopy fiber of $P_1 F(X) \to P_1 F(*)$. If the functor is already reduced, then the process of linearizing it can be made even more explicit. $P_1 F$ is reduced and hence is already linear. We have:

Proposition 3.7. *If F is a reduced homotopy functor satisfying stable first order excision then the linearization of F is simply $P_1 F$ and is given by*

$$P_1 F(X) = \operatorname*{hocolim}_n \Omega^n F(\Sigma^n X).$$

Proof. The proof of this proposition follows directly from the definition of $P_1 F$ that Goodwillie gives in [G1]. Specifically, $P_1 F$ is defined as the homotopy colimit of the diagram

$$F(X) \to TF(X) \to T(TF(X)) \to \dots$$

where

$$TF(X) = \operatorname{holim}(F(CX) \to F(\Sigma X) \leftarrow F(CX)).$$

CX is the cone on X. When F is reduced, we see that $TF(X)$ is equivalent to $\Omega F \Sigma X$, and the result follows.

Proposition 3.7 will be sufficient for our purposes since we will only need to linearize reduced functors.

If we have a functor of n variables, G, we may speak of its multilinearization. The multilinearization of G is simply the linearization of G with respect to each of its variables. That is, we hold all but one variable fixed and linearize G with respect to the unfixed variable and then repeat this process for each variable. As in the case of functors of one variable, if G is a reduced functor with respect to each of its variables, then we can give the multilinearization of G explicitly. This is a direct consequence of Proposition 3.7.

Corollary 3.8. *If G is a homotopy functor of n variables which satisfies first order excision and is reduced with respect to each of its variables, then the multilinearization of G is given by*

$$\widetilde{G}(X_1, \dots, X_n) = \operatorname*{hocolim}_{k_1, \dots, k_n} \Omega^{k_1 + \dots + k_n}(G)(\Sigma^{k_1} X_1, \dots, \Sigma^{k_n} X_n).$$

A functor which has been multilinearized or is linear in each of its variables is called a *multilinear* functor.

With the preceding definitions, we are now able to define the *nth differential* of a homotopy functor .

Definition 3.9. The *nth differential* of a homotopy functor F, denoted $D^{(n)}F$, is the functor of n variables defined by

$$D^{(n)}F(X_1, \dots, X_n) = \widetilde{\chi_n F}(X_1, \dots, X_n).$$

That is, $D^{(n)}F$ is the multilinearization of the nth cross effect of F.

$D^{(n)}F$ is an example of a symmetric multilinear functor.

Definition 3.10. A *symmetric multilinear functor* of n variables is a symmetric functor which is linear in each of its variables.

That $D^{(n)}F$ is a symmetric functor follows from the symmetry of the cross effect already established. Its multilinearity follows from the definition.

As stated at the beginning of the section, symmetric multilinear functors are naturally equivalent to spectra with Σ_n-action. Precisely, Goodwillie has shown (in [G3])

Proposition 3.11. *If G is a symmetric multilinear functor of n variables then it has the following form:*

$$G(X_1, \dots, X_n) = \Omega^\infty(\mathbf{C} \wedge X_1 \wedge \cdots \wedge X_n)$$

where $\mathbf{C}$ is a spectrum with Σ_n-action.

With this proposition we are now able to define the nth derivative of the functor F.

Definition 3.12. The *nth derivative* of the homotopy functor F is the spectrum with Σ_n-action associated with the symmetric multilinear functor $D^{(n)}F$, that is, the spectrum $\mathbf{C}$ such that

$$D^{(n)}F(X_1, \dots, X_n) \simeq \Omega^\infty(\mathbf{C} \wedge X_1 \wedge \cdots \wedge X_n).$$

We write $F^{(n)} = \mathbf{C}$.

We should note here that the definition of the derivative given above is the nth derivative of the functor F **at a point**. A more general definition can be given for derivatives at any space X. (See [G3].) For examples of derivatives, see [G1] and [G3].

We conclude this section by justifying the analogy drawn at the beginning of the section between the nth derivative of F as we have defined it and the coefficient of the nth term in the Taylor tower of F. The nth layer of the Taylor tower of F, D_nF, is a homogeneous functor of degree n. We stated that a homogeneous functor of degree n such as D_nF was naturally equivalent to a symmetric multilinear functor and could be written in the form

$$\Omega^\infty(\mathbf{C} \wedge X^{[n]})_{h\Sigma_n}$$

where $\mathbf{C}$ is a spectrum with Σ_n-action. Under the equivalence that Goodwillie establishes between homogeneous functors of degree n, symmetric multilinear functors and spectra with Σ_n-action, $D^{(n)}F$ is the symmetric multilinear functor equivalent to D_nF and $F^{(n)}$ is the spectra with Σ_n-action equivalent to D_nF. This is summarized below.

Proposition 3.13.

(a) *The symmetric multilinear functor equivalent to* D_nF *is the cross effect of* D_nF, $\chi_n D_n F$. *Furthermore, this functor is naturally equivalent to the multilinearization of* $\chi_n F$.

(b) D_nF *can be recovered from* $F^{(n)}$:

$$D_nF(X) = (F^{(n)} \wedge X^{[n]})_{h\Sigma_n}.$$

This process can be extended to all homogeneous functors of degree n to give an explicit means of constructing the equivalences between the three categories. See [G3], section 3 for details.

We are now ready to construct the derivatives of I.

4. The functor M_n

This section will be devoted to the construction of the functor M_n and the transformation T between $\chi_n I$ and M_n, first mentioned in section 2. We will show later that the multilinearization of M_n is equivalent to the multilinearization of $\chi_n I$, and from that determine the nth derivative of I.

Recall Theorem 2.2. Conditions (a) and (b) are sufficient to conclude that $\chi_n I$ and M_n have the same multilinearization. Specifically, we have

Proposition 4.1. *Let* F *and* G *be reduced functors of* n *variables. Let* $S : F \to G$ *be a natural transformation such that when* $X_1, X_2, \dots, X_n$ *are* k*-connected, then the resulting map*

$$S : F(X_1, X_2, \dots, X_n) \to G(X_1, X_2, \dots, X_n)$$

is $((n+1)k - c)$*-connected, where* c *is a constant which does not depend on* k. (F *and* G *agree to* n*th order via* S.) *If such a transformation exists, then the multilinearization of* F *is equivalent to the multilinearization of* G.

Proof. The multilinearizations of F and G evaluated at the spaces $X_1, \dots, X_n$ are equivalent to $\operatorname{hocolim}_{l\to\infty}\Omega^{ln}F(\Sigma^l X_1, \Sigma^l X_2, \dots, \Sigma^l X_n)$ and $\operatorname{hocolim}_{l\to\infty}\Omega^{ln}G(\Sigma^l X_1, \Sigma^l X_2, \dots, \Sigma^l X_n)$, respectively. If $X_1, \dots, X_n$ are k-connected then $\Sigma^l X_1, \Sigma^l X_2, \dots, \Sigma^l X_n$ are $(k+l)$-connected. Hence, the map

$$\Omega^{ln}S : \Omega^{ln}F(\Sigma^l X_1, \dots, \Sigma^l X_n) \to \Omega^{ln}G(\Sigma^l X_1, \dots, \Sigma^l X_n)$$

is $((n+1)(k+l) - c - ln) = nk + k + l - c$-connected. As l goes to infinity, so does the connectivity. Hence, in the limit, $\Omega^{ln}F(\Sigma^l X_1, \dots, \Sigma^l X_n)$ and $\Omega^{ln}G(\Sigma^l X_1, \dots, \Sigma^l X_n)$ are equivalent.

Corollary 4.2. *If* $\Omega T_n : \Omega\chi_n I(\Sigma X_1, \dots, \Sigma X_n) \to \Omega \mathrm{Map}_*(\Delta_n, \Sigma X_1 \wedge \cdots \wedge \Sigma X_n)$ *is* $(n+1)k - c$*-connected then*

$$\chi_n I(X_1, X_2, \dots, X_n) \to M_n(X_1, X_2, \dots, X_n)$$

is an equivalence after multilinearization.

Proof. The hypothesis implies that the map

$$\Omega^n T_n : \Omega^n \chi_n I(\Sigma X_1, \dots, \Sigma X_n) \to \Omega^n \mathrm{Map}_*(\Delta_n, \Sigma X_1 \wedge \cdots \wedge \Sigma X_n)$$

is $((n+1)k - (n-1) - c)$-connected. Proposition 4.1 implies that the multilinearizations of each side above are equivalent. But the multilinearizations of $\chi_n I$ and M_n are equivalent to those of $\Omega^n \chi_n I(\Sigma X_1, \dots, \Sigma X_n)$ and $\Omega^n \mathrm{Map}_*(\Delta_n, \Sigma X_1 \wedge \cdots \wedge \Sigma X_n)$ respectively. Hence, the multilinearizations of $\chi_n I$ and M_n are equivalent.

Hence, condition (b) of Theorem 2.2 guarantees that the spectra associated to the multilinearizations of $\chi_n I$ and M_n have the same homotopy type. Condition (a) of Theorem 2.2 ensures that the spectra have the same Σ_n-action. Thus, if we can prove Theorem 2.2, we will be able to recover the derivatives of I from the multilinearization of M_n. In this section we will define T and Δ_n. We will also verify condition (a) of Theorem 2.2.

If we use the inductive definition (Remark 3.2) to construct the total fiber of an n-cubical diagram, we see that the resulting space is equivalent to a collection of maps with certain compatibility properties. For example, the iterated fiber of a 1-cube, $f : X \to Y$, is, by definition,

$$\{(x, \gamma) \in X \times Y^I | \gamma(0) = f(x), \gamma(1) = y_0\}$$

where y_0 is the basepoint in Y. The exact description for any n-cube is given below in Definition 4.3, following remarks in section 0 of [G2]. This property of the total fiber will be used in constructing T and Δ_n.

Definition 4.3. Let $\mathbf{X}$ be an n-cube of spaces. For a subset U of $\underline{n}$, let $I^U = \{(t_1, t_2, \dots, t_n) \in I^n | t_i = 0 \text{ if } i \notin U\}$. A point in the total fiber of $\mathbf{X}$ is a collection of maps, $\Phi = \{\Phi_U\}_{U \subset \underline{n}}$, where $\Phi_U : I^U \to \mathbf{X}(U)$. Furthermore, these maps must satisfy the compatibility properties below:

(a) For $V \subset U$

$$\Phi_U|_{I^V} = \gamma_{V,U} \circ \Phi_V$$

where $\gamma_{V,U}$ is the map $\mathbf{X}(i_{V,U}) : \mathbf{X}(V) \to \mathbf{X}(U)$, and $i_{V,U}$ is the inclusion $V \hookrightarrow U$.

(b) $\Phi_U(t_1, \dots, t_n)$ is the basepoint in $\mathbf{X}(U)$ if $t_i = 1$ for some i.

Using 4.3 we will begin constructing T. First we show how to construct a map $T'_n : \chi_n I(X_1, X_2, \dots, X_n) \to \mathrm{Map}_*(I^{n(n-1)}, \prod_{i=1}^n X_i)$. Then we compose with the quotient map from the product $\prod_{i=1}^n X_i$ to the smash product $\bigwedge_{i=1}^n X_i$. This gives us a map from $\chi_n I(X_1, X_2, \dots, X_n)$ to $\mathrm{Map}_*(I^{n(n-1)}, \bigwedge_{i=1}^n X_i)$. Finally, we will identify a subspace of $I^{n(n-1)}$ which is always mapped to the basepoint in the smash product. Taking the quotient of $I^{n(n-1)}$ over this subspace yields the space we will call Δ_n and allows us to define the functor M_n as $\mathrm{Map}_*(\Delta_n, \bigwedge_{i=1}^n X_i)$. Furthermore, this yields the desired map from the cross effect to M_n, $T_n : \chi_n I(X_1, X_2, \dots, X_n) \to \mathrm{Map}_*(\Delta_n, \bigwedge_{i=1}^n X_i)$.

If we apply Definition 4.3 to $\chi_n I(X_1, X_2, \dots, X_n)$ then we see that a point Φ in $\chi_n I$ consists of maps $\{\Phi_U\}_{U \subset \{1, \dots, n\}}$ such that

$$\Phi_U : I^U \to \bigvee_{i \notin U} X_i$$

and conditions (a) and (b) of 4.3 are satisfied.

Consider the set $U_i = \{1, 2, \dots, \widehat{i}, \dots, n\}$. We have

$$\Phi_{U_i} : I^{U_i} \to X_i.$$

Note that I^{U_i} is an $(n-1)$-cube. Let

$$T_n' : \chi_n I(X_1, X_2, \dots, X_n) \to \mathrm{Map}_*(I^{n(n-1)}, \prod_{i=1}^n X_i)$$

be defined by

$$T_n'(\Phi) = \prod_{i=1}^n \Phi_{U_i} : \prod_{i=1}^n I^{U_i} \to \prod_{i=1}^n X_i.$$

If we compose with the quotient map from $\prod_{i=1}^n X_i$ to $\bigwedge_{i=1}^n X_i$, then we have

$$T_n'' : \chi_n I(X_1, X_2, \dots, X_n) \to \mathrm{Map}_*(I^{n(n-1)}, \bigwedge_{i=1}^n X_i)$$

defined by

$$T_n''(\Phi) = \bigwedge_{i=1}^n \tau_i.$$

Since we have now passed to the smash product, there will be many more points in $I^{n(n-1)}$ mapped by $T_n''(\Phi)$ to the basepoint in the new target $\bigwedge_{i=1}^n X_i$. Specifically, if $T_n'(\Phi)(t) \in (X_j \vee X_k) \times (\prod_{i=1, i\neq j,k}^n X_i)$ for some j and k, then $T_n''(\Phi)(t)$ is the basepoint in $\bigwedge_{i=1}^n X_i$. With this in mind, recall properties (a) and (b) of Definition 4.3. These properties tell us that if we restrict Φ_{U_i} to the boundary of I^{U_i} then either a point on the boundary is mapped by Φ_{U_i} to the basepoint in X_i (if one of the coordinates of the point is 1) or its image under Φ_{U_i} is the image under the retraction $X_i \vee X_j \to X_i$ of a point in $X_i \vee X_j$, for some j (if one of the coordinates of the boundary point is 0). Knowing this, we can identify a collection of subspaces of $I^{n(n-1)}$ which will always be mapped to the basepoint in $\bigwedge_{i=1}^n X_i$ by T_n''.

Let $t = [t_{ij}]_{1\leq i,j\leq n}$ where $t_{ii} = 0$ for all i denote a point in $I^{n(n-1)}$, so that $T''(\Phi)$ maps the $(n-1)$-cube, $\{(t_{i1}, t_{i2}, t_{i3}, \dots, t_{ii-1}, 0, t_{ii+1}, \dots, t_{in})\}$, to X_i. We wish to consider the following subspaces of $I^{n(n-1)}$.

Definition 4.4. For $1 \leq i < j \leq n$, let $W_{ij} = \{t \in I^{n(n-1)} | t_{ik} = t_{jk} \text{ for } 1 \leq k \leq n\}$.

As indicated, if one of the coordinates, t_{ij}, of $t \in I^{n(n-1)}$ is 0, then $\Phi_{U_i}(t_{i1}, \dots, t_{in})$ is the retraction to X_i of a point in $X_i \vee X_j$. The conditions placed on a point in W_{ij} guarantee that $\Phi_{U_i}(t_{i1}, \dots, t_{in})$ and $\Phi_{U_j}(t_{j1}, \dots, t_{jn})$ are retractions to X_i and X_j respectively of the same point in $X_i \vee X_j$. Hence, we state the following.

Lemma 4.5. *If $t \in W_{ij}$, then $T_n''(\Phi)(t) = e$, the basepoint in $\bigwedge_{i=1}^n X_i$.*

Proof. Let $t = [t_{kl}] \in W_{ij}$. The essence of the proof was described before the statement of the lemma. By property (a) of Definition 4.3 we have

$$\Phi_{U_i}|_{I^V} = \gamma_{V,U_i} \circ \Phi_V$$

and

$$\Phi_{U_j}|_{I^V} = \gamma_{V,U_j} \circ \Phi_V$$

where $V = \{1, \dots, \widehat{i}, \dots, \widehat{j}, \dots, n\}$. For a point $t \in W_{ij}$, we see that

$$\Phi_{U_i} \times \Phi_{U_j}((t_{i1}, \dots, t_{in}), (t_{j1}, \dots, t_{jn}))$$

is equal to

$$(\gamma_{V,U_i} \circ \Phi_V(t_{i1}, \dots, t_{in}), \gamma_{V,U_j} \circ \Phi_V(t_{i1}, \dots, t_{in}))$$

since $(t_{i1}, \dots, t_{in}) = (t_{j1}, \dots, t_{jn})$. That is,

$$\Phi_{U_i} \times \Phi_{U_j}((t_{i1}, \dots, t_{in}), (t_{j1}, \dots t_{jn}))$$

is the retraction of the same point in $X_i \vee X_j$ to both X_i and X_j. It follows that $T_n''(\Phi)(t) = e$.

There is another collection of subspaces of $I^{n(n-1)}$ which will be mapped by $T_n''(\Phi)$ to the basepoint in $\bigwedge_{i=1}^n X_i$. These are the points in $I^{n(n-1)}$ for which one of the coordinates is 1. We have the following result.

Lemma 4.6. *Let $Z = \{t \in I^{n(n-1)} | t_{ij} = 1$ for some $1 \le i, j \le n\}$. If $t \in Z$, and $\Phi \in \chi_n I(X_1, X_2, \dots, X_n)$, then $T_n''(\Phi)(t)$ is the basepoint in $\bigwedge_{i=1}^n X_i$.*

Proof. The proof follows immediately from property (b) of Definition 4.3.

Now we may define Δ_n, M_n, and T_n.

Definition 4.7. *Let $\Delta_n = I^{n(n-1)}/\{Z \cup \bigcup_{i<j} W_{ij}\}$, the quotient of $I^{n(n-1)}$ over $\{Z \cup \bigcup_{i<j} W_{ij}\}$.*

Definition 4.8. *M_n is a functor of n spaces defined as*

$$M_n(X_1, X_2, \dots, X_n) = \mathrm{Map}_*(\Delta_n, X_1 \wedge X_2 \wedge \cdots \wedge X_n).$$

Definition 4.9. *Let $T_n : \chi_n I(X_1, X_2, \dots, X_n) \to M_n(X_1, X_2, \dots, X_n)$ be the map T_n'' defined above. For $\Phi \in \chi_n I(X_1, X_2, \dots, X_n)$, $T_n''(\Phi)$ is a well-defined map by the previous lemmas.*

We conclude this section by noting that T_n is a Σ_n-equivariant transformation.

Theorem 2.2a. *The transformation T_n is Σ_n-equivariant, that is, it preserves the Σ_n-symmetry of $\chi_n I$ and M_n which permutes the spaces $X_1, X_2, \dots, X_n$ and images of the $(n-1)$-cubes in Δ_n.*

The proof follows from the definitions involved and is left to the reader.

5. The homotopy type of the derivatives

In this section, we determine the homotopy type of $\Omega\chi_n(\Sigma)(X_1, \dots, X_n)$ and Δ_n, and consequently, the homotopy type of the derivatives.

In proving Theorem 2.2b, we will be working with the cross effect of the functor $\Omega\Sigma$. Note that

$$\chi_n(\Omega\Sigma)(X_1, X_2, \dots, X_n) \simeq \chi_n\Omega(\Sigma X_1 \wedge \cdots \wedge \Sigma X_n) \simeq \Omega\chi_n I(X_1, X_2, \dots, X_n).$$

The latter equivalence comes from the fact that the loop space functor is a homotopy fiber and as such commutes with the homotopy fibers of χ_n.

It suffices to study the cross effect of $\Omega\Sigma$ because the multilinearizations of $\chi_n I$ and $\chi_n(\Omega\Sigma)$ are equivalent, by Proposition 4.1. The convenience of this approach is that the Hilton-Milnor theorem will allow us to determine the homotopy type of $\chi_n(\Omega\Sigma)(X_1, X_2, \dots, X_n)$ in the range we need. Specifically, we have the following.

Proposition 5.1. *If* $X_1, \dots, X_n$ *are* k*-connected spaces, then*

$$\pi_m\Big(\prod_{j=1}^{(n-1)!} \bigwedge_{i=1}^{n} X_i\Big) \cong \pi_m(\chi_n(\Omega\Sigma)(X_1, \dots, X_n))$$

for $0 \le m \le (n+1)(k+1) - 1$.

To prove Proposition 5.1, we recall the Hilton-Milnor theorem and the definitions needed to state it. The subsequent discussion follows that of Whitehead ([Wh], pp. 511-517). Hall ([Ha]) has shown that the free Lie algebra generated by the elements $x_1, x_2, \dots, x_n$ has an additive basis consisting of certain "basic products". The basic products are defined in terms of three numbers assigned to each product: the serial number, s, the rank, r, and the weight, w. The basic products are defined inductively by weight. The basic products of weight 1 are $x_1, x_2, \dots, x_n$ with $s(x_i) = i$ and $r(x_i) = 0$ for $i = 1, 2, \dots, n$. Suppose that the basic products of weight $\le m - 1$ have been defined, along with their serial numbers and rank, in such a way that if $w(u) < w(v)$, then $s(u) < s(v)$. Then, the basic products of weight m are all products, uv, where u and v are basic products satisfying the following conditions:

(i) $w(u) + w(v) = m$,
(ii) $s(v) < s(u)$,
(iii) $r(u) \le s(v)$.

For such a product, $r(uv) = s(v)$. If k is the largest serial number assigned to the products of weight less than or equal to $m - 1$, then the products of weight m can be assigned serial numbers in any order beginning with $k + 1$.

A formula due to Witt [Wi] tells us that the number of basic products involving x_j exactly k_j times is

$$(5.2) \qquad \frac{1}{k}\sum_{d|k_0} \mu(d)\frac{(k/d)!}{(k_1/d)!\cdots(k_n/d)!},$$

where $k = k_1 + \dots + k_n$, μ is the Möbius function, and k_0 is the greatest common divisor of $k_1, \dots, k_n$.

Given spaces $X_1, X_2, \dots, X_n$, one can define the basic products of these spaces by using the smash product. Let $w_k(X_1, \dots, X_n)$ denote the kth basic product of the spaces $X_1, X_2, \dots, X_n$. For a space X, recall that the space $J(X)$ is the reduced product space defined by James ([Ja]). One can define a map $h : \prod_{i=1}^{\infty} Jw_i(X_1, \dots, X_n) \to J(X_1 \vee \dots \vee X_n)$ (see [Wh] for details). With this we can state the Hilton-Milnor theorem ([Hi], [M]).

Theorem 5.3 (Hilton-Milnor). *The map*

$$h : \prod_{i=1}^{\infty} J(w_i(X_1, \dots, X_n)) \to J(X_1 \vee \dots \vee X_n)$$

is a homotopy equivalence.

Recall that for any space X, there is a weak homotopy equivalence, $J(X) \to \Omega\Sigma X$. This weak equivalence leads to the corollary of the Hilton-Milnor theorem stated below.

Corollary 5.4. *The spaces* $\prod_{i=1}^{\infty} \Omega\Sigma(w_i(X_1, \ldots, X_n))$ *and* $\Omega\Sigma(X_1 \vee \cdots \vee X_n)$ *have the same homotopy type.*

Proof of Proposition 5.1. Applying Corollary 5.4 we see that

$$\chi_n(\Omega\Sigma)(X_1, \ldots, X_n) \simeq \prod \Omega\Sigma(w_i(X_1, \ldots, X_n)).$$

The product on the right is taken over all basic products $w_i(X_1, \ldots, X_n)$ which include each X_j at least once. Clearly, the first basic products to satisfy this condition are those involving each space exactly once. By formula (5.2) the number of such products is $(n-1)!$. Note also that if $w_i(X_1, \ldots, X_n)$ is a basic product of weight t then $\Omega\Sigma w_i(X_1, \ldots, X_n)$ is $(t(k+1)-1)$-connected. Since $\prod_{j=1}^{(n-1)!} \Omega\Sigma(\bigwedge_{i=1}^{n} X_i)$ and $\chi_n(\Omega\Sigma)(X_1, \ldots, X_n)$ differ by basic products of weight $n+1$ and greater, it follows that

$$\pi_m\left(\prod_{j=1}^{(n-1)!} \Omega\Sigma\left(\bigwedge_{i=1}^{n} X_i\right)\right) \cong \pi_m(\chi_n(\Omega\Sigma)(X_1, \ldots, X_n))$$

for $0 \le m \le ((n+1)(k+1)-1)$. Consequently,

$$\pi_m\left(\prod_{j=1}^{(n-1)!} \left(\bigwedge_{i=1}^{n} X_i\right)\right) \cong \pi_m(\chi_n(\Omega\Sigma)(X_1, \ldots, X_n))$$

in the same range.

It should be noted here that, although the Hilton-Milnor theorem gives us the homotopy type of the nth cross effect in the range we need, the map involved does not respect the Σ_n-symmetry of $\chi_n(\Omega\Sigma)(X_1, X_2, \ldots, X_n)$. Hence, we have no way of recovering the Σ_n-action after the multilinearization of $\chi_n(\Omega\Sigma)(X_1, X_2, \ldots, X_n)$ from the information yielded by Hilton-Milnor. This is what first motivated the construction of the functor M_n.

We now turn our attention to the complex Δ_n. For the purpose of proving the connectivity of T_n, we will use an equivalent subcomplex of Δ_n, which we call $\widetilde{\Delta}_n$.

Definition 5.5. $\widetilde{\Delta}_n$ is the subspace of Δ_n consisting of all points $[t_{ij}]_{1 \le i, j \le n}$ in Δ_n such that $t_{ij} = 0$ when $j \neq 1$.

The homotopy type of $\widetilde{\Delta}_n$ is easily determined.

Proposition 5.6. $\widetilde{\Delta}_n$ *is homotopy equivalent to* $\bigvee_{(n-1)!} S^{n-1}$.

To prove Proposition 5.6 we will construct maps labelled by elements of the following set, G_n. We will refer to this set throughout the remainder of this paper.

Definition 5.7. G_n is the set of all bijections, $g : \underline{n} \to \underline{n}$, such that $g(1) = 1$. $\iota \in G_n$ will always denote the identity.

Proof of 5.6. Let $g \in G_n$. Define $h_g : I^{n-1} \to I^{n(n-1)}$ as follows.

$$h_g : (s_1, s_2, \ldots, s_{n-1}) \longmapsto \begin{pmatrix} 0 & 0 & \ldots & 0 \\ t_{21} & 0 & \ldots & 0 \\ t_{31} & 0 & \ldots & 0 \\ \vdots & & & \vdots \\ t_{n1} & 0 & \ldots & 0 \end{pmatrix}$$

where

$$t_{j1}(s_1, s_2, \dots, s_{n-1}) = \max\{s_l | l < g^{-1}(j)\}.$$

Specifically, the image of h_g in $I^{n(n-1)}$ is the $(n-1)$-cell in which $t_{g(2)1} < t_{g(3)1} < \cdots < t_{g(n)1}$. As a map of I^{n-1} into $I^{n(n-1)}$, h_g maps the boundary of I^{n-1} to the subspaces $W_{1g(2)}, W_{g(2)g(3)}, \dots, W_{g(n-1)g(n)}$, and Z of $I^{n(n-1)}$. That is, we have

$$h_g|_{s_i=0} : I^{n-1} \longrightarrow W_{g(i)g(i+1)}$$

and

$$h_g|_{s_j=1} : I^{n-1} \longrightarrow Z.$$

Furthermore, if $s_i \geq s_j$ for some $i < j$ then h_g takes the point to $W_{g(j)g(j+1)}$. To see these facts, note that in $\widetilde{\Delta}_n$, W_{kl} has the form

$$W_{kl} = \{t \in I^{n(n-1)} | t_{ij} = 0 \text{ for } j \neq 1, \text{ and } t_{k1} = t_{l1}\}.$$

If $s_i = 0$, then

$$\begin{aligned} t_{g(i+1)1}(s_1, \dots, s_{n-1}) &= \max\{s_l | l < i+1\} = \max\{s_l | l < i\} \\ &= t_{g(i)1}(s_1, \dots, s_{n-1}). \end{aligned}$$

Thus, $h_g(s_1, \dots, s_{n-1}) \in W_{g(i)g(i+1)}$ when $s_i = 0$.

Furthermore, if $s_j = 1$, then

$$t_{g(j+1)1} = \max\{s_l | l < j+1\} = 1.$$

So, $h_g(s_1, \dots, s_{n-1}) \in Z$ when any of the s_j's are 1. Let L be the set of all the points just identified above, whose images are in $(\bigcup_{i<j} W_{ij}) \cup Z$. That is, $L = \{(s_1, \dots, s_{n-1}) \in I^{n-1} | s_i = 0 \text{ or } 1 \text{ for some } i \text{ or } s_i \geq s_j \text{ for some } i < j\}$. From the above we can see that if we compose with the quotient map $q : I^{n(n-1)} \to \Delta_n$, then $\lambda_g = q \circ h_g$ maps I^{n-1}/L in a one-to-one manner to $\widetilde{\Delta}_n$. Note also that I^{n-1}/L is homotopy equivalent to S^{n-1} since L is equivalent to ∂I^{n-1}. Hence, the map

$$\bigvee_{g \in G_n} \lambda_g : \bigvee_{g \in G_n} S^{n-1} \to \widetilde{\Delta}_n$$

is a homotopy equivalence.

As claimed we also have the following proposition and its corollaries.

Proposition 5.8. *Δ_n and $\widetilde{\Delta}_n$ are weakly homotopy equivalent.*

Corollary 5.9. *Δ_n is equivalent to $\bigvee_{(n-1)!} S^{(n-1)}$.*

Corollary 5.10.

$$\mathrm{Map}_*(\Sigma\Delta_n, \Sigma X_1 \wedge \Sigma X_2 \wedge \cdots \wedge \Sigma X_n) \simeq \mathrm{Map}_*(\Sigma\widetilde{\Delta}_n, \Sigma X_1 \wedge \Sigma X_2 \wedge \cdots \wedge \Sigma X_n).$$

And, both of these spaces are equivalent to $\prod_{(n-1)!} \Omega^n \Sigma^n (X_1 \wedge X_2 \wedge \cdots \wedge X_n)$.

The proof of Proposition 5.8 is broken up into several lemmas. Basically, we show that the space $(\bigcup_{i<j} W_{ij}) \cup Z$ is equivalent to its intersection with $\widetilde{\Delta}_n$. We will rely on a nice relationship between graphs and the W_{ij}'s in order to better understand the structure of $(\bigcup_{i<j} W_{ij}) \cup Z$.

Lemma 5.11. *The intersection,* $\bigcap_{k=1}^{s} W_{i_k j_k}$ *in* $I^{n(n-1)}$, *for some collection of pairs,* $\{(i_k, j_k)\}_{k=1}^{s}$, *consists of one point, the zero matrix* $[0]$, *if and only if the graph,* Υ, *consisting of vertices* $1, 2, \ldots, n$ *and edges* $i_k - j_k$ *for each pair* $(i_k, j_k) \in \{(i_k, j_k)\}_{k=1}^{s}$ *is connected.*

Proof. We will first consider the case where, for each $1 \leq l \leq s$, the graphs consisting of vertices i_k, j_k, and edges $i_k - j_k$ for each pair $(i_k, j_k) \in \{(i_k, j_k)\}_{k=1}^{l}$ is connected, **and** for each $1 \leq q \leq s$, $i_q \in \{i_k, j_k\}_{k=1}^{q-1}$, and $j_q \notin \{i_k, j_k\}_{k=1}^{q-1}$. That is, adding the edge $i_k - j_k$ connects one new point, j_k, to the graph built out of the previous $k-1$ pairs. In this case we claim that for any point in the corresponding intersection, $\bigcap_{k=1}^{s} W_{i_k j_k}$, at least $n + s(s+1)$ of the coordinates must be equal to 0. To show that this is the case, we proceed inductively. If we have the graph $i - j$, then the claim holds since $t_{11}, t_{22}, \ldots, t_{nn}, t_{ij}$, and t_{ji} must equal 0 in W_{ij}. Adding the edge $i_s - j_s$ to the graph corresponding to the pairs $\{(i_k, j_k)\}_{k=1}^{s-1}$ means that for a point in $\bigcap_{k=1}^{s} W_{i_k j_k}$, $2s$ more coordinates must be zero in addition to the $n + (s-1)s$ coordinates which must be zero for points in $\bigcap_{k=1}^{s-1} W_{i_k j_k}$. Specifically, these new coordinates are $\{t_{j_s i_k}\}_{k=1}^{s}$ and $\{t_{i_k j_s}\}_{k=1}^{s}$. Thus, $n + 2s + s(s-1) = n + s(s+1)$ coordinates of a point in $\bigcap_{k=1}^{s} W_{i_k j_k}$ must be zero.

In general, we consider the connected components of Υ and label them $\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_a$. Furthermore, let W_{Υ_α} denote the intersection of the W_{ij}'s corresponding to the edges in Υ_α. Without loss of generality, we may assume that each Υ_α contains a minimal number of edges, i.e., if Υ_α has r_α vertices, then it has $(r_\alpha - 1)$ edges. (Adding an extra edge to Υ_α imposes no new conditions on the coordinates of a point in W_{Υ_α}.) Then, by the above, a point in W_{Υ_α} must have $n + r_\alpha(r_\alpha - 1)$ coordinates equal to 0. Furthermore, a point in $\bigcap_{k=1}^{s} W_{i_k j_k} = \bigcap_{\alpha=1}^{a} W_{\Upsilon_\alpha}$ must have at least $n + \sum_{\alpha=1}^{a} r_\alpha(r_\alpha - 1)$ coordinates equal to 0 and can have as many as $\sum_{1 \leq \alpha < \beta \leq a} 2 r_\alpha r_\beta$ non-zero coordinates. Thus, if Υ has only one connected component , then $n + \sum_{\alpha=1}^{a} r_\alpha(r_\alpha - 1) = n^2$ and $\bigcap_{k=1}^{s} W_{i_k j_k}$ contains only the zero matrix. If Υ has more than one connected component, then $\sum_{1 \leq \alpha < \beta \leq a} 2 r_\alpha r_\beta > 0$ so $\bigcap_{k=1}^{s} W_{i_k j_k}$ contains points with non-zero coordinates. With this we have proved the lemma.

Lemma 5.12. *The intersection,* $\bigcap_{k=1}^{s} \widetilde{W}_{i_k j_k}$, *in* $\widetilde{I}^{n(n-1)}$ *for some collection of pairs,* $\{(i_k, j_k)\}_{k=1}^{s}$, *is the zero matrix if and only if the graph,* Υ, *consisting of points* $1, 2, \ldots, n$ *and edges* $i_k - j_k$ *for each* $(i_k, j_k) \in \{(i_k, j_k)\}_{k=1}^{s}$ *is connected.*

Proof. Let $t = [t_{ij}]$ be a point in $\bigcap_{k=1}^{s} \widetilde{W}_{i_k j_k}$. It is easy to see that $t_{i_k 1} = 0$ if and only if the point i_k is part of the connected component of Υ containing 1. Hence, all the t_{i1}'s are equal to zero if and only if Υ is connected, otherwise $\bigcap_{k=1}^{n} \widetilde{W}_{i_k j_k}$ will contain points with nonzero coordinates.

Lemma 5.13. *If the graph,* Υ, *associated to the collection of pairs* $\{(i_k, j_k)\}_{k=1}^{s}$ *is not connected then* $\bigcap_{k=1}^{s} W_{i_k j_k}$ *and* $\bigcap_{k=1}^{s} \widetilde{W}_{i_k j_k}$ *are contractible to points in* Z *and* $\widetilde{Z}$ *respectively.*

Proof. By Lemma 5.12, if Υ is not connected then for a point in W_Υ, there is at least one coordinate which does not have to be 0. Let $t_1, \ldots, t_m$ be those coordinates which do not have to be 0. Let $H : \bigcap_{k=1}^{s} W_{i_k j_k} \times I \to \bigcap_{k=1}^{s} W_{i_k j_k}$ be the homotopy which takes each $t_l \in \{t_1, \dots, t_m\}$ and $t \in I$, to $t(1-t_l)+t_l$. The argument is similar for $\bigcap_{k=1}^{s} \widetilde{W}_{i_k j_k}$.

As easy consequences of these lemmas, we have the following corollaries.

Corollary 5.14. *Let Υ be the graph associated to the collection of pairs $\{(i_k, j_k)\}_{k=1}^{s}$, W_Υ be the corresponding intersection $\bigcap_{k=1}^{s} W_{i_k j_k}$, and $\widetilde{W}_\Upsilon$ be the corresponding intersection $\bigcap_{k=1}^{s} \widetilde{W}_{i_k j_k}$. If Υ is connected then $W_\Upsilon \cap Z$ and $\widetilde{W}_\Upsilon \cap \widetilde{Z}$ are empty.*

Corollary 5.15. *For any collection of pairs $\{(i_k, j_k)\}_{k=1}^{s}$, the inclusions $\bigcap_{k=1}^{s} \widetilde{W}_{i_k j_k} \hookrightarrow \bigcap_{k=1}^{s} W_{i_k j_k}$ and $(\bigcap_{k=1}^{s} \widetilde{W}_{i_k j_k}) \cap \widetilde{Z} \hookrightarrow (\bigcap_{k=1}^{s} W_{i_k j_k}) \cap Z$ are homotopy equivalences.*

The corollaries allow us to prove the next proposition.

Proposition 5.16. *The inclusion $\widetilde{W} \cup \widetilde{Z} \hookrightarrow W \cup Z$ is a weak homotopy equivalence.*

For the proof of this proposition we need the following version of the gluing lemma found in [Wa].

Lemma 5.17. *In the commutative diagram*

$$\begin{array}{ccccc} X_1 & \leftarrow & X_0 & \rightarrow & X_2 \\ \downarrow & & \downarrow & & \downarrow \\ Y_1 & \leftarrow & Y_0 & \rightarrow & Y_2 \end{array}$$

let the two left horizontal maps be cofibrations and suppose that all the vertical maps are weak equivalences. Then the map of pushouts $X_1 \cup_{X_0} X_2 \to Y_1 \cup_{Y_0} Y_2$ is also a weak equivalence.

Proof of 5.16. With repeated applications of the gluing lemma to diagrams of the form

$$\begin{array}{ccccc} \bigcap_{k=1, k\neq m}^{s} \widetilde{W}_{i_k j_k} & \leftarrow & \bigcap_{k=1}^{s} \widetilde{W}_{i_k j_k} & \rightarrow & \bigcap_{k=1, k\neq l}^{s} \widetilde{W}_{i_k j_k} \\ \downarrow & & \downarrow & & \downarrow \\ \bigcap_{k=1, k\neq m}^{s} W_{i_k j_k} & \leftarrow & \bigcap_{k=1}^{s} W_{i_k j_k} & \rightarrow & \bigcap_{k=1, k\neq l}^{s} W_{i_k j_k} \end{array}$$

and

$$\begin{array}{ccccc} \widetilde{Z} & \leftarrow & \widetilde{W}_\Upsilon \cap \widetilde{Z} & \rightarrow & \widetilde{W}_\Upsilon \\ \downarrow & & \downarrow & & \downarrow \\ Z & \leftarrow & W_\Upsilon & \rightarrow & W_\Upsilon \end{array}$$

we are able to show that $\widetilde{W} \cup \widetilde{Z} \hookrightarrow W \cup Z$ is a homotopy equivalence.

Since $\widetilde{I}^{n(n-1)} \hookrightarrow I^{n(n-1)}$ is also an equivalence, the quotients $\widetilde{\Delta}_n$ and Δ_n are equivalent and the proof of Proposition 5.8 is complete.

The proof of the second statement of Corollary 5.10 is as follows.

Proof of 5.10. As a consequence of 5.6, each of the spaces is equivalent to $\mathrm{Map}_*(\Sigma(\bigvee_{(n-1)!} S^{(n-1)}), \Sigma X_1 \wedge \cdots \wedge \Sigma X_n)$. Furthermore,

$$\mathrm{Map}_*(\Sigma(\bigvee_{(n-1)!} S^{n-1}), \Sigma X_1 \wedge \cdots \wedge \Sigma X_n)$$

is equivalent to $\prod_{(n-1)!} \mathrm{Map}_*(S^n, S^n \wedge (X_1 \wedge \cdots \wedge X_n))$, and this is equivalent to $\prod_{(n-1)!} \Omega^n \Sigma^n (X_1 \wedge X_2 \wedge \cdots \wedge X_n)$.

6. The connectivity of ΩT_n

We have shown that for $m \leq (n+1)(k+1)-1$, $\pi_m(\prod_{(n-1)!} \bigwedge_{i=1}^n X_i)$ is isomorphic to $\pi_m(\chi_n(\Omega\Sigma)(X_1, X_2, \ldots, X_n))$ when $X_1, X_2, \ldots, X_n$ are k-connected spaces. From Corollary 5.10 it follows that $\pi_m(\chi_n(\Omega\Sigma)(X_1, X_2, \ldots, X_n))$ and $\pi_m(\mathrm{Map}_*(\Sigma\Delta_n, X_1 \wedge X_2 \wedge \cdots \wedge X_n))$ are isomorphic for $m \leq (n+1)(k+1)-1$. Now it remains to show that $(\Omega T_n)_*$ induces this isomorphism. To do so, we will set up the commutative diagram below

(6.1)

$$\begin{array}{ccc}
\widetilde{H}_*(\bigvee_{g\in G_n}(\prod_{i=1}^n X_i)) & \xrightarrow{((\Omega T_n)\circ D)_*} & \widetilde{H}_*(\mathrm{Map}_*(\Sigma\widetilde{\Delta}_n, \Sigma X_1 \wedge \cdots \wedge \Sigma X_n)) \\
 & & \downarrow \cong \\
\downarrow & & \widetilde{H}_*(\prod_{i=1}^{(n-1)!}(\Omega^n\Sigma^n(X_1 \wedge \cdots \wedge X_n))) \\
 & & \downarrow \\
\widetilde{H}_*(\bigvee_{g\in G_n}(\bigwedge_{i=1}^n X_i)) & \longrightarrow & \bigoplus_{g\in G_n} \widetilde{H}_*(\Omega^n\Sigma^n(X_1 \wedge \cdots \wedge X_n))
\end{array}$$

The first step will be to define the maps in the diagram and compute the degree of those maps.

For each $g \in G_n$ we will construct a map C_g that maps $\prod_{i=1}^n X_i$ to $\chi_n(\Omega\Sigma)(X_1, \ldots, X_n)$. Like the map used to prove the Hilton-Milnor theorem (see [Hi], [M]), C_g will be defined via commutators. In order to construct C_g, we will need a few basic maps. These maps, A, B, C, and P, will provide standard homotopies from commutator loops to constant loops, and will allow us to build elements of $\chi_n(\Omega\Sigma)$ from elements of $\chi_k(\Omega\Sigma)$ for $k < n$.

Definition 6.2. $C : \Omega X \times \Omega Y \to \Omega(X \vee Y)$ is the commutator. That is, for $\alpha \in \Omega X$, and $\beta \in \Omega Y$, $C(\alpha, \beta)$ is the commutator loop $\alpha\beta\alpha^{-1}\beta^{-1}$.

Definition 6.3. $A : I^2 \to I$ is defined by

$$A(s,t) = \begin{cases}
\frac{4t}{s+1}, & \text{for } 0 \leq s \leq \frac{1}{2} \text{ and } 0 \leq t \leq (\frac{1}{4} + \frac{1}{4}s) \\
1, & \text{for } 0 \leq s \leq \frac{1}{2} \text{ and } (\frac{1}{4} + \frac{1}{4}s) \leq t \leq (\frac{1}{2} - \frac{1}{4}s) \\
\frac{3-4t}{s+1}, & \text{for } 0 \leq s \leq \frac{1}{2} \text{ and } (\frac{1}{2} - \frac{1}{4}s) \leq t \leq \frac{3}{4} \\
0, & \text{for } 0 \leq s \leq \frac{1}{2} \text{ and } \frac{3}{4} \leq t \leq 1 \\
0, & \text{for } \frac{1}{2} \leq s \leq 1 \text{ and } 0 \leq t \leq \frac{3}{4}(s - \frac{1}{2}) \\
\frac{8t-6s+3}{3}, & \text{for } \frac{1}{2} \leq s \leq 1 \text{ and } \frac{3}{4}(s - \frac{1}{2}) \leq t \leq \frac{3}{8} \\
\frac{-8t-6s+9}{3}, & \text{for } \frac{1}{2} \leq s \leq 1 \text{ and } \frac{3}{8} \leq t \leq \frac{9}{8} - \frac{3}{4}s \\
0, & \text{for } \frac{1}{2} \leq s \leq 1 \text{ and } \frac{9}{8} - \frac{3}{4}s \leq t \leq 1.
\end{cases}$$

Note that, if e is the constant loop at the basepoint in X, and $\alpha \in \Omega X$, then $\alpha(A(s,t))$ is a homotopy from the commutator loop $C(\alpha, e)$ to e. See Figure 1.

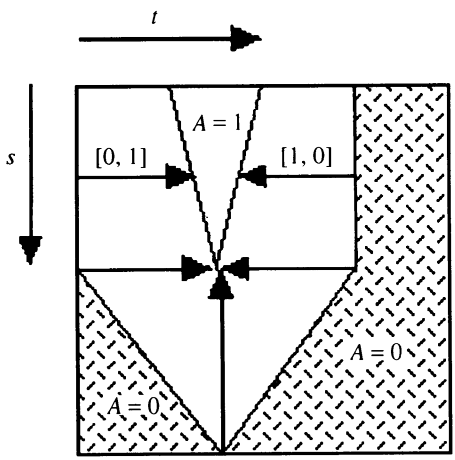

FIGURE 1

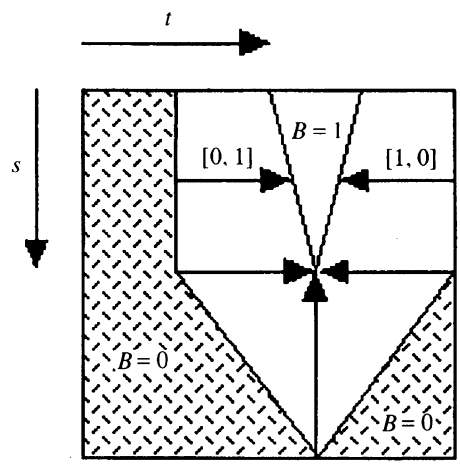

FIGURE 2

Definition 6.4. $B : I^2 \to I$ is defined by

$$B(s, t) = \begin{cases} 0, & \text{for } 0 \le s \le \frac{1}{2} \text{ and } 0 \le t \le \frac{1}{4} \\ \frac{4t-1}{s+1}, & \text{for } 0 \le s \le \frac{1}{2} \text{ and } \frac{1}{4} \le t \le \frac{1}{2} + \frac{1}{4}s \\ 1, & \text{for } 0 \le s \le \frac{1}{2} \text{ and } \frac{1}{2} + \frac{1}{4}s \le t \le \frac{3}{4} - \frac{1}{4}s \\ \frac{4-4t}{s+1}, & \text{for } 0 \le s \le \frac{1}{2} \text{ and } \frac{3}{4} - \frac{1}{4}s \le t \le 1 \\ 0, & \text{for } \frac{1}{2} \le s \le 1 \text{ and } 0 \le t \le \frac{3}{4}s - \frac{1}{8} \\ \frac{8t-6s+1}{3}, & \text{for } \frac{1}{2} \le s \le 1 \text{ and } \frac{3}{4}s - \frac{1}{8} \le t \le \frac{5}{8} \\ \frac{-8t-6s+11}{3}, & \text{for } \frac{1}{2} \le s \le 1 \text{ and } \frac{5}{8} \le t \le \frac{11}{8} - \frac{3}{4}s \\ 0, & \text{for } \frac{1}{2} \le s \le 1 \text{ and } \frac{11}{8} - \frac{3}{4}s \le t \le 1. \end{cases}$$

Note that, if e is the constant loop at the basepoint in X, and $\beta \in \Omega X$, then $\beta(B(s, t))$ is a homotopy from the commutator loop $C(e, \beta)$ to e. See Figure 2.

Finally, we define a map

$$P : \chi_{n-1}\Omega(X_1, X_2, \dots, X_{n-1}) \times \chi_1\Omega(X_n) \to \chi_n\Omega(X_1, X_2, \dots, X_n).$$

Essentially, P builds an element of $\chi_n\Omega(X_1, X_2, \dots, X_n)$ out of the commutator of a pair of elements from $\chi_{n-1}\Omega(X_1, X_2, \dots, X_{n-1})$ and $\chi_1\Omega(X_n)$. In the following e will always denote the constant loop at the basepoint.

Definition 6.5. For $\Phi \in \chi_{n-1}\Omega(X_1, X_2, \dots, X_{n-1})$ and $\beta \in \chi_1\Omega(X_n)$, $P : \chi_{n-1}\Omega(X_1, X_2, \dots, X_{n-1}) \times \chi_1\Omega(X_n) \to \chi_n\Omega(X_1, X_2, \dots, X_n)$ is the element $P(\Phi, \beta) = \{P(\Phi, \beta)_U\}_{U \in N}$ in $\chi_n\Omega(X_1, X_2, \dots, X_n)$. The maps $P(\Phi, \beta)_U : I^U \to \Omega(\bigvee_{i \notin U} X_i)$ are defined as below. t will denote the loop coordinate.

(a) If $n \notin U$,

$$P(\Phi, \beta)_U = \begin{cases} C(\Phi_U(2t_1, \dots, 2t_{n-1}), \beta_\varnothing)(t), & 0 \le t_1, \dots, t_{n-1} \le 1/2, \\ \beta_\varnothing(B(\max(2t_1 - 1, \dots, 2t_n - 1), t)), & \text{otherwise.} \end{cases}$$

(b) If $n \in U$,

$$P(\Phi, \beta)_U = \begin{cases} \Phi_{U-\{n\}}((2t_1, \dots, 2t_{n-1}))(A(t_n, t)), & 0 \le t_1, \dots, t_{n-1} \le 1/2, \\ e, & \text{otherwise.} \end{cases}$$

We leave it to the reader to check that $P(\Phi, \beta)$ as defined above satisfies all conditions necessary to be an element of $\chi_n\Omega(X_1, X_2, \dots, X_n)$.

Now we may define C_g.

Definition 6.6. For $g \in G_n$, we define $C_g \colon \prod_{i=1}^n X_i \to \chi_n(\Omega\Sigma)(X_1, X_2, \dots, X_n)$. The definition for any g will depend on the definition of C_ι where ι again denotes the identity function. C_ι is defined inductively. For $n = 1$, $C_\iota : X \to \chi_1\Omega\Sigma X$ is the function such that

$$(C_\iota(x))_\varnothing : I^\varnothing \to x(t)$$

where $x(t)$ is the loop in ΣX which takes t to $x \wedge t$. And,

$$(C_\iota(x))_{\{1\}} = e.$$

For $\iota \in G_n$, where $n > 1$, $C_\iota : \prod_{i=1}^n X_i \to \chi_n(\Omega\Sigma)(X_1, \dots, X_n)$ is defined as

$$C_\iota(x_1, \dots, x_n) = P(C_\iota(x_1, x_2, \dots, x_{n-1}), C_\iota(x_n)).$$

Finally, for any $g \in G_n$, $C_g : \prod_{i=1}^n X_i \to \chi_n(\Omega\Sigma)(X_1, \dots, X_n)$ is defined as

$$C_g(x_1, \dots, x_n) = C_\iota(x_{g^{-1}(1)}, x_{g^{-1}(2)}, \dots, x_{g^{-1}(n)}).$$

The maps C_g, though not explicitly the same, were inspired by the map of the Hilton-Milnor theorem. The Hilton-Milnor map takes a point in $\Omega\Sigma Y_j$, for some basic product Y_j, to a nested commutator of loops in $\Omega\Sigma(X_1 \vee X_2 \vee \cdots \vee X_n)$ determined by the grouping and ordering of the X_i's in Y_j. Essentially, C_g takes a point $(x_1, \dots, x_n)$ to the point in $\chi_n\Omega(\Sigma X_1, \Sigma X_2, \dots, \Sigma X_n)$ consisting of the nested commutator loop,

$$C(x_{g^{-1}(1)}, \dots, C(x_{g^{-1}(n-2)}(C(x_{g^{-1}(n-1)}, x_{g^{-1}(n)}))) \dots)$$

in $\Omega(\Sigma X_1 \vee \Sigma X_2 \vee \cdots \vee \Sigma X_n)$, determined by the surjection g, and homotopies from that loop to e in $\Omega(\bigvee_{i \notin U} \Sigma X_i)$.

We now wish to determine the connectivity of the composition $\Omega T_n \circ C_g$, as indicated in the commutative diagram at the beginning of this section. To do so, first note that for a point $(x_1, \dots, x_n)$ in $\prod_{i=1}^n X_i$, $(\Omega T_n) \circ C_g(x_1, \dots, x_n) \circ \Lambda_h$ (Λ_h is the suspension of the map λ_h defined in the proof of Proposition 5.6) is a map from S^n to $\Sigma^n(\bigwedge_{i=1}^n X_i)$. From the definition of C_g one can see that the image of a point $s \in S^n$ under this map will always have the form $\Gamma_{gh}(s) \wedge (x_1 \wedge \cdots \wedge x_n)$, where $\Gamma_{gh} \in \Omega^n\Sigma^n$ is a map determined by the choice of g and h. In other words, we can make the following definition.

Definition 6.7. $\Gamma_{gh} : S^n \to S^n$ is the map which makes the diagram below commutative. (q is the quotient map.)

$$\begin{array}{ccc} \prod_{i=1}^n X_i & \xrightarrow{(\Omega T_n)\circ C_g} & \mathrm{Map}_*(\Sigma\widetilde{\Delta}_n, \Sigma X_1 \wedge \cdots \wedge \Sigma X_n) \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle \Lambda_h^*} \\ \bigwedge_{i=1}^n X_i & \xrightarrow[x \mapsto \Gamma_{gh} \wedge x]{} & \Omega^n\Sigma^n(\bigwedge_{i=1}^n X_i) \end{array}$$

The advantage of considering the maps Γ_{gh} is that we can determine their degree for any choice of g and h in G_n.

Proposition 6.8. *For $g, h \in G_n$, the map $\Gamma_{gh} : S^n \to S^n$ has degree one if $h = g$. Otherwise, it is null-homotopic.*

The proof of Proposition 6.8 will proceed by induction on n. The heart of the proof will be defining a map ρ^* that makes the following diagram commute.

(6.9)
$$\begin{array}{ccc} \chi_{n-1}\Omega(X_1, \dots, X_{n-1}) \times \chi_1\Omega(X_n) & \xrightarrow{P} & \chi_n\Omega(X_1, X_2, \dots, X_n) \\ \Big\downarrow{\scriptstyle \Omega T_{n-1}\wedge\Omega T_1} & & \Big\downarrow{\scriptstyle \Omega T_n} \\ \mathrm{Map}_*(\Sigma\widetilde{\Delta}_{n-1} \wedge \Sigma\widetilde{\Delta}_1, \Sigma X_1 \wedge \Sigma X_n) & \xrightarrow{\rho^*} & \mathrm{Map}_*(\Sigma\widetilde{\Delta}_n, \Sigma X_1 \wedge \Sigma X_n). \end{array}$$

This diagram will allow us to compute the degree of the map in question by means of an equivalent map in $\mathrm{Map}_*(\Sigma\widetilde{\Delta}_{n-1} \wedge \widetilde{\Delta}_1, X_1 \wedge X_2 \wedge \cdots \wedge X_n)$, hence allowing us to set up an inductive argument to prove Proposition 6.8.

To define ρ^*, we will define a map $\rho : \Sigma\widetilde{\Delta}_n \to \Sigma\widetilde{\Delta}_{n-1} \wedge \widetilde{\Delta}_1$. ρ necessitates the use of the complex $\widetilde{\Delta}_n$, since it wasn't possible to construct such a map on Δ_n which was well-defined. (On the other hand, $\widetilde{\Delta}_n$ does not suffice for our purposes because the equivalence $\widetilde{\Delta}_n \to \Delta_n$ does not preserve the Σ_n-symmetry.)

Definition 6.10. Let $(s, t_{21}, t_{31}, \dots, t_{n1})$ be a point in $\Sigma\widetilde{\Delta}_n$, where s denotes the suspension coordinate and $t_{21}, t_{31}, \dots, t_{n1}$ denote the (possibly) non-zero coordinates of a point in $\widetilde{\Delta}_n$. $\rho : \Sigma\widetilde{\Delta}_n \to \Sigma\widetilde{\Delta}_{n-1} \wedge \widetilde{\Sigma\Delta}_1$ is defined by

$$\rho(s, t_{21}, t_{31}, \dots, t_{n1}) = (u \wedge (s_{21}, s_{31}, \dots, s_{(n-1)1})) \wedge v$$

where $(s_{21}, \dots, s_{(n-1)1})$ are the coordinates of a point in $\widetilde{\Delta}_{n-1}$, u is the suspension coordinate of $\Sigma\widetilde{\Delta}_{n-1}$, and v is a point in $\Sigma\widetilde{\Delta}_1$. They are defined as

$$\begin{aligned} s_{i1} &= \min(1, 2t_{i1}), \\ u &= A(0, s), \\ v &= B(\max(0, 2t_{n1} - 1), s). \end{aligned}$$

The map

$$\rho^* : \mathrm{Map}_*(\Sigma\widetilde{\Delta}_{n-1} \wedge \Sigma\widetilde{\Delta}_1, X_1 \wedge X_2 \wedge \cdots \wedge X_n) \to \mathrm{Map}_*(\Sigma\widetilde{\Delta}_n, X_1 \wedge X_2 \wedge \cdots \wedge X_n)$$

is defined as

$$\rho^* : \Theta \mapsto \Theta \circ \rho \qquad \text{for } \Theta \in \mathrm{Map}_*(\Sigma\widetilde{\Delta}_{n-1} \wedge \Sigma\widetilde{\Delta}_1, X_1 \wedge X_2 \wedge \cdots \wedge X_n).$$

With these definitions, it is easily shown that (6.9) is commutative. We omit the details.

The next step is to study the maps $\rho \circ \Lambda_g$.

Lemma 6.11. *Let $g \in G_n$. Consider the map $\rho \circ \Lambda_g : S^n \to \Sigma\widetilde{\Delta}_{n-1} \wedge \Sigma\widetilde{\Delta}_1$. When $g(n) = n$, let f denote the restriction of g to $\underline{\{n-1\}}$. Then,*

$$\rho \circ \Lambda_g \simeq \begin{cases} \Lambda_f \wedge \mathrm{id}_{S^1} & \text{if } g(n) = n, \\ e & \text{if } g(n) \neq n. \end{cases}$$

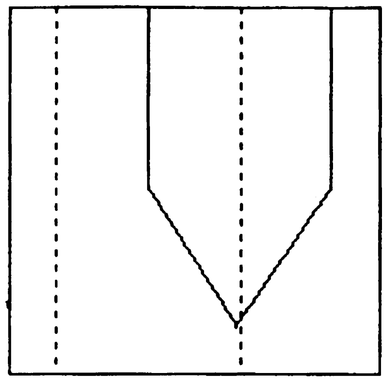

FIGURE 3

Proof. Let $t \wedge s_1 \wedge \cdots \wedge s_{n-1}$ be a point in S^n where S^n is considered as the smash product of n copies of S^1. If $g(n) = n$, then

$$\rho \circ \Lambda_g(t \wedge s_1 \wedge \cdots \wedge s_{n-1}) = (\min(1, 2t_{21}), \ldots, \min(1, 2t_{(n-1)1})) \wedge A(0, t) \wedge B(\max(0, 2t_{n1} - 1), t)$$

where

$$t_{j1} = \max\{s_l | l < g^{-1}(j)\}.$$

If $g(n) = n$, then

$$t_{n1} = \max\{s_1, \ldots, s_{n-1}\}$$

and $s_{n-1} \notin \{s_l | l < g^{-1}(j)\}$ for any $j \neq n$. (Thus, the image in $\widetilde{\Delta}_{n-1}$ does not depend on s_{n-1}.) Also note that if $\max(0, 2t_{n1} - 1) > 0$ then $s_j > 1/2$ for some j. If $j \neq n-1$, then $\min(1, 2t_{i1}) = 1$ for all i with $j \leq g^{-1}(i)$, and hence $\rho \circ \Lambda_g(t \wedge s_1 \wedge \cdots s_{n-1}) = e$. It follows that

$$\rho \circ \Lambda_g(t \wedge s_1 \wedge s_2 \wedge \cdots \wedge s_{n-1})$$

is equal to

$$(\min(1, 2t_{21}), \ldots, \min(1, 2t_{(n-1)1})) \wedge A(0, t) \wedge B(\max(0, 2s_{n-1} - 1), t).$$

The map $(\min(1, 2t_{21}), \min(1, 2t_{31}), \ldots, \min(1, 2t_{(n-1)1}))$ is homotopic to λ_f via the homotopy

$$(\min(1, (2-q)t_{21}), \min(1, (2-q)t_{31}), \ldots, \min(1, (2-q)t_{(n-1)1})).$$

To see that $A(0, t) \wedge B(\max(0, 2s_{n-1} - 1), t)$ is homotopic to the identity map on $S^2 = I^2/\partial I^2$, consider a point $(a, b) \in I^2$ such that $a, b \neq 0, 1$. The inverse image of $\{a\} \times I$ in I^2 has the form $\{a/4, (a+3)/4\} \times I$. The inverse image of $I \times \{b\}$ in I^2 under $B(\max(0, 2s_{n-1} - 1), t)$ is a path in I^2 from $(0, (b+1)/4)$ to $(0, (b+4)/4)$. It is easy to see that the inverse images of $I \times \{b\}$ and $\{a\} \times I$ intersect in a single point (see Figure 3). Hence the map has degree one and is homotopic to the identity map as claimed.

If $g(n) \neq n$, then $t_{g(n)1} = \max(s_1, s_2, \ldots, s_{n-1})$. In order to have a nontrivial point in the image of $\rho \circ \Lambda_g$ we must have $\min(1, 2t_{g(n)1}) \neq 1$, i.e.,

$s_i < 1/2$ for all i. If this is the case then $t_{n1} < 1/2$, and $\max(0, 2t_{n1} - 1) = 0$. Thus, if $s_i < 1/2$ for all i, then

$$\rho \circ \Lambda_g(t \wedge s_1 \wedge \cdots \wedge s_{n-1})$$

is equal to

$$(\min(1, 2t_{21}), \ldots, \min(1, 2t_{(n-1)1})) \wedge A(0, t) \wedge B(0, t).$$

But, $A(0, t) \wedge B(0, t)$ will always be the basepoint in S^2. Hence, $\rho \circ \Lambda_g$ is null homotopic.

With this we are now ready to prove Proposition 6.8.

Proof of Proposition 6.8. When $n = 1$, we need only consider one case, $\iota \in G_1$. One can easily see that $\Gamma_{\iota\iota}$ is the identity map on S^1, and hence the proposition holds here. Furthermore, for $\iota \in G_2$,

$$\Gamma_{\iota\iota}(t \wedge s_1) = A(0, t) \wedge B(\max(0, 2s_1 - 1), t)$$

since

$$\Omega T_2 \circ C_\iota(x_1, x_2) \circ \Lambda_g(t \wedge s_1) = A(0, t) \wedge x_1 \wedge (B(\max(0, 2s_1 - 1), t)) \wedge x_2.$$

We saw in the proof of Lemma 6.11 that this has degree one.

For $g, h \in G_n$, it follows from the definitions of C_g, T_n and Λ_g that

$$(\Omega T_n \circ C_g(x_1, x_2, \ldots, x_n)) \circ \Lambda_h = (\Omega T_n \circ C_\iota(x_1, x_2, \ldots, x_n)) \circ \Lambda_{g^{-1} \circ h}.$$

Furthermore,

$$\begin{aligned}(\Omega T_n \circ C_\iota(x_1, \ldots, x_n)) \circ \Lambda_{g^{-1}\circ h} &= (\Omega T_n \circ P(C_\iota(x_1, \ldots, x_n), C_\iota(x_n))) \circ \Lambda_{g^{-1}\circ h} \\ &= \Omega T_{n-1}(C_\iota(x_1, \ldots, x_{n-1})) \wedge \Omega T_1(C_\iota(x_n)) \circ (\rho \circ \Lambda_{g^{-1}\circ h}).\end{aligned}$$

If $g = h$, then $\rho \circ \Lambda_{g^{-1}\circ h} \simeq \Lambda_\iota \wedge \mathrm{id}_{S^1}$ by Lemma 6.11. So,

$$(\Omega T_n \circ C_g(x_1, \ldots, x_n)) \circ \Lambda_h$$

is equivalent to

$$(\Omega T_{n-1}(C_\iota(x_1, \ldots, x_{n-1})) \circ \Lambda_\iota) \wedge \Omega T_1(C_\iota(x_n)) \circ \mathrm{id}_{S^1}$$

which is equivalent to

$$\Gamma_{\iota\iota} \wedge x_1 \wedge \cdots \wedge x_{n-1} \wedge \Gamma_{\iota\iota} \wedge x_n.$$

Therefore Γ_{gg} has degree one, by induction.

If $g \neq h$, and $g^{-1} \circ h(n) \neq n$, then $\rho \circ \Lambda_{g^{-1}\circ h}$ is null homotopic, so Γ_{gh} is as well. If $g^{-1} \circ h(n) = n$, then $(\Omega T_{n-1} \circ C_\iota(x_1, x_2, \ldots, x_n)) \circ \Lambda_f$ (where f is the restriction of $g^{-1} \circ h$ to $\{\underline{n-1}\}$) can be reduced as above to a smash product of maps, one of which will have the form $\Omega T_{n-k}(C_\iota(x_1, \ldots, x_{n-k})) \circ (\rho \circ \Lambda_j)$ where $j(n-k) \neq n-k$. This is guaranteed by the fact that $g^{-1} \circ h \neq \iota$. Then Lemma 6.11 can be applied as above to determine that this component is null homotopic. Hence, we can conclude that Γ_{gh} is null homotopic when $g \neq h$.

With this we are able to prove Theorem 2.2b.

Proof of Theorem 2.2b. We will work with the commutative diagram of reduced homology groups (6.1).

$$\begin{tikzcd}
\widetilde{H}_m(\bigvee_{g\in G_n}(\prod_{i=1}^n X_i)) \arrow[r, "((\Omega T_n)\circ D)_*"] \arrow[dd, "q"] & \widetilde{H}_m(\mathrm{Map}_*(\Sigma\widetilde{\Delta}_n, \Sigma X_1\wedge\cdots\wedge\Sigma X_n)) \arrow[d, "\vee_{h\in G_n}\Lambda_h"] \\
 & \widetilde{H}_m(\prod_{h\in G_n}(\Omega^n\Sigma^n(X_1\wedge\cdots\wedge X_n))) \arrow[d, "r"] \\
\widetilde{H}_m(\bigvee_{g\in G_n}(\bigwedge_{i=1}^n X_i)) \arrow[r, "\Gamma"'] & \bigoplus_{h\in G_n}\widetilde{H}_m(\Omega^n\Sigma^n(X_1\wedge\cdots\wedge X_n)).
\end{tikzcd}$$

The map q is induced by the quotient map from $\prod_{i=1}^n X_i \to \bigwedge_{i=1}^n X_i$ and D denotes the map $(\bigvee_{g\in G_n} C_g)$.

We will determine the range in which the side and bottom arrows are injective and/or surjective. From there we will be able to deduce the connectivity of ΩT_n.

Let $\Gamma'_{gh} : \bigwedge_{i=1}^n X_i \to \Omega^n\Sigma^n(\bigwedge_{i=1}^n X_i)$ be the map that takes $x \in \bigwedge_{i=1}^n X_i$ to the map $\Gamma_{gh}\wedge x$. By Proposition 6.8 and the Freudenthal suspension theorem, Γ'_{gh} is $2n(k+1)-2$-connected when $g=h$ and is null homotopic otherwise. The map Γ can be represented by the $(n-1)!\times(n-1)!$ matrix

$$[(\Gamma'_{gh})_*]_{g,h\in G_n}.$$

By the Whitehead theorem and the preceding, it follows that Γ is an isomorphism for $m \le 2n(k+1)-2$.

By the Künneth theorem, r is an isomorphism for $m \le 2n(k+1)-2$, and q is a surjection. Therefore, $\Gamma\circ q$ is a surjection for $m \le 2n(k+1)-2$. Since the right arrows are both isomorphisms for $m \le 2n(k+1)-2$, $(\Omega T_n \circ \bigvee_{g\in G_n})_*$ is a surjection for $m \le 2n(k+1)-2$. In particular, ΩT_n is a surjection for $m \le 2n(k+1)-2$.

Since $\pi_m(\chi_n(\Omega\Sigma)(X_1, X_2, \dots, X_n))$ and $\pi_m(\mathrm{Map}_*(\Sigma\Delta_n, X_1\wedge X_2\wedge\cdots\wedge X_n))$ are isomorphic to $\pi_m(\prod_{(n-1)!}\bigwedge_{i=1}^n X_i)$ for $m \le (n+1)(k+1)-1$, we know that $\widetilde{H}_m(\chi_n(\Omega\Sigma)(X_1, X_2, \dots, X_n))$ and $\widetilde{H}_m(\mathrm{Map}_*(\Sigma\Delta_n, X_1\wedge X_2\wedge\cdots\wedge X_n))$ are isomorphic in the same range. Furthermore, the finiteness condition on the spaces $X_1, X_2, \dots, X_n$ guarantees that these homology groups are finitely generated abelian groups. Hence, the surjection ΩT_n must be an isomorphism for $m \le (n+1)(k+1)-1$. By the Whitehead theorem, ΩT_n is $(n+1)(k+1)-1$-connected.

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