

## STANDARD LYNDON BASES OF LIE ALGEBRAS AND ENVELOPING ALGEBRAS

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**ABSTRACT.** It is well known that the standard bracketings of Lyndon words in an alphabet  $A$  form a basis for the free Lie algebra  $\text{Lie}(A)$  generated by  $A$ . Suppose that  $\mathfrak{g} \cong \text{Lie}(A)/J$  is a Lie algebra given by a generating set  $A$  and a Lie ideal  $J$  of relations. Using a Gröbner basis type approach we define a set of “standard” Lyndon words, a subset of the set Lyndon words, such that the standard bracketings of these words form a basis of the Lie algebra  $\mathfrak{g}$ . We show that a similar approach to the universal enveloping algebra  $\mathfrak{u}$  naturally leads to a Poincaré-Birkhoff-Witt type basis of the enveloping algebra of  $\mathfrak{g}$ . We prove that the standard words satisfy the property that any factor of a standard word is again standard. Given root tables, this property is nearly sufficient to determine the standard Lyndon words for the complex finite-dimensional simple Lie algebras. We give an inductive procedure for computing the standard Lyndon words and give a complete list of the standard Lyndon words for the complex finite-dimensional simple Lie algebras. These results were announced in [LR].

### 1. LYNDON WORDS AND THE FREE LIE ALGEBRA

In this section we give a short summary of the facts about Lyndon words and the free Lie algebra which we shall use. All of the facts in this section are well known. A comprehensive treatment of free Lie algebras (and Lyndon words) appears in the book by C. Reutenauer [Re].

Let  $A$  be an ordered alphabet, and let  $A^*$  be the set of all words in the alphabet  $A$  (the free monoid generated by  $A$ ). Let  $|u|$  denote the length of the word  $u \in A^*$ , and let  $u < v$  denote that the word  $u$  is lexicographically smaller than the word  $v$ . A word  $l \in A^*$  is a Lyndon word if it is lexicographically smaller than all its cyclic rearrangements. Let  $l$  be a Lyndon word, and let  $m, n$  be words such that  $l = mn$  and  $n$  is the longest Lyndon word appearing as a proper right factor of  $l$ . Then  $m$  is also a Lyndon word [Lo, Proposition 5.1.3]. The standard bracketing of a Lyndon word is given (inductively) by

$$(1.1) \quad b[a] = a, \text{ for } a \in A, \quad b[l] = [b[m], b[n]],$$

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where  $l = mn$  and  $n$  is the longest Lyndon word appearing as a proper right factor of  $l$ . We shall use the following facts:

(1.2) ([Lo, Lemma 5.3.2 or Re, Theorem 5.1]) For each Lyndon word  $l$ ,

$$b[l] = l + \sum_{\substack{v > l \\ |v|=|l|}} a_v v,$$

for some integers  $a_v$ .

(1.3) ([Lo, Theorem 5.3.1 or Re, Theorem 4.9]) The elements  $b[l]$ , where  $l$  is a Lyndon word, are a basis of  $\text{Lie}(A)$ .

(1.4) ([Lo, Theorem 5.1.5 or Re, Corollary 4.7]) Every word  $w$  has a unique factorization  $w = l_1 \cdots l_k$ , such that the  $l_i$  are Lyndon words and  $l_1 \geq \cdots \geq l_k$ .

For each  $w \in A^*$  define

$$b[w] = b[l_1] \cdots b[l_k],$$

where  $w = l_1 \cdots l_k$ , the factors  $l_i$  are Lyndon words and  $l_1 \geq \cdots \geq l_k$ . The following result is essentially the same as [Re, Theorem 5.1]. The fact that the length of the words is preserved is clear from the proof given there.

(1.5) ([Re, Theorem 5.1]) For each  $w \in A^*$

$$b[w] = w + \sum_{\substack{v > w \\ |v|=|w|}} a_v v,$$

for some integers  $a_v$ .

The free Lie algebra  $\text{Lie}(A)$  with generating set  $A$  can be viewed as the span of the letters in  $A$  and all brackets of letters in  $A$ .  $\mathbb{Q}[A^*]$  is the associative algebra of  $\mathbb{Q}$ -linear combinations of words in the alphabet  $A$  where the product is juxtaposition. The algebra  $\mathbb{Q}[A^*]$  is graded by the length of the words. We shall have need of the following:

(1.6) ([Bou, II §3, Theorem 1 or Re, Theorem 0.5])  $\mathbb{Q}[A^*]$  is the enveloping algebra of  $\text{Lie}(A)$ .

(1.7) (Poincaré-Birkhoff-Witt Theorem, [Bou, I §3, Corollary 3 to Theorem 1]) If  $\mathfrak{g}$  is a Lie algebra and  $B$  is an ordered basis of  $\mathfrak{g}$ , then the set of products  $l_1 \cdots l_k$ ,  $l_i \in B$ ,  $l_1 \geq \cdots \geq l_k$ , is a basis of the enveloping algebra  $U\mathfrak{g}$  of  $\mathfrak{g}$ .

(1.8) ([Bou, I §3, Proposition 3]) Let  $\mathfrak{g}$  be a Lie algebra, and let  $J$  be a Lie ideal of  $\mathfrak{g}$ . Let  $U\mathfrak{g}$  be the enveloping algebra of  $\mathfrak{g}$ , and let  $I$  be the ideal in  $U\mathfrak{g}$  generated by  $J$ . Then the enveloping algebra of  $\bar{\mathfrak{g}} = \mathfrak{g}/J$  is  $U\bar{\mathfrak{g}} = U\mathfrak{g}/I$ .

The following well-known result follows easily from (1.1)–(1.5).

(1.9) **Theorem.** *Each of the following is a basis of  $\mathbb{Q}[A^*]$ .*

(B1) *The set of words  $A^*$ .*

(B2) *The set of products  $l_1 \cdots l_k$ , where the  $l_i$  are Lyndon words and  $l_1 \geq \cdots \geq l_k$ .*

(B3) *The set of products  $b[l_1] \cdots b[l_k]$ , where the  $l_i$  are Lyndon words and  $l_1 \geq \cdots \geq l_k$ .*

*Proof.* (B1) is a basis by definition of  $\mathbb{Q}[A^*]$  and (B2) = (B1) by (1.4). (B3) is a basis by (1.3) and the Poincaré-Birkhoff-Witt Theorem.  $\square$

2. STANDARD BASES

Order the words in  $A^*$  by setting

$$u \preceq w \quad \text{if} \quad \begin{cases} |u| < |w| \\ \text{or} \\ |u| = |w| \text{ and } u \geq v. \end{cases}$$

This is a total order on words with the additional property that there are a finite number of words less than any given word.

Suppose that  $J$  is a Lie ideal of  $\text{Lie}(A)$  and that  $I$  is the ideal in  $\mathbb{Q}[A^*]$  generated by  $J$ . Let

$$\mathfrak{g} = \text{Lie}(A)/J \quad \text{and} \quad U\mathfrak{g} = \mathbb{Q}[A^*]/I.$$

It follows from (1.6) and (1.8) that  $U\mathfrak{g}$  is the enveloping algebra of  $\mathfrak{g}$ . Define a Lyndon word to be *Lie-standard* with respect to  $J$  if its bracketing  $b[l]$  cannot be written as a sum of bracketings of strictly smaller Lyndon words modulo the ideal  $J$  of  $\text{Lie}(A)$  with respect to the ordering  $\preceq$ . Define a word  $w$  to be *standard* with respect to  $I$  if  $w$  cannot be written as a sum of strictly smaller words modulo the ideal  $I$ , again with respect to the ordering  $\preceq$ . Make the following notation:

- $L$  is the set of Lyndon words,
- $SL$  is the set of Lie-standard Lyndon words,
- $S$  is the set of standard words.

The standard words that we have defined are essentially a Gröbner basis. The following two theorems are the standard results from the Gröbner basis context.

(2.1) **Theorem.** *The set of elements  $b[l]$ , where  $l \in SL$ , is a basis of  $\mathfrak{g} = \text{Lie}(A)/J$ .*

*Proof.* The set of all  $b[l]$ , where  $l \in L$ , spans  $\mathfrak{g}$ . If  $l$  is not Lie-standard, then  $b[l]$  can be written as a linear combination of bracketings of Lyndon words modulo  $J$  which are smaller than  $l$ . If any of these words is not standard, express it as a sum of smaller words. Continue this process until all the words in the expansion are standard. The process must stop as the number of words smaller than any given word is finite. Thus the elements  $b[l]$ , where  $l \in SL$ , span  $\mathfrak{g}$ .

We now show that the set of Lie-standard Lyndon words is linearly independent. Suppose that there was a nontrivial relation among them. Then this relation expresses the maximal word as a linear combination of lower words modulo  $J$ , a contradiction to the standardness of the maximal word.  $\square$

(2.2) **Theorem.** *The set of words in  $S$  is a basis for  $U\mathfrak{g} = \mathbb{Q}[A^*]/I$ .*

*Proof.* The proof is exactly analogous to the proof of Theorem (2.1).  $\square$

We shall show that  $SL = S \cap L$ , i.e., the set of Lie-standard Lyndon words is the same as the set of standard Lyndon words (this is not a priori obvious).

(2.3) **Lemma.**  $S \cap L \subseteq SL$ .

*Proof.* Let  $m \in L$ . Suppose  $m \notin SL$ . Then

$$b[m] = \sum_{\substack{n \prec m \\ n \in L}} a_n b[n] + x,$$

for some  $x \in J$ . Using (1.2) on each side,

$$m + \sum_{v \prec m} b_v v = \sum_{n \prec m} a_n \left( n + \sum_{w \prec n} c_w w \right) + x,$$

for some integers  $b_v, a_n, c_w$ . Subtracting  $\sum_{v \prec m} b_v v$  from both sides,

$$m = \sum_{v \prec m} d_v v + x$$

for some integers  $d_v$ . Since  $x \in J \subseteq I$ , we have that  $m \notin S$ .  $\square$

(2.4) **Proposition.** *Any factor of a standard word is a standard word.*

*Proof.* Suppose that  $v$  is not standard, so that we have

$$v = \sum_{m \prec v} a_m m \pmod{I}.$$

Then

$$uvw = u \left( \sum_{m \prec v} a_m m + x \right) w,$$

where  $x \in I$ . Since  $I$  is an ideal  $uxw \in I$  and since  $umw \prec uvw$  for all  $m$ , we have that  $uvw$  is not standard.  $\square$

(2.5) **Corollary.** *If  $w \in S$ , then  $w$  has a unique factorization*

$$w = l_1 \cdots l_k, \quad l_i \in SL, \quad l_1 \geq \cdots \geq l_k.$$

(2.6) **Theorem.** *Let  $l = l_1 \cdots l_k, l_i \in SL$ , and  $l_1 \geq \cdots \geq l_k$ . Then*

$$b[l] = l_1 \cdots l_k + \sum_{m' \prec l} b_{m'} m'_1 \cdots m'_r \pmod{I},$$

where  $m' = m'_1 \cdots m'_r, m'_i \in SL$  for each  $i, m'_1 \geq \cdots \geq m'_r$ , and  $b_{m'} \in \mathbb{Z}$ .

*Proof.* By (1.5) and the definition of the ordering  $\prec$ ,

$$b[l] = l_1 \cdots l_k + \sum_{m \prec l} a_m m.$$

Expanding the sum in terms of standard words,

$$b[l] = l_1 \cdots l_k + \sum_{\substack{m' \in S \\ m' \preceq m \prec l}} b'_{m'} m' \pmod{I}.$$

The result now follows from Corollary (2.5) since each  $m'$  appearing in the sum has a unique factorization of the form  $m' = m'_1 \cdots m'_r, m'_i \in SL, m'_1 \geq \cdots \geq m'_r$ .  $\square$

(2.7) **Theorem.** *Each of the following sets is a basis of  $\mathbb{Q}[A^*]/I$ .*

(B1) *The set  $S$  of standard words with respect to  $I$ .*

(B2) The set of products  $b[l_1] \cdots b[l_k]$ , where  $l_i \in SL$  and  $l_1 \geq \cdots \geq l_k$ .

(B3) The set of products  $l_1 \cdots l_k$ , where  $l_i \in SL$  and  $l_1 \geq \cdots \geq l_k$ .

*Proof.* Statement (B1) is Theorem (2.2). (B2) is a basis by (2.1) and the Poincaré-Birkhoff-Witt Theorem. Theorem (2.6) gives a triangular relation between the elements of the set (B3) and the elements of the set (B2), which proves that (B3) is a basis.  $\square$

(2.8) **Corollary.**

(a) With notation as in Theorem (2.7),  $(B1) = (B3)$ .

(b)  $SL = S \cap L$ .

*Proof.* (a) Corollary (2.5) gives that  $(B1) \subseteq (B3)$ . Since these are both bases we must have  $(B3) = (B1)$  (express the basis (B3) in terms of the basis (B1)).

(b) Since  $(B3) = (B1)$ ,  $SL \subseteq S$ . Combining this with Lemma (2.3) we have that  $SL = S \cap L$ .  $\square$

The following proposition will help us to compute the standard Lyndon words by induction on the length of the words.

(2.9) **Proposition.** Let  $l$  be a standard Lyndon word. Then  $l$  is of the form  $l = l_1 l_2 \cdots l_k a$ , where

(1)  $l_i$  are standard Lyndon words for all  $1 \leq i \leq k$ ,

(2)  $l_i$  is a left factor of  $l_{i-1}$  for all  $i > 1$ ,

(3)  $a \in A$ .

*Proof.* Let  $m$  be the word  $l$  with the last letter removed. By (1.4)  $m$  has a factorization  $m = l_1 \cdots l_k$  into Lyndon words  $l_i$  such that  $l_i \geq l_{i+1}$  for all  $1 \leq i \leq k-1$ . By Proposition (2.4), each of the factors  $l_i$  is standard since they are factors of the standard word  $l$ .

It remains to prove that  $l_i$  is a left factor of  $l_{i-1}$  for all  $1 \leq i \leq k$ . Consider the following chain of inequalities. For  $i > 1$ ,

$$l_i \leq l_{i-1} \leq l_1 < l_1 \cdots l_k a \leq l_i \cdots l_k a,$$

where the last inequality follows since the right-hand side  $l_i \cdots l_k a$  is a right factor of the Lyndon word  $l_1 \cdots l_k a$ . It follows easily from  $l_i \leq l_{i-1} < l_i \cdots l_k a$  that  $l_i$  is a left factor of  $l_{i-1}$  (consider these as words in a dictionary).  $\square$

### 3. FINITE-DIMENSIONAL SIMPLE LIE ALGEBRAS

In this section we shall compute the standard Lyndon words corresponding to the finite-dimensional simple Lie algebras over  $\mathbb{C}$ . Each such algebra is determined by a Cartan matrix  $C$  with integer entries  $\langle \alpha_i, \alpha_j \rangle$ ,  $1 \leq i, j \leq n$ . A list of these Cartan matrices can be found in [Bou2, pp. 250–275]. We shall use the Bourbaki conventions for numbering.

Fix a Cartan matrix  $C$  corresponding to a finite-dimensional simple Lie algebra. Let

$$A = \{x_1, x_2, \dots, x_n, h_1, h_2, \dots, h_n, y_1, y_2, \dots, y_n\},$$

and let  $J$  be the ideal of Serre relations in  $\text{Lie}(A)$ , i.e., the ideal generated by

the elements

- (S1)  $[h_i, h_j] \quad (1 \leq i, j \leq n),$
- (S2)  $[x_i, y_i] - h_i, \quad [x_i, y_j] \quad \text{if } i \neq j,$
- (S3)  $[h_i, x_j] - \langle \alpha_j, \alpha_i \rangle x_j, \quad [h_i, y_j] + \langle \alpha_j, \alpha_i \rangle y_j,$
- (S<sub>ij</sub><sup>+</sup>)  $(\text{ad } x_i)^{-\langle \alpha_j, \alpha_i \rangle + 1}(x_j) \quad (i \neq j),$
- (S<sub>ij</sub><sup>-</sup>)  $(\text{ad } y_i)^{-\langle \alpha_j, \alpha_i \rangle + 1}(y_j) \quad (i \neq j).$

Here  $(\text{ad } a)^k(b) = [a, [a, [a, \dots, [a, b]] \dots]]$ . Let  $X = \{x_1, x_2, \dots, x_n\}$  be ordered by  $x_1 < x_2 < \dots < x_n$ , and let  $J^+$  be the Lie ideal in  $\text{Lie}(X)$  generated by the relations (S<sub>ij</sub><sup>+</sup>). Let  $Y = \{y_1, y_2, \dots, y_n\}$ ,  $y_1 < \dots < y_n$ , and let  $J^-$  be the Lie ideal in  $\text{Lie}(Y)$  generated by the relations (S<sub>ij</sub><sup>-</sup>). Define

$$\mathfrak{g} = \text{Lie}(A)/J, \quad \mathfrak{n}^+ = \text{Lie}(X)/J^+, \quad \text{and} \quad \mathfrak{n}^- = \text{Lie}(Y)/J^-.$$

Let  $\alpha_i$  be independent vectors. The  $\alpha_i$  are called the *simple roots*. The *root lattice* is the lattice  $Q = \sum_{i=1}^n \mathbb{Z}\alpha_i$ . Let  $Q^+ = \{\alpha = \sum_{i=1}^n a_i \alpha_i \in Q \mid a_i \geq 0\}$ . The *height* of a root  $\alpha = \sum_{i=1}^n a_i \alpha_i \in Q^+$  is  $ht(\alpha) = \sum_{i=1}^n a_i$ . The *weights* of words  $w = x_{i_1} \dots x_{i_k} \in X^*$  and  $\bar{w} = y_{i_1} \dots y_{i_k} \in Y^*$  are defined by

$$wt(l) = \alpha_{i_1} + \dots + \alpha_{i_k} \quad \text{and} \quad wt(\bar{l}) = -\alpha_{i_1} - \dots - \alpha_{i_k},$$

respectively. Note that the length of a word  $w$  such that  $wt(w) = \alpha$  is equal to  $ht(\alpha)$ .

With notation for standard bracketings as in (1.1) we define

$$\mathfrak{g}_\alpha = \mathbb{C}\text{-span}\{b[l] \mid l \in X^*, wt(l) = \alpha\}$$

and

$$\mathfrak{g}_{-\alpha} = \mathbb{C}\text{-span}\{b[\bar{l}] \mid \bar{l} \in Y^*, wt(\bar{l}) = -\alpha\},$$

for each  $\alpha \in Q^+$ . The set

$$\Phi^+ = \{\alpha \in Q^+ \mid \alpha \neq 0, \dim(\mathfrak{g}_\alpha) \neq 0\}$$

is the set of *positive roots*. Let  $\mathfrak{h}$  be the linear span of the generators  $h_i$ . The following facts about finite-dimensional simple Lie algebras  $\mathfrak{g}$  are standard [Hu, Theorems 18.3, 14.2, and 8.4]

- (3.1a)  $\mathfrak{g} \cong \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$
- (3.1b)  $\mathfrak{n}^+ \cong \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$  and  $\mathfrak{n}^- \cong \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha}.$
- (3.1c)  $\Phi^+$  is finite.
- (3.1d)  $\dim(\mathfrak{g}_\alpha) = 1$  for all  $\alpha \in \Phi^+.$

The following result follows easily from the above facts.

(3.2) **Proposition.** (a) For each  $\alpha \in \Phi^+$  there is a unique standard Lyndon word  $l_\alpha \in X^*$  with respect to the ideal  $J^+$  such that  $wt(l_\alpha) = \alpha$ .

(b) The words  $l_\alpha, \bar{l}_\alpha$  and the letters  $h_i$  are the standard Lyndon words in  $A^*$  with respect to the ideal  $J$ . These words form a basis of the finite-dimensional simple Lie algebra  $\mathfrak{g}$ .

*Proof.* (a) Since the standard Lyndon words in  $X^*$  with respect to the ideal  $J^+$  form a basis of  $\mathfrak{n}^+$ , it follows that  $\mathfrak{g}_\alpha$  is the subset of  $\mathfrak{n}^+$  spanned by the bracketings of standard Lyndon words of weight  $\alpha$  with letters in  $X$ . Similarly,

$\mathfrak{g}_{-\alpha}$  is the subspace of  $\mathfrak{n}^-$  spanned by all  $b[w]$  such that  $w$  is a standard Lyndon word with letters in  $Y$  and such that  $wt(t) = -\alpha$ . The statement in (a) now follows from (3.1a) and (3.1b).

(b) It follows from (3.1d) that for each  $\alpha \in \Phi^+$  there is a *unique* standard Lyndon word  $l_\alpha$  of weight  $\alpha$  and that  $\mathfrak{g}_\alpha = \mathbb{C}b[l_\alpha]$ . Furthermore, it is clear from the form of the relations in  $J^+$  and  $J^-$  that  $\mathfrak{g}_{-\alpha} = \mathbb{C}b[\bar{l}_\alpha]$ , where if  $l_\alpha = x_{i_1} \cdots x_{i_k}$ , then  $\bar{l}_\alpha$  is the word in  $Y^*$  given by  $\bar{l}_\alpha = y_{i_1} \cdots y_{i_k}$ .  $\square$

Our goal is to determine the standard Lyndon words  $l_\alpha$ , for all  $\alpha \in \Phi^+$ . This is done by induction on the lengths of the words (heights of the roots). The main tools are Propositions (2.4) and (2.9) from §2 and the tables of the positive roots for the finite-dimensional simple Lie algebras, [Bou2, pp. 250–275].

Let  $\alpha = \sum_{i=1}^n a_i \alpha_i \in \Phi^+$ . By Proposition (2.9) we know that  $l_\alpha$  is of the form

$$(3.3) \quad l_\alpha = l_{\beta_1} l_{\beta_2} \cdots l_{\beta_k} x_e,$$

where

- (1)  $l_{\beta_j}$  are standard Lyndon words for all  $1 \leq j \leq k$ ,
- (2)  $l_{\beta_j}$  is a left factor of  $l_{\beta_{j-1}}$  for all  $j > 1$ ,
- (3)  $x_e \in X$ .

In the following discussion we shall exclude the trivial case  $|l_\alpha| = 1$  so that  $k > 0$ . Because of (2), each of the factors  $l_{\beta_j}$  begins with the same letter, say  $x_b \in X$ ,  $1 \leq b \leq n$ , and  $x_b \neq x_e$  since  $l_\alpha$  is Lyndon; in fact, the letter  $x_b$  is the smallest letter in the word  $l_\alpha$ . Thus, since  $wt(x_b) = \alpha_b$ ,  $b$  is the smallest integer in  $\{1, \dots, n\}$  such that  $a_b \neq 0$ . Since  $x_b$  appears exactly  $a_b$  times in  $l_\alpha$  and it appears as the first letter of each of the factors  $l_{\beta_j}$ ,  $1 \leq j \leq k$ , it follows that  $k \leq a_b$ . A scan of the root tables for the finite-dimensional simple Lie algebras shows that  $a_b \leq 3$  and that  $a_b = 3$  for only one positive root (this root is in type  $G_2$ ).

For each  $1 \leq j \leq k$ , let  $\beta_j = wt(l_{\beta_j})$ . Then the factorization in (3.3) must satisfy the following:

- (1)  $\beta_j = wt(l_{\beta_j}) \in \Phi^+$  for all  $1 \leq j \leq k$ , since the factors  $l_{\beta_j}$  are standard Lyndon words.
- (2)  $\sum_{j=1}^k \beta_j + \alpha_e = \alpha$  where  $\alpha_e = wt(x_e)$ .
- (3) Since  $l_\alpha$  is Lyndon,  $x_e > x_b$ , where  $x_b$  is the first letter of the words  $l_{\beta_j}$  and  $x_e$  is the last letter of  $l_\alpha$ .
- (4)  $k \leq 2$ , except for a single root in type  $G_2$ .
- (5) If  $k = 2$ , then  $\beta_1 - \beta_2 \in Q^+$ , since  $l_{\beta_2}$  is a left factor of  $l_{\beta_1}$ .
- (6)  $l_\alpha$  is a Lyndon word.
- (7) All the Lyndon factors of  $l_\alpha$  are standard Lyndon words of smaller length and thus correspond to roots  $\gamma \in \Phi^+$  such that  $ht(\gamma) < ht(\alpha)$ .

Given these rules, it is easy to construct the standard Lyndon words by induction. Let  $\alpha \in \Phi^+$ , and assume that the standard Lyndon words  $l_\beta$  are known for all  $\beta \in \Phi^+$  such that  $ht(\beta) < ht(\alpha)$ . There are very few choices of roots  $\beta_j \in \Phi^+$ ,  $1 \leq j \leq k$ ,  $k \leq 2$ , such that  $\beta_1 - \beta_2 \in Q^+$  if  $k = 2$  and  $\alpha - \sum_j \beta_j = \alpha_e$ , where  $\alpha_e$ ,  $1 \leq e \leq n$ . The words  $l_{\beta_j}$  are all known since  $ht(\beta_j) < ht(\alpha)$ . Restricting to the cases where  $x_e$  is greater than the first letter of the words  $l_{\beta_j}$  leaves very few possibilities. In fact, one finds that for each

root  $\alpha \in \Phi^+$  (except the root  $\alpha = \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4$  in type  $F_4$ ) there is a unique word  $l_\alpha$  which satisfies conditions (1)–(7) above. Since there is a unique standard Lyndon word corresponding to the root  $\alpha$ , this word must be  $l_\alpha$ .

Consider the root  $\alpha = \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4$  in  $F_4$ . Applying (1)–(7) above leaves two possibilities for the word  $l_\alpha : w_1 = x_1x_2x_3x_4x_3x_2x_3x_4x_3$  and  $w_2 = x_1x_2x_3x_4x_3x_4x_2x_3x_3$ . Modulo the ideal  $J^+$  we can write  $w_1$  as a linear combination of standard Lyndon words which are smaller in the order  $\prec$  (greater in lexicographic order). This computation is as follows (we have suppressed the  $x$ 's in writing the words, and at each step we have underlined the letters which are being changed modulo the defining relations for the ideal  $J^+$ ):

$$\begin{aligned} w_1 &= 123432343 = \frac{1}{2}(123432433) + \frac{1}{2}(123432334) \\ &= \frac{1}{2} \left[ (123432433) + \frac{1}{3}(123423334) + (123433234) - \frac{1}{3}(123433324) \right] \\ &= \frac{1}{2} \left[ (123432433) + \frac{1}{3}(123243334) + (123433234) - \frac{1}{3}(123433324) \right] \\ &= \frac{1}{2} \left[ (123432433) + \frac{1}{6}[(132243334) + (\underline{122}343334)] \right. \\ &\qquad\qquad\qquad \left. + (123433234) - \frac{1}{3}(123433324) \right] \\ &= \frac{1}{2} \left[ (123432433) + \frac{1}{6}[(132243334) + [2 \cdot (212343334) - (221343334)]] \right. \\ &\qquad\qquad\qquad \left. + (123433234) - \frac{1}{3}(123433324) \right]. \end{aligned}$$

Thus  $w_1 = x_1x_2x_3x_4x_3x_2x_3x_4x_3$  is not a standard Lyndon word and  $l_\alpha = x_1x_2x_3x_4x_3x_4x_2x_3x_3$ .

Figure 1 gives the standard Lyndon words corresponding to each of the finite-dimensional simple Lie algebras. Let us explain how to read these diagrams. Each tree is rooted. A path

$$p_w = (i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_k)$$

in the tree consisting of a chain of successive vertices and edges moving away from the root determines a word  $w = x_{i_1} \dots x_{i_k} \in X^*$ . The trees are constructed (by applying the procedure described above) so that this word is always a standard word with respect to the ideal of Serre relations determined by the corresponding Cartan matrix. If the word is Lyndon, then we say that the path is Lyndon. In the discussion following Proposition (3.2) we have described how one proves (case by case) the following theorem.

**(3.4) Theorem.** *For each of the trees in Figure 1 the set of words determined by the Lyndon paths in the tree is the complete set of standard Lyndon words for the corresponding finite-dimensional simple Lie algebra.*

*Remark.* We have made some effort to compute the bracketing rule for the finite-dimensional simple Lie algebras in terms of the basis of standard Lyndon words. We have not yet succeeded in learning much from this exercise. We

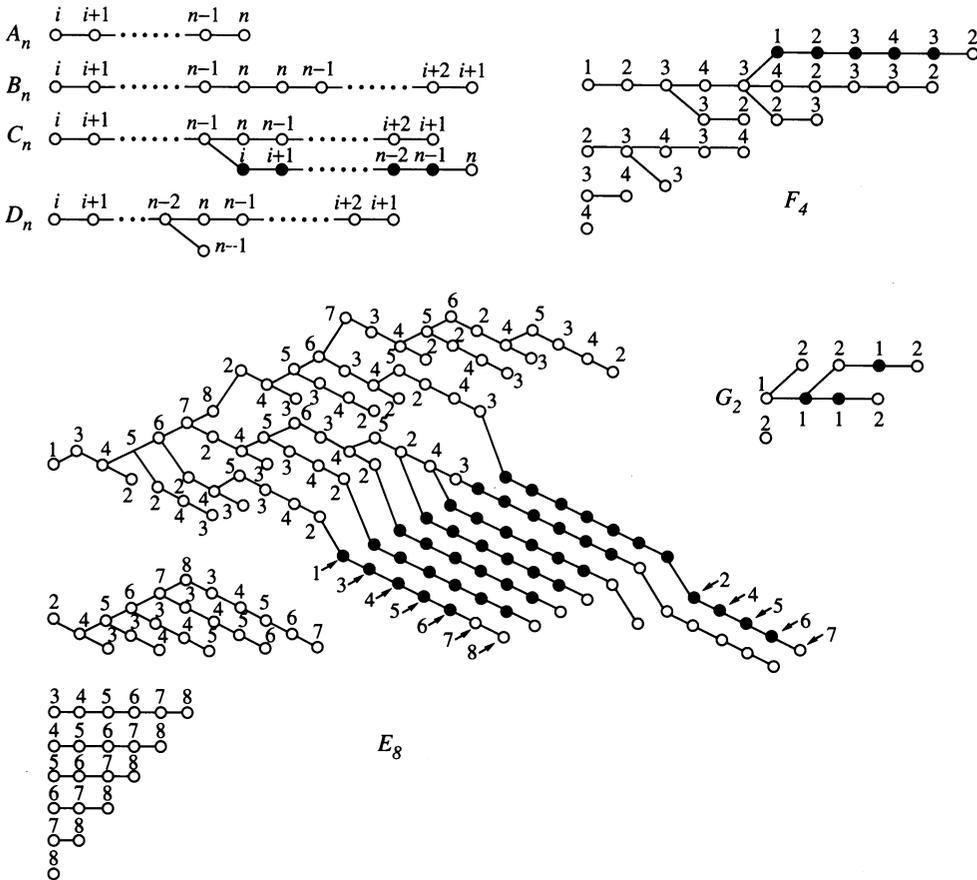


FIGURE 1. The trees giving the standard Lyndon paths. The root of each tree is the leftmost vertex. For  $A_n$ ,  $B_n$ ,  $C_n$ ,  $D_n$ , we give a generic tree with root  $i$ , where  $i = 1, 2, \dots, n$  ( $i = 1, 2, \dots, n - 1$  for  $D_n$ ). The standard paths for  $E_7$  (respectively  $E_6$ ) are the paths from the trees for  $E_8$  not containing 8 (respectively 7 and 8). The white vertices end Lyndon paths, while black vertices end non-Lyndon paths. The trees are designed so that for each root system all the ends of Lyndon paths corresponding to roots of the same height lie on a common vertical line.

make only the following remarks, with a bit of reservation, as the computations are complicated and difficult to check precisely. It seems that the standard Lyndon bases for Types  $A_n$ ,  $B_n$ ,  $C_n$ ,  $D_n$  are Chevalley bases. However, the standard Lyndon basis for type  $G_2$  is definitely not a Chevalley basis. In fact, in type  $F_4$  there are even some structure coefficients that are not integral.

*Remark.* It is clear that all of the results in §2 are valid for any Lie algebra given by generators and relations. Preliminary computations seem to indicate that it will be very instructive to study root multiplicities for Kac-Moody Lie algebras by way of standard Lyndon words.

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