SINGULAR LIMIT OF SOLUTIONS OF
\[ u_t = \Delta u^m - A \cdot \nabla (u^q/q) \quad \text{as} \quad q \to \infty \]

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Abstract. We will show that the solutions of
\[ u_t = \Delta u^m - A \cdot \nabla (u^q/q) \quad \text{in} \quad \mathbb{R}^n \times (0, T), \quad T > 0, \quad m > 1, \quad u(x, 0) = f(x) \quad \text{in} \quad L^1(\mathbb{R}^n) \quad \text{and} \quad L^\infty(\mathbb{R}^n) \]
converge weakly in \((L^\infty(G))^*\) for any compact subset \(G\) of \(\mathbb{R}^n \times (0, T)\) as \(q \to \infty\) to the solution of the porous medium equation \(v_t = \Delta v^m\) in \(\mathbb{R}^n \times (0, T)\) with
\[ v(x, 0) = g(x) \quad \text{where} \quad g \in L^1(\mathbb{R}^n), \quad 0 \leq g \leq 1, \quad \text{satisfies} \quad g(x) + (\tilde{g}(x))_{x_1} = f(x) \quad \text{in} \quad C'(\mathbb{R}^n) \quad \text{for some function} \quad \tilde{g}(x) \in L^1(\mathbb{R}^n), \quad \tilde{g}(x) \geq 0 \quad \text{such that} \quad g(x) = f(x), \quad \tilde{g}(x) = 0 \quad \text{whenever} \quad g(x) < 1 \quad \text{a.e.} \quad x \in \mathbb{R}^n. \]
The convergence is uniform on compact subsets of \(\mathbb{R}^n \times (0, T)\) if \(f \in C_0(\mathbb{R}^n)\).

In this paper we will study the asymptotic behaviour of nonnegative solutions \(u = u^{(q)}\) of the equation
\[
\begin{cases}
  u_t = \Delta u^m - A \cdot \nabla (u^q/q), & (x, t) \in \mathbb{R}^n \times (0, T), \\
  u(x, 0) = f(x) \geq 0, & f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n),
\end{cases}
\]
where \(0 \neq A = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n\) is a constant vector, \(T > 0, \quad m > 1, \quad \text{as} \quad q \to \infty\). Recently there is a lot of research on the above equation ([A],[DiK],[G1],[G2]) The equation arises in many physical applications such as the flow of water through a homogeneous isotropic rigid porous medium [G1]. When \(A = 0\), the above equation reduces to the well-known porous medium equation ([Ar],[P]). In the paper [CF], Caffarelli and A. Friedman studied the asymptotic behaviour of solutions of (0.1) when \(A = 0\) and showed that the solutions of (0.1) converge as \(m \to \infty\) if \(f\) satisfies (0.1) and the following conditions:
\[
\begin{align*}
  f & \in C^1 \text{ in supp } f, \\
  f(0) > 1, \quad f_c < 0 \text{ in } \mathbb{R}^n \setminus \{0\} \cap \text{supp } f, \\
  f_{r_{x_0}} \leq 0 \text{ in } \mathbb{R}^n \setminus B_1(0) \cap \text{supp } f \quad \forall x_0 \in B_{\epsilon_0}(0)
\end{align*}
\]
for some \(\epsilon_0 > 0\) where \(r_{x_0} = |x - x_0|, \quad B_r(0) = \{x : |x| < r\} \) and \(f_{r_{x_0}}\) is the radial derivative of \(f\) with center at \(x_0\).

This result has been extended in various directions by P. Bénilan, L. Boccardo and M. Herrero [BBH], P. E. Sacks [S2] in the case \(A = 0, \quad X. \quad Xu \quad [X]\) in the case of hyperbolic equations and K. M. Hui [H1], [H2] in the case of a porous...
medium equation with absorption and in the case of the generalized p-Laplacian equation.

For simplicity we will assume that $T = 1$ and $A = (1, 0, \ldots, 0)$ throughout the rest of the paper. We will show that as $q \to \infty$, the convection term in (0.1) disappears. More precisely, we will show that for fixed $m > 1$ the solutions $u = u(q)$ of (0.1) converge weakly in $(L^\infty(G))^*$ for any compact subset $G$ of $\mathbb{R}^n \times (0, 1)$ as $q \to \infty$. Moreover the limit $u(\infty) = \lim_{q \to \infty} u(q)$ satisfies the porous medium equation

$$
\left\{
\begin{aligned}
t & = \Delta u^m, & (x, t) & \in \mathbb{R}^n \times (0, 1), \\
v(x, t) & \leq g & \text{as } t & \to 0 \text{ in } \mathcal{D}'(\mathbb{R}^n),
\end{aligned}
\right.
$$

where $g \in L^1(\mathbb{R}^n)$, $0 \leq g \leq 1$, satisfies

$$
g(x) + (g(x))^{x_1} \leq f(x), \quad (x, t) \in \mathbb{R}^n \times (0, 1),
$$

for some function $\tilde{g}(x) \geq 0$, $\tilde{g}(x) \in L^1(\mathbb{R}^n)$ and $g(x) = f(x)$, $\tilde{g}(x) = 0$ whenever $g(x) < 1$ a.e. $x \in \mathbb{R}^n$. This extends the recent results obtained by M. Escobedo and E. Zuazua [EZ], who showed that the convection term was negligible compared with the other terms appearing in (0.1) for the case $m = 1$ and $q > 1 + 1/n$ as $t \to \infty$. Although we were not able to prove it, we suspect that the same result should remain valid when $A = A(x) \in L^\infty(\mathbb{R}^n)$.

We will first start with some definitions. For any open set $\Omega_0 \subset \mathbb{R}^n$, $h \in C(\mathbb{R})$, we say that $u$ is a solution (respectively subsolution, supersolution) of

$$
u_t = \Delta u^m - (h(u))_{x_1}
$$
in $\Omega_0 \times (0, 1)$ if $u$ is continuous and nonnegative in $\Omega_0 \times (0, 1)$, $u \in L^\infty([0, 1]; L^1(\Omega_0) \cap L^\infty(\Omega_0 \times (0, 1)))$ and satisfies

$$
\int_{\Omega} u^{m} \Delta \eta + \eta \frac{\partial \eta}{\partial t} + h(u) \eta_{x_1} dxdt = \int_{\Omega} u^{m} \frac{\partial \eta}{\partial N} d\sigma ds + \int_{\partial \Omega} u \eta dx |_{t_1}^{t_2}
$$

(respectively $\geq$, $\leq$) for all bounded open sets $\Omega \subset \Omega_0$ with $\partial \Omega \subset C^2$, $0 < \tau_1 \leq \tau_2 < 1$, $\eta \in C^\infty(\Omega \times [\tau_1, \tau_2])$, $\eta \equiv 0$ on $\partial \Omega \times [\tau_1, \tau_2]$ where $\partial / \partial N$ is the exterior normal derivative on $\partial \Omega$ and $d\sigma$ is the surface measure on $\partial \Omega$.

If $u$ is a solution of (0.4) in $\Omega_0 \times (0, 1)$, we say that $u$ has initial trace or initial value $du$ if

$$
\lim_{t \to 0} \int u(x, t) \eta(x) dx = \int \eta d\mu \quad \forall \eta \in C_c(\overline{\Omega_0}).
$$

We let $\rho \in C_0^\infty(\mathbb{R}^n)$, $\rho \geq 0$, $\int \rho = 1$ and for any $g$ we define

$$
g_{\varepsilon} = g * \rho_{\varepsilon}(x) = \int \rho_{\varepsilon}(x-y)g(y)dy, \quad \varepsilon > 0,
$$

where $\rho_{\varepsilon}(y) = \rho(y / \varepsilon) / \varepsilon^n$. For any $r > 0$, $x_0 \in \mathbb{R}^n$, let $B_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$. For any set $A \subset \mathbb{R}^n$, we let $\chi_A$ be the characteristic function of the set $A$. We will also assume $m > 1$, $q > m + 1$, and let $u(q)$ be the solution of (0.1) for the rest of the paper.

The plan of the paper is as follows. In section 1 we will state and prove the existence of solutions of (0.1). We will also prove a comparison theorem.
SINGULAR LIMIT OF SOLUTIONS OF $u_t = \Delta u^m - A \cdot \nabla (u^q/q)$ AS $q \to \infty$

for solutions of (0.1) and obtain some bounds on $u^{(q)}$ by constructing explicit supersolutions to (0.1). In section 2 we will first prove a comparison lemma for solutions of (0.3). We then prove the main theorem under the assumption $f \in C_0^1(R^n)$ (Theorem 2.9). Finally we will prove the main theorem (Theorem 2.10) by an approximation argument.

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We first state and prove an uniqueness theorem for solutions of (0.1).

**Theorem 1.1.** If $u_1^{(q)}$, $u_2^{(q)} \in L^\infty([0, 1]; L^1(R^n)) \cap L^\infty(R^n \times (0, 1)) \cap C(R^n \times (0, 1))$ are the solutions of

\[
(1.1) \quad u_t = \Delta u^m - (u^q/q)_x
\]

in $R^n \times (0, 1)$ with initial values $f_1$ and $f_2 \in L^1(R^n) \cap L^\infty(R^n)$ respectively, $f_1, f_2 \geq 0$, then there exists a constant $C > 0$ such that

(i) $\int_{R^n} (u_1^{(q)} - u_2^{(q)})_+(x, t)dx \leq e^{Ct} \int_{R^n} (f_1 - f_2)_+(x)dx,$

(ii) $\int_{R^n} |u_1^{(q)} - u_2^{(q)}|(x, t)dx \leq e^{Ct} \int_{R^n} |f_1 - f_2|(x)dx$

for all $0 < t < 1$. Hence $u_1^{(q)} \leq u_2^{(q)}$ if $f_1 \leq f_2$. In particular the solution of (1.1) in $R^n \times (0, 1)$ with initial value in $L^1(R^n) \cap L^\infty(R^n)$ is unique in the class $L^\infty([0, 1]; L^1(R^n)) \cap L^\infty(R^n \times (0, 1)) \cap C(R^n \times (0, 1)).$

**Proof.** The proof of the theorem is similar to the proof of Theorem 2.3 of [A].

By subtracting the equation for $u_1^{(q)}$ and $u_2^{(q)}$, we get

\[
\int_{B_R(0)} (u_1^{(q)} - u_2^{(q)})(x, t)\eta(x, t)dx = \int_{B_R(0)} (f_1 - f_2)(x)\eta(x, 0)dx
\]

\[
+ \int_0^t \int_{B_R(0)} (u_1^{(q)} - u_2^{(q)})(\eta_t + A\Delta \eta + B\eta_{x_1})dxd\tau
\]

\[- \int_0^t \int_{\partial B_R(0)} (u_1^{(q)m} - u_2^{(q)m})\frac{\partial \eta}{\partial N}d\sigma d\tau
\]

for all $0 < t < 1$, $\eta \in C^\infty(\overline{B_R(0)} \times [0, t])$, $R > 0$, such that $\eta \equiv 0$ on $\partial B_R(0) \times [0, t]$ where

\[
A = \begin{cases} 
\frac{u_1^{(q)m} - u_2^{(q)m}}{u_1^{(q)} - u_2^{(q)}} & \text{for } u_1^{(q)} \neq u_2^{(q)}, \\
m u_1^{(q)m} - 1 & \text{for } u_1^{(q)} = u_2^{(q)},
\end{cases}
\]

\[
B = \begin{cases} 
\frac{1}{q} \frac{u_1^{(q)q} - u_2^{(q)q}}{u_1^{(q)} - u_2^{(q)}} & \text{for } u_1^{(q)} \neq u_2^{(q)} , \\
 u_1^{(q)q} - 1 & \text{for } u_1^{(q)} = u_2^{(q)}. \n\end{cases}
\]

Since $u_1^{(q)}$, $u_2^{(q)} \in L^\infty(R^n \times (0, 1))$, there exists a constant $C_1 > 0$ such that

\[
\|u_1^{(q)}\|_{L^\infty(R^n),} \|u_2^{(q)}\|_{L^\infty(R^n)} \leq C_1
\]

$\Rightarrow B^2/2A \leq \frac{1}{2m} C_1^{2q - m - 1}, B/A \leq \frac{1}{m} C_1^{q - m}.$

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By an argument similar to section 4 of [A], there exists smooth functions \( A_{i,R} \) and \( B_{i,R} \) and constant \( c_i > 0 \) such that 
\[
c_i \leq A_{i,R} \leq mC_1^{m-1} + 1, \quad 0 \leq B_{i,R} \leq C_1^{q-1} + 1, \quad \frac{B_{i,R}}{2A_{i,R}} \leq \frac{C_2}{(C_1^{q-1}/2m)^{1/2}} + C_2, \quad B_{i,R}/A_{i,R} \leq (C_1^{q-m}/m)^{1/2} + C_3,
\]
and \( (A_{i,R} - A)/A_{i,R}^{1/2} \to 0 \) and \( B_{i,R} - B \to 0 \) in \( L^2(B_R(0) \times (0, 1)) \) as \( i \to \infty \) for all \( R > 0 \).

For any \( R_0 > 2, \ R > R_0 + 1, \ \lambda > C_2, \ \theta \in C_0^\infty(B_{R_0}(0)), \ 0 \leq \theta \leq 1, \) let \( \eta_{i,R} \) be the solution of
\[
\begin{aligned}
\begin{cases}
\eta_t + A_{i,R} \Delta \eta + B_{i,R} \eta_x - \lambda \eta = 0 & \text{for } (x, s) \in B_R(0) \times (0, t), \\
\eta(x, s) = 0 & \text{for } (x, s) \in \partial B_R(0) \times (0, t), \\
\eta(x, t) = \theta(x) & \text{for } x \in B_R(0).
\end{cases}
\end{aligned}
\]

Since \( 0 \leq \theta \leq 1 \), by the maximum principle \( 0 \leq \eta_{i,R} \leq 1 \). By Lemma 4.1 of [A], we have
\[
\int_0^t \int_{B_R(0)} A_{i,R}(\Delta \eta_{i,R})^2 dx dt + 2(\lambda - C_2) \int_0^t \int_{B_R(0)} |\nabla \eta_{i,R}|^2 dx dt 
\leq \int_{B_R(0)} |\nabla \theta|^2 dx.
\]

By the same argument as the proof of Theorem 2.1 (ii) of [PV], we see that for any \( \beta > 0 \), the function
\[
g(x, s) = e^{h(s)} \left( \frac{1 + R_0^2}{1 + |x|^2} \right)^\beta
\]
where \( h(s) = C'(t - s), \ C' = 4\beta(\beta + 1)(mC_1^{m-1} + 1) + \beta(C_1^{q-1} + 1) \), satisfies
\[
\begin{aligned}
\begin{cases}
g_s + A_{i,R} \Delta g + B_{i,R} g_x - \lambda g < 0 & \text{for } (x, s) \in B_R(0) \times (0, t), \\
g(x, s) \geq \eta_{i,R}(x, s) & \text{for } (x, s) \in B_R(0) \times \{t\} \cup \partial B_R(0) \times (0, t).
\end{cases}
\end{aligned}
\]

Hence by the maximum principle [LSU], \( g \geq \eta_{i,R} \) in \( B_R(0) \times (0, t) \). We next consider the function
\[
g^*(x, s) = ae^{h(s)} \Gamma(|x|), \quad R - \alpha \leq r \leq R, \ 0 \leq s \leq t,
\]
where \( \alpha = 1/2(C_3 + n - 1), \ \Gamma(r) = (R - r) - C_3(R - r)^2 \) and
\[
a = (1 + R_0^2)/(\Gamma(R - \alpha)(1 + (R - \alpha)^2)^\beta).
\]
Then \( g^* \geq 0, \ g^*_s = h'(s)g^* \leq 0 \) and
\[
\Delta g^* + (B_{i,R}/A_{i,R}) g_x \leq 0
\]
for all \( R - \alpha < r < R, \ 0 < s < t \) since \( R - \alpha \geq R_0 \geq 2 \). Hence \( g^* \) satisfies
\[
g^*_s + A_{i,R} \Delta g^* + B_{i,R} g_x^* - \lambda g^* < 0, \quad \text{for } (x, s) \in B_R(0) \setminus B_{R-\alpha}(0) \times (0, t).
with $g^*(x, s) \geq \eta_i, R(x, s)$ for all

$$(x, s) \in B_R(0) \setminus B_{R-\alpha}(0) \times \{t\} \cup (\partial B_R(0) \cup \partial B_{R-\alpha}(0)) \times (0, t].$$

By the maximum principle, $0 \leq \eta_i, R \leq g^*$ in $B_R(0) \setminus B_{R-\alpha}(0) \times (0, t)$. Since $g^* \equiv \eta_i, R \equiv 0$ on $\partial B_R(0) \times [0, t]$, we have

$$\|\partial \eta_i, R/\partial N\|_{L^{\infty}(\partial B_R(0) \times (0, t))} \leq \|\partial g^*/\partial N\|_{L^{\infty}(\partial B_R(0) \times (0, t))} \leq CR^{-2\beta}. \tag{1.4}$$

Putting $\eta = \eta_i, R$ in (1.2), we get by (1.3) and (1.4),

$$\int_{B_R(0)} (u_1^{(q)} - u_2^{(q)})(x, t)\theta(x)dx$$

$$= \int_{B_R(0)} (f_1 - f_2)(x)\eta_i, R(x, 0)dx + \int_0^t \int_{B_R(0)} (u_1^{(q)} - u_2^{(q)})(A - A_i, R)\Delta \eta_i, R$$

$$+ (B - B_i, R)(\eta_i, R)_x + \lambda \eta_i, R dxd\tau$$

$$- \int_0^t \int_{\partial B_R(0)} (u_1^{(q)m} - u_2^{(q)m})\partial \eta_i, R/\partial N d\sigma d\tau$$

$$\leq \int_{R^n} (f_1 - f_2)_+ dxdx + 2C_1\|\eta_i, R - A\|_{L^2(B_R(0) \times (0, t))}\|\nabla \theta\|_{L^2(B_R(0))}$$

$$+ (2C_1/(2(\lambda - C_2)^{1/2})\|B_i, R - B\|_{L^2(B_R(0) \times (0, t))}\|\nabla \theta\|_{L^2(B_R(0))}$$

$$+ \lambda \int_0^t \int_{R^n} (u_1^{(q)} - u_2^{(q)})_+ dxd\tau + CR^{-1-2\beta}$$

for all $\theta \in C_0^\infty(B_R_0(0))$, $0 \leq \theta \leq 1$, $0 < t < 1$. Choose now $\beta = n/2$ and let first $i \to \infty$ and then $R \to \infty, \lambda \to C_2$ in (1.5), we get

$$\int_{R^n} (u_1^{(q)} - u_2^{(q)})(x, t)\theta(x)dx \leq \int_{R^n} (f_1 - f_2)_+ dxdx + C_2 \int_0^t \int_{R^n} (u_1^{(q)} - u_2^{(q)})_+ dxd\tau$$

for all $\theta \in C_0^\infty(B_R_0(0))$, $0 \leq \theta \leq 1$, $R_0 > 2$. Putting $\theta = \chi_{\{u_1^{(q)} \geq u_2^{(q)}\} \cap B_{R_0-\epsilon}(0)} \rho_\epsilon$ into the above inequality and letting first $\epsilon \to 0$ and then $R_0 \to \infty$, we get

$$\int_{R^n} (u_1^{(q)} - u_2^{(q)})_+(x, t)dx$$

$$\leq \int_{R^n} (f_1 - f_2)_+ dxdx + C_2 \int_0^t \int_{R^n} (u_1^{(q)} - u_2^{(q)})_+ dxd\tau \quad \forall 0 < t < 1.$$

(i) then follows from the Gronwall's inequality. Similarly,

$$\int_{R^n} (u_1^{(q)} - u_2^{(q)})_-(x, t)dx \leq e^{C_2t} \int_{R^n} (f_1 - f_2)_- dxdx \quad \forall 0 < t < 1.$$

By combining the above inequality with (i), we get (ii).

**Corollary 1.2.** If $u_1^{(q)}$ is a subsolution and $u_2^{(q)}$ is a supersolution of (1.1) in $Q = D \times (0, 1)$ where $D = (-\infty, R_0) \times R^{n-1}$ for some $R_0 \in R$ (or $D = [R_0, R_1] \times R^{n-1}$ for some $R_0, R_1 \in R$, $R_0 < R_1$) with $u_1^{(q)}, u_2^{(q)} \in L^\infty([0, 1); L^1(D)) \cap L^\infty(D \times (0, 1)) \cap C(D \times (0, 1))$ with initial values $u_1^{(q)}(x, 0), u_2^{(q)}(x, 0)$ and boundary values satisfying

$$u_1^{(q)}(x, t) \leq u_2^{(q)}(x, t) \quad \forall (x, t) \in \partial_p Q$$
where $\partial_p Q = \{R_0\} \times \mathbb{R}^{n-1} \times (0,1) \cup (-\infty, R_0) \times \mathbb{R}^{n-1} \times \{0\}$ (respectively $\partial_p Q = \{R_0, R_1\} \times \mathbb{R}^{n-1} \times (0,1) \cup [R_0, R_1] \times \mathbb{R}^{n-1} \times \{0\}$), then

$$u^{(q)}(x, t) \leq u^{(q)}_2(x, t) \quad \forall (x, t) \in Q$$

**Proof.** The proof is the same as the proof of Theorem 1.1.

**Theorem 1.3.** The equation

\begin{equation}
\begin{cases}
    u_t = \Delta u^m - (u^d/q) x_i, & u \geq 0, \\
    u(x, 0) = f(x), & f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n),
\end{cases}
\end{equation}

has a unique solution $u^{(q)} \in L^\infty([0,1); \mathbb{R}^n) \cap L^\infty(\mathbb{R}^n \times (0,1)) \cap C(\mathbb{R}^n \times (0,1))$ with

\begin{equation}
\begin{array}{ll}
    \text{(i)} & \int u^{(q)}(x, t) dx = \int f dx \quad \forall 0 < t < 1, \\
    \text{(ii)} & \|u^{(q)}\|_{L^\infty(\mathbb{R}^n \times (0,1))} \leq \|f\|_{L^\infty(\mathbb{R}^n)}. \\
\end{array}
\end{equation}

**Proof.** The proof is similar to that of [ERV] and [DK]. Let $\psi \in C^\infty(\mathbb{R}^n)$, $0 \leq \psi \leq 1$, be such that $\psi(x) \equiv 1$ for all $|x| \leq 1/2$ and $\psi \equiv 0$ for all $|x| > 1$. For any $\varepsilon > 0$, $0 < \varepsilon < 1$, $R > 0$, let $f_{\varepsilon, R}(x) = f \ast \rho_\varepsilon(x) \cdot \psi(x/R) + \varepsilon$ and let $a_\varepsilon(s), b_\varepsilon(s) \in C^\infty(\mathbb{R})$ be such that $a_\varepsilon(s), b_\varepsilon(s) \geq 0$,

\begin{equation}
\begin{array}{ll}
    a_\varepsilon(s) &= \begin{cases}
        m s^{m-1} & \text{for } s > \|f\|_{L^\infty(\mathbb{R}^n)} + 2, \\
        m |s|^{m-1} & \text{for } s \leq \|f\|_{L^\infty(\mathbb{R}^n)} + 2, \\
        \varepsilon s^{m-1} & \text{for } s \leq \varepsilon/2,
    \end{cases} \\
    b_\varepsilon(s) &= \begin{cases}
        \|f\|_{L^\infty(\mathbb{R}^n)} + 2 \varepsilon^{q-1} & \text{for } s \leq \|f\|_{L^\infty(\mathbb{R}^n)} + 2, \\
        \varepsilon s^{q-1} & \text{for } s \leq \|f\|_{L^\infty(\mathbb{R}^n)} + 1, \\
        \varepsilon/2 s^{q-1} & \text{for } s \leq \varepsilon/2.
    \end{cases}
\end{array}
\end{equation}

By standard parabolic theory [LSU], there exists a unique solution $u_{\varepsilon, R}^{(q)}$ to the equation

\begin{equation}
\begin{cases}
    u_t = \text{div}(a_\varepsilon(u) \nabla u) - b_\varepsilon(u) u x_i, & \text{for } (x, t) \in B_R(0) \times (0,1), \\
    u(x, t) = \varepsilon, & \text{for } (x, t) \in \partial B_R(0) \times (0,1), \\
    u(x, 0) = f_{\varepsilon, R}(x), & \text{for } x \in B_R(0).
\end{cases}
\end{equation}

Since $\varepsilon \leq f_{\varepsilon, R} \leq \|f\|_{L^\infty(\mathbb{R}^n)} + \varepsilon$, by the maximum principle,

\begin{equation}
\varepsilon \leq u_{\varepsilon, R}^{(q)} \leq \|f\|_{L^\infty(\mathbb{R}^n)} + \varepsilon.
\end{equation}

Hence $a_\varepsilon(u_{\varepsilon, R}^{(q)}) = m u_{\varepsilon, R}^{(q)} m^{-1}$, $b_\varepsilon(u_{\varepsilon, R}^{(q)}) = u_{\varepsilon, R}^{(q)} q^{-1}$. Since (1.3) is a nondegenerate parabolic equation, by Schauder's estimate [LSU], $u_{\varepsilon, R}^{(q)} \in C^\infty(B_R(0) \times [0,1))$. Thus $u_{\varepsilon, R}^{(q)}$ satisfies (1.1) in $B_R(0) \times (0,1)$. Since $u_{\varepsilon, R}^{(q)}$ is uniformly bounded by $\|f\|_{L^\infty(\mathbb{R}^n)} + 1$, by the result of P. Sacks [S1], $\{u_{\varepsilon, R}^{(q)}\}_{R>0}$ has a convergent subsequent $\{u_{\varepsilon, R_j}^{(q)}\}_{j=1}^\infty$, $R_j \to \infty$ as $j \to 0$, such that $\{u_{\varepsilon, R_j}^{(q)}\}_{j=1}^\infty$ converges uniformly on compact subsets of $\mathbb{R}^n \times (0,1)$. Let $u_{\varepsilon}^{(q)} = \lim_{j \to \infty} u_{\varepsilon, R_j}^{(q)}$. Then $u_{\varepsilon}^{(q)} \in C(\mathbb{R}^n \times (0,1))$ and

\begin{equation}
\varepsilon \leq u_{\varepsilon}^{(q)} \leq \|f\|_{L^\infty(\mathbb{R}^n)} + \varepsilon.
\end{equation}
Putting $u = u^{(q)}_{\varepsilon, R_j}$ in (1.1) and letting $j \to 0$, we see that $u^{(q)}_{\varepsilon}$ satisfies (1.1) in $R^n \times (0, 1)$ with $u^{(q)}_{\varepsilon}(x, 0) = f \ast \rho_{\varepsilon}(x) + \varepsilon$. Thus $u^{(q)}_{\varepsilon} \in C^\infty(R^n \times (0, 1))$ by (1.11) and Schauder's estimates. Since $\|u^{(q)}_{\varepsilon}\|_{L^\infty(R^n \times (0, 1))} \leq \|f\|_{L^\infty(R^n)} + \varepsilon$, by [S1], $\{u^{(q)}_{\varepsilon}\}_{\varepsilon > 0}$ has a convergent subsequence $\{u^{(q)}_{\varepsilon_i}\}_{i=1}^{\infty}$, $\varepsilon_i \to 0$ as $i \to 0$, such that $\{u^{(q)}_{\varepsilon_i}\}_{i=1}^{\infty}$ converges uniformly on compact subsets of $R^n \times (0, 1)$.

Let $u^{(q)} = \lim_{\varepsilon \to 0} u^{(q)}_{\varepsilon}$. Then $u^{(q)} \in C(R^n \times (0, 1))$.

Putting $u = u^{(q)}_{\varepsilon_i}$ in (1.1) and letting $i \to 0$, we see that $u^{(q)}$ satisfies (1.1) in $R^n \times (0, 1)$. Moreover,

\begin{align*}
|\int_{R^n} u^{(q)}_{\varepsilon, R_j}(x, t)\eta(x)dx - \int_{R^n} f_{\varepsilon, R_j}\eta dx| &= \left| \int_0^t \int_{R^n} (u^{(q)}_{\varepsilon, R_j})_t(x, \tau)\eta(x)dxd\tau \right| \\
&= \left| \int_0^t \int_{R^n} \left[ \Delta u^{(q)}_{\varepsilon, R_j} - \frac{(u^{(q)}_{\varepsilon, R_j})}{q} \right] \eta dxd\tau \right| \\
&\leq (\|f\|_{L^\infty(R^n)} + 1)^m \|\Delta \eta\|_{L^1(R^n)} t + \frac{\|f\|_{L^\infty(R^n)} + 1)^q}{q} \|\eta_{x_1}\|_{L^1(R^n)} t
\end{align*}

for all $\eta \in C^\infty_0(R^n)$. Letting first $j \to 0$ and then $\varepsilon = \varepsilon_i \to 0$, $t \to 0$, we get

$$\lim_{l \to 0} \int_{R^n} u^{(q)}(x, t)\eta(x)dx = \int_{R^n} f\eta dx \quad \forall \eta \in C^\infty_0(R^n).$$

Hence $u^{(q)}$ has initial trace $f$ and $\|u^{(q)}\|_{L^\infty(R^n \times (0, 1))} \leq \|f\|_{L^\infty(R^n)}$ by (1.11). On the other hand, since $u^{(q)}_{\varepsilon_i}$ satisfies (1.1) in $R^n \times (0, 1),

\begin{align*}
\int_{B_R(0)} u^{(q)}_{\varepsilon}(x, t)\eta(x)dx &= \int_{B_R(0)} (f \ast \rho_{\varepsilon}(x) + \varepsilon)\eta(x, 0)dx \\
&+ \int_0^t \int_{B_R(0)} u^{(q)}_{\varepsilon}(\eta_t + A_{\varepsilon}\Delta \eta + B_{\varepsilon}\eta_{x_1})dxd\tau \\
&- \int_0^t \int_{\partial B_R(0)} u^{(q)}_{\varepsilon}m \frac{\partial \eta}{\partial N}d\sigma d\tau
\end{align*}

for all $0 < t < 1$, $\eta \in C^\infty(B_R(0) \times [0, t])$, $R > 0$ such that $\eta \equiv 0$ on $\partial B_R(0) \times [0, t]$ where $A_{\varepsilon} = u^{(q)}_{\varepsilon,m-1}$, $B_{\varepsilon} = u^{(q)}_{\varepsilon,q-1}/q$.

For any $R_0 > 2$, $R > R_0 + 1$, $\theta \in C^\infty_0(B_{R_0}(0))$, $0 \leq \theta \leq 1$, $\theta \equiv 1$ for $|x| \leq R_0 - 1$ let $\eta_{\varepsilon, R}$ be the solution of

- $\eta_0 + A_{\varepsilon}\Delta \eta + B_{\varepsilon}\eta_{x_1} = 0$ for $(x, s) \in B_R(0) \times (0, t),$
- $\eta(x, s) = 0$ for $(x, s) \in \partial B_R(0) \times (0, t),$
- $\eta(x, t) = \theta(x)$ for $x \in B_R(0)$.

By an argument similar to the proof of Theorem 1.1, we have $0 \leq \eta_{\varepsilon, R} \leq 1$,

$$\eta_{\varepsilon, R}(x, s) \leq e^{h(s)} \left( \frac{1 + R_0^2}{1 + |x|^2} \right)^n \quad \forall 0 \leq s \leq t,$$
where \( h(s) = C'(t-s), \) \( C' = 4n(n+1)(b^m_1+1)+n(b^m_2+1), \) \( b_1 = \|f\|_{L^\infty(R^n)} + 1, \) and
\[
\|\partial_N \eta_{e,R}/\partial N\|_{L^\infty(\partial B_R(0) \times (0,t))} \leq CR^{-2n}
\]
for some constant \( C > 0 \) depending only on \( R_0 \) and \( b_1 . \) Putting \( \eta = \eta_{e,R} \) into (1.13), we get
\[
(1.14) \quad \int_{B_R(0)} u^{(q)}_\varepsilon \theta(x) dx \leq \int f dx + C' R_0 R^{-n-1} + \varepsilon C R_0
\]
for some constant \( C_{R_0}, C'_{R_0} > 0 \) depending only on \( R_0 \) and \( b_1 . \) Letting \( R \to \infty, \varepsilon = \varepsilon_i \to 0, \)
\[
\int_{|x| \leq R_0} u^{(q)}(x,t) dx \leq \int_{R^n} u^{(q)}_\varepsilon(x,t) \theta(x) dx \leq \int f dx
\]
for all \( 0 < t < 1. \) Letting \( R_0 \to \infty, \)
\[
(1.15) \quad \int_{R^n} u^{(q)}(x,t) dx \leq \int f dx \quad \forall 0 < t < 1.
\]
Hence \( u^{(q)} \in L^\infty([0, 1]; L^1(R^n)) \cap L^\infty(R^n \times [0, 1]) \cap C(R^n \times [0, 1]) \) and satisfies (1.8). It remains to show (1.7). Since
\[
(1.15) \Rightarrow \int_0^1 \int_{R^n} u^{(q)}(x, \tau) dx d\tau \leq \int_{R^n} f dx
\]
\[
\Rightarrow \int_0^1 \int_{R/2 \leq |x| \leq R} u^{(q)}(x, \tau) dx d\tau \to 0 \quad \text{as} \ R \to \infty,
\]
putting \( \eta(x) = \psi(x/R), R > 0, \) in (1.12), we have
\[
\left| \int_{R^n} u^{(q)}_{\varepsilon,R_j}(x,t) \psi(x/R) dx - \int_{R^n} f_{\varepsilon,R_j}(x) \psi(x/R) dx \right|
\leq \frac{\left( \|f\|_{L^\infty(R^n)} + 1 \right)^{m-1}}{R^2} \|\Delta \psi\|_{L^\infty(R^n)} \int_0^1 \int_{R/2 \leq |x| \leq R} u^{(q)}_{\varepsilon,R_j}(x, \tau) dx d\tau
\]
\[
+ \frac{\left( \|f\|_{L^\infty(R^n)} + 1 \right)^{q-1}}{q R} \|\psi_x\|_{L^\infty(R^n)} \int_0^1 \int_{R/2 \leq |x| \leq R} u^{(q)}_{\varepsilon,R_j}(x, \tau) dx d\tau.
\]
By letting first \( j \to \infty \) and then \( \varepsilon = \varepsilon_i \to 0, R \to \infty, \) in the above inequality, we get (1.7). Since uniqueness of solution of (1.6) follows from Theorem 1.1. This completes the proof of the theorem.

Theorem 1.4. Let \( u^{(q)}_1, u^{(q)}_2, f_1, f_2 \) be as in Theorem 1.1. Then
\[
\int_{R^n} |u^{(q)}_1 - u^{(q)}_2|(x,t) dx \leq \int_{R^n} |f_1 - f_2| dx \quad \forall 0 < t < 1.
\]
Proof. By Theorem 1.1 and the proof of Theorem 1.3, there exist solutions \( u^{(q)}_{1,\varepsilon}, u^{(q)}_{2,\varepsilon} \in C^\infty(R^n \times (0, 1)) \cap L^\infty(R^n \times (0, 1)), 0 < \varepsilon < 1, \) of (1.6) with initial values \( u^{(q)}_{1,\varepsilon}(x,0) = f_1 * \rho_\varepsilon + \varepsilon, \) \( u^{(q)}_{2,\varepsilon}(x,0) = f_2 * \rho_\varepsilon + \varepsilon \) respectively such that \( u^{(q)}_{1,\varepsilon} \) and \( u^{(q)}_{2,\varepsilon} \) converges uniformly to \( u^{(q)}_1 \) and \( u^{(q)}_2 \) respectively on compact subsets of \( R^n \times (0, 1) \) as \( \varepsilon \to 0. \)
By a proof similar to the proof of (1.14), we have
\[
\int_{B_R(0)} (u_1^{(q)} - u_2^{(q)})(x, t)\theta(x)dx \leq \int_{\mathbb{R}^n} (f_1 - f_2)_+ dx + C_{R_0} R^{-1-n} + \varepsilon C_{R_0}
\]
for all \( \theta \in C_0^\infty(B_{R_0}(0)) \), \( R_0 > 1 \) where \( C_{R_0} \) and \( C_{R_0} > 0 \) are constants depending only on \( R_0 \). Letting \( R \to \infty \), we get
\[
\int_{\mathbb{R}^n} (u_1^{(q)} - u_2^{(q)})_+(x, t)\theta(x)dx \leq \int_{\mathbb{R}^n} (f_1 - f_2)_+ dx
\]
for all \( \theta \in C_0^\infty(B_{R_0}(0)) \), \( R_0 > 1 \). Putting \( \theta = \chi_{\{u_1^{(q)} \geq u_2^{(q)}\}} \) and letting first \( \varepsilon \to 0 \) and then \( R_0 \to \infty \),
\[
\int_{\mathbb{R}^n} (u_1^{(q)} - u_2^{(q)})_+(x, t)dx \leq \int_{\mathbb{R}^n} (f_1 - f_2)_+ dx \quad \forall 0 < t < 1.
\]
Similarly,
\[
\int_{\mathbb{R}^n} (u_1^{(q)} - u_2^{(q)})_-(x, t)dx \leq \int_{\mathbb{R}^n} (f_1 - f_2)_- dx \quad \forall 0 < t < 1.
\]
Combining the above two inequalities the theorem follows.

**Lemma 1.5.** If \( f \in C_0^1(\mathbb{R}^n) \) and \( f_\varepsilon = f + \varepsilon, \ 0 < \varepsilon < 1 \), then (1.1) has a unique solution \( u_\varepsilon^{(q)} \in C_0^\infty(\mathbb{R}^n \times (0, 1)) \cap C^1(\mathbb{R}^n \times [0, 1]) \) in \( \mathbb{R}^n \times (0, 1) \) with \( u_\varepsilon^{(q)}(x, 0) = f_\varepsilon(x) \) such that \( u_\varepsilon^{(q)} \) converges uniformly on compact subsets of \( \mathbb{R}^n \times (0, 1) \) to the solution \( u^{(q)} \) of (1.6) with \( u^{(q)}(x, 0) = f(x) \) as \( \varepsilon \to 0 \). Moreover
\[
\|u_{\varepsilon, x_k}\|_{L^\infty(\mathbb{R}^n)} \leq \|f_{x_k}\|_{L^\infty(\mathbb{R}^n)} \quad \forall k = 1, 2, \ldots, n.
\]

**Proof.** By Theorem 1.4 and an argument similar to the proof of Theorem 1.3, for any \( 0 < \varepsilon < 1 \) there exists a unique solution \( u_\varepsilon^{(q)} \in C_0^\infty(\mathbb{R}^n \times (0, 1)) \cap C^1(\mathbb{R}^n \times [0, 1]) \) to (1.1) in \( \mathbb{R}^n \times (0, 1) \) with \( u_\varepsilon^{(q)}(x, 0) = f(x) + \varepsilon \) and\n\[
\|u_{\varepsilon}^{(q)}\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^\infty(\mathbb{R}^n)} + \varepsilon
\]
such that \( u_\varepsilon^{(q)} \) converges uniformly on compact subsets of \( \mathbb{R}^n \times (0, 1) \) to the solution \( u^{(q)} \) of (1.6) with \( u^{(q)}(x, 0) = f(x) \) as \( \varepsilon \to 0 \).

Since \( u_\varepsilon^{(q)} \in C_0^\infty(\mathbb{R}^n \times (0, 1)) \cap C^1(\mathbb{R}^n \times [0, 1]) \), differentiating (1.1) with respect to \( x_k \) and writing \( z = u_{\varepsilon, x_k} \), we get
\[
\begin{cases}
z_t = \Delta(mu_{\varepsilon}^{(q)m-1}z) + (u_{\varepsilon}^{(q)q-1}z)x_1, & (x, t) \in \mathbb{R}^n \times (0, 1), \\
z(x, 0) = f_{x_k}(x), & x \in \mathbb{R}^n,
\end{cases}
\]
for all \( k = 1, 2, \ldots, n \). Since the above equation is nondegenerate by (1.16), by the maximum principle,
\[
\|z\|_{L^\infty(\mathbb{R}^n)} \leq \|f_{x_k}\|_{L^\infty(\mathbb{R}^n)} \quad \forall k = 1, 2, \ldots, n
\]
and the lemma follows.

**Lemma 1.6.** Let \( 0 \leq f \leq M \) with \( \text{supp} f \subset B_{R_1}(0) \) for some \( R_1 > 0 \). Then there exists \( R' > 0 \) depending only on \( m, R_1, M \) and is independent of \( q > m + 1 \) such that
\[
u^{(q)}(x, t) = 0 \quad \forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n, x_1 \leq -R', 0 \leq t < 1, q > m + 1,
\]
and
\[ 0 \leq u^{(q)}(x, t) \leq \left( \frac{x_1 + R' + 1}{t + (1/M^{q-1})} \right)^{1/q-1} \leq \left( \frac{x_1 + R' + 1}{t} \right)^{1/q-1} \]
for all \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n, x_1 \geq -R', 0 < t < 1, q > m + 1. \)

**Proof.** Let
\[
w(x_1, t) = \frac{1}{(t + t_0)^{1/m+1}} \left( a^2 - C_1 \left( \frac{x_1}{(t + t_0)^{1/m+1}} \right)^2 \right)^{1/m-1}, \quad x_1 \in \mathbb{R}, t \geq 0,
\]
be the Barenblatt solution for the porous medium equation \( w_t = (w^m)_{x_1} \) ([B], [HP]) where \( C_1 = \frac{m-1}{2m} \left( \frac{1}{(m+1)} \right), \ t_0 = \min \left( 1, \left( \frac{4C_1R^2_1}{(2m-1/m+1)\alpha^{-1}} \right)^{(m+1)/2} \right) \) and
\[
a = (C_1(2R_1/t_0^{1/m+1})^2 + (\alpha t_0^{1/m+1})^{m-1})^{1/2}.
\]
Then \( w \) is a supersolution of (1.1) in \( (-\infty, 0] \times \mathbb{R}^{n-1} \times (0, 1) \) with
\[
u(x + x_0) = f(x + x_0) \leq M \leq w(x_1, 0)
\]
for all \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n, x_1 \leq 0 \) where \( x_0 = (R_1, 0, \ldots, 0) \) and
\[
w(0, t) \geq \frac{1}{(1 + t_0)^{1/m+1}} a^{2/m-1}
\]
\[\geq \frac{1}{2^{1/m+1}} \left( C_1 \left( \frac{2R_1}{t_0^{1/m+1}} \right)^2 \right)^{1/m-1}
\]
\[\geq \frac{1}{2^{1/m+1}} \left( \frac{4R_1^2C_1}{4C_1R^2_1/(21/m + 1\alpha^{-1})} \right)^{1/m-1}
\]
\[= M \geq u(x_0, t)
\]
for all \( 0 < t < 1 \) by (1.8). Hence by applying the maximum principle (Corollary 1.2) to the functions \( u^{(q)}(\cdot + x_0, \cdot) \) and \( w \) in the region \( (-\infty, 0] \times \mathbb{R}^{n-1} \times (0, 1) \), we get
\[
u^{(q)}(x + x_0, t) \leq w(x_1, t) \quad \forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n, x_1 \leq 0, 0 \leq t < 1.
\]
Now for each \( 0 < t < 1, \ supp \ w(x_1, t) \subset B_{R_t}(0) \) where
\[
R_t = \frac{a}{C_1^{1/2}} (t + t_0)^{1/m+1} \leq \frac{2a}{C_1^{1/2}} \quad (= R_2 \ say).
\]
Hence
\[
u^{(q)}(x + x_0, t) \leq w(x_1, t) = 0 \quad \forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n,
\]
\[x_1 \leq -R_2, 0 \leq t < 1,
\]
\[\Rightarrow \nu^{(q)}(x, t) = 0 \quad \forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n, x_1 \leq -R', 0 \leq t < 1,
\]
\[q > m + 1,
\]
(1.17)
where \( R' = \max(R_2 - R_1, 0) \geq 0. \)
We next observe that
\[
\tilde{w}(x_1, t) = \left( \frac{x_1 + R' + 1}{t + (1/M^{q-1})} \right)^{1/q-1}, \quad q > m + 1,
\]
is a supersolution of (1.1) in \([-R', R_3] \times R^{n-1} \times (0, \infty)\) with
\[
\begin{cases}
    u^{(q)}(x, 0) \leq M \leq \tilde{w}(x_1, 0) & \text{for } x = (x_1, \ldots, x_n) \in R^n, \quad -R' \leq x_1 \leq R_3, \\
    u^{(q)}(x, t) \leq M \leq \tilde{w}(x_1, t) & \text{for } x = (x_1, \ldots, x_n) \in R^n, \\
    x_1 = -R' \text{ or } x_1 = R_3, \ 0 \leq t < 1,
\end{cases}
\]
for all \( R_3 > \max(2M^{q-1} - R' + 1, 0) \) by (1.17). Hence by applying Corollary 1.2 to the function \( u^{(q)} \) and \( \tilde{w} \) in the region \([-R', R_3] \times R^{n-1} \times (0, 1)\), we get
\[
u^{(q)}(x, t) \leq \tilde{w}(x_1, t)
\]
for all \( x = (x_1, \ldots, x_n) \in [-R', R_3] \times R^{n-1}, \ 0 \leq t < 1, \ q > m + 1, \ R_3 > \max(2M^{q-1} - R' + 1, 0) \). By letting \( R_3 \to \infty \), the lemma follows.

**Lemma 1.7.** Suppose \( f \) is as in Lemma 1.6. Let \( \Omega \subset R^n \) be a bounded open set with \( \partial \Omega \subset C^2 \) and \( \eta \in C^\infty(R^n \times (0, 1)) \). Then
\[
\int_{\tau_1}^{\tau_2} \int_{\Omega} \frac{u^{(q)}(x, t)}{q} \cdot \eta dx dt \to 0 \quad \text{as } q \to \infty
\]
for any \( 0 < \tau_1 \leq \tau_2 < 1 \).

**Proof.** By Lemma 1.6, there exists a constant \( R' > 0 \) such that
\[
u^{(q)}(x, t) \leq \left( \frac{|x_1| + R' + 1}{t} \right)^{1/q-1} \quad \forall x = (x_1, \ldots, x_n) \in R^n, \quad 0 < t < 1,
\]
and by Theorem 1.3 \( \|u^{(q)}\|_{L^\infty} \leq \|f\|_{L^\infty} \) for all \( q > m + 1 \). Hence
\[
\int_{\tau_1}^{\tau_2} \int_{\Omega} \frac{u^{(q)}(x, t)}{q} \cdot \eta dx dt = \int_{\tau_1}^{\tau_2} \int_{\Omega} \frac{u^{(q-2)}u^{(q)}(x, t)}{q} \cdot \eta dx dt
\]
\[
\leq \frac{\|\eta\|_{L^\infty} \|f\|_{L^\infty}^2}{q} \int_{\tau_1}^{\tau_2} \int_{\Omega} \left( \frac{|x_1| + R' + 1}{t} \right)^{-2/q-1} \, dx dt
\]
\[
\leq \frac{\|\eta\|_{L^\infty} \|f\|_{L^\infty}^2}{q} \frac{|\Omega|}{\tau_1^{1/q-1}} \frac{(R'' + R' + 1)}{\tau_1} \to 0
\]
as \( q \to \infty \) where \( R'' = \sup\{|x_1|: x = (x_1, \ldots, x_n) \in \Omega\} < \infty \).

**Lemma 1.8.** Let \( f \in C_0(R^n) \) and let \( p^{(q)}(x, t) = \int_0^t \frac{u^{(q)}(x, \tau)}{q} \, d\tau \). Then \( \{p^{(q)}\}_{q \geq m+1} \) is uniformly bounded on compact subsets of \( R^n \times [0, 1) \). For any sequence \( \{p^{(q)}\}_{q=m+1} \), \( q_i \to \infty \) as \( i \to \infty \), of \( \{p^{(q)}\}_{q \geq m+1} \), there exists a subsequence \( \{p^{(q_i)}\}_{i=1}^\infty \) of \( \{p^{(q)}\}_{i=1}^\infty \), a sequence of functions \( \{p_i\}_{i=1}^\infty \subset L^\infty_{\text{loc}}(R^n) \), \( \tilde{g} \in L^\infty_{\text{loc}}(R^n) \), \( p_j, \tilde{g} \geq 0 \), and a sequence \( \{\varepsilon_j\}_{j=1}^\infty \subset R, \varepsilon_j \to 0 \) as \( j \to \infty \), such that
\[
\left\{ p^{(q_i)}(\cdot, \varepsilon_j) \to p_j(\cdot) \text{ weakly in } (L^\infty(K))^* \text{ as } i \to \infty, \ \forall j = 1, 2, \ldots \right\}
\]
\[
\left\{ p_j(\cdot) \to \tilde{g}(\cdot) \text{ weakly in } (L^\infty(K))^* \text{ as } j \to \infty \right\}
\]
for any compact subset \( K \subset R^n \).
Proof. By Theorem 1.3, \( \|u^{(q)}\|_{L^\infty(R^n)} \leq \|f\|_{L^\infty(R^n)} \) for all \( q > m + 1 \) and by Lemma 1.6 there exists \( R' > 0 \) such that
\[
0 \leq u^{(q)}(x, \tau) \leq \left( \frac{|x_1| + R' + 1}{\tau} \right)^{(q-1)/2} \quad \forall x = (x_1, x_2, \ldots, x_n) \in R^n, \\
0 < \tau < 1, \ q > m + 1.
\]
Hence
\[
0 \leq p^{(q)}(x, \tau) = \int_0^\tau \frac{u^{(q)}(x, \tau)}{q} \, d\tau 
\leq \frac{\|f\|_{L^\infty(R^n)}^2}{q} \int_0^\tau \left( \frac{|x_1| + R' + 1}{\tau} \right)^{q-2/q} 
\leq \frac{q-1}{q} \|f\|_{L^\infty(R^n)}^2 \left( \frac{|x_1| + R' + 1}{\tau} \right)^{q-2/q-1} 
\leq \|f\|_{L^\infty(R^n)}^2 \left( \frac{|x_1| + R' + 1}{\tau} \right)^{q-2/q-1}
\]
for all \( x = (x_1, x_2, \ldots, x_n) \in R^n, \ 0 < \tau < 1, \ q > m + 1 \). Thus \( \{p^{(q)}\}_{q>m+1} \) is uniformly bounded on compact subsets of \( R^n \times [0, 1) \). So any sequence \( \{p^{(q)}\}_{i=1}^\infty \) of \( \{p^{(q)}\}_{q>m+1} \) will have a subsequence \( \{p^{(q_i)}\}_{i=1}^\infty \) such that \( \{p^{(q_i)}\}_{i=1}^\infty \) converges weakly in \( (L^\infty(K))^* \) for any compact subset \( K \subset R^n \).

Let \( p_1(\cdot) = \lim_{i \to \infty} p^{(q_i)}(\cdot, 1/2) \). Then \( \{p^{(q_i)}(\cdot, 1/2)\} \) has a subsequence \( \{p^{(q_i')}\}_{i=1}^\infty \) such that \( p^{(q_i')}(x, 1/2) \to p_1(x) \) a.e. \( x \in R^n \) as \( i \to \infty \). Without loss of generality we may assume \( p^{(q_i)}(x, 1/2) \to p_1(x) \) a.e. \( x \in R^n \) as \( i \to \infty \). We may also assume that \( q_1 < q_{1,1} \). Since \( \{p^{(q_{1,1})}(\cdot, 1/3)\}_{i=1}^\infty \) is uniformly bounded on compact subsets of \( R^n \), \( \{p^{(q_{1,1})}(\cdot, 1/3)\}_{i=1}^\infty \) converges weakly in \( (L^\infty(K))^* \) for any compact set \( K \subset R^n \). Let \( p_2(\cdot) = \lim_{i \to \infty} p^{(q_{i,1})}(\cdot, 1/3) \). We may assume without loss of generality that \( p^{(q_{i,1})}(x, 1/3) \to p_2(x) \) a.e. \( x \in R^n \) as \( i \to \infty \) and \( q_{1,1} < q_{2,1} \).

Repeating the argument, for each \( j = 2, 3, \ldots, \) we can find a subsequence \( \{p^{(q_j)}(x, 1/(j+1))\}_{i=1}^\infty \) of \( \{p^{(q_{j-1})}(x, 1/(j+1))\}_{i=1}^\infty \) with \( q_{j,1} > q_{j-1,1} \) and a function \( p_j \in L^\infty_{\text{loc}}(R^n) \) such that \( p^{(q_{j,1})}(x, 1/(j+1)) \to p_j(x) \) weakly in \( (L^\infty(K))^* \) for every compact set \( K \subset R^n \) as \( i \to \infty \) and \( p^{(q_{j,1})}(x, 1/(j+1)) \to p_j(x) \) a.e. \( x \in R^n \) as \( i \to \infty \).

Let \( q'_j = q_{j,1} \). Then for each \( j = 1, 2, \ldots, \), \( \{p^{(q'_j)}(\cdot, 1/(j+1))\}_{i=1}^\infty \) is a subsequence of \( \{p^{(q_{j,1})}(\cdot, 1/(j+1))\}_{i=1}^\infty \). Hence \( p^{(q'_j)}(x, 1/(j+1)) \to p_j(x) \) weakly in \( (L^\infty(K))^* \) for every compact set \( K \subset R^n \) as \( i \to \infty \) and \( p^{(q'_j)}(x, 1/(j+1)) \to p_j(x) \) a.e. \( x \in R^n \) as \( i \to \infty \). Thus \( \{p_j\}_{j=1}^\infty \) is also uniformly bounded on every compact subset of \( R^n \). So there exists a subsequence \( \{p_{j_k}\}_{k=1}^\infty \) of \( \{p_j\}_{j=1}^\infty \) and a function \( \tilde{g} \in L^\infty_{\text{loc}}(R^n) \) such that \( p_{j_k} \to \tilde{g} \) weakly in \( (L^\infty(K))^* \) for any compact subset \( K \subset R^n \). Letting \( \varepsilon_k = 1/(j_k + 1) \), the lemma follows.

In this section we will first establish some technical lemmas and prove the main theorem (Theorem 2.10) under the assumption that \( f \in C^1_0(R^n) \) (Theorem 2.9). The main theorem will then follow by an approximation argument.
Theorem 2.1. Suppose \( f \in C_0(R^n) \). For any sequence \( \{u^{(q)}_i\}_{i=1}^{\infty} \), \( q_i \to \infty \) as \( i \to \infty \), of \( \{u^{(q)}_i\}_{q_i>m+1} \), there exists a subsequence \( \{u^{(q)}_i\}_{i=1}^{\infty} \) of \( \{u^{(q)}_i\}_{i=1}^{\infty} \) and a \( u^{(\infty)} \in C(R^n \times (0,1)) \), \( 0 \leq u^{(\infty)} \leq 1 \), such that \( u^{(q)}_i \to u^{(\infty)} \) uniformly on compact subsets \( R \times (0,1) \) as \( i \to \infty \). Moreover \( u^{(\infty)} \) satisfies (0.2) with initial trace \( g \in L^1(R^n) \), \( 0 \leq g \leq 1 \), satisfying (0.3) for some function \( \overline{g} \in L^\infty_1(R^n) \), \( \overline{g} \geq 0 \).

Proof. The proof is a modification of the proof of Theorem 4 of [H1]. We first observe that \( u^{(q)} \) is uniformly bounded by \( ||f||_{L^\infty} \) by Theorem 1.3 and there exists \( R' > 0 \) such that

\[
0 < u^{(q)}(x,t) < \frac{|x_i|+R'+1}{t}^{1/q-1}, \quad \forall x = (x_1, x') \in R^n, 0 < t < 1, q > m+1,
\]

by Lemma 1.6. If \( \gamma(s) = s^{q/m}/q \), then \( \gamma(u^{(q)}m) = \frac{u^{(q)}m}{q} \) and

\[
\gamma'(u^{(q)}m(x,t)) = \frac{1}{m(u^{(q)}m(x,t))^{q-m}} \leq \frac{1}{m} \left( \frac{|x_i|+R'+1}{t} \right)^{q-m/q-1} \leq \frac{1}{m} \left( \frac{|x_i|+R'+1}{t} \right) \forall x = (x_1, x') \in R^n, 0 < t < 1, q > m+1,
\]

by (2.1). Hence both \( u^{(q)}m \) and \( \gamma'(u^{(q)}m) \) are uniformly bounded on compact subsets of \( R^n \times (0,1) \) for \( q > m+1 \). By the result of P. Sacks [S1], \( u^{(q)} \) is uniformly Hölder continuous on every compact subset of \( R^n \times (0,1) \). Hence \( \{u^{(q)}\}_{q=1}^{\infty} \) has a convergent subsequence \( \{u^{(q)}_i\}_{i=1}^{\infty} \) such that \( u^{(q)}_i \to u^{(\infty)} \) uniformly on every compact subset of \( R^n \times (0,1) \). Without loss of generality we may assume that \( u^{(q)}_i \) converges uniformly on every compact subset of \( R^n \times (0,1) \). Let \( u^{(\infty)} = \lim_{i \to \infty} u^{(q)}_i \). Then \( u^{(\infty)} \in C(R^n \times (0,1)) \).

Putting \( q = q_i \) and letting \( i \to \infty \) in (2.1), we get \( 0 \leq u^{(\infty)} \leq 1 \). Putting \( h(u) = u^{(q)}/q_i \), \( u = u^{(q)} \) in (0.5) and letting \( i \to \infty \) we see that, by Lemma 1.6, \( u^{(\infty)} \) satisfies

\[
\int_{\Omega} u^{(\infty)} \Delta \eta + u \frac{\partial \eta}{\partial t} \mathrm{d}x \mathrm{d}t = \int_{\Omega} u^{(\infty)} \frac{\partial \eta}{\partial N} \mathrm{d}s \mathrm{d}s + \int_{\Omega} u \eta \mathrm{d}x \bigg|_{t_1}^{t_2}
\]

for all bounded open sets \( \Omega \subset R^n \) with \( \partial \Omega \subset C^2 \), \( 0 \leq \tau_1 \leq \tau_2 < 1 \), \( \eta \in C^\infty(\Omega \times [\tau_1, \tau_2]) \), \( \eta \equiv 0 \) on \( \partial \Omega \times [\tau_1, \tau_2] \). Hence \( u^{(\infty)} \) is a solution of the equation \( u_t = \Delta u^m \) in \( R^n \times (0,1) \). Since \( ||u^{(\infty)}||_{L^\infty} \leq ||f||_{L^\infty} \), \( u^{(\infty)} \) has an initial trace \( d\mu \) by [DK] and \( d\mu \) is absolutely continuous with respect to the Lebesgue measure. Hence \( d\mu = g(x)\mathrm{d}x \) for some function \( g \geq 0 \). Since \( 0 \leq u^{(\infty)} \leq 1 \) and

\[
\lim_{i \to 0} u^{(\infty)}(x,t) = g(x) \text{ a.e. } x \in R^n
\]

by the result of [DFK], \( 0 \leq g \leq 1 \). Since

\[
\int_{R^n} u^{(q)}(x,t)\mathrm{d}x = \int_{R^n} f(x)\mathrm{d}x, \quad \forall 0 \leq t \leq 1, i = 1, 2, \ldots,
\]
Letting $i \to \infty$, we get by Fatou's lemma,
\[
\int_{\mathbb{R}^n} u^{(\infty)}(x, t) dx \leq \int_{\mathbb{R}^n} f(x) dx, \quad \forall 0 < t \leq 1.
\]
Letting $t \to 0$, we get by Fatou's lemma and (2.3),
\[
\int_{\mathbb{R}^n} g(x) dx \leq \int_{\mathbb{R}^n} f(x) dx.
\]
Hence $g \in L^1(\mathbb{R}^n)$. Let $p^{(q)}$ be as in Lemma 1.8 and $\Omega$ be a bounded open subset of $\mathbb{R}^n$ with $\partial \Omega \in C^2$. Then by Lemma 1.8 there exists a constant $C_1 > 0$ such that $\|p^{(q)}\|_{L^\infty(\Omega \times [0, 1])} \leq C_1$ for all $q > m + 1$ and there exists a subsequence $\{p^{(q_i)}\}_{i=1}^\infty$ of $\{p^{(q_i)}\}_{i=1}^\infty$, a sequence of functions $\{p_j\}_{j=1}^\infty \subset L^\infty_{\text{loc}}(\mathbb{R}^n)$, $\tilde{g} \in L^\infty_{\text{loc}}(\mathbb{R}^n)$, $p_j, \tilde{g} \geq 0$, and a sequence $\{\varepsilon_j\}_{j=1}^\infty \subset \mathbb{R}$, $\varepsilon_j \to 0$ as $j \to \infty$, such that (1.18) holds. Hence for any $0 < \tau_2 < 1$, $\eta \in C^\infty_0(\mathbb{R}^n)$,
\[
\left| \int_0^{\tau_2} \int_{\Omega} \frac{u^{(q_i)}(x, \tau)}{q_i} \eta_{x_i} dx d\tau - \int_{\Omega} \tilde{g} \eta_{x_i} dx \right| \leq \left\| \eta_{x_i} \right\|_{L^\infty(\mathbb{R}^n)} \int_{\Omega} \int_0^\tau \frac{u^{(q_i)}(x, \tau)}{q_i} \eta_{x_i} dx d\tau
\]
\[
\quad + \left| \int_{\Omega} \left( \int_0^{\varepsilon_j} \frac{u^{(q_i)}(x, \tau)}{q_i} d\tau \right) \eta_{x_i} dx - \int_{\mathbb{R}^n} \tilde{g} \eta_{x_i} dx \right|
\]
\[
\quad \leq \left\| \eta_{x_i} \right\|_{L^\infty(\mathbb{R}^n)} \int_{\Omega} \int_0^\tau \frac{u^{(q_i)}(x, \tau)}{q_i} \eta_{x_i} dx d\tau
\]
\[
\quad + \left| \int_{\Omega} p^{(q_i)}(x, \varepsilon_j) \eta_{x_i}(x) dx - \int_{\Omega} p_j(x) \eta_{x_i}(x) dx \right| + \left| \int_{\Omega} p_j(x) \eta_{x_i}(x) dx - \int_{\Omega} \tilde{g}(x) \eta_{x_i}(x) dx \right|
\]
Letting first $i \to \infty$ and then $j \to \infty$, we get by Lemma 1.7 and Lemma 1.8,
\[
\limsup_{i \to \infty} \left| \int_0^{\tau_2} \int_{\Omega} \frac{u^{(q_i)}(x, \tau)}{q_i} \eta_{x_i} dx d\tau - \int_{\Omega} \tilde{g} \eta_{x_i} dx \right| = 0
\]
(2.4)
\[
\Rightarrow \lim_{i \to \infty} \int_0^{\tau_2} \int_{\Omega} \frac{u^{(q_i)}(x, \tau)}{q_i} \eta_{x_i} dx d\tau = \int_{\Omega} \tilde{g} \eta_{x_i} dx.
\]
Putting $h(u) = u^{q_i}/q_i$, $u = u^{(q_i)}$, in (0.5) and letting $\tau_1 \to 0$, we have
\[
\int_{R^n} u^{(q_i)}(\Delta \eta) dx + \int_{R^n} u^{(q_i)}(x, \tau_2) \eta(x) dx - \int_{R^n} f \eta dx
\]
for all $\eta \in C^\infty_0(\mathbb{R}^n)$, $0 < \tau_2 < 1$. Letting $i \to \infty$, we get by (2.4) and Lebesgue dominated convergence theorem,
\[
\int_0^{\tau_2} \int_{R^n} u^{(\infty)}(\Delta \eta) dx d\tau + \int_{R^n} \tilde{g} \eta_{x_i} dx d\tau = \int_{R^n} u^{(\infty)}(x, \tau_2) \eta(x) dx - \int_{R^n} f \eta dx
\]
for all $\eta \in C^\infty_0(\mathbb{R}^n)$. Letting $\tau_2 \to 0$,
\[
\int \tilde{g} \eta_{x_i} dx = \int g \eta dx - \int f \eta dx \quad \forall \eta \in C^\infty_0(\mathbb{R}^n)
\]
\[
\Rightarrow g + \tilde{g}_{x_i} = f \quad \text{in } \mathcal{D}'(\mathbb{R}^n).
\]
This completes the proof of Theorem 2.1.

We will now let
\[ S(g) = \left\{ x_0 \in \mathbb{R}^n : \lim_{h \to 0} \frac{1}{|B_h(0)|} \int_{B_h(x_0)} |g(x) - g(x_0)| dx = 0 \right\}, \]
\[ G(u^{(\infty)}, g) = \left\{ x \in \mathbb{R}^n : \lim_{t \to 0} u^{(\infty)}(x, t) = g(x) \right\}. \]

**Lemma 2.2.** Let \( f, u^{(\infty)}, u^{(q_i)}, g \) be as in Theorem 2.1 and let \( S^* = S(g) \cap G(u^{(\infty)}, g) \cap \{ g < 1 \} \). If \( x_0 \in S^* \) is such that \( g(x_0) \leq \theta < 1 \), then for any \( \theta_1 \in (\theta, 1) \) and \( \delta > 0 \), there exists \( q_0 > m + 1, \varepsilon_0 > 0, 0 < \varepsilon_0 < 1/2 \), such that
\[ \inf_{|x-x_0| \leq \delta} u^{(q_i)}(x, t) \leq \theta_1 \quad \forall 0 < t < \varepsilon_0, q_i' \geq q_0. \]

**Proof.** The proof is similar to the proof of Theorem 3.3 of [CF]. Suppose the lemma is not true. Then there exists \( \theta_1 \in (\theta, 1), \delta > 0, \) and \( \{ \varepsilon_i \}_{i=1}^\infty, 0 < \varepsilon_i < 1/2, i = 1, 2, \ldots, \varepsilon_i \to 0 \) as \( i \to \infty \) and a subsequence \( \{ u^{(q_i')} \}_{i=1}^\infty \) of \( \{ u^{(q_i)} \}_{i=1}^\infty \) such that
\[ \inf_{|x-x_0| \leq \delta} u^{(q_i')} (x, \varepsilon_i) > \theta_1. \]

Let \( \bar{u}^{(q_i')} \) be the solution of (1.1) in \( \mathbb{R}^n \times (0, 1) \) with initial value \( \bar{u}^{(q_i')} (x, 0) = \theta_1 \chi_{B_\delta(x_0)} \) where \( \chi_{B_\delta(x_0)} \) is the characteristic function of the set \( B_\delta(x_0) \). By Theorem 1.1,
\[ \bar{u}^{(q_i')} (x, t) \leq u^{(q_i')} (x, t + \varepsilon_i) \quad \forall x \in \mathbb{R}^n, 0 < t \leq 1/2 \]
\[ \Rightarrow \int \int \bar{u}^{(q_i')} (x, t) \eta(x, t) dx dt \leq \int \int u^{(q_i')} (x, t + \varepsilon_i) \eta(x, t) dx dt \]
\[ = \int \int u^{(q_i')} (x, t) \eta(x, t - \varepsilon_i) dx dt \]
for all \( \eta \in C_0^\infty(\mathbb{R}^n \times (0, 1/2)) \) and \( \varepsilon_i \) sufficiently small. By Theorem 2.1, \( \{ \bar{u}^{(q_i')} \}_{i=1}^\infty \) has a convergent subsequence converging uniformly on compact subsets of \( \mathbb{R}^n \times (0, 1) \). Without loss of generality, we may assume that \( \{ \bar{u}^{(q_i')} \}_{i=1}^\infty \) converges uniformly on compact subsets of \( \mathbb{R}^n \times (0, 1) \). Let \( \bar{u}^{(\infty)} = \lim_{i \to \infty} \bar{u}^{(q_i')} \). Since \( 0 \leq u^{(q_i')} \leq \theta_1 < 1 \), letting \( i \to \infty \) in (2.5), we get by Lebesgue dominated convergence theorem
\[ \int \int \bar{u}^{(\infty)}(x, t) \eta(x, t) dx dt \leq \int \int u^{(\infty)}(x, t) \eta(x, t) dx dt \]
\[ \Rightarrow \bar{u}^{(\infty)}(x, t) \leq u^{(\infty)}(x, t) \quad \forall x \in \mathbb{R}^n, 0 < t < 1/2 \]
since \( u^{(\infty)}, \bar{u}^{(\infty)} \in C(\mathbb{R}^n \times (0, 1/2)) \)
\[ \Rightarrow \int_{\mathbb{R}^n} \bar{u}^{(\infty)}(x, t) \eta(x) dx \leq \int_{\mathbb{R}^n} u^{(\infty)}(x, t) \eta(x) dx \quad \forall \eta \in C_0^\infty(\mathbb{R}^n) \]
\[ \Rightarrow \int_{\mathbb{R}^n} \theta_1 \chi_{B_\delta(x_0)}(x) dx \leq \int_{\mathbb{R}^n} g(x) \eta(x) dx \quad \text{as } t \to 0 \quad \forall \eta \in C_0^\infty(\mathbb{R}^n) \]
\[ \Rightarrow \theta < \theta_1 \leq g(x_0) \]
since \( x_0 \in S(g) \). Thus contradiction arise and the lemma follows.
Lemma 2.3. Suppose $f \in C^1_0(R^n)$. Let $u^{(\infty)}$, $u^{(q'_i)}$, $g$ be as in Theorem 2.1 and let $S^*$ be as in Lemma 2.2. If $x_0 \in S^*$ is such that $g(x_0) \leq \theta < 1$, then for any $\theta_1 \in (0, 1)$, there exists $q_0 > m + 1$, $\varepsilon_0 \in (0, 1/2)$ depending only on $\theta$, $\theta_1$ and $\|f_{x_k}\|_{L^\infty(R^n)}$, $k = 1, 2, \ldots, n$, such that

$$u^{(q'_i)}(x, t) \leq \theta_1 \quad \forall x \in B_{\delta}(x_0), \ 0 < t \leq \varepsilon_0, \ q'_i \geq q_0,$$

where $\delta = (\theta_1 - \theta)/4(\sqrt{n} \max_{1 \leq k \leq n} \|f_{x_k}\|_{L^\infty(R^n)} + 1)$.

Proof. The proof is similar to the proof of Theorem 2.4 of [H2]. Let

$$\delta = (\theta_1 - \theta)/4(\sqrt{n} \max_{1 \leq k \leq n} \|f_{x_k}\|_{L^\infty(R^n)} + 1).$$

Then by Lemma 2.2, there exists $q_0 > m + 1$, $\varepsilon_0 > 0$, $0 < \varepsilon_0 < 1/2$, such that

$$\inf_{|x-x_0| \leq \delta} u^{(q'_i)}(x, t) \leq \frac{\theta_1 + \theta}{2} \quad \forall 0 < t \leq \varepsilon_0, \ q'_i \geq q_0.$$

Hence for each $q'_i \geq q_0$ and $0 < t \leq \varepsilon_0$, there exists an $x_t \in B_{\delta}(x_0)$ such that

$$u^{(q'_i)}(x_t, t) \leq \frac{\theta_1 + \theta}{2} \quad \forall 0 < t \leq \varepsilon_0.$$

For any $0 < \varepsilon < 1$, let $f_\varepsilon = f + \varepsilon$ and let $u^{(q'_i)}_\varepsilon$ be the solution of (1.1) in $R^n \times (0, 1)$ with $u^{(q'_i)}_\varepsilon(x, 0) = f_\varepsilon(x)$ given by Lemma 1.5. Then by Lemma 1.5,

$$|u^{(q'_i)}_\varepsilon(x, t) - u^{(q'_i)}(x_t, t)| \leq \frac{\theta_1 + \theta}{2} \quad \forall 0 < t \leq \varepsilon_0, \ q'_i \geq q_0.$$

Lemma 2.4. Suppose $f \in C^1_0(R^n)$. Let $g$, $\tilde{g}$ be as in Theorem 2.1 and let $S^*$ be as in Lemma 2.2. Then

$$\begin{cases} g(x) = f(x), \\ \tilde{g}(x) = 0 \end{cases}$$

for all $x \in S^* \cap S(\tilde{g})$.

Proof. Let $u^{(\infty)}$, $u^{(q'_i)}$ be as in Theorem 2.1. By Theorem 2.1 we may assume without loss of generality that $u^{(q'_i)}$ converges uniformly to $u^{(\infty)}$ on compact subsets of $R^n \times (0, 1)$ as $i \to \infty$. We also let $p^{(q'_i)}$, $p_j$, $\varepsilon_j$ be as in Lemma 1.8. Suppose $x_0 \in S^* \cap S(\tilde{g})$. Then there exists $\theta$, $\theta_1 > 0$ such that
$$g(x_0) \leq \theta < \theta_1 < 1.$$ By Lemma 2.3, there exists $q_0 > m + 1$, $\delta > 0$, $\varepsilon_0 > 0$, $0 < \varepsilon_0 < 1/2$ such that

$$u(q')(x, t) \leq \theta_1 \quad \forall x \in B_\delta(x_0), \quad 0 < t \leq \varepsilon_0, \quad q'_i \geq q_0.$$

Hence

$$\left| \int_{R^n} u(q'_i)(x, t)\eta(x)dx - \int_{R^n} f(x)\eta(x)dx \right|$$

$$= \left| \int_0^t \int_{R^n} [u(q'_i)m\Delta\eta + \frac{u(q'_i)q'_i}{q'_i} - \eta_{x_i}]dxdt \right|$$

$$\leq \theta_1^n\|\Delta\eta\|_{L^1(R^n)}t + \frac{\theta_1^n}{q'_i}\|\eta_{x_i}\|_{L^1(R^n)}t \quad \forall q'_i \geq q_0, \quad 0 < t \leq \varepsilon_0, \quad \eta \in C_0^\infty(B_\delta(x_0)).$$

Letting $i \to \infty$,

$$\left| \int_{R^n} u(\infty)(x, t)\eta(x)dx - \int_{R^n} f(x)\eta(x)dx \right|$$

$$\leq \theta_1^n\|\Delta\eta\|_{L^1(R^n)}t + \frac{\theta_1^n}{q'_i}\|\eta_{x_i}\|_{L^1(R^n)}t \quad \forall q'_i \geq q_0, \quad 0 < t \leq \varepsilon_0, \quad \eta \in C_0^\infty(B_\delta(x_0)).$$

Letting $t \to 0$,

$$\int_{R^n} g\eta dx = \int_{R^n} f\eta dx \quad \forall \eta \in C_0^\infty(B_\delta(x_0)) \Rightarrow g(x_0) = f(x_0)$$

since $x_0 \in S(g)$. Similarly

$$\int_{B_\delta(x_0)} p(q'_i)(x, e_j)dx$$

$$= \int_{B_\delta(x_0)} \int_0^{e_j} u(q'_i)q'_i d\tau dx \leq \frac{\theta_1^n}{q'_i}\|B_\delta(x_0)\|e_j \to 0 \text{ as } i \to 0 \quad \forall j = 1, 2, \ldots$$

$$\Rightarrow \int_{B_\delta(x_0)} p_j(x)dx = 0 \quad \text{by Fatou's lemma since } p_j \geq 0$$

$$\Rightarrow \int_{B_\delta(x_0)} \tilde{g}(x)dx = 0 \quad \text{by Fatou's lemma since } \tilde{g} \geq 0$$

$$\Rightarrow \tilde{g} \equiv 0 \text{ on } B_\delta(x_0)$$

$$\Rightarrow \tilde{g}(x_0) = 0 \text{ since } x_0 \in S(\tilde{g}).$$

**Lemma 2.5.** Suppose $f \in C_0^1(R^n)$ and let $g$, $\tilde{g}$ be as in Theorem 2.1. Then there exists $r' > 0$ such that

$$g(x) = f(x), \quad \tilde{g}(x) = 0$$

a.e. $x \in R^n \setminus B_r(0)$

**Proof.** Let $u(\infty)$, $u(q'_i)$ be as in Theorem 2.1, $S^*$ be as in Lemma 2.2 and let $S_1 = S(g) \cap S(\tilde{g}) \cap G(\infty, g)$, $S_2 = S(g) \cap G(u(\infty), g)$. For any $0 < \theta < 1$, $r > 0$, let $A_{\theta, r} = \{x \in R^n \setminus B_r(0) : g(x) \geq \theta\}$. We now fix $\theta$, $\theta_1 \in (0, 1)$ such that $\theta < \theta_1$. Choose a constant $\theta' > 0$ such that $\theta < \theta' < \theta_1$ and let

$$\delta = \min((\theta' - \theta)/4(\sqrt{n} \max_{1 \leq k \leq n} \|f_{x_k}\|_{L^\infty(R^n)} + 1)), 1).$$
Since $g \in L^1(\mathbb{R}^n)$,
\[ \int_{|x| \geq r} g \, dx \to 0 \text{ as } r \to 0. \]
Thus there exists $r_0 > 0$ such that
\[ \int_{|x| \geq r_0} g \, dx \leq \frac{1}{2} \theta |B_\delta(0)| \]
\[ \Rightarrow \theta |A_{\theta, r_0}| \leq \frac{1}{2} \theta |B_\delta(0)| \]
\[ \Rightarrow |A_{\theta, r_0}| \leq \frac{1}{2} |B_\delta(0)|. \]

Let $r' = r_0 + 1$. Since $|\mathbb{R}^n \setminus S_1| = 0$ by the result of [DFK] and Chapter 1 of [S], (2.6) holds for a.e. $x \in A_{\theta, r'}$ by Lemma 2.4. Hence in order to prove the lemma, it suffices to show that (2.6) holds for a.e. $x \in A_{\theta, r'} \cap S_1$. Let $y_0 \in A_{\theta, r'} \cap S_1$. If $|B_\delta(y_0) \cap A_{\theta, r'}| = 0$, then
\[ g(z) > 0 \text{ a.e. } z \in B_\delta(y_0) \Rightarrow |A_{\theta, r_0}| \geq |B_\delta(y_0)| \]
since $B_\delta(y_0) \subset \mathbb{R}^n \setminus B_{r_0}(0)$. This contradicts (2.7). Thus $|B_\delta(y_0) \cap A_{\theta, r'}| \neq 0$.
Since $|(B_\delta(y_0) \cap A_{\theta, r'}) \setminus (B_\delta(y_0) \cap A_{\theta, r'} \cap S_2)| = 0$, $B_\delta(y_0) \cap A_{\theta, r'} \cap S_2 \neq \emptyset$ and there exists $x_0 \in B_\delta(y_0) \cap A_{\theta, r'} \cap S_2 \subset S^*$. By Lemma 2.3, there exists $q_0 > m + 1$ and $\varepsilon_0 > 0$, $0 < \varepsilon_0 < 1/2$, such that
\[ u(q_i(x, t)) \leq \varepsilon' \ \forall x \in B_\delta(x_0), \ 0 < t \leq \varepsilon_0, \ q_i \geq q_0. \]
Letting $i \to \infty$,
\[ u(\infty)(x, t) \leq \varepsilon' \ \forall x \in B_\delta(x_0), \ 0 < t \leq \varepsilon_0 \]
\[ \Rightarrow \int_{\mathbb{R}^n} u(\infty)(x, t) \eta(x) \, dx \leq \varepsilon' \int_{\mathbb{R}^n} \eta(x) \, dx \ \forall \eta \in C_0(B_\delta(x_0)) \]
\[ \Rightarrow \int_{\mathbb{R}^n} g \eta \, dx \leq \varepsilon' \int_{\mathbb{R}^n} \eta \, dx \ \forall \eta \in C_0(B_\delta(x_0)) \quad \text{as } t \to 0 \]
\[ \Rightarrow g(y_0) \leq \varepsilon' < 1 \]
since $y_0 \in S(g) \cap B_\delta(x_0)$. Hence $y_0 \in S^* \cap S(\bar{g})$. Thus (2.6) holds for $x = y_0$ by Lemma 2.4 and the lemma follows.

**Corollary 2.6.** Suppose $f \in C_0^1(\mathbb{R}^n)$ and $\bar{g}$ is as in Theorem 2.1. Then $\bar{g} \in L^1(\mathbb{R}^n)$.

**Proof.** The lemma follows directly from Lemma 2.5 and the fact that $\bar{g} \in L^1_\infty(\mathbb{R}^n)$.

**Lemma 2.7.** For any $0 \leq f_1, f_2, g_1, g_2, \bar{g}_1, \bar{g}_2 \in L^1(\mathbb{R}^n)$, $0 \leq g_1, g_2 \leq 1$, $\bar{g}_1, \bar{g}_2 \geq 0$, if
\[ g_i + (\bar{g}_i)x_1 = f_i \quad \text{in } \mathcal{D}'(\mathbb{R}^n) \]
and
\[ g_i(x) = f_i(x), \ \bar{g}_i(x) = 0 \quad \text{whenever } g_i(x) < 1 \text{ a.e. } x \in \mathbb{R}^n \]
for $i = 1, 2$, then
\[ \int_{|x| \leq R'} \int_{\mathbb{R}^{n-1}} |\bar{g}_1 - \bar{g}_2|(x_1, x') \, dxdx' \leq 2R' \|f_1 - f_2\|_{L^1(\mathbb{R}^n)} \quad \forall R' > 0. \]
Proof. We will use a modification of an argument of [SX]. By (2.8),
\[(g_1 - g_2) + (\bar{g}_1 - \bar{g}_2)_{x_k} = f_1 - f_2 \quad \text{in } B'(R^n)\]
(2.10)
\[\Rightarrow \int_{R^n} \left[ (g_1 - g_2) \eta - (\bar{g}_1 - \bar{g}_2) \eta_{x_k} \right] dx\]
Putting \(\eta(x) = \rho_k(\xi - x)\) in (2.10), we get
(2.11)
\[(\bar{g}_1 - \bar{g}_2)_{x_k}(\xi) = (f_1 - f_2)(\xi) - (g_1, e - g_2, e)(\xi) \quad \forall \xi = (\xi_1, \ldots, \xi_n) \in R^n.
\]
For any \(k = 1, 2, \ldots\), we let \(p_k(\cdot) \in C_0^\infty(R), 0 \leq p_k \leq 1\), be such that \(p_k(x) \equiv 1\) for \(x \geq 1/k\), \(p_k(x) \equiv 0\) for \(x \leq 1/2k\) and \(\|p_{k,x}\|_{L^\infty} \leq 5k\). Then for all \(z_1, y_1 \in R\),
\[
\int_{R^{n-1}} (\bar{g}_1 - \bar{g}_2)(z_1, x')p_k(\bar{g}_1 - \bar{g}_2)(z_1, x')dxdx'\]
\[= \int_{R^{n-1}} \int_{y_1}^{z_1} \frac{\partial}{\partial x_1} \left[ (\bar{g}_1 - \bar{g}_2)(x_1, x')p_k(\bar{g}_1 - \bar{g}_2)(x_1, x') \right] dx_1 dx'dx'\]
\[
= \int_{R^{n-1}} \int_{y_1}^{z_1} \left[ \frac{\partial}{\partial x_1} (\bar{g}_1 - \bar{g}_2) \right] p_k(\bar{g}_1 - \bar{g}_2)
+ (\bar{g}_1 - \bar{g}_2)p_k'(\bar{g}_1 - \bar{g}_2) \frac{\partial}{\partial x_1} (\bar{g}_1 - \bar{g}_2) dxdx'\]
(2.12)
\[
\Rightarrow \int_{R^{n-1}} (\bar{g}_1 - \bar{g}_2)(z_1, x')p_k(\bar{g}_1 - \bar{g}_2)(z_1, x')dxdx'\]
+ \[
\int_{R^{n-1}} \int_{y_1}^{z_1} (g_1 - g_2)(x_1, x')p_k(\bar{g}_1 - \bar{g}_2)(x_1, x')dxdx'\]
\[= \int_{R^{n-1}} (\bar{g}_1 - \bar{g}_2)(y_1, x')p_k(\bar{g}_1 - \bar{g}_2)(y_1, x')dxdx'\]
+ \[
\int_{R^{n-1}} \int_{y_1}^{z_1} (f_1 - f_2)(x_1, x')p_k(\bar{g}_1 - \bar{g}_2)(x_1, x')dxdx'\]
+ \[
\int_{R^{n-1}} \int_{y_1}^{z_1} (g_1 - g_2)p_k'(\bar{g}_1 - \bar{g}_2)dxdx'\]
\[
\cdot [(f_1, e - f_2, e) - (g_1, e - g_2, e)](x_1, x')dxdx'\]
by (2.11). Since \(\bar{g}_1, \bar{g}_2 \in L^1(R^n),\)
\[
\int_{R^n} (\bar{g}_1, e - \bar{g}_2, e) \cdot |p_k(\bar{g}_1, e - \bar{g}_2, e)| dx \leq \int_{R^n} (\bar{g}_1 + \bar{g}_2) dx < \infty.
\]
Hence there exists a sequence \(\{y_1^j\}_{j=1}^\infty \subset R, \ y_1^j \rightarrow -\infty \ \text{as} \ \ j \rightarrow \infty \) such that
\[
\int_{R^{n-1}} (\bar{g}_1, e - \bar{g}_2, e)(y_1^j, x')p_k(\bar{g}_1, e - \bar{g}_2, e)(y_1^j, x')dxdx' \rightarrow 0 \ \ \text{as} \ \ j \rightarrow \infty .
\]
Putting $y_1 = y'_1$ in (2.12) and letting $j \to \infty$, we get

\[
\int_{R^n-1} (\bar{g}_1, e - \bar{g}_2, e)(z_1, x')p_k(\bar{g}_1, e - \bar{g}_2, e)(z_1, x')dx' \\
+ \int_{R^n-1} \int_{-\infty}^{z_1} (g_1, e - g_2, e)(x_1, x')p_k(\bar{g}_1, e - \bar{g}_2, e)(x_1, x')dx_1dx' \\
= \int_{R^n-1} \int_{-\infty}^{z_1} (\bar{g}_1, e - \bar{g}_2, e)p_k'(\bar{g}_1, e - \bar{g}_2, e) \\
\cdot [(f_1, e - f_2, e) - (g_1, e - g_2, e)](x_1, x')dx_1dx' \\
(2.13) + \int_{R^n-1} \int_{-\infty}^{z_1} (f_1, e - f_2, e)(x_1, x')p_k(\bar{g}_1, e - \bar{g}_2, e)(x_1, x')dx_1dx'.
\]

Since $\tilde{g}_1, \tilde{g}_2 \in L^1(R^n)$,

\[
\int_R \left| \int_{R^n-1} (\tilde{g}_1, e - \tilde{g}_2, e)(z_1, x')p_k(\tilde{g}_1, e - \tilde{g}_2, e)(z_1, x')dx' \right| dz_1 \\
= \int_{R^n} |(\tilde{g}_1, e - \tilde{g}_2, e) - (\tilde{g}_1 - \tilde{g}_2)|p_k(\tilde{g}_1, e - \tilde{g}_2, e)dx \\
+ \int_{R^n} (\tilde{g}_1 - \tilde{g}_2) \cdot |p_k(\tilde{g}_1, e - \tilde{g}_2, e) - p_k(\tilde{g}_1 - \tilde{g}_2)|dx \\
\leq \int_{R^n} |\tilde{g}_1, e - \tilde{g}_1|dx + \int_{R^n} |\tilde{g}_2, e - \tilde{g}_2|dx \\
+ \int_{R^n} (\tilde{g}_1 + \tilde{g}_2) \cdot |p_k(\tilde{g}_1, e - \tilde{g}_2, e) - p_k(\tilde{g}_1 - \tilde{g}_2)|dx \\
\to 0 \quad \text{as } e \to 0
\]

by the Lebesgue dominated convergence theorem and Theorem 2 in Chapter 3 of [S]. Hence there exists a sequence $\{e_j\}_{j=1}^{\infty} \subset R, e_j \to 0$ as $j \to \infty$, such that

\[
\int_{R^n-1} (\bar{g}_1, e_j - \bar{g}_2, e_j)(z_1, x')p_k(\bar{g}_1, e_j - \bar{g}_2, e_j)(z_1, x')dx' \\
\to \int_{R^n-1} (\bar{g}_1 - \bar{g}_2)(z_1, x')p_k(\bar{g}_1 - \bar{g}_2)(z_1, x')dx'.
\]
a.e. \( z_1 \in \mathbb{R} \) as \( j \to \infty \). On the other hand,

\[
\left| \int_{R^n-1}^{z_1} \int_{-\infty}^{\infty} (\tilde{g}_1, e - \tilde{g}_2, e) p_k^\prime(\tilde{g}_1, e - \tilde{g}_2, e) \cdot \left[ (g_1, e - g_2, e) - (f_1, e - f_2, e) \right](x_1, x') dx_1 dx' \right|
\]

\[
- \int_{R^n-1}^{z_1} \int_{-\infty}^{\infty} (\tilde{g}_1 - \tilde{g}_2) p_k^\prime(\tilde{g}_1 - \tilde{g}_2) \left[ (g_1 - g_2) - (f_1 - f_2) \right](x_1, x') dx_1 dx' \right|
\]

\[
\leq \int_{R^n} \left[ (\tilde{g}_1, e - \tilde{g}_2, e) p_k^\prime(\tilde{g}_1, e - \tilde{g}_2, e) \right] (g_1, e - g_2, e) - (g_1 - g_2) dx
\]

\[
+ \int_{R^n} \left[ (\tilde{g}_1, e - \tilde{g}_2, e) p_k^\prime(\tilde{g}_1, e - \tilde{g}_2, e) \right] (f_1, e - f_2, e) - (f_1 - f_2) dx
\]

\[
+ \int_{R^n} \left[ (\tilde{g}_1, e - \tilde{g}_2, e) p_k^\prime(\tilde{g}_1, e - \tilde{g}_2, e) \right] (g_1 - g_2) dx
\]

\[
+ \int_{R^n} \left[ (\tilde{g}_1, e - \tilde{g}_2, e) p_k^\prime(\tilde{g}_1, e - \tilde{g}_2, e) \right] (f_1 - f_2) dx
\]

\[
\leq 5 \int_{R^n} \left[ (g_1, e - g_1) + |g_2, e - g_2| + |f_1, e - f_1| + |f_2, e - f_2| \right] dx
\]

\[
+ \int_{R^n} \left[ (\tilde{g}_1, e - \tilde{g}_2, e) p_k^\prime(\tilde{g}_1, e - \tilde{g}_2, e) \right] (g_1 - g_2) dx
\]

\[
\cdot (g_1 + g_2 + f_1 + f_2) dx
\]

\[
\to 0 \quad \text{as } e \to 0
\]

by the Lebesgue dominated convergence theorem since the integrand of the last integral above is bounded by \( 5(g_1 + g_2 + f_1 + f_2) \in L^1(R^n) \) and tends to 0 as \( k \to \infty \). Similarly

\[
\int_{R^n-1}^{z_1} \int_{-\infty}^{\infty} \left( g_1, e - g_2, e \right)(x_1, x') p_k(\tilde{g}_1, e - \tilde{g}_2, e)(x_1, x') dx_1 dx'
\]

\[
\to \int_{R^n-1}^{z_1} \int_{-\infty}^{\infty} \left( g_1 - g_2 \right)(x_1, x') p_k(\tilde{g}_1 - \tilde{g}_2)(x_1, x') dx_1 dx' \quad \text{as } e \to 0
\]

and

\[
\int_{R^n-1}^{z_1} \int_{-\infty}^{\infty} \left( f_1, e - f_2, e \right)(x_1, x') p_k(\tilde{g}_1, e - \tilde{g}_2, e)(x_1, x') dx_1 dx'
\]

\[
\to \int_{R^n-1}^{z_1} \int_{-\infty}^{\infty} \left( f_1 - f_2 \right)(x_1, x') p_k(\tilde{g}_1 - \tilde{g}_2)(x_1, x') dx_1 dx' \quad \text{as } e \to 0.
\]
Putting \( \varepsilon = \varepsilon_j \) in (2.13) and letting \( j \to \infty \), we get
(2.14)
\[
\int_{R^n} (\bar{g}_1 - \bar{g}_2)(z_1, x')p_k(\bar{g}_1 - \bar{g}_2)(z_1, x')dx' \\
+ \int_{R^n} \int_{-\infty}^{z_1} (g_1 - g_2)(x_1, x')p_k(\bar{g}_1 - \bar{g}_2)(x_1, x')dx_1dx' \\
= \int_{R^n} \int_{-\infty}^{z_1} (\bar{g}_1 - \bar{g}_2)p_k(\bar{g}_1 - \bar{g}_2)((f_1 - f_2) - (g_1 - g_2))(x_1, x')dx_1dx' \\
+ \int_{R^n} \int_{-\infty}^{z_1} (f_1 - f_2)(x_1, x')p_k(\bar{g}_1 - \bar{g}_2)(x_1, x')dx_1dx' \\
\leq I_1 + \int_{R^n} (f_1 - f_2)_+ dx \quad \text{a.e. } z_1 \in R
\]
Since \( p_k'(s) = 0 \) for \( s \leq 1/2k \) or \( s \geq 1/k \), \( I_1 \) is bounded by
\[
\int_{R^n} |(\bar{g}_1 - \bar{g}_2)(x)| \|p_k\rceil_{L^\infty} \cdot (g_1 + g_2 + f_1 + f_2)(x) \cdot \chi_{A_k}(x, x')dx \\
\leq \int_{R^n} \frac{1}{k} \cdot 5k \cdot (g_1 + g_2 + f_1 + f_2)(x) \cdot \chi_{A_k}(x)dx \\
\leq 5 \int_{R^n} (g_1 + g_2 + f_1 + f_2)(x) \cdot \chi_{A_k}(x)dx \\
\to 0 \quad \text{as } k \to \infty
\]
by the Lebesgue dominated convergence theorem since \( g_1, g_2, f_1, f_2 \in L^1(R^n) \) and
\[
(g_1 + g_2 + f_1 + f_2)(x)\chi_{A_k}(x) \to 0 \quad \text{as } k \to \infty \text{ a.e. } x \in R^n
\]
where \( A_k = \{ x \in R^n : 1/2k \leq (g_1 - g_2)(x) \leq 1/k \} \). Hence by letting \( k \to \infty \) in (2.14), we get
\[
\int_{R^n} (\bar{g}_1 - \bar{g}_2)_+(z_1, x')(z_1, x')dx' \\
+ \int_{R^n} \int_{-\infty}^{z_1} (g_1 - g_2)(x_1, x')\text{sign}_+(\bar{g}_1 - \bar{g}_2)(x_1, x')dx_1dx' \\
\leq \int_{R^n} (f_1 - f_2)_+ dx \\
\text{a.e. } z_1 \in R. \quad \text{Since } (g_1 - g_2)(x)\text{sign}_+(\bar{g}_1(x) - \bar{g}_2(x)) \geq 0 \text{ a.e. } x \in R^n \text{ by (2.9)},
\]
\[
\int_{R^n} (\bar{g}_1 - \bar{g}_2)_+(z_1, x')(z_1, x')dx' \leq \int_{R^n} (f_1 - f_2)_+ dx \quad \text{a.e. } z_1 \in R \\
\Rightarrow \int_{|x_1| \leq R'} \int_{R^n} (\bar{g}_1 - \bar{g}_2)_+(x_1, x')dx'dx_1 \leq 2R' \int_{R^n} (f_1 - f_2)_+ dx \quad \forall R' > 0.
\]
Similarly
\[
\int_{|x_1| \leq R'} \int_{R^n} (\bar{g}_1 - \bar{g}_2)_-(x_1, x')dx'dx_1 \leq 2R' \int_{R^n} (f_1 - f_2)_- dx \quad \forall R' > 0.
\]
Thus
\[
\int_{|x_1| \leq R'} \int_{R^n} |\bar{g}_1 - \bar{g}_2|(x_1, x')(x_1, x')dx'dx_1 \leq 2R' \int_{R^n} |f_1 - f_2|dx \quad \forall R' > 0.
\]
Corollary 2.8. Let \( 0 \leq f \in L^1(\mathbb{R}^n) \). Then there exists at most one function \( g, g \in L^1(\mathbb{R}^n), 0 \leq g \leq 1 \), and one function \( \bar{g} \in L^1(\mathbb{R}^n), \bar{g} \geq 0 \) satisfying

\[
\begin{align*}
\left\{ \begin{array}{ll}
g + (\bar{g})_x = f & \text{in } \mathcal{D}'(\mathbb{R}^n), \\
g(x) = f(x), \bar{g}(x) = 0 & \text{whenever } g(x) < 1 \text{ a.e. } x \in \mathbb{R}^n.
\end{array} \right.
\]

As a consequence of Theorem 2.1, Lemmas 2.4, 2.5, Corollary 2.8 and the uniqueness theorem (Theorem 6.13) of [DK], we have

**Theorem 2.9.** Suppose \( f \in C^0_0(\mathbb{R}^n) \). Then there exists a unique function \( u^{(\infty)} \in C(\mathbb{R}^n \times (0, 1)), 0 \leq u^{(\infty)} \leq 1 \), such that \( u^{(q)} \) converges uniformly to \( u^{(\infty)} \) on compact subsets of \( \mathbb{R}^n \times (0, 1) \) as \( q \to \infty \). Moreover \( u^{(\infty)} \) satisfies (0.2) with initial value \( g \in L^1(\mathbb{R}^n), 0 \leq g \leq 1 \), satisfying (2.15) and (2.6) for some function \( \bar{g} \in L^1(\mathbb{R}^n), \bar{g} \geq 0 \).

We are now ready to state and prove the main theorem.

**Theorem 2.10.** For any \( m > 1 \) fixed, there exists a unique function \( u^{(\infty)} \in C(\mathbb{R}^n \times (0, 1)), 0 \leq u^{(\infty)} \leq 1 \) such that \( u^{(q)} \) converges weakly to \( u^{(\infty)} \) in \( (L^{\infty}(G))^* \) for any compact subset \( G \) of \( \mathbb{R}^n \times (0, 1) \) as \( q \to \infty \). Moreover \( u^{(\infty)} \) satisfies (0.2) with initial value \( g \in L^1(\mathbb{R}^n), 0 \leq g \leq 1 \), satisfying (2.15) and (2.6) for some function \( \bar{g} \in L^1_{\text{loc}}(\mathbb{R}^n), \bar{g} \geq 0 \). The convergence is uniform on every compact subsets of \( \mathbb{R}^n \times (0, 1) \) if \( f \in C^0_0(\mathbb{R}^n) \).

**Proof.** Since \( f \in L^{\infty}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \), we can choose a sequence \( \{f_j\}_{j=1}^{\infty} \subseteq C^0_0(\mathbb{R}^n) \) such that \( \|f_j\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^\infty(\mathbb{R}^n)} + 1 \), \( \|f_j\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)} + 1 \) for all \( j = 1, 2, \ldots \) and \( \|f_j - f\|_{L^1(\mathbb{R}^n)} \to 0 \) as \( j \to \infty \).

For all \( j = 1, 2, \ldots \), let \( u^{(q)}_j \) be the solution of (1.1) in \( \mathbb{R}^n \times (0, 1) \) with initial value \( u^{(q)}_j(x, 0) = f_j(x) \). By Theorem 2.9, for each \( j = 1, 2, \ldots \), there exists an unique function \( u^{(\infty)}_j \) such that \( u^{(q)}_j \) converges uniformly on compact subsets of \( \mathbb{R}^n \times (0, 1) \) to \( u^{(\infty)}_j \) as \( q \to \infty \). Moreover \( u^{(\infty)}_j \) satisfies (0.2) with initial value \( g_j \in L^1(\mathbb{R}^n), 0 \leq g_j \leq 1 \), satisfying

\[
\begin{align*}
\left\{ \begin{array}{ll}
\int_{\mathbb{R}^n} g_j \leq \int_{\mathbb{R}^n} f_j \leq \int_{\mathbb{R}^n} f \, dx + 1, \\
g_j + (\tilde{g}_j)_x = f & \text{in } \mathcal{D}'(\mathbb{R}^n) \text{ for some } \tilde{g}_j \in L^1(\mathbb{R}^n), \tilde{g}_j \geq 0, \\
g_j(x) = f(x), \tilde{g}_j(x) = 0 & \text{whenever } g_j(x) < 1 \text{ a.e. } x \in \mathbb{R}^n.
\end{array} \right.
\]

for all \( j = 1, 2, \ldots \). Since \( \|u^{(q)}_j\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^\infty(\mathbb{R}^n)} \), any sequence \( \{u^{(q)}_i\}_{i=1}^{\infty} \), \( q_i \to \infty \) as \( i \to \infty \), of \( \{u^{(q)}_j\}_{j=1}^{\infty} \) has a subsequence \( \{u^{(q)}_{i_j}\}_{j=1}^{\infty} \) such that \( \{u^{(q)}_{i_j}\}_{j=1}^{\infty} \) converges weakly in \( (L^{\infty}(G))^* \) for any compact subset \( G \) of \( \mathbb{R}^n \times (0, 1) \) as \( i \to \infty \). Let \( u^{(\infty)} = \lim_{i \to \infty} u^{(q)}_{i_j} \). Without loss of generality we may assume that \( u^{(q)}_j(x, t) \to u^{(\infty)}(x, t) \) a.e. \((x, t) \in \mathbb{R}^n \times (0, 1)\) as \( i \to \infty \). By Theorem 1.4,

\[
\int_{\mathbb{R}^n} |u^{(q)}_j(x, t) - u^{(q)}_{i_j}(x, t)| \, dx \leq \int_{\mathbb{R}^n} |f_j - f_i|(x) \, dx \quad \forall i, j = 1, 2, \ldots
\]

\[
\Rightarrow \int_{\tau_1}^{\tau_2} \int_{\mathbb{R}^n} |u^{(q)}_j - u^{(q)}_{i_j}|(x, t) \, dx \, dt
\]

\[
\leq (\tau_2 - \tau_1) \int_{\mathbb{R}^n} |f_j - f_i|(x) \, dx \quad \forall 0 < \tau_1 \leq \tau_2 < 1.
\]
Letting $i \to \infty$, we get by Fatou's lemma,
\[
\int_{\tau_1}^{\tau_2} \int_{\mathbb{R}^n} |u^{(\infty)}_j(x, t) - u^{(\infty)}(x, t)| \, dx \, dt \leq (\tau_2 - \tau_1) \int_{\mathbb{R}^n} |f_j - f|(x) \, dx \to 0 \text{ as } j \to \infty
\]
for all $0 < \tau_1 \leq \tau_2 < 1$.

Hence $u^{(\infty)}$ is the limit of the functions \( \{u^{(\infty)}_j\}_{j=1}^\infty \) in \( L^1_{\text{loc}}(\mathbb{R}^n \times (0, 1)) \) as $j \to \infty$. Thus $u^{(\infty)}$ is unique and $u^{(q)}$ converges weakly to $u^{(\infty)}$ in $(L^\infty(G))^*$ for any compact subset $G$ of $\mathbb{R}^n \times (0, 1)$ as $q \to \infty$. This together with Theorem 2.1 implies that $u^{(q)}$ converges uniformly to $u^{(\infty)}$ on every compact subsets of $\mathbb{R}^n \times (0, 1)$ as $q \to \infty$ if $f \in C_0(\mathbb{R}^n)$.

Moreover \( \{u^{(\infty)}_j\}_{j=1}^\infty \) has a subsequence converging a.e. $(x, t) \in \mathbb{R}^n \times (0, 1)$ to $u^{(\infty)}$. Without loss of generality we may assume that $u^{(\infty)}_j(x, t) \to u^{(\infty)}(x, t)$ a.e. $(x, t) \in \mathbb{R}^n \times (0, 1)$ as $j \to \infty$.

On the other hand since $u^{(\infty)}_j$ satisfies (0.2) and
\[
|u^{(\infty)}_j(x, t)| \leq \|f_j\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^\infty(\mathbb{R}^n)} + 1 \quad \forall (x, t) \in \mathbb{R}^n \times (0, 1),
\]
(2.17)
\[
\Rightarrow \|u^{(\infty)}_j\|_{L^\infty(\mathbb{R}^n \times (0, 1))} \leq \|f\|_{L^\infty(\mathbb{R}^n)} + 1 \quad \forall j = 1, 2, \ldots,
\]
as $q \to \infty$ by Theorem 1.3, by the result of [S1] \( \{u^{(\infty)}_j\}_{j=1}^\infty \) has a subsequence \( \{u^{(\infty)}_k\}_{k=1}^\infty \) converging uniformly on compact subsets of $\mathbb{R}^n \times (0, 1)$. Hence we may assume without loss of generality that \( \{u^{(\infty)}_j\}_{j=1}^\infty \) converges uniformly on compact subsets of $\mathbb{R}^n \times (0, 1)$ to $u^{(\infty)}$. Thus $u^{(\infty)} \in C(\mathbb{R}^n \times (0, 1))$.

Putting $h(u) = 0$, $u = u^{(\infty)}$ in (0.5) and letting $j \to \infty$, we see that $u^{(\infty)}$ satisfies (0.2). By (2.17) and the result of [DK], $u^{(\infty)}$ has an initial trace $d\mu$ and $d\mu$ is absolutely continuous with respect to the Lebesgue measure. Hence $d\mu = g(x) \, dx$ for some $g \geq 0$, $g \in L^1(\mathbb{R}^n)$. By (2.16) and Lemma 2.7,
\[
\int_{|x_1| \leq R'} \int_{R^n-1} |\tilde{g}_j - \tilde{g}_{j'}|(x_1, x') \, dx' \, dx_1 \leq 2R' \|f_j - f_{j'}\|_{L^1(\mathbb{R}^n)} \to 0
\]
as $j, j' \to \infty$ \quad $\forall R' > 0$.

Hence \( \{	ilde{g}_j\}_{j=1}^\infty \) is a Cauchy sequence in $L^1_{\text{loc}}(\mathbb{R}^n)$ and there exists $\tilde{g} \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that $\tilde{g}_j \to \tilde{g}$ in $L^1_{\text{loc}}(\mathbb{R}^n)$ as $j \to \infty$. Without loss of generality we may assume that \( \tilde{g}_j(x) \to \tilde{g}(x) \) a.e. $x \in \mathbb{R}^n$. By the proof of Theorem 2.1, $u^{(\infty)}_j$ satisfies, for all $\eta \in C_0^\infty(\mathbb{R}^n)$, $0 < \tau_2 < 1$,
\[
\int_0^{\tau_2} \int_{\mathbb{R}^n} u^{(\infty)}_j \Delta \eta \, dx \, dt + \int_{\mathbb{R}^n} \tilde{g}_j \eta x_1 \, dx = \int_{\mathbb{R}^n} u^{(\infty)}_j(x, t) \eta(x) \, dx - \int_{\mathbb{R}^n} f_j \eta \, dx
\]
\[
\Rightarrow \int_0^{\tau_2} \int_{\mathbb{R}^n} u^{(\infty)} \Delta \eta \, dx \, dt + \int_{\mathbb{R}^n} \tilde{g} \eta x_1 \, dx
\]
\[
= \int_{\mathbb{R}^n} u^{(\infty)}(x, t) \eta(x) \, dx - \int_{\mathbb{R}^n} f \eta \, dx \quad \text{as } j \to \infty
\]
\[
\Rightarrow \int_{\mathbb{R}^n} \tilde{g} \eta x_1 \, dx = \int_{\mathbb{R}^n} g(x) \eta(x) \, dx - \int_{\mathbb{R}^n} f \eta \, dx \quad \text{as } \tau_2 \to 0
\]
\[
\Rightarrow g + \tilde{g}_j = f \quad \text{in } \mathcal{D}'(\mathbb{R}^n).
Thus

\[ \left| \int (g - g_j) \eta \, dx \right| = \left| \int (\tilde{g} - \tilde{g}_j) \eta \, dx + \int (f - f_j) \eta \, dx \right| \]

\[ \leq \| \eta \|_{L^\infty(\mathbb{R}^n)} \int |\eta|_{L^\infty(\mathbb{R}^n)} \int_{|x| \leq R'} \int_{\mathbb{R}^n-1} |\tilde{g} - \tilde{g}_j|(x_1, x') \, dx' \, dx_1 \]

\[ + \| \eta \|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} |f - f_j| \, dx \]

\[ \rightarrow 0 \]

as \( j \rightarrow \infty \) for all \( \eta \in C_0^\infty(\mathbb{R}^n) \) such that \( \text{supp} \ \eta \subset B_{R'}(0) \) for some \( R' > 0 \).

Hence \( g_j \) converges weakly to \( g \) in \( \mathcal{D}'(\mathbb{R}^n) \) as \( j \rightarrow \infty \). We may assume without loss of generality that \( g_j(x) \rightarrow g(x) \) and \( \tilde{g}_j(x) \rightarrow \tilde{g}(x) \) a.e. \( x \in \mathbb{R}^n \).

Let

\[ E = \{ x \in \mathbb{R}^n : g_j(x) \rightarrow g(x) \text{ and } \tilde{g}_j \rightarrow \tilde{g}(x) \text{ as } j \rightarrow \infty \}, \]

\[ E_0 = E \cap \{ g < 1 \} \cap \left( \bigcap_{j=1}^{\infty} (S(g_j) \cap S(\tilde{g}_j) \cap G(u_j^{(\infty)}, g_j)) \right). \]

For any \( x_0 \in E_0 \), since \( g_j(x_0) \rightarrow g(x_0) \) as \( j \rightarrow \infty \), there exists \( j_0 \in \mathbb{Z}^+ \) such that \( g_j(x_0) < 1 \ \forall j \geq j_0 \). So \( g_j(x_0) = f(x_0) \) and \( \tilde{g}_j(x_0) = 0 \) for all \( j \geq j_0 \) by Lemma 2.4. Letting \( j \rightarrow \infty \), we have \( g(x_0) = f(x_0) \) and \( \tilde{g}(x_0) = 0 \). Since \( |\{ g < 1 \} \setminus E_0| = 0 \), \( g(x_0) = f(x_0) \) and \( \tilde{g}(x_0) = 0 \) a.e. \( x_0 \in \{ g < 1 \} \) and the theorem follows.

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