SOME INEQUALITIES OF ALGEBRAIC POLYNOMIALS
WITH NONNEGATIVE COEFFICIENTS

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ABSTRACT. Let \( S_n \) be the collection of all algebraic polynomials of degree \( \leq n \) with nonnegative coefficients. In this paper we discuss the extremal problem

\[
\sup_{p_n(x) \in S_n} \frac{\int_a^b (p'_n(x))^2 \omega(x) \, dx}{\int_a^b p_n^2(x) \omega(x) \, dx}
\]

where \( \omega(x) \) is a positive and integrable function. This problem is solved completely in the cases

(i) \([a, b] = [-1, 1], \, \omega(x) = (1 - x^2)^\alpha, \, \alpha > -1;\)
(ii) \([a, b] = [0, \infty), \, \omega(x) = x^\alpha e^{-x}, \, \alpha > -1;\)
(iii) \((a, b) = (-\infty, \infty), \, \omega(x) = e^{-ax^2}, \, \alpha > 0.\)

The second case was solved by Varma for some values of \( \alpha \) and by Milovanović completely. We provide a new proof here in this case.

1. INTRODUCTION

In this paper we investigate the following extremal problem

\[
\sup_{p_n(x) \in S_n} \frac{\int_a^b (p'_n(x))^2 \omega(x) \, dx}{\int_a^b p_n^2(x) \omega(x) \, dx}
\]

where

\[
S_n = \left\{ p_n(x): p_n(x) = \sum_{i=0}^n a_i x^i, \, a_i \geq 0, \, 0 \leq i \leq n \right\},
\]

and \( \omega(x): (a, b) \to \mathbb{R} \) is a positive and integrable function.

In the case \([a, b] = [0, \infty), \, \omega(x) = x^\alpha e^{-x}, \, \alpha > -1 \), the extremal problem (1) was initiated and solved by Varma [10] in the cases \( 0 \leq \alpha \leq 1/2 \) and \((\sqrt{5} - 1)/2 \leq \alpha < \infty. \) Later, it was solved completely by Milovanović [4] for \(-1 \leq \alpha < \infty. \)

In this note we consider the above extremal problem (1) for different weight functions on different intervals. Throughout this paper, we denote \( S_n \) the collection of all algebraic polynomials of degree \( \leq n \) with nonnegative coefficients.

In Section 2, we provide the complete answer to the case \([a, b] = [-1, 1], \, \omega(x) = (1 - x^2)^\alpha, \, \alpha > -1. \) In the case \( \alpha = 0 \), this result is an analogue of a
Theorem of Lorentz [3] in the $L_\infty$ norm. Indeed, that theorem holds for a wider class (Lorentz class) of polynomials, which was studied extensively by Scheick [7]. For some subsets of Lorentz class of polynomials, the extremal problem (1) was discussed by Milovanović and Petković [5] for the Jacobi weight.

In Section 3, we give a new proof of Milovanović's Theorem [4]. In our last section, Section 4, we consider the weight function $\omega(x) = e^{-\alpha x^2}$, $\alpha > 0$, on the interval $(-\infty, \infty)$.

The corresponding extremal problem for the unrestricted polynomials was discussed in Dörfler [1], [2], Mirsky [6] and Turán [8], which are Markov type inequalities in $L_2$ norm.

2. THE WEIGHT $\omega(x) = (1 - x^2)^\alpha$

In this section, we discuss the extremal problem in the $L_2$ norm under the weight function $\omega(x) = (1 - x^2)^\alpha$, $\alpha > -1$, on $[-1, 1]$. For some special values of $\alpha$, we obtain several corollaries corresponding to some classic weight functions. The main result in this section is the following theorem.

**Theorem 2.1.** Let $p_n(x) \in S_n$, $\alpha > -1$; then

$$
(2) \quad \int_{-1}^{1} (p'_n(x))^2(1 - x^2)^\alpha \, dx \leq \frac{2n + 2\alpha + 1}{2n - 1} n^2 \int_{-1}^{1} p_n^2(x)(1 - x^2)^\alpha \, dx
$$

with equality when $p_n(x) = x^n$.

**Proof.** Since $p_n(x) \in S_n$, we can write

$$
p_n(x) = \sum_{i=0}^{n} a_i x^i
$$

with $a_i \geq 0$, $0 \leq i \leq n$. Then

$$
p'_n(x) = \sum_{i=1}^{n} ia_i x^{i-1}
$$

and

$$
\int_{-1}^{1} p_n^2(x)(1 - x^2)^\alpha \, dx = \sum_{i=0}^{n} \sum_{j=0}^{n} a_i a_j \int_{-1}^{1} x^{i+j}(1 - x^2)^\alpha \, dx,
$$

$$
\int_{-1}^{1} (p'_n(x))^2(1 - x^2)^\alpha \, dx = \sum_{i=0}^{n} \sum_{j=0}^{n} a_i a_j i j \int_{-1}^{1} x^{i+j-2}(1 - x^2)^\alpha \, dx.
$$

Let

$$
b_{ij} = \int_{-1}^{1} x^{i+j}(1 - x^2)^\alpha \, dx
$$

$$
= \frac{1 - (-1)^{i+j+1}}{2} B \left( \frac{i + j + 1}{2}, \alpha + 1 \right)
$$

where $B(x, y)$ is the Beta function and

$$
c_{ij} = i j \int_{-1}^{1} x^{i+j-2}(1 - x^2)^\alpha \, dx
$$

$$
= i j \frac{1 - (-1)^{i+j+1}}{2} B \left( \frac{i + j - 1}{2}, \alpha + 1 \right)
$$
for $1 \leq i, j \leq n$, $c_{ij} = 0$ if $i = 0$ or $j = 0$. Now denote

$$B = (b_{ij})_{0 \leq i, j \leq n}, \quad C = (c_{ij})_{0 \leq i, j \leq n},$$

and

$$a = (a_0, a_1, \ldots, a_n)^T;$$

then we can derive that

$$\int_{-1}^{1} p_n^2(x)(1 - x^2)\alpha dx = \sum_{i=0}^{n} \sum_{j=0}^{n} a_i a_j b_{ij} = a^T B a,$$

$$\int_{-1}^{1} (p_n'(x))^2(1 - x^2)\alpha dx = \sum_{i=0}^{n} \sum_{j=0}^{n} a_i a_j c_{ij} = a^T C a.$$

Now it suffices to consider the following extremal problem:

$$\sup_{a \in R^{n+1}_+} \frac{a^T C a}{a^T B a}$$

where $R^{n+1}_+ = \{a: a = (a_0, a_1, \ldots, a_n)^T, a_i \geq 0, 0 \leq i \leq n\}$. Or find the least $\lambda$ such that

$$\frac{a^T C a}{a^T B a} \leq \lambda, \quad \text{for all } a \in R^{n+1}_+,$$

which is

$$\lambda (\lambda B - C) a \geq 0, \quad \text{for all } a \in R^{n+1}_+.$$

Observe that $b_{ij} \geq 0$, $c_{ij} \geq 0$, $0 \leq i, j \leq n$. If we can find a smallest $\lambda$ such that all the elements of $\lambda B - C$ are nonnegative, then we obtain (4) automatically. Notice also that the matrices $B$ and $C$ have the same structure; thus it suffices to find $\lambda$ such that

$$\lambda b_{ij} - c_{ij} \geq 0, \quad \text{when } b_{ij} \neq 0,$$

i.e.,

$$\lambda \geq \frac{c_{ij}}{b_{ij}} = \frac{ij(i + j + 2\alpha + 1)}{i + j - 1}, \quad 1 \leq i, j \leq n.$$

If we consider $c_{ij}/b_{ij}$ as a function of two continuous variables $i$ and $j$, then we have

$$\partial_i \left( \frac{ij(i + j + 2\alpha + 1)}{i + j - 1} \right) = \frac{j[i^2 + (j - 1)(2i + j + 2\alpha + 1)]}{(i + j - 1)^2} \geq 0$$

and similarly

$$\partial_j \left( \frac{ij(i + j + 2\alpha + 1)}{i + j - 1} \right) = \frac{i[j^2 + (i - 1)(2j + i + 2\alpha + 1)]}{(i + j - 1)^2} \geq 0;$$

thus this is an increasing function of $i$ and $j$, and we can pick up

$$\lambda = \frac{ij(i + j + 2\alpha + 1)}{i + j - 1} \bigg|_{i=n, j=n} = \frac{2n + 2\alpha + 1}{2n - 1} n^2.$$

To see that $\lambda$ is the best one, we can consider $p_n(x) = x^n$ or $a^T = (0, 0, \ldots, 0, 1)$. This completes the proof of the theorem. \hfill \square

For some special values of $\alpha$, we have the following corollaries.
Corollary 2.2. Let $p_n(x) \in S_n$; then
\[ \int_{-1}^{1} (p'_n(x))^2 \, dx \leq \frac{2n+1}{2n-1} n^2 \int_{-1}^{1} p_n^2(x) \, dx \]
with equality when $p_n(x) = x^n$.

Corollary 2.3. Let $p_n(x) \in S_n$; then
\[ \int_{-1}^{1} (p'_n(x))^2 (1-x^2)^{-1/2} \, dx \leq \frac{2n}{2n-1} n^2 \int_{-1}^{1} p_n^2(x)(1-x^2)^{-1/2} \, dx \]
with equality when $p_n(x) = x^n$.

Corollary 2.4. Let $p_n(x) \in S_n$; then
\[ \int_{-1}^{1} (p'_n(x))^2 (1-x^2)^{-1/2} \, dx \leq \frac{2n+2}{2n-1} n^2 \int_{-1}^{1} p_n^2(x)(1-x^2)^{-1/2} \, dx \]
with equality when $p_n(x) = x^n$.

In the case $\alpha = 1$, a similar result was proved by Varma [9] for polynomials having real roots.

3. The weight $\omega(x) = x^\alpha e^{-x}$

We give a new proof of Milovanović’s Theorem [4] in this section. Indeed we use the same argument as was used in the proof of Theorem 2.1. This time, we consider the weight function $\omega(x) = x^\alpha e^{-x}$, $\alpha > -1$, on the interval $[0, \infty)$.

Theorem 3.1. Let $p_n(x) \in S_n$, $\alpha > -1$; then
\[ \int_{0}^{\infty} (p'_n(x))^2 x^\alpha e^{-x} \, dx \leq C_n(\alpha) \int_{0}^{\infty} p_n^2(x) x^\alpha e^{-x} \, dx \]
where
\[ C_n(\alpha) = \begin{cases} 1/[(2 + \alpha)(1 + \alpha)], & -1 < \alpha \leq \alpha_n, \\ n^2/[(2n + \alpha)(2n + \alpha - 1)], & \alpha_n \leq \alpha < \infty, \end{cases} \]
and
\[ \alpha_n = \frac{1}{2} (n + 1)^{-1} [(2n^2 + 2n + 1)^{1/2} - 3n + 1]. \]
Moreover, $C_n(\alpha)$ is the best possible constant.

Proof. Let $p_n(x) = \sum_{i=0}^{n} a_i x^i$, $a_i \geq 0$, $0 \leq i \leq n$, then
\[ \int_{0}^{\infty} p_n^2(x) x^\alpha e^{-x} \, dx = \sum_{i=0}^{n} \sum_{j=0}^{n} a_i a_j \int_{0}^{\infty} x^{i+j+\alpha} e^{-x} \, dx = \sum_{i=0}^{n} \sum_{j=0}^{n} a_i a_j b_{ij} = a^T B a \]
where
\[ b_{ij} = \int_{0}^{\infty} x^{i+j+\alpha} e^{-x} \, dx = \Gamma(i + j + \alpha + 1), \]
\[ B = (b_{ij})_{0 \leq i, j \leq n}. \]
And similarly, we have
\[ \int_0^\infty (p_n'(x))^2 x^\alpha e^{-x} dx = \sum_{i=0}^n \sum_{j=0}^n a_i a_j c_{ij} = a^T Ca \]
where
\[ c_{ij} = \begin{cases} ij \Gamma(i + j + \alpha - 1), & 1 \leq i, j \leq n, \\ 0, & i = 0 \text{ or } j = 0, \end{cases} \]
\[ C = (c_{ij})_{0 \leq i, j \leq n}. \]

Therefore, we need to find the least \( \lambda \) such that
\[ \lambda b_{ij} - c_{ij} \geq 0, \quad \text{for } 1 \leq i, j \leq n. \]
That is, the maximum value of the function
\[ f(i, j) := \frac{c_{ij}}{b_{ij}} = \frac{ij}{(i + j + \alpha)(i + j + \alpha - 1)}. \]
Let \( k = i + j \); then
\[ f(i, j) = \frac{ij}{(i + j + \alpha)(i + j + \alpha - 1)} = \frac{i(k - i)}{(k + \alpha)(k + \alpha - 1)} =: g(i, k). \]

If we consider \( g \) as a function of two continuous variables \( i \) and \( k \), then we have
\[ \frac{\partial g(i, k)}{\partial i} = \frac{k - 2i}{(k + \alpha)(k + \alpha - 1)}. \]
Therefore, \( g(i, k) \) takes on its maximum value at \( i = k/2 \) if we fix \( k \) (consider it as a function of \( i \) alone). Now it suffices to consider the maximum value of the function
\[ h(k) := g \left( \frac{k}{2}, k \right) = \frac{k^2}{4(k + \alpha)(k + \alpha - 1)}. \]
Following the exactly same argument of Milovanović [4, p. 425], we can see that the best possible value of \( \lambda \) is \( C_n(\alpha) \). We omit the details. This completes the proof. \( \square \)

Remark. The same idea also seems to work for other \( L_p \) norms when \( p \) is an integer, but they become more and more complicated as \( p \) is bigger and bigger. We will not formulate them here. However, for the \( L_1 \) norm, the result is simple.

Theorem 3.2. Let \( p_n(x) \in S_n, \alpha > -1; \) then
\[ \int_0^\infty p_n'(x) x^\alpha e^{-x} dx \leq \lambda_n(\alpha) \int_0^\infty p_n(x) x^\alpha e^{-x} dx \]
where
\[ \lambda_n(\alpha) = \begin{cases} 1/(1 + \alpha), & -1 < \alpha \leq 0, \\ n/(n + \alpha), & 0 \leq \alpha < \infty. \end{cases} \]
Moreover, \( \lambda_n(\alpha) \) is the best possible constant.
4. **The weight** $\omega(x) = e^{-ax^2}$

In this section we discuss the weight function $\omega(x) = e^{-ax^2}$, $\alpha > 0$, on the whole real line. The corresponding result is the following theorem.

**Theorem 4.1.** Let $p_n(x) \in S_n$, $\alpha > 0$; then

\[
\int_{-\infty}^{\infty} (p_n'(x))^2 e^{-ax^2} \, dx \leq \frac{2\alpha}{2n-1} n^2 \int_{-\infty}^{\infty} p_n^2(x) e^{-ax^2} \, dx
\]

with equality when $p_n(x) = x^n$.

**Proof.** Let $p_n(x) = \sum_{i=0}^{n} a_i x^i \in S_n$; then

\[
\int_{-\infty}^{\infty} p_n^2(x) e^{-ax^2} \, dx = \sum_{i=0}^{n} \sum_{j=0}^{n} a_i a_j b_{ij} = a^T B a
\]

where

\[
b_{ij} = \int_{-\infty}^{\infty} x^{i+j} e^{-ax^2} \, dx = (1 - (-1)^{i+j+1})(i+j)! 2^{-(i+j)/2-1} \alpha^{-(i+j+1)/2} \sqrt{\pi} ,
\]

and

\[
b = (b_{ij})_{0 \leq i, j \leq n},
\]

and

\[
\int_{-\infty}^{\infty} (p_n'(x))^2 e^{-ax^2} \, dx = \sum_{i=0}^{n} \sum_{j=0}^{n} a_i a_j c_{ij} = a^T C a
\]

where

\[
c_{ij} = i j \int_{-\infty}^{\infty} x^{i+j-2} e^{-ax^2} \, dx = (1 - (-1)^{i+j+1}) i j (i+j-3)!! 2^{-(i+j)/2} \alpha^{-(i+j-1)/2} \sqrt{\pi} ,
\]

and

\[
c = (c_{ij})_{0 \leq i, j \leq n}.
\]

For $i + j$ even, let

\[
f(i, j) := \frac{c_{ij}}{b_{ij}} = 2\alpha \frac{i j}{i+j-1} , \quad 1 \leq i, j \leq n;
\]

then considering $f$ as a function of two continuous variables $i$ and $j$, we can obtain

\[
\frac{\partial f(i, j)}{\partial i} = \frac{2\alpha j (j-1)}{(i+j-1)^2} \geq 0 \quad \text{for } 1 \leq i, j \leq n,
\]

and

\[
\frac{\partial f(i, j)}{\partial j} = \frac{2\alpha i (i-1)}{(i+j-1)^2} \geq 0 , \quad \text{for } 1 \leq i, j \leq n.
\]

Therefore, $f(i, j)$ attains its maximum value at $i = n$, $j = n$, which implies the desired result. $\square$

**Added in Proof.** After this manuscript was written, the author learned that Professor A. K. Varma [11] had written a paper on the same subject. There are some overlaps between his results and our results in §§2 and 3, but we do use different methods.
REFERENCES


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