SOME INEQUALITIES OF ALGEBRAIC POLYNOMIALS
WITH NONNEGATIVE COEFFICIENTS

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Abstract. Let $S_n$ be the collection of all algebraic polynomials of degree $\leq n$ with nonnegative coefficients. In this paper we discuss the extremal problem

$$\sup_{p_n(x) \in S_n} \frac{\int_a^b (p'_n(x))^2 \omega(x) \, dx}{\int_a^b p_n^2(x) \omega(x) \, dx}$$

where $\omega(x)$ is a positive and integrable function. This problem is solved completely in the cases

(i) $[a, b] = [-1, 1], \, \omega(x) = (1 - x^2)^\alpha, \, \alpha > -1$;
(ii) $[a, b) = [0, \infty), \, \omega(x) = x^\alpha e^{-x}, \, \alpha > -1$;
(iii) $(a, b) = (-\infty, \infty), \, \omega(x) = e^{-ax^2}, \, \alpha > 0$.

The second case was solved by Varma for some values of $\alpha$ and by Milovanović completely. We provide a new proof here in this case.

1. Introduction

In this paper we investigate the following extremal problem

$$\sup_{p_n(x) \in S_n} \frac{\int_a^b (p'_n(x))^2 \omega(x) \, dx}{\int_a^b p_n^2(x) \omega(x) \, dx}$$

where

$$S_n = \left\{ p_n(x): p_n(x) = \sum_{i=0}^n a_i x^i, \, a_i \geq 0, \, 0 \leq i \leq n \right\},$$

and $\omega(x): (a, b) \to \mathbb{R}$ is a positive and integrable function.

In the case $[a, b) = [0, \infty), \, \omega(x) = x^\alpha e^{-x}, \, \alpha > -1$, the extremal problem (1) was initiated and solved by Varma [10] in the cases $0 \leq \alpha \leq 1/2$ and $(\sqrt{5} - 1)/2 \leq \alpha < \infty$. Later, it was solved completely by Milovanović [4] for $-1 < \alpha < \infty$.

In this note we consider the above extremal problem (1) for different weight functions on different intervals. Throughout this paper, we denote $S_n$ the collection of all algebraic polynomials of degree $\leq n$ with nonnegative coefficients. In Section 2, we provide the complete answer to the case $[a, b] = [-1, 1], \, \omega(x) = (1 - x^2)^\alpha, \, \alpha > -1$. In the case $\alpha = 0$, this result is an analogue of a...
theorem of Lorentz [3] in the $L_{\infty}$ norm. Indeed, that theorem holds for a wider class (Lorentz class) of polynomials, which was studied extensively by Scheick [7]. For some subsets of Lorentz class of polynomials, the extremal problem (1) was discussed by Milovanović and Petković [5] for the Jacobi weight.

In Section 3, we give a new proof of Milovanović's Theorem [4]. In our last section, Section 4, we consider the weight function $\omega(x) = e^{-\alpha x^2}$, $\alpha > 0$, on the interval $(-\infty, \infty)$.

The corresponding extremal problem for the unrestricted polynomials was discussed in Dörfler [1], [2], Mirsky [6] and Turán [8], which are Markov type inequalities in $L_2$ norm.

2. The weight $\omega(x) = (1 - x^2)^\alpha$

In this section, we discuss the extremal problem in the $L_2$ norm under the weight function $\omega(x) = (1 - x^2)^\alpha$, $\alpha > -1$, on $[-1, 1]$. For some special values of $\alpha$, we obtain several corollaries corresponding to some classic weight functions. The main result in this section is the following theorem.

**Theorem 2.1.** Let $p_n(x) \in S_n$, $\alpha > -1$; then

$$\int_{-1}^{1} (p'_n(x))^2 (1 - x^2)^\alpha \, dx \leq \frac{2n + 2\alpha + 1}{2n - 1} n^2 \int_{-1}^{1} p_n^2(x) (1 - x^2)^\alpha \, dx$$

with equality when $p_n(x) = x^n$.

**Proof.** Since $p_n(x) \in S_n$, we can write

$$p_n(x) = \sum_{i=0}^{n} a_i x^i$$

with $a_i \geq 0$, $0 \leq i \leq n$. Then

$$p'_n(x) = \sum_{i=1}^{n} i a_i x^{i-1}$$

and

$$\int_{-1}^{1} p_n^2(x) (1 - x^2)^\alpha \, dx = \sum_{i=0}^{n} \sum_{j=0}^{n} a_i a_j \int_{-1}^{1} x^{i+j} (1 - x^2)^\alpha \, dx,$$

$$\int_{-1}^{1} (p'_n(x))^2 (1 - x^2)^\alpha \, dx = \sum_{i=0}^{n} \sum_{j=0}^{n} a_i a_j i j \int_{-1}^{1} x^{i+j-2} (1 - x^2)^\alpha \, dx.$$

Let

$$b_{ij} = \int_{-1}^{1} x^{i+j} (1 - x^2)^\alpha \, dx$$

$$= \frac{1 - (-1)^{i+j+1}}{2} B \left( \frac{i + j + 1}{2}, \alpha + 1 \right)$$

where $B(x, y)$ is the Beta function and

$$c_{ij} = i j \int_{-1}^{1} x^{i+j-2} (1 - x^2)^\alpha \, dx$$

$$= i j \frac{1 - (-1)^{i+j+1}}{2} B \left( \frac{i + j - 1}{2}, \alpha + 1 \right)$$

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for $1 \leq i, j \leq n$, $c_{ij} = 0$ if $i = 0$ or $j = 0$. Now denote $B = (b_{ij})_{0 \leq i, j \leq n}$, $C = (c_{ij})_{0 \leq i, j \leq n}$, and
\[
a = (a_0, a_1, \ldots, a_n)^T;
\]
then we can derive that
\[
\int_{-1}^{1} p_n^2(x)(1 - x^2)^{a} dx = \sum_{i=0}^{n} \sum_{j=0}^{n} a_i a_j b_{ij} = a^T B a,
\]
\[
\int_{-1}^{1} (p'_n(x))^2(1 - x^2)^a dx = \sum_{i=0}^{n} \sum_{j=0}^{n} a_i a_j c_{ij} = a^T C a.
\]
Now it suffices to consider the following extremal problem:
\[
(3) \quad \sup_{a \in R^{n+1}_{+}} \frac{a^T C a}{a^T B a}
\]
where $R^{n+1}_{+} = \{a: a = (a_0, a_1, \ldots, a_n)^T, a_i \geq 0, 0 \leq i \leq n\}$. Or find the least $\lambda$ such that
\[
\frac{a^T C a}{a^T B a} \leq \lambda, \quad \text{for all } a \in R^{n+1}_{+},
\]
which is
\[
(4) \quad a^T (\lambda B - C) a \geq 0, \quad \text{for all } a \in R^{n+1}_{+}.
\]
Observe that $b_{ij} \geq 0$, $c_{ij} \geq 0$, $0 \leq i, j \leq n$. If we can find a smallest $\lambda$ such that all the elements of $\lambda B - C$ are nonnegative, then we obtain (4) automatically. Notice also that the matrices $B$ and $C$ have the same structure; thus it suffices to find $\lambda$ such that
\[
\lambda b_{ij} - c_{ij} \geq 0, \quad \text{when } b_{ij} \neq 0,
\]
i.e.,
\[
\lambda \geq \frac{c_{ij}}{b_{ij}} = \frac{ij(i + j + 2\alpha + 1)}{i + j - 1}, \quad 1 \leq i, j \leq n.
\]
If we consider $c_{ij}/b_{ij}$ as a function of two continuous variables $i$ and $j$, then we have
\[
\frac{\partial}{\partial i} \left( \frac{ij(i + j + 2\alpha + 1)}{i + j - 1} \right) = \frac{j[i^2 + (j - 1)(2i + j + 2\alpha + 1)]}{(i + j - 1)^2} \geq 0
\]
and similarly
\[
\frac{\partial}{\partial j} \left( \frac{ij(i + j + 2\alpha + 1)}{i + j - 1} \right) = \frac{i[j^2 + (i - 1)(2j + i + 2\alpha + 1)]}{(i + j - 1)^2} \geq 0;
\]
thus this is an increasing function of $i$ and $j$, and we can pick up
\[
\lambda = \left. \frac{ij(i + j + 2\alpha + 1)}{i + j - 1} \right|_{i=n, j=n} = \frac{2n + 2\alpha + 1}{2n - 1} n^2.
\]
To see that $\lambda$ is the best one, we can consider $p_n(x) = x^n$ or $a^T = (0, 0, \ldots, 0, 1)$. This completes the proof of the theorem. □

For some special values of $\alpha$, we have the following corollaries.
Corollary 2.2. Let $p_n(x) \in S_n$; then
\[
\int_{-1}^{1} (p'_n(x))^2 dx \leq \frac{2n + 1}{2n - 1} n^2 \int_{-1}^{1} p_n^2(x) dx
\]
with equality when $p_n(x) = x^n$.

Corollary 2.3. Let $p_n(x) \in S_n$; then
\[
\int_{-1}^{1} (p'_n(x))^2 (1 - x^2)^{-1/2} dx \leq \frac{2n + 2}{2n - 1} n^2 \int_{-1}^{1} p_n^2(x) (1 - x^2)^{-1/2} dx
\]
with equality when $p_n(x) = x^n$.

Corollary 2.4. Let $p_n(x) \in S_n$; then
\[
\int_{-1}^{1} (p''_n(x))^2 (1 - x^2)^{-1/2} dx \leq \frac{2n + 2}{2n - 1} n^2 \int_{-1}^{1} p_n^2(x) (1 - x^2)^{-1/2} dx
\]
with equality when $p_n(x) = x^n$.

In the case $\alpha = 1$, a similar result was proved by Varma [9] for polynomials having real roots.

3. The weight $\omega(x) = x^\alpha e^{-x}$

We give a new proof of Milovanović's Theorem [4] in this section. Indeed we use the same argument as was used in the proof of Theorem 2.1. This time, we consider the weight function $\omega(x) = x^\alpha e^{-x}$, $\alpha > -1$, on the interval $[0, \infty)$.

Theorem 3.1. Let $p_n(x) \in S_n$, $\alpha > -1$; then
\[
\int_0^\infty (p'_n(x))^2 x^\alpha e^{-x} dx \leq C_n(\alpha) \int_0^\infty p_n^2(x) x^\alpha e^{-x} dx
\]
where
\[
C_n(\alpha) = \begin{cases} 1/[(2 + \alpha)(1 + \alpha)], & -1 < \alpha \leq \alpha_n, \\ n^2/[(2n + \alpha)(2n + \alpha - 1)], & \alpha_n \leq \alpha < \infty, \end{cases}
\]
and
\[
\alpha_n = \frac{1}{2} (n + 1)^{-1} [((17n^2 + 2n + 1)^{1/2} - 3n + 1].
\]
Moreover, $C_n(\alpha)$ is the best possible constant.

Proof. Let $p_n(x) = \sum_{i=0}^n a_i x^i$, $a_i \geq 0$, $0 \leq i \leq n$, then
\[
\int_0^\infty p_n^2(x) x^\alpha e^{-x} dx = \sum_{i=0}^n \sum_{j=0}^n a_ia_j \int_0^\infty x^{i+j+\alpha} e^{-x} dx
\]
\[
= \sum_{i=0}^n \sum_{j=0}^n a_ia_j b_{ij} = a^T B a
\]
where
\[
b_{ij} = \int_0^\infty x^{i+j+\alpha} e^{-x} dx = \Gamma(i + j + \alpha + 1),
\]
\[
B = (b_{ij})_{0 \leq i, j \leq n}.
\]
And similarly, we have
\[
\int_0^\infty (p'_n(x))^2 x^\alpha e^{-x} dx = \sum_{i=0}^{n} \sum_{j=0}^{n} a_i a_j c_{ij} = a^T C a
\]
where
\[
c_{ij} = \begin{cases} 
  ij \Gamma(i + j + \alpha - 1), & 1 \leq i, j \leq n, \\
  0, & i = 0 \text{ or } j = 0,
\end{cases}
\]
\[C = (c_{ij})_{0 \leq i, j \leq n}.
\]
Therefore, we need to find the least \(\lambda\) such that
\[\lambda b_{ij} - c_{ij} \geq 0, \quad \text{for } 1 \leq i, j \leq n.
\]
That is, the maximum value of the function
\[
f(i, j) := \frac{c_{ij}}{b_{ij}} = \frac{ij}{(i + j + \alpha)(i + j + \alpha - 1)}.
\]
Let \(k = i + j\); then
\[
f(i, j) = \frac{ij}{(i + j + \alpha)(i + j + \alpha - 1)} = \frac{i(k - i)}{(k + \alpha)(k + \alpha - 1)} =: g(i, k).
\]
If we consider \(g\) as a function of two continuous variables \(i\) and \(k\), then we have
\[
\frac{\partial g(i, k)}{\partial i} = \frac{k - 2i}{(k + \alpha)(k + \alpha - 1)}.
\]
Therefore, \(g(i, k)\) takes on its maximum value at \(i = k/2\) if we fix \(k\) (consider it as a function of \(i\) alone). Now it suffices to consider the maximum value of the function
\[
h(k) := g \left( \frac{k}{2}, k \right) = \frac{k^2}{4(k + \alpha)(k + \alpha - 1)}.
\]
Following the exactly same argument of Milovanović [4, p. 425], we can see that the best possible value of \(\lambda\) is \(C_n(\alpha)\). We omit the details. This completes the proof. \(\Box\)

Remark. The same idea also seems to work for other \(L_p\) norms when \(p\) is an integer, but they become more and more complicated as \(p\) is bigger and bigger. We will not formulate them here. However, for the \(L_1\) norm, the result is simple.

Theorem 3.2. Let \(p_n(x) \in S_n, \alpha > -1\); then
\[
(9) \quad \int_0^\infty p'_n(x)x^\alpha e^{-x} dx \leq \lambda_n(\alpha) \int_0^\infty p_n(x)x^\alpha e^{-x} dx
\]
where
\[
\lambda_n(\alpha) = \begin{cases} 
  1/(1 + \alpha), & -1 < \alpha \leq 0, \\
  n/(n + \alpha), & 0 \leq \alpha < \infty.
\end{cases}
\]
Moreover, \(\lambda_n(\alpha)\) is the best possible constant.
4. THE WEIGHT $\omega(x) = e^{-ax^2}$

In this section we discuss the weight function $\omega(x) = e^{-ax^2}$, $\alpha > 0$, on the whole real line. The corresponding result is the following theorem.

**Theorem 4.1.** Let $p_n(x) \in S_n$, $\alpha > 0$; then

$$\int_{-\infty}^{\infty} (p_n'(x))^2 e^{-ax^2} \, dx \leq \frac{2\alpha}{2n-1} n^2 \int_{-\infty}^{\infty} p_n^2(x) e^{-ax^2} \, dx$$

with equality when $p_n(x) = x^n$.

**Proof.** Let $p_n(x) = \sum_{i=0}^{n} a_i x^i \in S_n$; then

$$\int_{-\infty}^{\infty} p_n^2(x) e^{-ax^2} \, dx = \sum_{i=0}^{n} \sum_{j=0}^{n} a_i a_j b_{ij} = a^T B a$$

where

$$b_{ij} = \int_{-\infty}^{\infty} x^{i+j} e^{-ax^2} \, dx$$

$$= (1 - (-1)^{i+j+1})(i + j - 1)!!2^{-(i+j)/2-1}\alpha^{-(i+j+1)/2}\sqrt{\pi},$$

$$B = (b_{ij})_{0 \leq i, j \leq n},$$

and

$$\int_{-\infty}^{\infty} (p_n'(x))^2 e^{-ax^2} \, dx = \sum_{i=0}^{n} \sum_{j=0}^{n} a_i a_j c_{ij} = a^T C a$$

where

$$c_{ij} = ij \int_{-\infty}^{\infty} x^{i+j-2} e^{-ax^2} \, dx$$

$$= (1 - (-1)^{i+j+1})ij(i + j - 3)!!2^{-(i+j)/2}\alpha^{-(i+j-1)/2}\sqrt{\pi},$$

$$C = (c_{ij})_{0 \leq i, j \leq n}.$$

For $i + j$ even, let

$$f(i, j) := \frac{c_{ij}}{b_{ij}} = 2\alpha \frac{ij}{i + j - 1}, \quad 1 \leq i, j \leq n;$$

then considering $f$ as a function of two continuous variables $i$ and $j$, we can obtain

$$\frac{\partial f(i, j)}{\partial i} = 2\alpha j(j - 1) \frac{i}{(i + j - 1)^2} \geq 0 \quad \text{for } 1 \leq i, j \leq n,$$

and

$$\frac{\partial f(i, j)}{\partial j} = 2\alpha i(i - 1) \frac{j}{(i + j - 1)^2} \geq 0, \quad \text{for } 1 \leq i, j \leq n.$$

Therefore, $f(i, j)$ attains its maximum value at $i = n$, $j = n$, which implies the desired result. □

*Added in Proof.* After this manuscript was written, the author learned that Professor A. K. Varma [11] had written a paper on the same subject. There are some overlaps between his results and our results in §§2 and 3, but we do use different methods.
REFERENCES


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