INTERPRETATION OF LAVRENTIEV PHENOMENON
BY RELAXATION: THE HIGHER ORDER CASE

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ABSTRACT. We consider integral functionals of the calculus of variations of the form
\[ F(u) = \int_0^1 f(x, u, u', \ldots, u^{(n)}) \, dx \]
defined for \( u \in W^{n, \infty}(0, 1) \), and we show that the relaxed functional \( \overline{F} \) with respect to the weak \( W^{n, 1}_{\text{loc}}(0, 1) \) convergence can be written as
\[ \overline{F}(u) = \int_0^1 f(x, u, u', \ldots, u^{(n)}) \, dx + L(u) , \]
where the additional term \( L(u) \), the Lavrentiev Gap, is explicitly identified in terms of \( F \).

1. Introduction

In 1926 M. Lavrentiev (see [L]) first demonstrated this surprising result: given a variational integral of a two-point Lagrange problem, which is sequentially weakly lower semicontinuous on the admissible class of absolutely continuous functions, its infimum on the dense subclass of \( C^1 \) admissible functions may be strictly greater than its minimum value on the full admissible class. Some years later Manià (see [M]) gave an example of this phenomenon with a polynomial integrand. In recent years there have been additional works by several authors; for further bibliographical references the reader can see for instance [BuM].

In this paper we follow the Buttazzo and Mizel [BuM] approach which consists in studying the Lavrentiev Phenomenon from the point of view of relaxation theory. More precisely let \( X \) be a topological space, \( Y \subset X \) a dense subset, \( F : X \to [0, +\infty] \) a given functional, and define
\[
\begin{align*}
\overline{F}_X &= \sup\{G : X \to [0, +\infty] : G \text{ l.s.c., } G \leq F \text{ on } X\} , \\
\overline{F}_Y &= \sup\{G : X \to [0, +\infty] : G \text{ l.s.c., } G \leq F \text{ on } Y\} , \\
L(u) &= \begin{cases} 
\overline{F}_Y(u) - \overline{F}_X(u) & \text{if } \overline{F}_X(u) < +\infty , \\
0 & \text{otherwise ,}
\end{cases}
\end{align*}
\]

We call this nonnegative functional \( L \) (notice that \( \overline{F}_X \leq \overline{F}_Y \)) the "Lavrentiev Gap" associated to \( F \), \( X \) and \( Y \). In their paper Buttazzo and Mizel [BuM]...
considered integral functionals of the form
\[ G(u) = \int_0^1 f(x, u(x), u'(x)) \, dx \]
with \( X = W^{1,1}(0,1) \) and \( Y = W^{1, +\infty}(0,1) \), and gave a characterization of \( L \) in term of the "Value Function" \( V \) (see §2 below). Then they obtained an explicit representation of \( L \) for a large class of integrands.

In this paper we extend the results of [BuM] to integral functionals depending on higher order derivatives, of the form
\[ G(u) = \int_0^1 f(x, u(x), \ldots, u^{(n)}(x)) \, dx \]
with \( X = W^{n,1}(0,1) \) and \( Y = W^{n, +\infty}(0,1) \). More precisely, in §2 we obtain a characterization of \( L \) in terms of the "Value Function" \( V \); in §3 we provide an explicit representation of \( L \) for some integrands which satisfy a "homogeneity condition", and an integrand of Mania type (see [M], [BM1], [BM2]) is analyzed in detail by following this approach. Our results deal with regular integrands (in a sense to be specified), but we want to point out an interesting result involving autonomous second order integrands (see Cheng [C], Cheng and Mizel [CM]) showing the nonoccurrence of the gap phenomenon when the integrand satisfies some continuity assumptions, with an example of a nonvanishing gap when a constraint of the form \( \{u \geq 0\} \) is added.

2. The representation theorem

Let \( \Omega \) be the interval \((0,1)\); we consider the following spaces:

- \( W^{n,1}(0,1) \) the space of all functions \( u : \Omega \to \mathbb{R} \) which are absolutely continuous together with their \((n-1)\) derivatives;
- \( W^{n, \infty}[0, 1] \) the space of all functions \( u : \Omega \to \mathbb{R} \) which are Lipschitz continuous together with their \((n-1)\) derivatives;
- \( W^{n, \infty}_{\text{loc}}[0, 1] \) the space of all functions \( u : \Omega \to \mathbb{R} \) which are Lipschitz continuous together with their \((n-1)\) derivatives on every interval \([\delta, 1]\), with \( \delta > 0 \);
- \( \mathcal{A}_{\infty} \) the space of all function \( u \in W^{n,1}(0,1) \cap W^{n, \infty}_{\text{loc}}[0,1] \) such that \( u^{(i)}(0) = 0 \) for \( i = 0, \ldots, (n-1) \).

Let \( f : \Omega \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) be a function such that:

1. \( f(x, s, z) \) is of Carathéodory type (i.e. measurable in \( X \) and continuous in \((s, z)\));
2. \( f(x, s, \cdot) \) is convex on \( \mathbb{R} \) for every \((x, s) \in \Omega \times \mathbb{R}^n \);
3. there exists a function \( \omega : \Omega \times \mathbb{R} \times \mathbb{R} \to [0, +\infty[ \), with \( \omega(x, t, \tau) \) integrable in \( x \) and integrable in \( t, \tau \), such that
\[ 0 \leq f(x, s, z) \leq \omega(x, |s|, |z|) \quad \forall (x, s, z) \in \Omega \times \mathbb{R}^n \times \mathbb{R}. \]
For every \( u \in \mathcal{A}_\infty \), define
\[
F(u) = \int_0^1 f(x, u, \ldots, u^{(n)}) \, dx,
\]
\[
G(u) = \begin{cases} 
F(u) & \text{if } u \in W^{n, \infty}[0, 1], \\
+\infty & \text{otherwise,}
\end{cases}
\]
and denote by \( \overline{G} \) the functional
\[
\overline{G} = \max \left\{ H : \mathcal{A}_\infty \to [0, +\infty] : H \text{ seq. } w^{-1}.l.s.c., H \leq G \right\}.
\]

Our goal is to give a representation formula for \( \overline{G} \) over \( \mathcal{A}_\infty \).

Since \( F \) is sequentially weakly lower semicontinuous on \( W^{n, 1}_{\text{loc}}(0, 1) \) (briefly seq. w- \( W^{n, 1}_{\text{loc}}(0, 1) \)-l.s.c.) (see [B]), we have
\[
\overline{G}(u) \geq F(u) \quad \forall u \in \mathcal{A}_\infty,
\]
and then
\[
\overline{G}(u) = F(u) + L(u) \quad \forall u \in \mathcal{A}_\infty
\]
for a suitable functional \( L \geq 0 \). We call the functional \( L \) the "Lavrentiev Gap" relative to \( G \) over the space \( \mathcal{A}_\infty \). Obviously we have that \( \overline{G} \leq G \). Then
\[
\overline{G}(u) = F(u) \quad \forall u \in W^{n, \infty}[0, 1];
\]
i.e. \( L(u) = 0 \) for every \( u \in W^{n, \infty}[0, 1] \). In order to identify the functional \( L \) we introduce the "Value Function" \( V(x, s) \) defined for every \( (x, s) \in \Omega \times \mathbb{R}^n \) by:
\[
V(x, s) = \inf \left\{ \int_0^x f(t, u, \ldots, u^{(n)}) \, dt : u \in W^{n, \infty}[0, 1], u^{(i)}(0) = 0, \right. \\
\left. u^{(i)}(x) = s_i, \ i = 0, \ldots, (n - 1) \right\}
\]
and its lower semicontinuous envelope with respect to \( s = (s_0, \ldots, s_{n-1}) \), given by
\[
W(x, s) = \liminf_{\xi \to s} V(x, \xi).
\]

We now state a representation result for the Lavrentiev Gap \( L \).

**Theorem 2.1.** If the integrand \( f(x, s, z) \) satisfies the hypotheses above, then
\[
L(u) = \liminf_{x \to 0^+} W(x, u(x), \ldots, u^{(n-1)}(x)) \quad \text{for every } u \in \mathcal{A}_\infty.
\]

In order to achieve the proof of Theorem 2.1 we need some lemmas. For the sake of simplicity in the following we set
\[
M(u) = \liminf_{x \to 0^+} W(x, u(x), \ldots, u^{(n-1)}(x))
\]
and, when no confusion is possible, we use the notation \( \overline{u}(x) \) to indicate the vector \( (u^{(i)}(x))_{i=0}^{n-1} \).
Lemma 2.2. Take $u, u_h \in \mathcal{A}_\infty$ with $u_h \in W^{n, \infty}[0, 1]$ and assume that $u_h \rightharpoonup u$ weakly in $W^{n, 1}_{\text{loc}}(0, 1)$. Then
\[ F(u) + M(u) \leq \liminf_{h \to +\infty} G(u_h). \]

Proof. Take $\delta > 0$; for every $h \in \mathbb{N}$, by the definitions of $V(x, s)$ and $W(x, s)$ we get
\[
G(u_h) = \int_0^\delta f(x, u_h, \ldots, u_h^{(n)}) \, dx + \int_\delta^1 f(x, u_h, \ldots, u_h^{(n)}) \, dx
\geq V(\delta, \bar{u}_h(\delta)) + \int_\delta^1 f(x, u_h, \ldots, u_h^{(n)}) \, dx
\geq W(\delta, \bar{u}_h(\delta)) + \int_\delta^1 f(x, u_h, \ldots, u_h^{(n)}) \, dx.
\]
As $h \to +\infty$, taking into account that $W$ is seq. w-$W^{n, 1}_{\text{loc}}$-l.s.c. and that the assumptions on the integrand $f$ provide the seq. w-$W^{n, 1}_{\text{loc}}$-l.s.c. of the integral term, we get
\[
\liminf_{h \to +\infty} G(u_h) \geq \liminf_{h \to +\infty} \left[ W(\delta, \bar{u}_h(\delta)) + \int_\delta^1 f(x, u_h, \ldots, u_h^{(n)}) \, dx \right]
\geq W(\delta, \bar{u}(\delta)) + \int_\delta^1 f(x, u, \ldots, u^{(n)}) \, dx.
\]
Finally, as $\delta \to 0$ we obtain
\[
\liminf_{h \to +\infty} G(u_h) \geq \liminf_{\delta \to 0} \left[ W(\delta, \bar{u}(\delta)) + \int_\delta^1 f(x, u, \ldots, u^{(n)}) \, dx \right]
\geq M(u) + \int_0^1 f(x, u, \ldots, u^{(n)}) \, dx
= M(u) + F(u),
\]
and the lemma is proved. \(\Box\)

Lemma 2.3. The functional $F + M$ is seq. w-$W^{n, 1}_{\text{loc}}$-l.s.c. on $\mathcal{A}_\infty$.

Proof. Taking $u, u_h \in \mathcal{A}_\infty$ with $u_h \rightharpoonup u$ weakly in $W^{n, 1}_{\text{loc}}$, we have to show that
\[ F(u) + M(u) \leq \liminf_{h \to +\infty} [F(u_h) + M(u_h)]. \]
Assume that the right-hand side is finite (otherwise there is nothing to prove), and consider a sequence $(x_h)$ in $\Omega$ with $x_h \to 0$ such that
\[ W(x_h, \bar{u}_h(x_h)) \geq M(u_h) + \frac{1}{h} \quad \forall h \in \mathbb{N}. \tag{2.1} \]
It is now possible to find a sequence $(s_h)$ in $\mathbb{R}^n$, with $s_h \to 0$ such that
\[ |s_h - \bar{u}_h(x_h)| \leq \frac{1}{h}; \tag{2.2} \]
\[ V(x_h, s_h) \leq W(x_h, \bar{u}_h(x_h)) + \frac{1}{h}. \tag{2.3} \]
Moreover, denoting by $P_{n-1}$ the polynomial of degree $n-1$ such that $\overline{P}_{n-1}(x_h) = s_h - \overline{u}_h(x_h)$, it is easy to see that, since $f$ is of Carathéodory type, the sequence $(s_h)$ can be taken such that

\begin{equation}
\int_{x_h}^{1} f(x, \overline{u}_h + P_{n-1}, u_h^{(n)}) \, dx \leq \int_{x_h}^{1} f(x, \overline{u}_h, u_h^{(n)}) \, dx + \frac{1}{h}.
\end{equation}

Finally, let $v_h \in W^n, \infty[0, x_h]$ be such that $\overline{v}_h(0) = 0$, $\overline{v}_h(x_h) = s_h$ and

\begin{equation}
\int_{0}^{x_h} f(x, \overline{v}_h, v_h^{(n)}) \, dx \leq V(x_h, s_h) + \frac{1}{h}.
\end{equation}

Setting

\begin{equation}
w_h(x) = \begin{cases} u_h(x) + P_{n-1}(x) & \text{if } x > x_h, \\ v_h(x) & \text{if } 0 < x \leq x_h, \end{cases}
\end{equation}

we have $w_h \in W^n, \infty[0, 1]$, $\overline{w}_h(0) = 0$, and

\begin{equation}
w_h \xrightarrow{h \to +\infty} u \text{ w-}W^{n-1}_{\text{loc}}(0, 1).
\end{equation}

Therefore, by using Lemma 2.2 and (2.1)–(2.6), we obtain

\begin{align*}
F(u) + M(u) & \leq \liminf_{h \to +\infty} F(w_h) \\
& = \liminf_{h \to +\infty} \left[ \int_{0}^{x_h} f(x, \overline{v}_h, v_h^{(n)}) \, dx + \int_{x_h}^{1} f(x, \overline{u}_h + \overline{P}_{n-1}(x_h), u_h^{(n)}) \, dx \right] \\
& \leq \liminf_{h \to +\infty} \left[ \left( V(x_h, s_h) + \frac{1}{h} \right) + \left( \int_{x_h}^{1} f(x, \overline{u}_h, u_h^{(n)}) \, dx + \frac{1}{h} \right) \right] \\
& \leq \liminf_{h \to +\infty} \left[ \left( W(x_h, \overline{u}_h(x_h)) + \frac{1}{h} \right) + F(u_h) + \frac{2}{h} \right] \\
& \leq \liminf_{h \to +\infty} \left[ \left( M(u_h) + \frac{1}{h} \right) + F(u_h) + \frac{3}{h} \right] \\
& = \liminf_{h \to +\infty} [M(u_h) + F(u_h)],
\end{align*}

and the lemma is proved. \(\square\)

**Proof of Theorem 2.1.** It is easy to see that

\[ M(u) = 0 \text{ for every } u \in \mathcal{A}_\infty \cap W^n, \infty[0, 1]; \]

hence we have $F + M \leq G$ on $\mathcal{A}_\infty$. By Lemma 2.3 we have $F + M \leq \overline{G}$ on $\mathcal{A}_\infty$, so it remains to prove that

\[ \overline{G} \leq F(u) + M(u) \text{ for every } u \in \mathcal{A}_\infty. \]

To this aim, fix $u \in \mathcal{A}_\infty$ and take a sequence $(x_h)$ in $\Omega$, $x_h \to 0$, such that

\begin{equation}
M(u) = \lim_{h \to +\infty} W(x_h, \overline{u}(x_h)).
\end{equation}
By definition of $W$ and the assumptions on the integrand $f$ we may find a sequence $(s_h)$ in $\mathbb{R}^n$, $s_h \rightarrow 0$, such that for every $h \in \mathbb{N}

(2.8) \quad |s_h - \bar{u}(x_h)| \leq \frac{1}{h},

(2.9) \quad V(x_h, s_h) \leq W(x_h, \bar{u}(x_h)) + \frac{1}{h},

(2.10) \quad \int_{x_h}^{1} f(x, \bar{u} + P_{n-1}, u^{(n)}) \, dx \leq \int_{x_h}^{1} f(x, \bar{u}, u^{(n)}) \, dx + \frac{1}{h},

where $P_{n-1}$ is as in the proof of Lemma 2.3. Finally, let $v_h \in W^{n, \infty}[0, x_h]$ be a sequence such that $v_h(0) = 0$, $v_h(x_h) = s_h$ and

(2.11) \quad \int_{0}^{x_h} f(x, \bar{v}_h, v_h^{(n)}) \, dx \leq V(x_h, s_h) + \frac{1}{h}.

As in the proof of Lemma 2.3, we define the sequence

$$w_h(x) = \begin{cases} u_h(x) + P_{n-1}(x) & \text{if } x > x_h, \\ v_h(x) & \text{if } 0 \leq x \leq x_h. \end{cases}$$

Then $w_h \in W^{n, \infty}[0, 1]$, $\bar{w}_h(0) = 0$ and

$$w_h \underset{h \rightarrow +\infty}{\rightharpoonup} u \text{ strongly } W^{n, 1}_{\text{loc}}(0, 1).$$

Hence, by using (2.7)—(2.11), we obtain

$$\overline{G}(u) \leq \liminf_{h \rightarrow +\infty} G(w_h)$$

$$= \liminf_{h \rightarrow +\infty} \left[ \int_{0}^{x_h} f(x, \bar{v}_h, v_h^{(n)}) \, dx + \int_{x_h}^{1} f(x, \bar{u} + P_{n-1}(x_h), u^{(n)}) \, dx \right]$$

$$\leq \liminf_{h \rightarrow +\infty} \left[ \left( V(x_h, s_h) + \frac{1}{h} \right) + \left( \int_{x_h}^{1} f(x, \bar{u}, u^{(n)}) \, dx + \frac{1}{h} \right) \right]$$

$$\leq \liminf_{h \rightarrow +\infty} \left[ \left( W(x_h, \bar{u}(x_h)) + \frac{1}{h} \right) + F(u) + \frac{2}{h} \right]$$

$$= M(u) + F(u);$$

so $M = L$, and the theorem is completely proved. \(\square\)

**Remark 2.4.** Fix a subset $\beta$ of $\{0, 1, \ldots, n - 1\}$ and consider the class $\mathcal{A}^\beta_{\infty}$ of all functions $u \in W^{n, 1}(0, 1) \cap W_{\text{loc}}^{n, \infty}[0, 1]$ such that $u^{(i)}(0) = 0$ for $i \in \beta$. We denote by $\overline{G}_{\beta}$ the functional

$$\overline{G}_{\beta} = \max\{H : \mathcal{A}^\beta_{\infty} \rightarrow [0, +\infty] : H \text{ seq. w-} W_{\text{loc}}^{n, 1}\text{-l.s.c., } H \leq G\}.$$

As in the previous case, we have

$$\overline{G}_{\beta}(u) = F(u) + L_{\beta}(u) \quad \forall u \in \mathcal{A}_{\infty}$$

for a suitable functional $L_{\beta} \geq 0$, the "Lavrentiev Gap" relative to $G$ over the space $\mathcal{A}^\beta_{\infty}$. In order to identify the functional $L_{\beta}$ we introduce the Value...
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Function $V_\beta(x, s)$ defined for every $(x, s) \in \Omega \times \mathbb{R}^k$ by:

$$V_\beta(x, s) = \inf \left\{ \int_0^X f(t, u, \ldots, u^{(n)}) \, dt : u \in W^{n, \infty}[0, 1], \, u^{(j)}(0) = 0, \right.$$

$$u^{(j)}(x) = s_j, \quad j = 0, 1, \ldots, k - 1 \}
$$

and its lower semicontinuous envelope with respect to $s = (s_{i_0}, \ldots, s_{i_{k-1}})$, given by

$$W_\beta(x, s) = \liminf_{\xi \to s} V_\beta(x, \xi).$$

By repeating step by step the proof of Theorem 2.1, we obtain the following result:

**Theorem 2.5.** If the integrand $f(x, s, z)$ satisfies the assumptions of Theorem 2.1, then

$$L_\beta(u) = \liminf_{x \to 0^+} W_\beta(x, u^{(0)}(x), \ldots, u^{(k-1)}(x)) \quad \text{for every } u \in \mathcal{A}_\beta^0.$$

Note that the polynomial $P_{n-1}$ may be chosen, in this case, such that

$$P^{(r)}_{n-1} = \begin{cases} u_r^{(r)}(x_h) - (s_h)^r & \text{if } r \in \beta, \\ 0 & \text{otherwise.} \end{cases}$$

### 3. Some examples

In this section we give an explicit representation formula for a class of second order integrands $f$ (we mean that $f$ is a function depending on $x, u, u', u''$). We introduce the so-called "invariance property" for second order integrands (analogous to the one introduced in [HM1] for first order integrands, and to the one of [CM] for second order autonomous integrands):

there exists $\gamma \in ]1, 2[$ such that for every $t > 0$ and $(x, s, z, w) \in \Omega \times \mathbb{R}^3 f(tx, t^\gamma s, t^{\gamma-1} z, t^{\gamma-2} w) = f(x, s, z, w).$

We want to analyze a class of second order integrands $f(x, s, z, w)$ that satisfies this invariance property only in an asymptotic sense near the relevant singular abscissa. Let us take $\delta > 1, \tau \in [1, \delta[; \text{ we suppose the integrand } f: \Omega \times \mathbb{R} \to \mathbb{R} \text{ has the form}

$$f(x, s, z, w) = x^{\tau-1} a(x, s) b(x, z)|w|^\delta,$$

with $a(x, s), b(x, z)$ nonnegative, continuous functions such that, setting $\gamma = 2 - \frac{1}{\delta}$, for every $y \in \Omega$ the functions $m_y, n_y, M_y, N_y: \Omega \to \mathbb{R}$ defined by

$$m_y(s) = \inf\{a(x, x^\gamma s) : x \leq y\}, \quad n_y(s) = \inf\{b(x, x^\gamma s) : x \leq y\},$$

$$M_y(s) = \sup\{a(x, x^\gamma s) : x \leq y\}, \quad N_y(s) = \sup\{b(x, x^\gamma s) : x \leq y\}$$

are locally bounded. Take now $x, y \in \Omega, \, x \leq y$, and consider the following functionals:

$$F_x(u) = \int_0^x f(t, u, u', u'') \, dt,$$

$$F_{x, y}(u) = \int_0^x t^{\tau-1} t^\gamma u) m_y(t^{1-\gamma} u) |u''|^\delta \, dt,$$

$$F_{x, y}(u) = \int_0^x t^{\tau-1} M_y(t^{1-\gamma} u) |u''|^\delta \, dt.$$
We suppose that there exists \( \bar{y} \in \Omega \) such that, for every \( x \in \{ x \in \Omega : x \leq \bar{y} \} \), we have

\[
(3.1) \quad F_{x,\bar{y}}^*(u) < +\infty \quad \text{whenever} \quad F_x(u) < +\infty .
\]

Obviously for every \( x, y \in \Omega \) with \( x \leq y \)

\[
F_{x,y}(u) \leq F_x(u) \leq F_{y,y}(u) \quad \forall u \in \mathcal{A}_\infty^1;
\]

then for every \( x \in \Omega \)

\[
\sup_{y \in \Omega} F_{x,y}(u) \leq F_x(u) \leq \inf_{y \leq \bar{y}} F_{x,y}(u) \quad \forall u \in \mathcal{A}_\infty^1,
\]

say

\[
(3.2) \quad \lim_{y \to 0^+} F_{x,y}(u) \leq F_x(u) \leq \lim_{y \to 0^+} F_{x,y}(u) \quad \forall u \in \mathcal{A}_\infty^1.
\]

Define, for \( y \leq \bar{y} \),

\[
\lim_{y \to 0^+} m_x(s) = m_0(s), \quad \lim_{y \to 0^+} M_x(s) = M_0(s),
\]

\[
\lim_{y \to 0^+} n_x(s) = n_0(s), \quad \lim_{y \to 0^+} N_x(s) = N_0(s);
\]

by the assumptions (3.1) we apply the Monotone and Lebesgue Convergence Theorems to (3.2) obtaining

\[
(3.3) \quad F_0,x(u) \leq F_x(u) \leq F_0^0(u) \quad \forall u \in \mathcal{A}_\infty^1
\]

where

\[
F_0,x(u) = \int_0^x t^{-1}m_0(t^{-1}u)n_0(t^{-1}u')|u''|^\delta \, dt,
\]

\[
F_0^0(u) = \int_0^x t^{-1}M_0(t^{-1}u)N_0(t^{-1}u')|u''|^\delta \, dt.
\]

We suppose also that \( m_0(s) = P \leq Q = M_0(s) \), with \( P, Q \in [0, +\infty[ \).

**Theorem 3.1.** Under the previous assumptions, for every

\[
u \in \mathcal{A}_\infty^1 = \{ u \in W^{2,1}(0, 1) \cap W^{2,\infty}_{\text{loc}}[0, 1] : u'(0) = 0 \}
\]

we have

\[
P \delta k^{\delta-1} \left| \int_0^{\liminf_{x \to 0^+} u'(x)x^{1-\gamma}} n_0(\xi)|\xi|^{\delta-1} \, d\xi \right|
\]

\[
\leq L(u) \leq Q \delta k^{\delta-1} \left| \int_0^{\liminf_{x \to 0^+} u'(x)x^{1-\gamma}} N_0(\xi)|\xi|^{\delta-1} \, d\xi \right|,
\]

where \( k = \frac{\delta(\gamma-1)}{\delta-1} \).

In order to achieve the proof of Theorem 3.1, we need a lemma.
Lemma 3.2. Let \( h(Z) \) be the solution of the minimum problem
\[
\inf\{G(u) : u \in W^{2, \infty}(0, 1), u'(0) = 0, u'(1) = Z\},
\]
where
\[
G(u) = \int_0^1 x^{r-1} n(x^{1-\gamma} u'(x)) |u''(x)|^\delta \, dx.
\]
We have
\[
h(Z) = \delta k^{\delta-1} \left| \int_0^Z n(\xi) |\xi|^{\delta-1} \, d\xi \right|,
\]
where \( k = \frac{\delta(\gamma-1)}{\delta-1} \) and the function \( h(Z) \) is the solution of the equation
\[
(\gamma - 1) Z h'(Z) = \sup\{Qh'(Z) - n(Z)|Q|^\delta : Q \in \mathbb{R}\},
\]
\( h(0) = 0 \).

Proof. By explicitly carrying out the maximization, the equation (3.5) becomes
\[
h'(Z) = \delta k^{\delta-1} n(Z)|Z|^{\delta-2}
\]
and by direct integration
\[
h(Z) = \delta k^{\delta-1} \left| \int_0^Z n(\xi) |\xi|^{\delta-1} \, d\xi \right|.
\]
Let us take \( u \in \mathcal{A}(x, z) = \{u \in W^{2, \infty} : u'(0) = 0, u'(x) = z\} \); from (3.5), setting \( Z(t) = t^{1-\gamma} u'(t) \) and \( Q(t) = t^{2-\gamma} u''(t) \) we obtain
\[
(\gamma - 1) Z(t) h'(Z(t)) \geq |Q(t)| h'(Z(t)) - n(Z(t))|Q|^\delta.
\]
Then
\[
t^{-1} n(Z(t))|Q|^\delta \geq t^{-1} h'(Z(t))[Q(t) + (1 - \gamma)Z(t)]
\]
\[
= h'(Z(t))Z'(t) = (h \circ Z)'(t)
\]
(for the last equality see [MM]). Integrating on \( ]0, x[ \) yields
\[
I(u) = \int_0^x t^{r-1} n(t^{1-\gamma} u'(t)) |u''(t)|^\delta \, dt
\]
\[
= \int_0^x t^{-1} n(t^{1-\gamma} u'(t)) |t^{r/\delta} u''(t)|^\delta \, dt
\]
\[
= \int_0^x t^{-1} n(Z(t)) |Q|^\delta \, dt
\]
\[
\geq \int_0^x (h \circ Z)'(t) \, dt
\]
\[
= h(Z(x)) - \lim_{t \to 0^+} h(Z(t))
\]
\[
= h(Z(x))
\]
(in fact \( u \in W^{2, \infty}[0, x] \) with \( u'(0) = 0 \) implies
\[
\lim_{t \to 0^+} t^{1-\gamma} u'(t) = 0 \quad \forall \gamma \in [1, 2[,
\]
and hence \( \lim_{t \to 0^+} h(Z(t)) = 0 \)). It follows that
\[
W(x, z) = \inf\{I(u) : u \in \mathcal{A}(x, z)\} \geq h(x^{1-\gamma} z) = h(Z).
\]
Consider now the sequence \((u_\varepsilon) \subset \mathcal{A}(x, z)\) defined by
\[
u_\varepsilon(0) = 0, \quad \nu'_\varepsilon(t) = \begin{cases} \left(\frac{t}{\varepsilon}\right)^k z & \text{if } t \geq \varepsilon, \\ t \frac{e^{k-1}}{e^\varepsilon} z & \text{if } t < \varepsilon. \end{cases}
\]

Taking \(\varepsilon\) sufficiently small we have
\[
W(x, z) \leq I(u_\varepsilon) = \int_0^\varepsilon t^{n-1} n(t^{1-\gamma} u_\varepsilon')|u_\varepsilon''|\delta \, dt + \int_\varepsilon^x t^{n-1} n(t^{1-\gamma} u_0)|u_0''|\delta \, dt,
\]
where \(u'_0(t) = \left(\frac{t}{\varepsilon}\right)^k, u_0(0) = 0\); passing to the limit for \(\varepsilon \to 0\) the first integral tends to 0, and hence
\[
W(x, s) \leq I(u_0).
\]

At this point we can easily verify that \(I(u_0) = h(x^{1-\gamma} z)\), and the proof of the lemma is then complete. \(\square\)

**Proof of Theorem 3.1.** We fix \(u \in \mathcal{A}^1\); by Theorem 2.5
\[
L_1(u) = \lim_{x \to 0} W_1(x, u'(x))
\]
where \(W_1(x, z) = \liminf_{x \to z} V_1(x, q)\) and
\[
V_1(x, z) = \inf\{F_x(u) : u \in W^{2, \infty}(0, 1), u'(0) = 0, u'(x) = z\} = \inf\{F_x(u) : u \in \mathcal{A}(x, z)\},
\]
where \(\mathcal{A}(x, z) = \{u \in W^{2, \infty}(0, x) : u'(0) = 0, u'(x) = z\}\).

Let us introduce the Value Functions relative to the functionals \(F_0, x, F_x^0\) given by
\[
V_0(x, z) = \inf\{F_0(x, u) : u \in \mathcal{A}(x, z)\},
\]
\[
V_0^0(x, z) = \inf\{F_0^0(x, u) : u \in \mathcal{A}(x, z)\};
\]
obviously, for every \(x \in \Omega\) and for every \(z \in \mathbb{R}\), we have by (3.3)
\[
(3.8) \quad V_0(x, z) \leq V_1(x, z) \leq V_0^0(x, z).
\]

Setting \(S = s x^{-\gamma}, Z = z x^{1-\gamma}\) and
\[
G_0(u) = P \int_0^1 t^{n-1} n_0(t^{1-\gamma} u')|u''|\delta \, dt,
\]
\[
G_0^0(u) = Q \int_0^1 t^{n-1} N_0(t^{1-\gamma} u')|u''|\delta \, dt,
\]
\[
\mathcal{A}(Z) = \{u \in W^{2, \infty}(0, 1) : u'(0) = 0, u'(1) = Z\},
\]
by the change of variable \(t = x y\) we get
\[
V_0(x, z) = H_0(Z) = \inf\{G_0(u) : u \in \mathcal{A}(Z)\},
\]
\[
V_0^0(x, z) = H_0^0(Z) = \inf\{G_0^0(u) : u \in \mathcal{A}(Z)\},
\]
so that inequalities (3.8) become
\[
(3.9) \quad H_0(Z) \leq V_1(x, z) \leq H_0^0(Z)
\]
for every \(x \in \Omega\) and for every \(z \in \mathbb{R}\).
By Lemma 3.2 we have that $H_0(Z)$ and $H^0(Z)$ are given by
\begin{align*}
H_0(Z) &= P\delta k^{\delta-1} \left| \int_0^Z n_0(\xi)|\xi|^\delta d\xi \right|,
(3.10) \\
H^0(Z) &= Q\delta k^{\delta-1} \left| \int_0^Z N_0(\xi)|\xi|^\delta d\xi \right|;
\end{align*}
and inserting (3.10) into (3.9) we obtain the inequality (3.4), that is the thesis.

\textbf{Example 3.3.} Consider the functional
\[ F(u) = \int_0^1 f(x, u(x), u'(x), u''(x)) \, dx, \]
where the integrand $f$ has the following form, with $1 < p < 2$ and $0 < q < 1$,
\[ f(x, s, z, w) = (s-x^p)^2(z-x^q)^2|w|^\delta = x^{2(p+q)}(sx^{-p} - 1)^2(zx^{-q} - 1)^2|w|^\delta = x^{2(p+q)}a(x, s)b(x, z)|w|^\delta, \]
where we set
\[ a(x, s) = (sx^{-p} - 1)^2, \]
\[ b(x, z) = (zx^{-q} - 1)^2. \]
If $\delta \leq 1$ we can easily verify that the Lavrentiev Gap $L(u)$ is identically equal to 0: for every $u \in W^{2,1}(0, 1)$ with $u'(0) = 0$ we construct the sequence in $W^{2,\infty}$
\[ u'_\varepsilon(x) = \begin{cases} u'(x), & \text{if } x_\varepsilon < x, \\ \frac{u'(x_\varepsilon)}{x_\varepsilon}, & \text{if } 0 < x < x_\varepsilon, \end{cases} \]
(3.11)
\[ u_\varepsilon(0) = u(0), \]
where $x_\varepsilon \in [0, 1]$ is a sequence with limit 0 as $\varepsilon \to 0$, and we verify that $F(u_\varepsilon) \to F(u)$ as $\varepsilon \to 0$. Here, for simplicity, we restrict our attention to the case $\delta > \frac{1+2(p+q)}{2-p}$. With the notation above
\[ \tau = 1 + 2(p+q), \quad \gamma = 2 - \frac{\tau}{\delta} = 2 - \frac{1+2(p+q)}{\delta}. \]
This integrand $f$ has as "zero cost curves" the functions $z_1(x) = x^p$, $z_2(x) = (q+1)^{-1}x^{q+1}$; by the assumption on $p$ and $q$ we obtain $z_1(x), z_2(x) \in W^{2,1}(0, 1) \setminus W^{2,\infty}[0, 1]$. When $\delta > \frac{1+2(p+q)}{1-q}$, we have $\gamma > p$, $\gamma > q$ and then
\[ m_0(s) = n_0(s) = M_0(s) = N_0(s) = 1; \]
hence for every fixed $u \in \mathcal{A}_\infty$, we obtain
\[ L_1(u) = k^{\delta-1} \liminf_{x \to 0^+} \left| \frac{u'(x)}{x^{-\gamma-1}} \right|; \]
this functional is not identically equal to 0: for instance, $L_1(x^p) = +\infty$ and $L_1((q+1)^{-1}x^{q+1}) = +\infty$. 

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When $\delta = \frac{1+2(p+q)}{1-q}$, by a computation similar to the previous case, we obtain

$$m_0(s) = M_0(s) = 1, \quad n_0(s) = N_0(s) = (s - 1)^2.$$  

Then, for every fixed $u \in A^1_{\infty}$ we have

$$L_1(u) = \delta k^{\delta-1} \liminf_{x \to 0^+} \int_0^u (\xi - 1)^2 |\xi|^{\delta-1} d\xi;$$

also in this case this functional is not identically equal to 0: for instance $L_1(x^p) = +\infty$, while $L_1((q + 1)^{-1} x^{q+1}) = 2k^{\delta-1}/(\delta + 1)(\delta + 2)$.

Finally, when $\delta < \frac{1+2(p+q)}{1-q}$, Theorem 3.1 does not apply because the functions $n_y, N_y$ are not locally bounded. However it is possible to show that in this case the gap phenomenon does not occur (see [Be]): for every $u \in A^1_{W^{2,1}}$ we construct $u_\varepsilon$ in $W^2, \infty(0, 1)$ by (3.11) and we prove that, if $F(u) < +\infty$, then $F(u_\varepsilon) \to F(u)$ as $\varepsilon \to 0$, i.e.

$$\int_0^1 f(x, u, u', u'') dx < +\infty \Rightarrow L(u) = 0.$$

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