ON C*-ALGEBRAS
ASSOCIATED TO THE CONJUGATION REPRESENTATION
OF A LOCALLY COMPACT GROUP

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ABSTRACT. For a locally compact group $G$, let $\gamma_G$ denote the conjugation representation of $G$ in $L^2(G)$. In this paper we are concerned with nuclearity of $C^*$-algebras associated to $\gamma_G$ and the question of when these are of bounded representation type.

INTRODUCTION

Let $G$ be a locally compact group with left Haar measure and $C^*(G)$ the group $C^*$-algebra of $G$. For any unitary representation $\pi$ of $G$, there are two $C^*$-algebras associated to $\pi$. The first one is $\pi(C^*(G))$, which henceforth will be denoted $C^*(\pi)$, and the second one is $C^*(\pi(G))$, the $C^*$-algebra generated by the set of operators $\pi(x)$, $x \in G$, on the Hilbert space of $\pi$. If $G_d$ stands for the same group $G$ endowed with the discrete topology and $i_G : G_d \to G$ for the identity, then $C^*(\pi(G)) = C^*(\pi \circ i_G)$. Thus, investigating $C^*(\pi(G))$ naturally involves $G_d$.

For $\pi$ the left regular representation $\lambda_G$ of $G$, $C^*(\lambda_G)$ is called the reduced group $C^*$-algebra which is usually denoted by $C_r^*(G)$. It has been a matter of enormous interest in harmonic analysis and is one of the most important examples in the general theory of $C^*$-algebras. Very recently, Bédos [2] has drawn attention to $C^*(\lambda_G \circ i_G)$ and has shown that amenability of $G$ and of $G_d$ can both be characterized in terms of $C^*(\lambda_G \circ i_G)$.

In this paper we study $C^*$-algebras associated to the conjugation representation $\gamma_G$ of $G$ on $L^2(G)$ which is defined by

$$\gamma_G(x)f(y) = \delta(x)^{1/2}f(xy^{-1}yx), \quad f \in L^2(G), \ x, y \in G,$$

where $\delta$ denotes the modular function of $G$. We show that nuclearity of either $C^*(\gamma_G)$ or $C^*(\gamma_G \circ i_G)$ forces $G_d$ to be amenable (Theorem 1.2). Conversely, if $G_d$ is amenable then $C^*(\gamma_G)$ and $C^*(\gamma_G \circ i_G)$ are isomorphic (Theorem 1.7) and nuclear. Unfortunately, in this regard nothing substantial can be said about $C^*(\gamma_G)$ for arbitrary $G$ except that, of course, amenability of $G$ implies that $C^*(\gamma_G)$ is nuclear.

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These results will be applied in §2, where we deal with the question of when any one of the \( C^* \)-algebras \( C^*(\gamma_G) \), \( C^*(\gamma_{G_d}) \) and \( C^*(\gamma_G \circ i_G) \) is of bounded representation type, that is, possesses only finite-dimensional irreducible representations and there is an upper bound for the dimensions. Clearly, since \( \gamma_G \) is trivial on \( Z(G) \), the centre of \( G \), such conditions can only be reflected by the structure of the factor group \( G/Z(G) \). It turns out that, for a compactly generated Lie group \( G \), any one of the above \( C^* \)-algebras being of bounded representation type is equivalent to the existence of an abelian subgroup of finite index in \( G/Z(G) \) (Theorem 2.10).

The conjugation representation is of interest not least because of its connections to questions on inner invariant means on \( L^\infty(G) \) (compare [17], [18] and [13]) and the structure of \( G/Z(G) \) [14]. However, so far it is much less understood than the left regular representation. The main difficulty arising is that, even for finite groups, the support of \( \gamma_G \) is generally strictly contained in the dual of \( G/Z(G) \) and is intricate to determine (compare [11], [12], [13], [20], and [22]).

**Preliminaries and notation**

Let \( G \) be a locally compact group. We use the same letter, for example \( \pi \), for a unitary representation of \( G \) and for the corresponding \( * \)-representation of \( C^*(G) \), and \( \mathcal{H}(\pi) \) always denotes the Hilbert space of \( \pi \). Let \( \ker \pi \) be the \( C^* \)-kernel of \( \pi \). If \( S \) and \( T \) are sets of unitary representations of \( G \), then \( S \) is weakly contained in \( T \) \( (S \preceq T) \) if \( \bigcap_{\sigma \in S} \ker \sigma \supseteq \bigcap_{\tau \in T} \ker \tau \) or, equivalently, if any positive definite function associated to \( S \) can be uniformly approximated on compact subsets of \( G \) by sums of positive definite functions associated to \( T \). \( S \) and \( T \) are weakly equivalent \( (S \sim T) \) if \( S \preceq T \) and \( T \preceq S \). The dual space \( \widehat{G} \) is the set of equivalence classes of irreducible representations of \( G \), endowed with the Jacobson topology. As general references to dual spaces and representation theory we mention [5] and [7].

For any representation of \( \pi \) of \( G \), the support of \( \pi \) is the closed subset \( \text{supp} \pi = \{ p \in \widehat{G} ; p \prec \pi \} \) of \( \widehat{G} \). In particular, the support of the left regular representation \( \lambda_G \) is the reduced dual \( \widehat{G}_r \).

Recall that amenability of \( G \) is equivalent to a number of different conditions: \( C^*(\lambda_G) = C^*(G) \), \( \widehat{G}_r = \widehat{G} \), or \( 1_G \preceq \lambda_G \), where \( 1_G \) is the trivial one-dimensional representation of \( G \). Concerning amenability we refer to [8], [23] and [24].

Also, we remind the reader that a \( C^* \)-algebra \( A \) is called nuclear if there exists exactly one \( C^* \)-norm on the algebraic tensor product \( A \otimes B \) for every \( C^* \)-algebra \( B \). For properties equivalent to nuclearity and a short overview on this concept we refer to [23, §1.31].

Let \( N \) be a closed normal subgroup of \( G \). Then every representation of \( G/N \) can be lifted to a representation of \( G \), and in this sense will also be regarded as a representation of \( G \). In particular \( (G/N)^\sim \subseteq \widehat{G} \). If \( H \) is a subgroup of \( G \), and \( \sigma \) and \( \pi \) are representations of \( H \) and \( G \), respectively, then \( \text{ind}_H^G \sigma \) denotes the representation of \( G \) induced by \( \sigma \) and \( \pi \mid H \) the restriction of \( \pi \) to \( H \). A readable account of the theory of induced representations can be found in [7, Chapter 11]. We will use throughout the fact that inducing and restricting representations are continuous with respect to Fell's topology [6].
Next, let 
\[ \{e\} = Z_0(G) \subseteq Z(G) = Z_1(G) \subseteq Z_2(G) \subseteq \cdots \]
be the ascending central series and \( G_f \) the finite conjugacy class subgroup of \( G \). For any two subsets \( M, N \) of \( G \) we denote by \( C_M(N) \) the centralizer of \( N \) in \( M \). If \( M = G \) we often omit the index. Using this notation, for discrete groups \( G \), \( \gamma_G \) is weakly equivalent to the set \( \{ \text{ind}_{\gamma_G(a)} C(a); a \in G\} \) (see [13, p. 27]).

For general \( G \) the only available description of \( \text{supp} \gamma_G \) is as follows. Let \( G \) be a \( \sigma \)-compact locally compact group, and suppose that \( C^*(\lambda_G) \) is nuclear. Then by [11, Theorem]
\[ \text{supp} \gamma_G = \bigcup_{\pi \in \hat{G}} \text{supp}(\pi \otimes \pi). \]

1. \( C^*(\gamma_G), C^*(\gamma_G \circ i_G), \) AND AMENABILITY

We start with a lemma which will be used in the proof of Theorem 1.2 below as well as in §2.

**Lemma 1.1.** For any locally compact group \( G \) and \( i_G : G_d \to G \) the identity
\[ \lambda_{G_d/(G_d)f} \ll \gamma_G \circ i_G. \]

**Proof.** The proof is an adaptation of the proof of Theorem 1.8 in [13]. Let \( D = G_d \) and recall that \( \lambda_{D/D_f} \) is the GNS-representation defined by the characteristic function \( \chi_{D_f} \) of \( D_f \). Therefore it suffices to show that given any finite subset \( F \) of \( D \), there exists a positive definite function \( \varphi \) associated to \( \gamma_G \circ i_G \) such that \( \varphi | F = \chi_{D_f} | F \). Set \( F_1 = F \cap D_f \) and \( F_2 = F \setminus F_1 \). Then, by the proof of [13, Theorem 1.8], there exists \( a \in C(F_1) \) such that \( x^{-1}ax \neq a \) for all \( x \in F_2 \).

\( C(F_1) \) is a closed subgroup of finite index in \( G \), and hence is open. Thus we find an open neighbourhood \( V \) of \( a \) in \( G \) such that \( V \subseteq C(F_1) \) and \( x^{-1}Vx \cap V = \emptyset \) for all \( x \in F_2 \). Observe that \( \delta(x) = 1 \) for all \( x \in F_1 \) since \( x^{-1}Vx = V \). Now, let \( f = |V|^{-1/2} \chi_V \) and
\[ \varphi(x) = \langle \gamma_G(x)f, f \rangle = \delta(x)^{1/2} |V|^{-1} \int_V \chi_V(x^{-1}yx) \, dy. \]
It follows that \( \varphi(x) = 1 \) for \( x \in F_1 \) as \( V \subseteq C(F_1) \), and \( \varphi(x) = 0 \) for \( x \in F_2 \) since \( x^{-1}Vx \cap V = \emptyset \) for \( x \in F_2 \).

**Theorem 1.2.** For a locally compact group \( G \) the following are equivalent.

(i) \( G_d \) is amenable.
(ii) \( C^*(\gamma_G \circ i_G) \) is nuclear.
(iii) \( C^*(\gamma_G) \) is nuclear.

**Proof.** (i) \( \Rightarrow \) (ii) and (i) \( \Rightarrow \) (iii) are obvious, since amenability of \( G_d \) implies that \( C^*(\gamma_G) \) is nuclear, and hence so are the quotients \( C^*(\gamma_G \circ i_G) \) and \( C^*(\gamma_{G_d}) \) of \( C^*(G_d) \) (compare [4, Corollary 4]).

Since \( \lambda_{G_d/(G_d)f} \ll \gamma_G \circ i_G \) (Lemma 1.1), \( C^*(\lambda_{G_d/(G_d)f}) \) is a quotient of \( C^*(\gamma_G \circ i_G) \). Thus (ii) implies nuclearity of \( C^*(\lambda_{G_d/(G_d)f}) \), and by [16, Theorem 4.2] this forces \( G_d/(G_d)_f \) to be amenable. Now groups with finite conjugacy classes are well known to be amenable (see [24, Proposition 12.9 or Corollary]...
14.26]). As the class of amenable groups is closed under forming extensions by amenable groups, \( G_d \) turns out to be amenable. (iii) \( \Rightarrow \) (i) follows in the same way by appealing to Theorem 1.8 of [13] instead of Lemma 1.1.

For \( \gamma_G \) replaced by the left regular representation, Theorem 1.2 has been established in [2, Theorem 3].

**Lemma 1.3.** Let \( G \) and \( H \) be locally compact groups, and let \( j : H \to G \) be a continuous and injective homomorphism with dense range. Then \( \hat{G} \circ j \subseteq \hat{H} \), and \( \hat{G} \circ j \) is dense in \( \hat{H} \) provided that \( H \) is discrete and amenable.

**Proof.** Let \( \pi_1, \pi_2 \) be representations of \( G \). If \( \pi_1 \) and \( \pi_2 \) are equivalent, then \( \pi_1 \circ j \) and \( \pi_2 \circ j \) are equivalent representations of \( H \). Conversely, if \( \pi_1 \circ j \) and \( \pi_2 \circ j \) are equivalent, then since \( j(H) \) is dense in \( G \) and representations are strongly continuous, it follows immediately that \( \pi_1 \) and \( \pi_2 \) are equivalent. Moreover, for a representation \( \pi \) of \( G \), \( \pi \) is irreducible if and only if \( \pi \circ j \) is irreducible. Thus \( \pi \to \pi \circ j \) induces an injective mapping from \( \hat{G} \) into \( \hat{H} \).

It is easy to see that the Dirac function \( \delta_e \) on \( H \) can be pointwise approximated by positive definite functions associated to \( \lambda_G \circ j \) [3, Proposition 1]. For \( H \) discrete, this shows that \( \lambda_H \prec \lambda_G \circ j \), and hence \( \hat{G} \circ j \) is dense in \( \hat{H} \) if, in addition, \( H \) is amenable. \( \square \)

**Corollary 1.4.** Suppose that \( H \) is amenable and discrete, and let \( G \) and \( j \) be as in Lemma 1.3. Then

\[
\{(\pi \circ j) \otimes (\pi \circ j) ; \pi \in \hat{G}\} \sim \{\rho \otimes \rho ; \rho \in \hat{H}\}.
\]

**Proof.** Let \( P \) and \( R \) denote the set of representations on the left and on the right, respectively. It is clear from \( \hat{G} \circ j \subseteq \hat{H} \) that \( P \prec R \). On the other hand, since \( \hat{G} \circ j \) is dense in \( \hat{H} \) by Lemma 1.3, for \( \rho \in \hat{H} \) every coordinate function of the form

\[
x \rightarrow \langle (\rho \otimes \bar{\rho})(x)(\xi_1 \otimes \xi_2), \eta_1 \otimes \eta_2 \rangle = \langle \rho(x)\xi_1, \eta_1 \rangle \langle \bar{\rho}(x)\xi_2, \eta_2 \rangle,
\]

where \( \xi_1, \eta_1 \in \mathcal{H}(\rho) \) and \( \xi_2, \eta_2 \in \mathcal{H}(\rho) \), can be approximated on finite subsets of \( H \) by a product of functions each of which is a finite sum of positive definite functions associated to \( \pi \circ j \) and \( \pi \circ j, \pi \in \hat{G} \), respectively. It follows that \( \rho \otimes \bar{\rho} \prec P \). \( \square \)

We have to compare \( \gamma_{G_d} \) and \( \gamma_G \circ i_G \) with respect to weak equivalence. As mentioned in the proof of Lemma 1.3, for every locally compact group \( G \), \( \lambda_{G_d} \prec \lambda_G \circ i_G \). In general, however, \( \gamma_{G_d} \) need not be weakly contained in \( \gamma_G \circ i_G \). We will further comment on this in Lemma 1.8 and Remarks 1.9. But at least we have

**Corollary 1.5.** Suppose that \( G \) is \( \sigma \)-compact and \( H \) is amenable and discrete, and let \( j \) be as in Lemma 1.3. Then \( \gamma_H \prec \gamma_G \circ j \).

**Proof.** Since \( G \) is amenable and \( \sigma \)-compact, \( \gamma_G \sim \{\pi \otimes \pi ; \pi \in \hat{G}\} \) by the theorem of [11]. Corollary 1.4 yields

\[
\gamma_G \circ j \sim \{(\pi \circ j) \otimes (\pi \circ j) ; \pi \in \hat{G}\} \sim \{\rho \otimes \bar{\rho} ; \rho \in \hat{H}\},
\]

and this latter set weakly contains \( \gamma_H \) [11, Corollary 1]. \( \square \)
Lemma 1.6. Let $G$ be a second countable group such that $G_d$ is amenable. Then $\gamma_G \circ i_G \prec \gamma_{G_d}$.

Proof. There exists a countable dense subset in $G$ as $G$ is second countable. Thus every finite subset of $G$ is contained in some countable dense subgroup $H$ of $G$. For any such $H$, $\{\rho \otimes \rho; \rho \in \hat{H}_d\} \sim \gamma_{H_d}$, and hence by Corollary 1.4,

$$\gamma_G \circ j_H \sim \{(\pi \circ j_H) \otimes (\pi \circ j_H); \pi \in \hat{G}\} \sim \{\rho \otimes \rho; \rho \in \hat{H}_d\} \sim \gamma_{H_d},$$

where $j_H$ denotes the inclusion $H_d \to G$. On the other hand, $\gamma_{H_d}$ is a subrepresentation of $\gamma_{G_d}|H_d$ and

$$\langle \gamma_G \circ i_G(x)f, f \rangle = \langle \gamma_G \circ j_H(x)f, f \rangle$$

for all $x \in H$ and $f \in L^2(G)$. This proves $\gamma_G \circ i_G \prec \gamma_{G_d}$. □

Theorem 1.7. Let $G$ be a locally compact group. If $G_d$ is amenable, then $\gamma_G \circ i_G \prec \gamma_{G_d}$, and $C^*(\gamma_G \circ i_G)$ and $C^*(\gamma_{G_d})$ are isomorphic.

Proof. We first reduce to the $\sigma$-compact case. To that end, let $\mathcal{H}$ denote the set of all $\sigma$-compact open subgroups $H$ of $G$, and suppose that we already know $\gamma_H \circ i_H \prec \gamma_{H_d}$ for every $H \in \mathcal{H}$. To show that $\gamma_G \circ i_G \prec \gamma_{G_d}$, let a finite subset $F$ of $G$ and $f \in L^2(G)$ be given and consider the function $\varphi(x) = \langle \gamma_G(x)f, f \rangle$. Choose $H \in \mathcal{H}$ such that $F \subseteq H$ and $f|G\backslash H = 0$. Then $\varphi(x) = \langle \gamma_H(x)f|H, f|H \rangle$ for all $x \in H$. Since $\gamma_H \circ i_H \prec \gamma_{H_d}$, $\varphi$ can be approximated on $F$ by sums of positive definite functions associated to $\gamma_{H_d}$. It follows that

$$\gamma_G \circ i_H \prec \gamma_{H_d} \prec \gamma_{G_d}|H_d,$$

and hence $\gamma_G \circ i_G \prec \gamma_{G_d}$. That, conversely, $\gamma_{G_d} \prec \gamma_G \circ i_G$ is seen in the same way.

Recall next that, by Corollary 1.5, $\gamma_H \circ i_H \prec \gamma_{H_d}$ for each $H \in \mathcal{H}$. From Lemma 1.6 we know that conversely $\gamma_H \circ i_H \prec \gamma_{H_d}$, provided that $H$ is second countable. Thus it remains to extend this to the case of a $\sigma$-compact group $H$.

Being $\sigma$-compact, $H$ is a projective limit of second countable groups $H_\alpha = H/K_\alpha$, $\alpha \in A$, where the $K_\alpha$ are compact. Now, the set

$$\{f \in C_c(H); \text{ for some } \alpha \in A, f(xk) = f(x) \text{ for all } x \in H \text{ and } k \in K_\alpha \}$$

is dense in $C_c(H)$ in the inductive limit topology. Therefore it suffices to approximate a function $x \to \langle \gamma_H(x)f, f \rangle$, where $f \in C_c(H)$ is constant on cosets of some $K = K_\alpha$, on finite subsets of $H$ by sums of positive definite functions associated to $\gamma_{H_d}$. Define $g$ on $H/K$ by $g(xK) = f(x)$ for $x \in H$. Then

$$\langle \gamma_H(x)f, f \rangle = \langle \gamma_{H/K}(xK)g, g \rangle,$$

and by Lemma 1.6 the function on the right can be approximated on finite subsets of $H/K$ by sums of positive definite functions associated to $\gamma_{(H/K)_d}$. Now, $(H/K)_d = H_d/K_d$, and by [20, Lemma 1.1], $\gamma_{H_d/K_d} \prec \gamma_{H_d}$ since $H_d$ is amenable. This shows that $\gamma_H \circ i_H \prec \gamma_{H_d}$ and finishes the proof. □

Obviously, if $G_d$ is amenable, then so is $G$. As to the regular representation, it has been observed in [2, Theorem 3] that if $G$ is amenable and $\lambda_{G_d} \sim \lambda_G \circ i_G$, then $G_d$ is amenable. In fact, under these assumptions,

$$1_{G_d} = 1_G \circ i_G \prec \lambda_G \circ i_G \sim \lambda_{G_d}.$$
Although it is conceivable, we do not know whether, as a converse to Theorem 1.7, amenability of $G$ and $\gamma_{G_d} \sim \gamma_G \circ i_G$ imply that $G_d$ is amenable.

We conclude this section by returning to the question of when $\gamma_{G_d} \prec \gamma_G \circ i_G$. Recall that a locally compact group is said to be an [SIN]-group if $G$ has a system of neighbourhoods $V$ of the identity such that $x^{-1}Vx = V$ for all $x \in G$.

**Lemma 1.8.** If $G$ is an [SIN]-group, then $\gamma_{G_d} \prec \gamma_G \circ i_G$.

**Proof.** It suffices to approximate the function $x \mapsto \chi_{C(a)}(x) = \langle \gamma_{G_d}(x)\delta_a, \delta_a \rangle$, $a \in G$, on finite subsets $F$ of $G$ by positive definite functions associated to $\gamma_G \circ i_G$. Now, given such an $F$, there exists an invariant symmetric neighbourhood $V$ of $e$ in $G$ such that $x^{-1}ax \notin V^2a$ for all $x \in F \setminus C(a)$. Let $\varphi = |V|^{-1/2}\chi_V$; then it is easily verified that

$$\langle \gamma_G(x)\varphi, \varphi \rangle = |V|^{-1}\int_V \chi_V(x^{-1}vax) \, dv$$

is equal to 1 for all $x \in C(a)$ and equal to 0 for all $x \in F \setminus C(a)$. □

**Remarks 1.9.** (i) Suppose that $C^*(\lambda_G)$ is nuclear and that $\gamma_{G_d} \prec \gamma_G \circ i_G$. Then $G$ is amenable. This can be seen as follows. Since

$$l_{G_d} \prec \gamma_{G_d} \prec \gamma_G \circ i_G \prec \lambda_G \circ i_G$$

[11, Proposition 1], there is a homomorphism of $C^*(\lambda_G(G)) = C^*(\lambda_G \circ i_G)$ onto $\mathbb{C}$. By [2, Theorem 1] this implies that $G$ is amenable. In particular, for any noncompact connected semisimple Lie group $G$, $\gamma_{G_d}$ is not weakly contained in $\gamma_G \circ i_G$.

(ii) By Lemma 1.8 for $G$ compact, $\gamma_{G_d} \prec \gamma_G \circ i_G$. Moskowitz [22] has shown that, for $G$ a compact connected Lie group, $\text{supp} \gamma_G = (G/Z(G))^\wedge$. This can be used to compare the sets $\text{supp} (\gamma_G \circ i_G)$, $(\text{supp} \gamma_G) \circ i_G$, and $\text{supp} \gamma_{G_d}$. An illustrating example let us look at $G = SO(3)$. Then $(\text{supp} \gamma_G) \circ i_G = \widehat{G} \circ i_G$, and $\widehat{G} \circ i_G$ fails to be dense in $\widehat{G}$ (see [3, Corollary 1]).

Considering $G_d$, it follows from [13, Corollary 1.9] that $\text{supp} \gamma_{G_d} = (G_d)^\wedge \cup \{1_{G_d}\}$ since $(G_d)^\wedge$ is trivial and the centralizer of each matrix in $SO(3) \setminus \{E\}$ has a subgroup of index 2, which is conjugate to $SO(2)$. Thus $\text{supp} \gamma_{G_d} \cap (\text{supp} \gamma_G) \circ i_G = \{1_{G_d}\}$ and $\text{supp} \gamma_{G_d}$ is strictly contained in $\text{supp}(\gamma_G \circ i_G)$, since $1_{G_d}$ is the only finite-dimensional representation in $\text{supp} \gamma_{G_d}$.

2. *When is $C^*(\gamma_G)$ of bounded representation type?*

Let $A$ be a $C^*$-algebra and $\widehat{A}$ its dual space. $A$ is said to be of bounded representation type (b.r.t.) if every $\pi \in \widehat{A}$ is finite dimensional and if there is an upper bound for these dimensions. The analogous notion applies to representations. Thus, a representation $\rho$ of $A$ is of b.r.t. if $\rho(A)$ is of b.r.t. Moreover, a locally compact group $G$ is of bounded representation type if $C^*(G)$ has this property. The first paper dealing with such groups that we are aware of is [15]. Groups of b.r.t. have finally been identified by Moore [21] as precisely those which have an abelian subgroup of finite index.

In this section we are interested in the question of when the $C^*$-algebras $C^*(\gamma_G)$, $C^*(\gamma_{G_d})$ and $C^*(\gamma_G \circ i_G)$ are of bounded representation type. For
such a particular representation, this appears to be a rather intricate problem. We succeeded in resolving it for compactly generated Lie groups, where by Lie group we mean a locally compact group $G$ whose connected component $G_0$ of the identity is open and is an analytic group. However, we were unable to characterize non-finitely-generated discrete groups $G$ or totally disconnected compact groups $G$ with $C^*({\gamma}_G)$ of b.r.t.

It is worth commenting here on the same question for the left regular representation. Now, for any locally compact group $H$, $C^*({\lambda}_H)$ being of b.r.t. implies that $H$ has an abelian subgroup of finite index. Indeed, this follows from [26, Satz 2] and can also be deduced from Moore’s results [21]. As to $C^*({\lambda}_H \circ i_H)$, notice that by [2, Lemma 2] ${\lambda}_H$ is weakly contained in ${\lambda}_H \circ i_H$, so that $C^*({\lambda}_H)$ is of b.r.t. provided that $C^*({\lambda}_H \circ i_H)$ is.

Remarks 2.1. (i) If $\gamma_G$ is of bounded representation type (b.r.t.), then $\gamma_G|H$ is of b.r.t. for every closed subgroup $H$ of $G$. Indeed, let

$$T = \bigcup_{\pi \in \supp \gamma_G} \supp(\pi|H) \subseteq \hat{H},$$

and suppose that $\dim \pi \leq d$ for all $\pi \in \supp \gamma_G$. Then $\dim \tau \leq d$ for all $\tau \in T$, and hence for all $\tau \in \bar{T}$. On the other hand, $\bar{T} = \supp(\gamma_G|H)$ since $T$ is weakly equivalent to $\gamma_G|H$.

(ii) Let $H$ be an open subgroup of $G$. If $\gamma_G$ is of b.r.t., then so is $\gamma_H$. In fact, by (i) $\gamma_G|H$ is of b.r.t., and $\gamma_H$ is a subrepresentation of $\gamma_G|H$ as $L^2(H)$ is a subspace of $L^2(G)$. Notice, however, that in general for a closed subgroup $H$ of $G$, $\gamma_H$ need not even be weakly contained in $\gamma_G|H$ (see [14]).

(iii) If $\gamma_G$ is of b.r.t. and $C^*({\lambda}_G)$ is nuclear, then $G$ is amenable. The nuclearity assumption guarantees that $\gamma_G \prec {\lambda}_G$ [11, Proposition 1]. Now, it is well known that $G$ is amenable provided that $\lambda_G$ weakly contains a finite-dimensional representation. Recall that $C^*({\lambda}_G) (C^*(G)$, as a matter of fact) is nuclear if $G/G_0$ is amenable.

If $N$ is a closed normal subgroup of $G$, then $G$ acts on $\hat{N}$ by $(x, \lambda) \rightarrow \lambda^x$, where $\lambda^x(n) = \lambda(x^{-1}nx)$ for $x \in G$ and $n \in N$, and $G_\lambda$ denotes the stability subgroup of $\lambda$ in $G$ under this action.

Lemma 2.2. Let $G$ be a locally compact group, and suppose that $\supp \gamma_G$ contains a dense subset of finite-dimensional representations. Let $N$ be a closed normal subgroup of $G$ such that $N/N \cap Z(G)$ is a vector group. Then there exists a closed subgroup $H$ of finite index in $G$ such that $N \subseteq Z_\Sigma(H)$.

Proof. Let $\Pi = \{\pi \in \supp \gamma_G; \dim \pi < \infty\}$ and $\Lambda = \bigcup_{\pi \in \Pi} \supp(\pi|N)$. By hypothesis, $\gamma_G|N \sim \Pi|N \sim \Lambda$, so that $\Lambda$ separates the points of $\hat{V} = N/N \cap Z(G)$. $\hat{V}$ and hence $\bar{V}$ being a vector group, $\Lambda$ contains a basis $\{\lambda_1, \ldots, \lambda_m\}$ of $\bar{V}$. Now, $H = \bigcap_{j=1}^m G_{\lambda_j}$ has finite index in $G$ and $\lambda_j = \lambda_j$ for all $h \in H$ and $1 \leq j \leq m$. Since continuous automorphisms of vector groups are linear, it follows that $\lambda^h = \lambda$ for all $\lambda \in \hat{V}$ and $h \in H$. This implies that $N/N \cap Z(G) \subseteq Z(H/N \cap Z(G))$ and hence $N \subseteq Z_\Sigma(H)$. $\square$

Lemma 2.3. Let $G$ and $\gamma_G$ be as in Lemma 2.2. Let $N$ be a closed normal subgroup of $G$ such that $N/N \cap Z(G) = \mathbb{T}^m$ for some $m \in \mathbb{N}$. Then $N \subseteq Z_\Sigma(H)$ for some subgroup $H$ of finite index in $G$. 

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Proof. Let Π and Λ be as in the proof of the previous lemma. Then Λ generates \((N/N \cap Z(G))^\sim = Z^m\), so that \(G_1\) has finite index in \(G\) for each \(λ \in Z^m\). As \(Z^m\) is finitely generated, we find a subgroup \(H\) of finite index in \(G\) such that \(λ^h = λ\) for all \(λ \in Z^m\) and all \(h \in H\). This proves that \(N/N \cap Z(G) \subseteq Z(H/N \cap Z(G))\) and hence \(N \subseteq Z_2(H)\). □

Lemma 2.4. Let \(K\) be a compact connected normal subgroup of the Lie group \(G\). If \(γ_G\) is of b.r.t., then the commutator subgroup \(K'\) of \(K\) is contained in the centre of \(G\).

Proof. It suffices to show that \(K' \subseteq Z(H)\) for every \(σ\)-compact open subgroup \(H\) of \(G\). Since \(γ_H\) is of b.r.t. for every such \(H\), we can assume that \(G\) is \(σ\)-compact and hence second countable as it is a Lie group. Recall that by [19, Lemma 3.1], for any second countable group \(G\), \(γ_G\) is unitarily equivalent to the restriction of \(\text{ind}_{\Delta_G}^{G \times G} 1_{Δ_G}\) to \(Δ_G\) where \(Δ_G\) denotes the diagonal subgroup of \(G \times G\). Since \(K\) is compact and \(G\) is second countable, \(Δ_K\) and \(Δ_G\) are regularly related in \(G \times G\) in the sense of Mackey. Therefore, by [6, Theorem 5.3], with \(Δ_G^u = uΔ_Gu^{-1}\) for \(u \in G \times G\),

\[
γ_G | K = \text{ind}_{Δ_G}^{G \times G} 1_{Δ_G} | Δ_K \sim \{\text{ind}_{u^{-1}Δ_Gu \cap Δ_K}^{Δ_K} 1_{u^{-1}Δ_Gu \cap Δ_K} ; u \in G \times G\}
\]

\[
= \{\text{ind}_{C(a) \cap K}^{K} 1_{C(a) \cap K} ; a \in G\} = \{\text{ind}_{C(a)}^{K} ; a \in G\}.
\]

Fix \(a \in G\), and let \(N(a)\) denote the greatest normal subgroup of \(K\) contained in \(C_K(a)\). There exist finitely many \(x_1, \ldots, x_m \in K\) such that

\[
N(a) = \bigcap_{j=1}^{m} x_j^{-1}C_K(a)x_j
\]

(compare [1, Proposition 2.1]). By [6, Theorem 5.5] the \(m\)-fold tensor product \((γ_G | K)^{σm}\) weakly contains

\[
\text{ind}_{C(a) \cap K}^{K} 1_{C(a) \cap K} \bigotimes_{j=1}^{m} x_j^{-1}C_K(a)x_j = \text{ind}_{N(a)}^{K} 1_{N(a)}.
\]

Now tensor products of representations of b.r.t. are again of b.r.t. [25, Lemma 5]. Thus \(\text{ind}_{N(a)}^{K} 1_{N(a)}\) is of b.r.t., and since \(K\) is connected this yields that \(K/N(a)\) is abelian. It follows that

\[
K/ \left( \bigcap_{a \in G} (C(a) \cap K) \right) = K/ \bigcap_{a \in G} N(a)
\]

is abelian. This proves \(K' \subseteq \bigcap_{a \in G} C(a) = Z(G)\). □

Proposition 2.5. Let \(G\) be a Lie group and \(N\) a connected closed normal subgroup of \(G\). If \(C^*(γ_G)\) is of b.r.t., then there exists a subgroup \(H\) of finite index in \(G\) such that \(N \subseteq Z_6(H)\).

Proof. Let \(M = N \cap Z(G)\). Since \(γ_G | N\) separates the points of \(N/M\), \(N/M\) is a maximally almost periodic connected Lie group. By the Freudenthal-Weil theorem [5, Théorème 16.4.6] \(N/M\) is a direct product of a vector group \(W\) and a compact connected Lie group \(K\).

Let \(q : G \to G/M\) be the quotient homomorphism. As \(K\) is normal in \(G/M\), it follows from Lemma 2.4 that \(K' \subseteq Z(G/M)\) and hence \(q^{-1}(K') \subseteq \)
$\mathbb{Z}_2(G)$. Applying Lemma 1.1 in [14] twice gives $\gamma_{G/q^{-1}(K')} \prec \gamma_G$, so that $\gamma_{G/q^{-1}(K')}^2$ is of b.r.t. Now, $K/K'$ is a normal torus in $G/q^{-1}(K')$. It follows from Lemma 2.3 that $K/K' \subseteq \mathbb{Z}_2(H_1/q^{-1}(K'))$ for some subgroup $H_1$ of finite index in $G$. Thus $q^{-1}(K') \subseteq \mathbb{Z}_2(H_1)$.

Now, moving to $G/q^{-1}(K)$, similar arguments apply to the normal vector subgroup $W$ of $G/q^{-1}(K)$. Again, since continuous automorphisms of vector groups are linear, $W \cap \mathbb{Z}(G/q^{-1}(K))$ is a vector group and hence so is $W/W \cap \mathbb{Z}(G/q^{-1}(K))$. Lemma 2.2 yields that $W \subseteq \mathbb{Z}_2(H_2/q^{-1}(K))$ for some subgroup $H_2$ of finite index in $G$ containing $q^{-1}(K)$. With $H = H_1 \cap H_2$, we obtain that $N \subseteq \mathbb{Z}_6(H)$. □

Remark 2.6. Let $D$ be a discrete group with $\gamma_D$ of b.r.t. Then, since $\lambda_{D/D_f} < \gamma_D$ [13, Theorem 1.8], $\lambda_{D/D_D}$ is of b.r.t. and therefore $D/D_f$ has an abelian subgroup of finite index (compare [26, Satz 1]). In particular, $D$ is amenable. It is worthwhile to remind the reader that in order to conclude that a discrete group $G$ is almost abelian it is only required that $\lambda_G$ is of type I [10].

Corollary 2.7. If $G$ is a Lie group with $C^*(\gamma_G)$ of b.r.t., then $G_d$ is amenable and $C^*(\gamma_{G_d})$ and $C^*(\gamma_G \circ i_G)$ are both of b.r.t.

Proof. By Proposition 2.5, $G_0 \subseteq \mathbb{Z}_m(H)$ for some $m \in \mathbb{N}$ and some subgroup $H$ in $G$ of finite index. In particular, $G_0$ is nilpotent. Let $D = G/G_0$; then repeated application of [14, Lemma 1.1] gives $\gamma_D \prec \gamma_G$. Thus $\gamma_D$ is of b.r.t., and hence $D$ is amenable (Remark 2.6). Since $(G_0)_d$ and $G_0$ are amenable, $G_d$ is amenable.

By what we have seen in Theorem 1.7, $\gamma_{G_d} \sim \gamma_G \circ i_G$, and $G^*(\gamma_{G_d})$ and $C^*(\gamma_G \circ i_G)$ are isomorphic. Thus it remains to recognize that $\gamma_{G_d}$ is of b.r.t. But this follows because $\gamma_G$ is of b.r.t. and $\text{supp} \gamma_{G_d}$ is contained in the closure of $(\text{supp} \gamma)_G \circ i_G$ in $\tilde{G}_d$. □

Corollary 2.8. For a connected group $G$, $C^*(\gamma_G)$ is of bounded representation type if and only if $G$ is 2-step nilpotent.

Proof. Clearly, if $G/Z(G)$ is abelian, then every $\pi \in \text{supp} \gamma_G$ is one-dimensional. Conversely, suppose that $G$ is connected and $\gamma_G$ is of b.r.t. Then $G$ is a projective limit of Lie groups $G_i = G/K_i$, $i \in I$, where the $K_i$ are compact, and every $\gamma_{G_i}$ is of b.r.t. Let $q_i : G \to G_i$ denote the quotient homomorphism. Since $Z(G) = \bigcap_{i \in I} q_i^{-1}(Z(G_i))$, $G$ is 2-step nilpotent if all $G_i$ are. Therefore we can assume that $G$ is a Lie group.

By Corollary 2.7, $\gamma_{G_f}$ is of b.r.t., and hence $G/G_f$ has an abelian subgroup of finite index. For any $x \in G_f$, $C(x)$ is a closed subgroup of finite index in $G$, so that $x \in Z(G)$. It follows that $\mathbb{G}_f \subseteq Z(G)$, and $G/\mathbb{G}_f$ has a closed abelian subgroup of finite index. $G$ being connected, we obtain that $G/Z(G)$ is abelian. □

Lemma 2.9. Let $D$ be a discrete group such that $\gamma_D$ is of b.r.t. For $x \in D$ let $N(x)$ denote the greatest normal subgroup of $D$ contained in $C(x)$. Suppose that for some finite subset $F$ of $D$, $\bigcap_{x \in F} N(x) = Z(D)$. Then $D/Z(D)$ has an abelian subgroup of finite index.
Proof. Since \( \text{ind}^D_{C(x)} 1_{C(x)} \prec \gamma_D \) for each \( x \in D \), all these quasi-regular representations are of b.r.t. The kernel of \( \text{ind}^D_{C(x)} 1_{C(x)} \) is \( N(x) \) as is easily verified.

Now, \( \text{ind}^D_{C(x)} 1_{C(x)} \) being of b.r.t. is equivalent to the algebra generated by the operators \( \text{ind}^D_{C(x)} 1_{C(x)}(y) \), \( y \in D \), on \( L^2(D/C(x)) \) satisfying a standard polynomial identity (see [15] and [21]).

Therefore, by Satz 1 of [26], the factor group \( D/N(x) \), which is isomorphic to \( \text{ind}^D_{C(x)} 1_{C(x)}(D) \), has an abelian subgroup \( A(x)/N(x) \) of finite index. With

\[
A = \bigcap_{x \in F} A(x)
\]

it follows that \( A \) has finite index in \( D \) and

\[
A' \subseteq \bigcap_{x \in F} A(x)' \subseteq \bigcap_{x \in F} N(x) = Z(D). \quad \Box
\]

Theorem 2.10. For a compactly generated Lie group \( G \) the following conditions are equivalent:

(i) \( C^*(\gamma_G) \) is of bounded representation type.
(ii) \( C^*(\gamma_{G_d}) \) is of bounded representation type.
(iii) \( C^*(\gamma_G \circ i_G) \) is of bounded representation type.
(iv) \( G/Z(G) \) possesses an abelian subgroup of finite index.

Proof. (iv) \( \Rightarrow \) (i), (ii), (iii) are clear since all three representations \( \gamma_G \), \( \gamma_{G_d} \), and \( \gamma_G \circ i_G \) are trivial on \( Z(G) = Z(G_d) \). (i) \( \Rightarrow \) (ii) and (i) \( \Rightarrow \) (iii) are consequences of Corollary 2.7.

Notice next that (iii) \( \Rightarrow \) (ii). In fact, if \( C^*(\gamma_{G_d}) \) is of b.r.t., then so is \( C^*(\lambda_{G_d}G/G_d) \) by Lemma 1.1. This implies that \( G_d/(G_d)_f \) is almost abelian and hence \( G_d \) is amenable. Theorem 1.8 now shows that \( C^*(\gamma_{G_d}) \) is of b.r.t.

It remains to show (ii) \( \Rightarrow \) (iv). For that we want to apply Lemma 2.9. Thus we have to produce a finite subset \( F \) of \( G \) such that \( \bigcap_{x \in F} N(x) = Z(G) \).

To construct \( F \) let \( Z_0 = Z(G) \cap G_0 \) and notice that \( \gamma_G | G_0 \) separates the points of \( G_0/Z_0 \) and is of b.r.t. by Remarks 2.1 (i). Therefore \( G_0/Z_0 \) is a maximally almost periodic connected Lie group. It follows from the Freudenthal-Weil theorem (see [5, Théorème 16.4.6]) that \( G_0/Z_0 \) is a direct product of a compact Lie group \( K \) and some \( \mathbb{R}^m \). Now, \( \gamma_{G/Z_0} \) is of b.r.t. and \( K \) is normal in \( G/Z_0 \). An application of Lemma 2.4 yields that \( K \) is 2-step nilpotent. As is well known this implies that \( K \), being a compact connected Lie group, is a torus \( T^n \).

Let \( q : G \to G/G_0 \) and \( h : G \to G/Z_0 \) denote the quotient homomorphisms. Choose a finite subset \( F_1 \) of \( G \) such that \( q(F_1) \) generates \( G/G_0 \) as a group. Both \( \mathbb{R}^m \) and \( T^n \) contain finitely generated dense subgroups. Thus there exist finite subsets \( F_2 \) and \( F_3 \) of \( G_0 \) such that \( h(F_2) \) and \( h(F_3) \) generate a dense subgroup of \( \mathbb{R}^m \) and \( T^n \), respectively. Finally, let \( F = F_1 \cup F_2 \cup F_3 \). It is now obvious that \( F \cup Z_0 \) generates a dense subgroup of \( G \), whence \( C(F) = Z(G) \). This completes the proof. \( \Box \)

One might well expect that Theorem 2.10 remains true when the assumption that \( G \) be compactly generated is dropped. However, as mentioned earlier, we did not succeed in proving that if \( G \) is a (not necessarily finitely generated)
discrete group with $\mathcal{C}^*(\gamma_G)$ of b.r.t., then $G/Z(G)$ must be almost abelian. This is surprising since we already know that $G/G_f$ is almost abelian (Remark 2.6). The point is that it seems to be difficult to handle discrete groups with finite conjugacy classes (so-called [FC]-groups). We finish the paper by looking at a special class of [FC]-groups.

**Example 2.11.** Let $G$ be the restricted direct product of finite groups $G_i$, $i \in I$. We claim that the following conditions are equivalent:

(i) $\mathcal{C}^*(\gamma_G)$ is of bounded representation type.

(ii) $\dim \pi < \infty$ for every $\pi \in \supp(\gamma_G)$.

(iii) $G_i$ is 2-step nilpotent for almost all $i \in I$.

Condition (iii) implies that $G/Z_2(G)$ is finite, whence (i) follows. To verify (ii) $\Rightarrow$ (iii), first consider a finite group $F$. Suppose that $\supp(\sigma \otimes \tau) \subseteq (F/F')^\sim$ for all $\sigma \in \tilde{F}$. Then $\sigma | F'$ has to be a multiple of a $G$-invariant character for all $\sigma \in \tilde{F}$, and this yields $F' \subseteq Z(F)$. Thus, if $F$ fails to be 2-step nilpotent, then for at least one $\sigma \in \tilde{F}$, $\sigma \otimes \tau$ has an irreducible subrepresentation of dimension $\geq 2$.

Now, suppose that $G_i$ is not 2-step nilpotent for all $i$ in some infinite subset $J$ of $I$. For each $i \in J$, choose $\sigma_i \in \tilde{G}_i$ and $\tau_i \in \supp(\sigma_i \otimes \tau_i)$ with $\dim \tau_i \geq 2$. For $i \in I \setminus J$, let $\sigma_i = \tau_i = 1_{G_i}$. The infinite tensor products $\pi = \bigotimes_{i \in I} \sigma_i$ and $\rho = \bigotimes_{i \in I} \tau_i$ are irreducible [9, §11], $\rho$ is infinite dimensional, and $\rho \in \supp(\pi \otimes \pi)$. This contradicts (ii).

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**REFERENCES**


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