POWER REGULAR OPERATORS

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Abstract. We show that for a wide class of operators $T$ on a Banach space, including the class of decomposable operators, the sequence $\{\|T^n x\|^{1/n}\}_{n=1}^{\infty}$ converges for every $x$ in the space to the spectral radius of the restriction of $T$ to the subspace $\bigvee_{n=0}^{\infty}(T^n x)$.

1. Introduction

Throughout this paper, $X$ will denote a complex Banach space and $\mathcal{L}(X)$ the Banach algebra of bounded linear operators on $X$. For an operator $T$ in $\mathcal{L}(X)$, we denote as usual by $\sigma(T)$ its spectrum and by $r(T)$ its spectral radius. By Gelfand's formula for the spectral radius

$$r(T) = \lim_{n \to \infty} \|T^n\|^{1/n}$$

for all $T$ in $\mathcal{L}(X)$.

It is well known that if $\{w_n\}_{n=1}^{\infty}$ is a sequence of nonnegative numbers which is submultiplicative (that is, $w_{m+n} \leq w_m w_n$ for all $m$ and $n$), then $\lim_{n \to \infty} w_n^{1/n}$ exists. Hence, the existence of the limit in the right-hand side of Gelfand's formula can be deduced from the fact that for every $T$ in $\mathcal{L}(X)$, the sequence $\{\|T^n\|\}_{n=1}^{\infty}$ is submultiplicative. On the other hand, for $T$ in $\mathcal{L}(X)$ and $x$ in $X$, the sequence $\{\|T^n x\|\}_{n=1}^{\infty}$ is in general not submultiplicative. Nevertheless, we shall show in this paper that for a wide class of operators $T$ in $\mathcal{L}(X)$, the sequence $\{\|T^n x\|^{1/n}\}_{n=1}^{\infty}$ is convergent for all $x$ in $X$. We shall call such operators power-regular.

We shall prove, in particular, that all decomposable operators (see definition in §3) are power-regular. By [4] this class includes all spectral operators in Dunford's sense (hence all normal operators) and all operators with totally disconnected (hence countable) spectrum. We shall also prove that every operator $T$ in $\mathcal{L}(X)$ for which the set $\{\lambda; \lambda \in \sigma(T)\}$ has empty interior in $[0, \infty)$ is power-regular. This class clearly contains all operators with spectrum included in a countable union of circles with centers at the origin. Moreover, we shall show that for every operator $T$ in $\mathcal{L}(X)$ which belongs to one of these classes, the sequence $\{\|T^n x\|^{1/n}\}_{n=1}^{\infty}$ converges for all $x$ in $X$ to the spectral radius of the restriction of $T$ to the subspace $\bigvee_{n=0}^{\infty}(T^n x)$.

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3101
Power-regularity of compact operators and selfadjoint operators can also be deduced from the results in [11, §9]. In [5] power-regularity is also established for a more general class of operators with countable spectrum and for normal operators.

We recall that the local spectral radius of an operator $T$ in $\mathcal{L}(X)$ at a vector $x$ in $X$ is defined by (cf. [5] and [13])

$$r(x, T) = \lim sup_{n \to \infty} \|T^n x\|^{1/n}.$$ 

As shown in [5], for every $T$ in $\mathcal{L}(X)$, the equality $r(x, T) = r(T)$ holds for quasi-all $x$ in $X$ (that is, for all $x$ in the complement of a set of first category). Thus if $T$ is power-regular,

$$r(x, T) = \lim_{n \to \infty} \|T^n x\|^{1/n}$$

for all $x$ in $X$, and the limit is equal to $r(T)$ for quasi-all $x$ in $X$.

We mention also that by [13], for every $T$ in $\mathcal{L}(X)$, $\{\|T^n x\|^{1/n}\}_{n=1}^\infty$ converges to $r(T)$ for all $x$ in a dense subset of $X$. For Hilbert spaces this follows also from [3, Theorem 2.1]. On the other hand, it is proved in [5] that if for $T$ in $\mathcal{L}(X)$ and $x$ in $X$ the sequence $\{\|T^n x\|^{1/n}\}_{n=1}^\infty$ does not converge, then its set of limit points is a closed interval.

It is easy to construct weighted shifts which are not power-regular (cf. [5] and [9]). We shall also see in §6 that the backward shift on $l^2$ is not power-regular.

In §2 we establish a general criterion for power-regularity from which most of our subsequent results are derived. In §3 we prove power-regularity of decomposable operators and deduce several corollaries. In §4 we introduce the class of radially decomposable operators and prove power-regularity of operators belonging to the subclass of radially super-decomposable operators. We show that this subclass contains all operators $T$ in $\mathcal{L}(X)$ for which the set $\{\lambda; \lambda \in \sigma(T)\}$ has empty interior in $[0, \infty)$. In §5 we give two direct and elementary proofs of power-regularity of operators on a finite-dimensional space. Finally, in §6, we present some additional facts and examples and raise several problems.

## 2. A general criterion

In this section we prove a general criterion for power-regularity which is applied in the sequel to establish power-regularity of operators in some concrete classes, in particular in those mentioned in the previous section. We first need some notation.

Let $T$ be an operator in $\mathcal{L}(X)$. We denote as usual by $\text{Lat}(T)$ the collection of all closed subspaces of $X$ which are invariant under $T$, and for a subspace $M$ in $\text{Lat}(T)$ we denote by $T_M$ the restriction of $T$ to $M$. We shall denote by $q(T)$ the minimum of the set $\{|\lambda|; \lambda \in \sigma(T)\}$ (for $X = \{0\}$, we define $q(T) = \infty$). For every $x$ in $X$ we shall denote by $r_x(T)$ the spectral radius of the restriction of $T$ to the subspace $\bigvee_{n=0}^\infty \{T^n x\}$ (for $x = 0$, we define $r_x(T) = 0$). Note that by the spectral radius formula,

$$r(x, T) \leq r_x(T) \leq r(T)$$

for all $x$ in $X$.

We can now state our general criterion for power-regularity.
Theorem 2.1. Let $T$ be an operator in $\mathcal{L}(X)$, and assume that for every $0 < t_1 < t_2 < \infty$ there exist a complex Banach space $Y$, an operator $S$ in $\mathcal{L}(Y)$, and a bounded linear operator $J$ from $X$ into $Y$, such that setting $M = \ker J$, the following three conditions hold:

(I) $J^*T = SJ$;
(II) $r(T_M) \leq t_2$;
(III) $\rho(S) \geq t_1$.

Then for all $x$ in $X$,

$$\lim_{n \to \infty} \|T^n x\|^{1/n} = r_x(T).$$

Remarks. (1) Condition (I) implies that $M \subseteq \text{Lat}(T)$.
(2) For $t_2 > r(T)$ the above conditions hold for every $T$ in $\mathcal{L}(X)$, with $Y = X$, $S = t_2 I$, and $J = 0$.

The following fact is needed for the proof of the theorem.

Lemma 2.2. If $Y$ is a complex Banach space and $S$ is an operator in $\mathcal{L}(Y)$, then for every $y \neq 0$ in $Y$,

$$\rho(S) \leq \liminf_{n \to \infty} \|T^n y\|^{1/n}.$$ 

Proof. The inequality is clear if $\rho(S) = 0$. Assume that $\rho(S) > 0$. Then $S$ is invertible and $r(S^{-1}) = (\rho(S))^{-1}$. Hence, noticing that for every $y$ in $Y$

$$\|y\| \leq \|S^{-n}\| \|S^n y\|, \quad n = 1, 2, \ldots,$$

by using the spectral radius formula for $S^{-1}$ we obtain that if $y \neq 0$,

$$1 = \lim_{n \to \infty} \|y\|^{1/n} \leq (\rho(S))^{-1} \liminf_{n \to \infty} \|S^n y\|^{1/n},$$

and the lemma is proved.

Proof of Theorem 2.1. Let $x \in X$. If $r_x(T) = 0$ the assertion is clear. Assume that $r_x(T) > 0$, and consider numbers $t_1$ and $t_2$ such that $0 < t_1 < t_2 < r_x(T)$. Let $S$ and $J$ be the operators which satisfy the conditions of the theorem for $t_1$ and $t_2$. We claim that $Jx \neq 0$. In fact, assuming that $x \in M = \ker J$ and remembering that $M \subseteq \text{Lat}(T)$, we obtain that $\bigvee_{n=0}^{\infty} \{T^n x\} \subseteq M$, and therefore $r_x(T) \leq r(T_M)$, which by condition (II) contradicts the choice of $t_2$. So $Jx \neq 0$ (and in particular $J \neq 0$). Thus by Lemma 2.2 and condition (III),

$$t_1 \leq \rho(S) \leq \liminf_{n \to \infty} \|S^n Jx\|^{1/n}.$$ 

Therefore, observing that by condition (I),

$$J T^n x = S^n J x, \quad n = 1, 2, \ldots,$$

we obtain that

$$t_1 \leq \liminf_{n \to \infty} \|S^n J x\|^{1/n} \leq \liminf_{n \to \infty} (\|J\|^{1/n} \|T^n x\|^{1/n}) = \liminf_{n \to \infty} \|T^n x\|^{1/n};$$

and since $t_1$ is an arbitrary number less than $r_x(T)$, we deduce that

$$r_x(T) \leq \liminf_{n \to \infty} \|T^n x\|^{1/n}.$$ 

But as noticed at the beginning of this section,

$$r(x, T) = \limsup_{n \to \infty} \|T^n x\|^{1/n} \leq r_x(T),$$
and we conclude that

$$\lim_{n \to \infty} \|T^n x\|^{1/n} = r_x(T).$$

This completes the proof of the theorem.

3. Decomposable operators

In this section we shall establish power-regularity of decomposable operators. We recall that according to [1] an operator $T$ in $\mathcal{L}(X)$ is decomposable if for every cover of the complex plane by a pair of open sets $U$ and $V$, there exist subspaces $M$ and $K$ in $\text{Lat}(T)$ such that $M + K = X$, $\sigma(T_M) \subseteq U$, and $\sigma(T_K) \subseteq V$.

The main ingredients in the proof of power-regularity of decomposable operators are Theorem 2.1 in the preceding section and Theorem 12.15 in [7].

In the proof we shall need an additional notation. For an operator $T$ in $\mathcal{L}(X)$ and a subspace $M$ in $\text{Lat}(T)$, we shall denote by $T^M$ the canonical operator induced by $T$ on the quotient space $X/M$.

Theorem 3.1. If $T$ is a decomposable operator in $\mathcal{L}(X)$, then for all $x$ in $X$,

$$\lim_{n \to \infty} \|T^n x\|^{1/n} = r_x(T).$$

Proof. Let $0 < t_1 < t_2 < \infty$, and consider the disc $G = \{ \lambda \in \mathbb{C}; |\lambda| < t_2 \}$. Since $T$ is decomposable, it follows from [7, Theorem 12.15] that there exists a subspace $M$ in $\text{Lat}(T)$, such that $\sigma(T_M) \subseteq \overline{G}$ and $\sigma(T^M) \cap G = \emptyset$. This is equivalent to the conditions $r(T_M) \leq t_2 \leq q(T^M)$. From this it is readily verified that the conditions of Theorem 2.1 hold for $t_1$ and $t_2$, with $Y = X/M$, $S = T^M$, and $J$ the canonical map of $X$ onto $X/M$, and the proof is complete.

It follows from Theorem 3.1 that all operators considered in [4] are power-regular. In particular, from [4, pp. 33, 67, and 185] we obtain the following.

Corollary 3.2. If $T$ is an operator in $\mathcal{L}(X)$, then each of the following conditions implies that the conclusion of Theorem 3.1 holds for $T$.

1. $T$ is a spectral operator in Dunford’s sense (hence, in particular, if $T$ is a normal operator in Hilbert space).
2. $\sigma(T)$ is totally disconnected (hence, in particular, if $\sigma(T)$ is countable).
3. $\sigma(T)$ is included in the real line, and the integral

$$\int_0^1 \ln \ln \sup_{|\text{Im}\lambda| > y} \|(T - \lambda I)^{-1}\| \, dy$$

is convergent. (This condition is satisfied, in particular, if $\|(T - \lambda I)^{-1}\| \leq c \exp(b/|\text{Im}\lambda|^{\alpha})$, for $\text{Im}\lambda \neq 0$, where $\alpha$, $b$, $c$ are positive constants.)

Another corollary is concerned with Banach algebras. We shall say that a Banach algebra $B$ is power-regular, if $\lim_{n \to \infty} \|x^n y\|^{1/n}$ exists for all $x$ and $y$ in $B$.

The following is an immediate consequence of Theorem 3.1 and [4, Theorem 2.6, p. 201].
Corollary 3.3. Every commutative semi-simple regular Banach algebra is power-regular.

4. Radially decomposable operators

According to [7, Theorem 12.5], if $T$ is a decomposable operator, then for every open set $G$ in the complex plane, there exists a subspace $M$ in $\text{Lat}(T)$ such that $\sigma(TM) \subset G$ and $\sigma(TM) \cap G = \emptyset$. In the proof of Theorem 3.1 we used only the fact that this holds when $G$ is an open disc with center at the origin. This suggests that power-regularity might be true for operators $T$ in $\mathcal{L}(X)$ which satisfy a condition that is considerably weaker than decomposability, namely, that for every $0 < t_1 < t_2 < \infty$, there exists subspaces $M$ and $K$ in $\text{Lat}(T)$, such that $M + K = X$, $q(TM) \geq t_1$, and $r(T_K) \leq t_2$. We shall call operators which satisfy this condition radially decomposable.

We conjecture that all radially decomposable operators are power-regular. This would follow from the proof of Theorem 3.1, if one could show that the conclusion of Theorem 12.15 in [7] holds for these operators for open discs with centers at the origin.

We prove in this section power-regularity of operators which belong to a somewhat more restricted class. Before describing it, we mention that, according to [12], an operator $T$ in $\mathcal{L}(X)$ is called super-decomposable if for every cover of the complex plane by a pair of open sets $U$ and $V$, there exists an operator $A$ in $\mathcal{L}(X)$ which commutes with $T$ such that $\sigma(T_{AX}) \subset U$ and $\sigma(T_{(I-A)X}) \subset V$.

Motivated by this terminology, we shall say that an operator $T$ in $\mathcal{L}(X)$ is radially super-decomposable if for every $0 < t_1 < t_2 < \infty$, there exists an operator $A$ in $\mathcal{L}(X)$ which commutes with $T$ such that $q(T_{AX}) \geq t_1$ and $r(T_{(I-A)X}) \leq t_2$.

It is clear that a radially super-decomposable operator is radially decomposable and that a super-decomposable operator is decomposable and radially super-decomposable.

Theorem 4.1. If $T$ is a radially super-decomposable operator in $\mathcal{L}(X)$, then for all $x$ in $X$,

$$\lim_{n \to \infty} \|T^n x\|^{1/n} = r_x(T).$$

Proof. Let $0 < t_1 < t_2 < \infty$, and consider an operator $A$ in $\mathcal{L}(X)$ which commutes with $T$ such that $q(T_{AX}) \geq t_1$ and $r(T_{(I-A)X}) \leq t_2$. Observing that $\ker A \subset (I - A)X$, we see that $r(T_{\ker A}) \leq r(T_{(I-A)X})$; and this implies that the conditions of Theorem 2.1 hold for $t_1$ and $t_2$, with $Y = AX$, $S = T_Y$, and $J = A$. This completes the proof.

Remark. It follows from [12, Propositions 2.1 and 2.2] that all operators that satisfy the conditions of Corollary 3.2 are super-decomposable, and therefore the corollary also follows from Theorem 4.1. By [12, Corollary 2.4] the same is true for Corollary 3.3.

Theorem 4.2. If $T$ is an operator in $\mathcal{L}(X)$ such that the set $\{ |\lambda| ; \lambda \in \sigma(T) \}$ has empty interior in $[0, \infty)$, then the conclusion of Theorem 4.1 holds for $T$.

Proof. We shall show that $T$ is radially super-decomposable. Let $0 < t_1 < t_2$, and consider the set $B = \{ |\lambda| ; \lambda \in \sigma(T) \}$. Since $B$ has empty interior
in \([0, \infty)\), there exist \(t \notin B\) such that \(t_1 < t < t_2\), and therefore the set 
\[\tau = \{\lambda \in \mathbb{C}; |\lambda| \geq t\} \cap \sigma(T)\]
is a spectral set for \(T\) (that is, an open and closed subset of \(\sigma(T)\)). Let \(A\) denote the corresponding spectral projection \(E(\tau, T)\) (see [6, p. 573]). It is well known [6, Chapter 7] that \(A\) commutes with \(T\), 
\(\sigma(T_A) = \tau\), and \(\sigma(T(1-A)^{1/2}) = \sigma(T) \setminus \tau\), and therefore \(q(T_A) \geq t > t_1\) and 
\(r(T(1-A)^{1/2}) \leq t < t_2\). This shows that \(T\) is radially super-decomposable, and 
the assertion follows from Theorem 4.1.

The following is an immediate consequence of Theorem 4.2.

**Corollary 4.3.** If \(T\) is an operator in \(L(X)\) such that \(\sigma(T)\) is included in a 
countable union of circles with centers at the origin, then \(T\) is power-regular.

Corollary 4.3 implies that all operators considered in [2] which are annihilated by a nonzero analytic function are power-regular, since by [2, Theorem 3(a)], if \(T\) is such an operator, then the set \(\{\lambda; \lambda \in \sigma(T)\}\) is countable. This includes, in particular, operators of class \(C_0\), that is, completely nonunitary contractions in Hilbert space, which are annihilated by a nonzero bounded analytic function in the unit disc. For this class power-regularity follows also from Theorem 3.1, since by a result of Foiaş [8], operators of class \(C_0\) are decomposable.

## 5. Finite-dimensional spaces

If \(X\) is finite dimensional, then every operator in \(L(X)\) has finite spectrum 
and hence is power-regular by Corollaries 3.2 or 4.3. This also follows from 
[11, p. 116] where the more general case of compact operators is considered. 
In that proof the Jordan canonical form is used. We give here two direct and 
elementary proofs.

**Theorem 5.1.** If \(X\) is finite dimensional and \(T\) is in \(L(X)\), then for all \(x\) in 
\(X\),

\[
\lim_{n \to \infty} \|T^n x\|^{1/n} = r_x(T).
\]

**First proof.** Let \(x \in X\). By considering the restriction of \(T\) to the subspace 
spanned by the vectors \(x, Tx, T^2x, \ldots\), we may assume that \(x\) is a cyclic 
vector for \(T\). Let \(\lambda\) be an eigenvalue of \(T\) such that \(|\lambda| = r(T)\), and let \(v\) be 
a corresponding unit eigenvector. Since \(x\) is a cyclic vector for \(T\), there exists 
a polynomial \(p\) such that \(p(T)x = v\), and therefore for every positive integer \(n\),

\[
(r(T))^n = |\lambda|^n = \|p(T)T^n x\| \leq \|p(T)\| \|T^n x\|.
\]

Hence noticing that \(p(T) \neq 0\) (since \(v \neq 0\)), we obtain that

\[
r(T) \leq \liminf_{n \to \infty} \|p(T)\|^{1/n} \|T^n x\|^{1/n} = \liminf_{n \to \infty} \|T^n x\|^{1/n}.
\]

Combining this with the fact that

\[
\limsup_{n \to \infty} \|T^n x\|^{1/n} \leq \lim_{n \to \infty} \|T^n\|^{1/n} = r(T),
\]

the assertion follows.

**Second proof.** Assume again that \(x\) is a cyclic vector for \(T\). Let \(\mathcal{A}\) be the 
algebra generated in \(L(X)\) by \(T\) and the identity operator, that is, 
\(\mathcal{A} = \text{span}\{T^n; n = 0, 1, \ldots\}\). Consider the linear mapping \(L: \mathcal{A} \to X\) defined by
L(S) = Sx, S ∈ \mathcal{A}. Since x is a cyclic vector for T, L is an isomorphism between the finite-dimensional Banach spaces \mathcal{A} and X. Therefore, there exists a constant c > 0 such that
\[ c^{-1}||S|| \leq ||Sx|| \leq c||S||, \]
for all S in \mathcal{A}; hence in particular,
\[ c^{-1}||T^n|| \leq ||T^n x|| \leq c||T^n||, \quad n = 1, 2, \ldots. \]

This implies that
\[ \lim_{n \to \infty} ||T^n x||^{1/n} = \lim_{n \to \infty} ||T^n||^{1/n} = r(T), \]
and the proof is complete.

Remarks. (1) The second proof is valid also for finite-dimensional spaces over the real field (in this case the spectral radius of an operator is defined by Gelfand’s formula).

(2) For general X and all T in \mathcal{L}(X), it is easily verified that for every t > 0, the set \{x ∈ X; r(x, T) < t\} is a linear space (which is not generally closed in X). Hence if X is finite dimensional and T is in \mathcal{L}(X), then \lim_{n \to \infty} ||T^n x||^{1/n} = r(T) for all x in X, except (if r(T) > 0) for x in the subspace \{x ∈ X; r(x, T) < r(T)\}. Thus, in particular, this equality holds with probability one (with respect to normalized area measure) for x in the Euclidean unit sphere of X.

6. Additional facts and problems

It is easily verified that power-regularity is preserved by similarity and (finite) direct sums but is not in general preserved by sums and products. In fact, every Hilbert space operator A is the sum of two power-regular operators \(\frac{1}{2}(A + A^*)\) and \(\frac{1}{2}(A - A^*)\) (recall that selfadjoint operators are power-regular), and every weighted shift is the product of a diagonal operator and an isometry (which are clearly power-regular), but as already mentioned in §1, there exist weighted shifts which are not power-regular. For commuting operators the situation is less clear.

Problem 1. Are the sum and product of two commuting power-regular operators also power-regular?

We show next that power-regularity is not preserved in general by adjoint operators. Consider the unilateral shift S on \(l^2\) (that is, \(S e_n = e_{n+1}, \ n = 0, 1, \ldots\), where \(\{e_n\}_{n=0}^\infty\) is the standard orthonormal basis in \(l^2\)). Since S is an isometry, it is power-regular. We claim that its adjoint S* (the backward shift) is not power-regular. To see this, consider the sequence \(\{a_n\}_{n=0}^\infty\) in \(l^2\) defined by \(a_0 = 1\) and \(a_n = \exp(-2k), \ 2^{k-1} \leq n < 2^k, \ k = 1, 2, \ldots\). Then \(x = \{(a_n^2 - a_{n-1}^2)^{1/2}\}_{n=0}^\infty\) is in \(l^2\), and \(||S^nx|| = a_n, \ n = 1, 2, \ldots\). Since the sequence \(\{a_n^{1/n}\}_{n=1}^\infty\) is not convergent, this shows that S* is not power-regular.

It is clear that power-regularity is preserved by restrictions to invariant subspaces. Hence, for example, power-regularity of subnormal operators follows from that of normal operators. On the other hand, the preceding example implies that power-regularity is not preserved in general by passing to quotient spaces. In fact, if T is the unitary operator on \(l^2(\mathbb{Z})\) defined by \(T\{a_n\}_{n=-\infty}^\infty = \left\{a_n^{1/n}\right\}_{n=1}^\infty\).
\{a_{n+1}\}_{n=-\infty}^{\infty}, \text{ then the subspace } M, \text{ which consists of all sequences } \{c_n\}_{n=-\infty}^{\infty} \text{ in } l^2(\mathbb{Z}) \text{ such that } c_n = 0 \text{ for } n \geq 0, \text{ is in } \text{lat}(T), \text{ and } T^M \text{ can be identified in an obvious way with the backward shift } S^*. \n
For all operators \( T \) in \( \mathcal{L}(X) \) for which power-regularity was proved in the preceding sections, the equality
\[
\lim_{n \to \infty} \| T^n x \|^{1/n} = r_x(T)
\]
was also established for all \( x \) in \( X \). We do not know whether this is always the case.

**Problem 2.** Assume that \( T \) is a power-regular operator in \( \mathcal{L}(X) \). Is the above equality satisfied for all \( x \) in \( X \)?

For general operators the equality is not always true, even if the limit on the left-hand side exists. To see this, consider again the backward shift \( S^* \) on \( l^2 \), and let \( x \) denote the sequence \( \{1/(n+1)!\}_{n=0}^{\infty} \) in \( l^2 \). It follows from [10, p. 282] that \( x \) is a cyclic vector for \( S^* \), and therefore \( r_x(S^*) = r(S^*) = 1 \). On the other hand, a simple estimate shows that
\[
\|S^{*n}x\| \leq \frac{1}{n!}, \quad n = 1, 2, \ldots,
\]
and therefore \( \lim_{n \to \infty} \|S^{*n}x\|^{1/n} = 0 \). Hence the equality is not satisfied in this case.

It follows from Corollary 4.3 that every operator whose spectrum is included in a circle with center at the origin is power regular. We do not know which of these is true for operators with spectrum included in a smooth curve, even for curves of very special type.

**Problem 3.** Is every operator in \( \mathcal{L}(X) \) with spectrum included in a circle or in the real line power-regular?

We conclude with some comments about power-regular Banach algebras. It is clear that a closed subalgebra of a power-regular Banach algebra is also power-regular. Thus, by Corollary 3.3, every closed subalgebra of a commutative semisimple regular Banach algebra is power-regular. Also, for a general commutative Banach algebra \( B \) with identity, one can show that for every \( y \) in \( B \) whose Gelfand transform does not vanish on any nonempty open subset of the maximal ideal space of \( B \), and for all \( x \) in \( B \),
\[
\lim_{n \to \infty} \|x^ny\|^{1/n} = r(x)
\]
(where \( r(x) \) denotes the spectral radius of \( x \)). Thus, in particular, if every nonzero element \( y \) of \( B \) has this property, then \( B \) is power-regular. Therefore, every Banach algebra with identity of analytic functions on some domain in the complex plane, which is dense in its maximal ideal space, is power-regular.

In a lecture at the Banach center we raised the question whether every commutative semisimple Banach algebra with identity is power-regular. Vladimir Müller constructed an example which shows that this is not the case.

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