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A RIGHT COUNTABLY SIGMA-CS RING WITH ACC OR DCC ON PROJECTIVE PRINCIPAL RIGHT IDEALS IS LEFT ARTINIAN AND QF-3

DINH VAN HUYNH

ABSTRACT. A module M is called a CS module if every submodule of M is essential in a direct summand of M. A ring R is said to be right (countably) Σ-CS if any direct sum of (countably many) copies of the right R-module R is CS. It is shown that for a right countably Σ-CS ring R the following are equivalent: (i) R is right Σ-CS, (ii) R has ACC or DCC on projective principal right ideals, (iii) R has finite right uniform dimension and ACC or DCC holds on projective uniform principal right ideals of R, (iv) R is semiperfect. From results of Oshiro [12], [13], under these conditions, R is left artinian and QF-3. As a consequence, a ring R is quasi-Frobenius if it is right countably Σ-CS, semiperfect and no nonzero projective right ideals are contained in its Jacobson radical.

1. INTRODUCTION

Let R be a ring and M_R be a right R-module. Then M_R is called (countably) Σ-injective if every direct sum of (countably many) copies of M is injective. A ring R is called right (countably) Σ-injective if R_R is a (countably) Σ-injective module. (Countably) Σ-CS modules and right (countably) Σ-CS rings are defined similarly.

By a significant result of Faith [5] (see also [6, Proposition 20.3A]), an injective module M_R is Σ-injective iff M_R is countably Σ-injective iff R satisfies ascending chain condition (briefly, ACC) on annihilators of subsets from M.

Unlike countably Σ-injective modules, countably Σ-CS modules do not supply any chain condition in the ring, in general, as it was shown in Dung-Smith [4] that any right self-injective von Neumann regular ring is right countably Σ-CS. However, by Oshiro [12] and [13], a right Σ-CS ring is left artinian and QF-3. (For details of QF-3 rings we refer to Tachikawa [16].)

While the structure of right countably Σ-CS rings in general is unknown, in this note we show that in some cases Σ-CS and countably Σ-CS are equivalent. Precisely the following theorem holds where DCC is the abbreviation of descending chain condition.

Theorem 1. *For a ring R the following conditions are equivalent:*
(a) *R is right Σ-CS.*

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(b) *R is right countably Σ-CS and ACC (DCC) holds on projective principal right ideals of R.*

(c) *R is right countably Σ-CS having finite right uniform dimension and ACC (DCC) holds on projective uniform principal right ideals.*

(d) *R is right countably Σ-CS and semiperfect.*

Note that in (c) of Theorem 1, because R is right CS, the condition "having finite right uniform dimension" includes the assumption that $R/J(R)$ has finite right uniform dimension (e.g. R is semilocal) where $J(R)$ is the Jacobson radical of R.

Corollary 2. *A ring R is quasi-Frobenius (briefly, QF) if and only if R is right countably Σ-CS, semiperfect and no nonzero projective right ideal of R is contained in the Jacobson radical of R.*

It is well known that any right perfect two-sided self-injective ring is QF (see Osofsky [14] or Kato [10]). However the question on one-sided self-injectity remains open even assuming that the ring is semiprimary which is now known as Faith's Conjecture:

(FC) Any right self-injective semiprimary ring is QF.

Many authors have been working on this; however it remains unproved. Let R be a right self-injective semiprimary ring. If R is right countably Σ-CS, then R is QF by Corollary 2, since the right self-injectivity does not allow R to have nonzero projective right ideal in its Jacobson radical. Hence to check (FC) it is enough to show that $R_R^{(\mathbb{N})}$ is CS. Furthermore, from the consideration in [3], a right self-injective semiperfect ring S is QF if and only if any uniform submodule of $S_S^{(\mathbb{N})}$ is *contained* in a finitely generated submodule of $S_S^{(\mathbb{N})}$. This reduces the study of (FC) to the consideration of uniform submodules of the module $R_R^{(\mathbb{N})}$. On the other hand, it is shown in Armendariz-Park [2] that if R is a right self-injective ring such that $R/\operatorname{Soc}(R_R)$ has ACC on right annihilators, then R is semiprimary. However, in this case it is also unknown whether R is QF or not. This means Faith's Conjecture is still unproved even if we additionally assume that $R/\operatorname{Soc}(R_R)$ has ACC on right annihilators. As is known, a right self-injective ring with ACC on right or left annhilators is QF by Faith [5].

2. The proofs

Throughout we consider associative rings with identity and all modules are unitary. For a module M over a ring R we write M_R to indicate that M is a right R-module. The Jacobson radical and the injective hull of a module M are denoted respectively by $J(M)$ and $E(M)$.

Let A be a set and M be a right R-module. Then $M^{(A)}$ denotes the direct sum of $|A|$ copies of M. Let $N = \bigoplus_{i\in I} N_i$ be a submodule of a module M. Then N is called a local direct summand of M if, for each finite subset F of I, $\bigoplus_{i\in F} N_i$ is a direct summand of M.

A module M is called a CS module if every submodule of M is essential in a direct summand of M. The module M is called (countably) Σ-CS if $M^{(A)}$ ($M^{(\mathbb{N})}$) is CS for any set A (for the set $\mathbb{N}$ of positive integers). If R_R is (countably) Σ-CS, then R is said to be a right (countably) Σ-CS ring. Oshiro [12] considered right Σ-CS rings under the name right co-H rings.

The texts by Anderson-Fuller [1], Faith [6], Goodearl-Warfield [7], Kasch [9], Mohamed-Müller [11] and Wisbauer [17] are general references for module and ring-theoretic notions not defined here.

Let R be a right countably Σ-CS ring such that

$$R = e_1R \oplus \cdots \oplus e_nR \tag{1}$$

where $\{e_1, \dots, e_n\}$ is a set of orthogonal idempotents of R and each e_iR is uniform. For convenience we put

$$S(R) = \{e_1R, \dots, e_nR\}.$$

We keep this notion and assumption of R throughout the following lemmas.

Lemma 3. *Let L be a countably generated uniform right R-module such that L contains a copy of some e_iR in $S(R)$. Then L is embedded in some e_kR of $S(R)$. If R is semiperfect, then this embedding is an isomorphism.*

Proof. Since L is countably generated, there is an epimorphism g of $R_R^{(\mathbb{N})}$ onto L. Put

$$P := R_R^{(\mathbb{N})} = \bigoplus_{i\in\mathbb{N}} P_i \tag{2}$$

where each P_i is isomorphic to some e_jR in $S(R)$. Since by assumption P is CS, $\operatorname{Ker}(g)$ is essential in a direct summand U of P and we have $P = U \oplus V$ for some submodule V of P. It follows $L_R \simeq (U/\operatorname{Ker}(g)) \oplus V$. Since L contains a copy of some e_iR in $S(R)$, L cannot be a singular module. From this and since L is uniform, we must have $U/\operatorname{Ker}(g) = 0$; i.e. L is isomorphic to the direct summand V of P. If R is semiperfect, then each P_i in (2) has local endomorphism ring. Hence we may use [9, 7.3.4] to see that V is isomorphic to some e_kR of $S(R)$ and so is L.

Now we consider the general case. By Zorn's Lemma there is a subset I of $\mathbb{N}$ which is maximal with respect to $V \cap P(I) = 0$ where $P(I) = \bigoplus_{i\in I} P_i$. Let us assume that there are distinct i, j in $\mathbb{N}$ which are not in I. Then by the maximality of I,

$$V_1 = P_i \cap (P(I) \oplus V) \neq 0 \quad \text{and} \quad V_2 = P_j \cap (P(I) \oplus V) \neq 0.$$

Put $X = (P_i \oplus P_j) \cap (P(I) \oplus V)$. Then $P(I) \cap X = 0$ and so X is embedded in V; i.e. X is uniform. But on the other hand, $V_1 \oplus V_2 \subseteq X$, a contradiction. Hence there is only one i in $\mathbb{N}$ such that $i \notin I$. It follows that V is embedded in P_i of (2). But P_i is isomorphic to some e_kR in $S(R)$. Therefore V is embedded in e_kR and so is L. $\square$

Lemma 4. (i) *If R is semiperfect or R has ACC or DCC on projective uniform principal right ideals, then for each e_iR in $S(R)$ there is an e_kR in $S(R)$ such that $E(e_iR)$ is isomorphic to e_kR.*

(ii) *If R has ACC or DCC on projective uniform principal right ideals, then each e_iR in $S(R)$ is quasi-injective.*

Proof. (i) Let $E_i = E(e_iR)$ and assume on the contrary that there is some e_iR with infinitely generated E_i. Then, certainly, there is a countably generated submodule M of E_i containing e_iR and M is not finitely generated. For convenience we say that M is "infinite-countably" generated.

If R is semiperfect, then M is isomorphic to some e_kR in $S(R)$ by Lemma 3, a contradiction. Hence E_i is finitely generated for each $i = 1, \dots, n$. Then again by Lemma 3, each E_i is isomorphic to some e_kR in $S(R)$. We are done in this case.

Now we consider the case that R has ACC or DCC on projective uniform principal right ideals. By Lemma 3, M is embedded in some e_kR of $S(R)$. Then by the injectivity of E_i, the inverse mapping of this embedding extends to a monomorphism g of e_kR to E_i. Hence

$$e_iR \subset M \subset P_1' := g(e_kR) \subset E_i.$$

Clearly, P_1', being isomorphic to e_kR, contains M properly and since E_i is infinitely generated, we have $E_i \neq P_1'$.

Assume inductively that we already found m $(m \geq 1)$ projective submodules P_k' $(k = 1, \dots, m)$ of E_i with $P' \subset P_2' \subset \cdots \subset P_m'$ where each P_k' is isomorphic to some $e_{i_k}R$ in $S(R)$. Since $P_m' \neq E_i$ and E_i is infinitely generated, there is an infinite-countably generated submodule M' of E_i containing P_m'. By Lemma 3, M' is embedded in some e_jR in $S(R)$. Then by the above argument we find a submodule P_{m+1}' of E_i containing M' and P_{m+1}' is isomorphic to e_jR. Clearly, $P_{m+1}' \neq M'$. Hence we have a strictly ascending chain $P_1' \subset \cdots \subset P_{m+1}' \neq E_i$. This induction process shows that in E_i there is an infinite strictly ascending chain

$$P_1' \subset \cdots \subset P_t' \subset \cdots \tag{3}$$

of submodules P_t' each of which is isomorphic to some e_tR in $S(R)$.

(a) R has DCC on projective uniform principal right ideals. Since the set $S(R)$ is finite, there are P_i' and P_j' in (3) with $P_i' \neq P_j'$ but $P_i' \simeq P_j' \simeq e_tR$ for some e_tR in $S(R)$. This shows that e_tR is embedded properly in itself; i.e. e_tR contains a proper submodule isomorphic to e_tR. This embedding of e_tR in itself produces an infinite strictly descending chain of projective uniform principal right ideals of R in e_tR, a contradiction. Thus in this case each E_i is finitely generated. Hence by Lemma 3 each E_i must be isomorphic to some e_kR in $S(R)$, as desired.

(b) R has ACC on projective uniform principal right ideals. Let U be the union of all P_t' in (3). Then U is an infinite-countably generated submodule of E_i. By Lemma 3, U is embedded in some e_jR of $S(R)$. Then the inverse mapping of this embedding extends to a monomorphism h of e_jR into E_i with $h(e_jR) \supset U$. This and (3) show that e_jR contains an infinite strictly ascending chain of projective uniform principal right ideals of R, a contradiction. Hence each E_i must be finitely generated. By Lemma 3, each E_i is embedded in some e_kR in $S(R)$. But each E_i is injective, so we have $E_i \simeq e_kR$, as desired.

(ii) Assume that R has ACC or DCC on projective uniform principal right ideals. We will show that each e_iR is quasi-injective. (Note that this does not imply that R is right self-injective!) For this purpose let $\widehat{e_iR}$ be the quasi-injective hull of e_iR. By [17, 17.9], $\widehat{e_iR}$ is a submodule of an epimorphic image N of $(e_iR)^{(A)}$ for some set A. By [1, Proposition 16.13], $\widehat{e_iR}$ is N-injective and hence it is a direct summand of N. From this it is easy to see that there is an epimorphism

$$f\colon (e_iR)^{(A)} \to \widehat{e_iR}. \tag{4}$$

If $\widehat{e_iR}$ is countably generated, then from (4) we see that there is a countable subset B of A such that $\widehat{e_iR}$ is an epimorphic image of $(e_iR)^{(B)}$. Then by the same argument as in Lemma 3 we see that $\widehat{e_iR}$ is embedded in e_iR, proving $\widehat{e_iR} = e_iR$, i.e., e_iR is quasi-injective. We are done in this case.

Assume that $\widehat{e_iR}$ is uncountably generated. Then in (4) A must be uncountable. There is an infinite-countably generated submodule M^* of $\widehat{e_iR}$ containing e_iR. Hence it is easy to see that A contains a countable subset I with $f(e_iR)^{(I)} \supseteq M^*$. Then using the epimorphism

$$(f \mid (e_iR)^{(I)})\colon (e_iR)^{(I)} \to f((e_iR)^{(I)})$$

and the fact that $(e_iR)^{(I)}$ is CS we can show, by a similar argument as for Lemma 3, that $f((e_iR)^{(I)})$ is embedded in e_iR. Then by the e_iR-injectivity of $\widehat{e_iR}$, the inverse mapping of this embedding extends to a monomorphism g of e_iR into $\widehat{e_iR}$. Clearly. $P_1^* := g(e_iR) \supset M^* \supset e_iR$ and $P_1^* \neq e_iR$. Hence e_iR is embedded in itself and this produces an infinite strictly descending chain of projective uniform principal right ideals of R in e_iR, a contradiction if we assume the DCC. Hence in the DCC case, $\widehat{e_iR}$ must be countably generated and then, as shown above, we have $\widehat{e_iR} = e_iR$, as desired. It remains to consider the ACC case.

Since $P_1^* \neq \widehat{e_iR}$, we may repeat our argument and use an induction proof as in (i) to get an infinite strictly ascending chain of submodules P_k^* in $\widehat{e_iR}$:

$$P_1^* \subset \cdots \subset P_k^* \subset \cdots$$

where each P_k^* is isomorphic to e_iR. Let U^* be the union of these P_k^*'s. Then U_R^* is infinite-countably generated. By the above argument for considering M^*, now applied to U^*, we find a submodule P^* of $\widehat{e_iR}$ with $P^* \simeq e_iR$ and $P^* \supset U^*$. This shows that in e_iR there is an infinite strictly ascending chain of projective uniform principal right ideals of R, a contradiction. Thus $\widehat{e_iR}$ must be countably generated and so we can find a countable set A for which (4) holds. Hence, as concluded above, we have $\widehat{e_iR} = e_iR$, proving the quasi-injectivity of e_iR. □

Lemma 5. *Under assumptions of Lemma* 4 (i), $E = E(R_R)$ *is* Σ*-injective.*

Proof. By a result of Faith [5] mentioned in the introduction, it is enough to show that E is countably Σ-injective. For convenience we put

$$Q = E^{(\mathbb{N})} = \bigoplus_{i\in\mathbb{N}} Q_i \tag{5}$$

where, by Lemma 4 (i), each Q_i is isomorphic to some injective e_jR in $S(R)$. Moreover, because of this we easily verify that Q_R is a direct summand of a direct sum of countably many copies of R_R. Hence Q_R is a CS module. First we show that (5) complements direct summands; i.e. if B is a direct summand of Q_R, then there is a subset N' of $\mathbb{N}$ such that $Q = B \oplus Q(N')$, where

here and below we denote by $Q(J)$ the direct sum of Q_i in (5) with all $i \in J$ whenever J is a subset of $\mathbb{N}$.

Thus we assume that B is a direct summand of Q. By Zorn's Lemma, there is a subset H of $\mathbb{N}$ which is maximal with respect to $B \cap Q(H) = 0$. Since each Q_i is uniform, it is clear that $C = B \oplus Q(H)$ is essential in Q. Put $K_1 = K_2 = Q$. Then like Q, $K_1 \oplus K_2$ is also a direct summand of a direct sum of countably many copies of R_R. Hence $K_1 \oplus K_2$ is a CS module, since R is right countably Σ-CS. We have

$$K_1 \oplus K_2 = (B \oplus D) \oplus K_2$$

for some submodule D of K_1. Since K_2 contains a direct summand K' which is isomorphic to $Q(H)$, we see that $C = B \oplus Q(H)$ is isomorphic to $B \oplus K'$, a direct summand of $K_1 \oplus K_2$. Hence C is a CS module. Moreover, if V is a uniform direct summand of C, then V is isomorphic to a uniform direct summand V' of $B \oplus K'$ and hence one of $K_1 \oplus K_2$. Since each Q_i is injective, uniform, so its endomorphism ring is local. Then we may use [9, 7.3.4] to see that V' is isomorphic to some Q_i in (5). In particular, V' is injective and so is V.

Now assume that $C \neq Q$. Then there is a Q_k in (5) with $Q_k \nsubseteq C$. Put $T = Q_k \cap C$. Then T is a uniform submodule of C. Let T^* be a maximal essential extension of T in C. Then T^* is a direct summand of C. Moreover, by the previous consideration, T^* is injective. Hence $Q_k + T^* = T^* \oplus T'$ for some submodule T' of $Q_k + T^*$. If $T' = 0$, $T^* \subseteq Q_k$, implying $Q_k = T^* \subseteq C$, a contradiction. Hence $T' \neq 0$. By modularity we have $C \cap (Q_k + T^*) = (C \cap Q_k) + T^* = T + T^* = T^* = C \cap (T^* \oplus T') = T^* \oplus (C \cap T')$. Therefore $C \cap T' = 0$, a contradiction to the fact that C is essential in Q. Thus $C = Q$, proving that (5) complements direct summands. From this we may use [11, Theorem 2.25] to obtain that local direct summands of Q are direct summands. We apply this below to show that Q is injective.

Let U be a right ideal of R and f be an R-homomorphism of U to Q. We may assume that $f \neq 0$ and U is essential in R_R. Put $M = E \oplus Q$. Note that $U \subseteq R \subseteq E$. Since $Q = E^{(\mathbb{N})}$, it is clear that $M \simeq Q$ and so M is CS and local direct summands of M are direct summands. There exists a direct summand V^* of M such that the submodule $V = \{x - f(x)\,;\, x \in U\}$ is essential in V^*. Put

$$M = V^* \oplus M^* \tag{6}$$

for some submodule M^* of M. We have $V^* \cap Q = 0$ and moreover $V^* \oplus Q$ is essential in M. Let π be the projection of M onto M^* given by (6). Then clearly, $\pi^* = (\pi|Q)$ is a monomorphism. It follows that $\{\pi^*(Q_i)\,;\, i \in \mathbb{N}\}$ is an independent set of injective submodules of M^* and so $W = \bigoplus_{i\in\mathbb{N}} \pi^*(Q_i)$ is a local direct summand of M^* and also of M. Hence W is a direct summand of M^*, say $M^* = W \oplus Y$. If $Y \neq 0$, there is $0 \neq x \in (V^* \oplus Q) \cap Y$ and so $x = u + v$ $(u \in V^*, u \in Q)$. Hence $x = \pi(x) = \pi(u) + \pi(v) = \pi(v) \in W$, a contradiction. Hence $Y = 0$ and therefore $\pi(M) = W = \pi(Q)$, implying $M = V^* \oplus Q$.

Let π' be the projection of $V^* \oplus Q$ onto Q. Then $(\pi'|R)$ (R is taken from the direct summand E of $E \oplus Q$) is an extension of f from R_R to Q_R, proving the injectivity of Q. □

Proof of Theorem 1. (a) $\Rightarrow$ (b). Assume (a). Then by [12], R is semiprimary. Hence R has DCC and by [8] R has ACC on principal right ideals. Thus (a) implies (b).

(b) $\Rightarrow$ (c) Assume (b). Clearly, it is enough to show that R has finite right uniform dimension; this means there is no infinite direct sum of nonzero right ideals in R. Assume on the contrary that R contains an infinite direct sum of nonzero right ideals. Since R is right CS, by a standard argument we find an infinite set $\{f_i\}_{i=1}^{\infty}$ of nonzero orthogonal idempotents f_i in R. Then the right ideals $R_i = f_1R \oplus \cdots \oplus f_iR$, $i = 1, 2, \ldots$, form an infinite strictly ascending chain of projective principal right ideals of R. This is a contradiction if we assume that R has ACC on projective principal right ideals for (b). Hence R must have finite right uniform dimension in this case.

For the DCC case of (b), put $f_i^* = f_1 + \cdots + f_i$, $i = 1, 2, \ldots$. Then $R = f_i^*R \oplus (1 - f_i^*)R$. Clearly, each $(1 - f_i^*)R$ is nonzero and they form an infinite strictly descending chain of projective principal right ideals in R, a contradiction. Thus R must have finite right uniform dimension, as desired.

(c) $\Rightarrow$ (a). Assume (c). Then by a standard argument we can show that R_R has a direct decomposition of form (1). By Lemma 5, $E(R_R)$ is then Σ-injective. Hence by the mentioned result of Faith in the Introduction, and since $R_R^{(\mathbb{N})}$ is contained in $E^{(\mathbb{N})}$, R has ACC on the annihilators of subsets from $R_R^{(\mathbb{N})}$. From this and since R is right countably Σ-CS, every local direct summand of $R_R^{(\mathbb{N})}$ is a direct summand by [11, Proposition 2.18]. Moreover, by Lemma 4 (ii), $R_R^{(\mathbb{N})}$ is a direct sum of quasi-injective uniform modules (and therefore whose endomorphism rings are local). Hence we may use [11, Theorem 2.25] to see that this decomposition of $R_R^{(\mathbb{N})}$ complements direct summands. By [1, Theorem 28.14], R is right perfect. Thus by [15, Theorem II], R is right Σ-CS, proving (a).

(a) $\Rightarrow$ (d) is clear by [12].

(d) $\Rightarrow$ (a). Assume (d). Since R is semiperfect and right CS, it is easy to see that R_R has a direct decomposition of the form (1). Moreover, each e_iR has a local endomorphism ring. Hence the module $R_R^{(\mathbb{N})}$ is a direct sum of uniform modules with local endomorphism rings. Therefore we may use the argument of proving (c) $\Rightarrow$ (a) to verify that R is right Σ-CS, as desired. □

Proof of Corollary 2. One direction is clear (see [6, Theorem 24.20]). Assume conversely that R is right countably Σ-CS, semiperfect and no nonzero projective right ideals are contained in $J(R)$. By Theorem 1, R is right Σ-CS. In particular R_R has a decomposition of form (1). Hence by Lemma 4 (i), each $E(e_iR)$ is isomorphic to some e_kR in $S(R)$. If $E(e_iR) \neq e_iR$, then e_iR is embedded in $J(e_kR) \subseteq J(R)$, a contradiction to our assumption. It follows that $e_iR = E(e_iR)$ for each e_iR; i.e. R is right self-injective. Since R is right Σ-CS, R is QF by [12, Theorem 4.3]. □

From the considerations in this paper, especially from the proof of Lemma 4, we immediately obtain the following result:

Proposition 6. *A right countably Σ-CS ring R is right Σ-CS if and only if R has finite right uniform dimension and each projective uniform principal right ideal of R is not embedded properly in itself.* □

It would be interesting to know whether Corollary 2 holds also for semilocal rings. After submitting this paper we received a preprint "Σ-Extending Modules" of J. Clark and R. Wisbauer, in which they showed, among others, that a right countably Σ-CS ring with ACC on right annihilators is right Σ-CS. This together with Theorem 1 yields a conclusion that in a right countably Σ-CS ring the ACC on right annihilators and the ACC on projective principal right ideals are equivalent. In general, these two kinds of ACC are quite different (see, for example, that the ring in Faith's Conjecture has ACC and DCC on principal right and principal left ideals, but if we could show that it has ACC on right annihilators, then (FC) would be established!).

We would like to ask the following questions:

(Q_1) Is a right countably Σ-CS ring with finite right uniform dimension necessarily right Σ-CS?

(Q_2) Is a right countably Σ-CS ring necessarily right Σ-CS if (i) R has ACC on left annihilators or (ii) R has ACC on (projective) principal left ideals?

From the results of Faith mentioned in the Introduction, a ring R is QF iff R is right countably Σ-injective iff R is left countably Σ-injective. Of course, a similar result does not hold for right countably Σ-CS rings. However from the considerations in this note we easily verify that for a semiperfect right self-injective ring R the following are equivalent: (i) R is QF, (ii) R is right countably Σ-CS, (iii) R is left countably Σ-CS.

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INSTITUTE OF MATHEMATICS, P.O. BOX 631 BOHO, HANOI, VIETNAM