A RIGHT COUNTABLY SIGMA-CS RING WITH ACC OR DCC ON PROJECTIVE PRINCIPAL RIGHT IDEALS IS LEFT ARTINIAN AND QF-3

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Abstract. A module $M$ is called a CS module if every submodule of $M$ is essential in a direct summand of $M$. A ring $R$ is said to be right (countably) $\Sigma$-CS if any direct sum of (countably many) copies of the right $R$-module $R$ is CS. It is shown that for a right countably $\Sigma$-CS ring $R$ the following are equivalent: (i) $R$ is right $\Sigma$-CS, (ii) $R$ has ACC or DCC on projective principal right ideals, (iii) $R$ has finite right uniform dimension and ACC or DCC holds on projective uniform principal right ideals of $R$, (iv) $R$ is semiperfect. From results of Oshiro [12], [13], under these conditions, $R$ is left artinian and QF-3. As a consequence, a ring $R$ is quasi-Frobenius if it is right countably $\Sigma$-CS, semiperfect and no nonzero projective right ideals are contained in its Jacobson radical.

1. Introduction

Let $R$ be a ring and $M_R$ be a right $R$-module. Then $M_R$ is called (countably) $\Sigma$-injective if every direct sum of (countably many) copies of $M$ is injective. A ring $R$ is called right (countably) $\Sigma$-injective if $R_R$ is a (countably) $\Sigma$-injective module. (Countably) $\Sigma$-CS modules and right (countably) $\Sigma$-CS rings are defined similarly.

By a significant result of Faith [5] (see also [6, Proposition 20.3A]), an injective module $M_R$ is $\Sigma$-injective iff $M_R$ is countably $\Sigma$-injective iff $R$ satisfies ascending chain condition (briefly, ACC) on annihilators of subsets from $M$.

Unlike countably $\Sigma$-injective modules, countably $\Sigma$-CS modules do not supply any chain condition in the ring, in general, as it was shown in Dung-Smith [4] that any right self-injective von Neumann regular ring is right countably $\Sigma$-CS. However, by Oshiro [12] and [13], a right $\Sigma$-CS ring is left artinian and QF-3. (For details of QF-3 rings we refer to Tachikawa [16].)

While the structure of right countably $\Sigma$-CS rings in general is unknown, in this note we show that in some cases $\Sigma$-CS and countably $\Sigma$-CS are equivalent. Precisely the following theorem holds where DCC is the abbreviation of descending chain condition.

Theorem 1. For a ring $R$ the following conditions are equivalent:
(a) $R$ is right $\Sigma$-CS.
(b) \( R \) is right countably \( \Sigma \)-CS and ACC (DCC) holds on projective principal right ideals of \( R \).

(c) \( R \) is right countably \( \Sigma \)-CS having finite right uniform dimension and ACC (DCC) holds on projective uniform principal right ideals.

(d) \( R \) is right countably \( \Sigma \)-CS and semiperfect.

Note that in (c) of Theorem 1, because \( R \) is right CS, the condition “having finite right uniform dimension” includes the assumption that \( R/J(R) \) has finite right uniform dimension (e.g. \( R \) is semilocal) where \( J(R) \) is the Jacobson radical of \( R \).

**Corollary 2.** A ring \( R \) is quasi-Frobenius (briefly, QF) if and only if \( R \) is right countably \( \Sigma \)-CS, semiperfect and no nonzero projective right ideal of \( R \) is contained in the Jacobson radical of \( R \).

It is well known that any right perfect two-sided self-injective ring is QF (see Ososfksy [14] or Kato [10]). However the question on one-sided self-injectivity remains open even assuming that the ring is semiprimary which is now known as Faith’s Conjecture:

\[(FC) \quad \text{Any right self-injective semiprimary ring is QF.}\]

Many authors have been working on this; however it remains unproved. Let \( R \) be a right self-injective semiprimary ring. If \( R \) is right countably \( \Sigma \)-CS, then \( R \) is QF by Corollary 2, since the right self-injectivity does not allow \( R \) to have nonzero projective right ideal in its Jacobson radical. Hence to check (FC) it is enough to show that \( R^{(N)}_{R} \) is CS. Furthermore, from the consideration in [3], a right self-injective semiperfect ring \( S \) is QF if and only if any uniform submodule of \( S^{(N)}_{R} \) is contained in a finitely generated submodule of \( S^{(N)}_{R} \). This reduces the study of (FC) to the consideration of uniform submodules of the module \( R^{(N)}_{R} \). On the other hand, it is shown in Armendariz-Park [2] that if \( R \) is a right self-injective ring such that \( R/Soc(RR) \) has ACC on right annihilators, then \( R \) is semiprimary. However, in this case it is also unknown whether \( R \) is QF or not. This means Faith’s Conjecture is still unproved even if we additionally assume that \( R/Soc(RR) \) has ACC on right annihilators. As is known, a right self-injective ring with ACC on right or left annihilators is QF by Faith [5].

2. The proofs

Throughout we consider associative rings with identity and all modules are unitary. For a module \( M \) over a ring \( R \) we write \( M_{R} \) to indicate that \( M \) is a right \( R \)-module. The Jacobson radical and the injective hull of a module \( M \) are denoted respectively by \( J(M) \) and \( E(M) \).

Let \( A \) be a set and \( M \) be a right \( R \)-module. Then \( M^{(A)} \) denotes the direct sum of \( |A| \) copies of \( M \). Let \( N = \bigoplus_{i \in I} N_{i} \) be a submodule of a module \( M \). Then \( N \) is called a local direct summand of \( M \) if, for each finite subset \( F \) of \( I \), \( \bigoplus_{i \in F} N_{i} \) is a direct summand of \( M \).

A module \( M \) is called a CS module if every submodule of \( M \) is essential in a direct summand of \( M \). The module \( M \) is called (countably) \( \Sigma \)-CS if \( M^{(A)}(M^{(N)}) \) is CS for any set \( A \) (for the set \( N \) of positive integers). If \( R_{R} \) is
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(countably) Σ-CS, then \( R \) is said to be a right (countably) Σ-CS ring. Oshiro [12] considered right Σ-CS rings under the name right co-H rings.

The texts by Anderson-Fuller [1], Faith [6], Goodearl-Warfield [7], Kasch [9], Mohamed-Müller [11] and Wisbauer [17] are general references for module and ring-theoretic notions not defined here.

Let \( R \) be a right countably Σ-CS ring such that

\[
R = e_1R \oplus \cdots \oplus e_nR
\]

where \( \{e_1, \ldots, e_n\} \) is a set of orthogonal idempotents of \( R \) and each \( e_iR \) is uniform. For convenience we put

\[
S(R) = \{e_1R, \ldots, e_nR\}.
\]

We keep this notion and assumption of \( R \) throughout the following lemmas.

**Lemma 3.** Let \( L \) be a countably generated uniform right \( R \)-module such that \( L \) contains a copy of some \( e_iR \) in \( S(R) \). Then \( L \) is embedded in some \( e_kR \) of \( S(R) \). If \( R \) is semiperfect, then this embedding is an isomorphism.

**Proof.** Since \( L \) is countably generated, there is an epimorphism \( g \) of \( R^\infty \) onto \( L \). Put

\[
P := R^\infty = \bigoplus_{i \in \mathbb{N}} P_i
\]

where each \( P_i \) is isomorphic to some \( e_iR \) in \( S(R) \). Since by assumption \( P \) is CS, \( \text{Ker}(g) \) is essential in a direct summand \( U \) of \( P \) and we have \( P = U \oplus V \) for some submodule \( V \) of \( P \). It follows \( L_R \simeq (U/\text{Ker}(g)) \oplus V \). Since \( L \) contains a copy of some \( e_iR \) in \( S(R) \), \( L \) cannot be a singular module. From this and since \( L \) is uniform, we must have \( U/\text{Ker}(g) = 0 \); i.e. \( L \) is isomorphic to the direct summand \( V \) of \( P \). If \( R \) is semiperfect, then each \( P_i \) in (2) has local endomorphism ring. Hence we may use [9, 7.3.4] to see that \( V \) is isomorphic to some \( e_kR \) of \( S(R) \) and so is \( L \).

Now we consider the general case. By Zorn’s Lemma there is a subset \( I \) of \( \mathbb{N} \) which is maximal with respect to \( V \cap P(I) = 0 \) where \( P(I) = \bigoplus_{i \in I} P_i \). Let us assume that there are distinct \( i, j \) in \( \mathbb{N} \) which are not in \( I \). Then by the maximality of \( I \),

\[
V_1 = P_i \cap (P(I) \oplus V) \neq 0 \quad \text{and} \quad V_2 = P_j \cap (P(I) \oplus V) \neq 0.
\]

Put \( X = (P_i + P_j) \cap (P(I) \oplus V) \). Then \( P(I) \cap X = 0 \) and so \( X \) is embedded in \( V \); i.e. \( X \) is uniform. But on the other hand, \( V_1 \oplus V_2 \subseteq X \), a contradiction. Hence there is only one \( i \) in \( \mathbb{N} \) such that \( i \notin I \). It follows that \( V \) is embedded in \( P_i \) of (2). But \( P_i \) is isomorphic to some \( e_kR \) in \( S(R) \). Therefore \( V \) is embedded in \( e_kR \) and so is \( L \). □

**Lemma 4.** (i) If \( R \) is semiperfect or \( R \) has ACC or DCC on projective uniform principal right ideals, then for each \( e_iR \) in \( S(R) \) there is an \( e_kR \) in \( S(R) \) such that \( E(e_iR) \) is isomorphic to \( e_kR \).

(ii) If \( R \) has ACC or DCC on projective uniform principal right ideals, then each \( e_iR \) in \( S(R) \) is quasi-injective.

**Proof.** (i) Let \( E_i = E(e_iR) \) and assume on the contrary that there is some \( e_iR \) with infinitely generated \( E_i \). Then, certainly, there is a countably generated
submodule $M$ of $E_i$ containing $e_iR$ and $M$ is not finitely generated. For convenience we say that $M$ is "infinite-countably" generated.

If $R$ is semiperfect, then $M$ is isomorphic to some $e_kR$ in $S(R)$ by Lemma 3, a contradiction. Hence $E_i$ is finitely generated for each $i = 1, \ldots, n$. Then again by Lemma 3, each $E_i$ is isomorphic to some $e_kR$ in $S(R)$. We are done in this case.

Now we consider the case that $R$ has ACC or DCC on projective uniform principal right ideals. By Lemma 3, $M$ is embedded in some $e_kR$ of $S(R)$. Then by the injectivity of $E_i$, the inverse mapping of this embedding extends to a monomorphism $g$ of $e_kR$ to $E_i$. Hence

$$e_iR \subset M \subset P'_i := g(e_kR) \subset E_i.$$  

Clearly, $P'_i$, being isomorphic to $e_kR$, contains $M$ properly and since $E_i$ is infinitely generated, we have $E_i \neq P'_i$.

Assume inductively that we already found $m$ ($m \geq 1$) projective submodules $P'_k$ ($k = 1, \ldots, m$) of $E_i$ with $P'_1 \subset P'_2 \subset \cdots \subset P'_m$ where each $P'_k$ is isomorphic to some $e_kR$ in $S(R)$. Since $P'_m \neq E_i$ and $E_i$ is infinitely generated, there is an infinite-countably generated submodule $M'$ of $E_i$ containing $P'_m$. By Lemma 3, $M'$ is embedded in some $e_jR$ in $S(R)$. Then by the above argument we find a submodule $P'_{m+1}$ of $E_i$ containing $M'$ and $P'_{m+1}$ is isomorphic to $e_jR$. Clearly, $P'_{m+1} \neq M'$. Hence we have a strictly ascending chain $P'_1 \subset \cdots \subset P'_{m+1} \neq E_i$. This induction process shows that in $E_i$ there is an infinite strictly ascending chain

$$(3) \quad P'_1 \subset \cdots \subset P'_i \subset \cdots$$

of submodules $P'_i$ each of which is isomorphic to some $e_iR$ in $S(R)$.

(a) $R$ has DCC on projective uniform principal right ideals. Since the set $S(R)$ is finite, there are $P'_i$ and $P'_j$ in (3) with $P'_i \neq P'_j$ but $P'_i \simeq P'_j \simeq e_iR$ for some $e_iR$ in $S(R)$. This shows that $e_iR$ is embedded properly in itself; i.e. $e_iR$ contains a proper submodule isomorphic to $e_iR$. This embedding of $e_iR$ in itself produces an infinite strictly descending chain of projective uniform principal right ideals of $R$ in $e_iR$, a contradiction. Hence each $E_i$ is finitely generated. Hence by Lemma 3 each $E_i$ must be isomorphic to some $e_kR$ in $S(R)$, as desired.

(b) $R$ has ACC on projective uniform principal right ideals. Let $U$ be the union of all $P'_i$ in (3). Then $U$ is an infinite-countably generated submodule of $E_i$. By Lemma 3, $U$ is embedded in some $e_jR$ of $S(R)$. Then the inverse mapping of this embedding extends to a monomorphism $h$ of $e_jR$ into $E_i$ with $h(e_jR) \supset U$. This and (3) show that $e_jR$ contains an infinite strictly ascending chain of projective uniform principal right ideals of $R$, a contradiction. Hence each $E_i$ must be finitely generated. By Lemma 3, each $E_i$ is embedded in some $e_kR$ in $S(R)$. But each $E_i$ is injective, so we have $E_i \simeq e_kR$, as desired.

(ii) Assume that $R$ has ACC or DCC on projective uniform principal right ideals. We will show that each $e_iR$ is quasi-injective. (Note that this does not imply that $R$ is right self-injective!) For this purpose let $\overline{e_iR}$ be the quasi-injective hull of $e_iR$. By [17, 17.9], $\overline{e_iR}$ is a submodule of an epimorphic image $N$ of $(e_iR)^{(A)}$ for some set $A$. By [1, Proposition 16.13], $e_iR$ is $N$-injective and hence it is a direct summand of $N$. From this it is easy to see
that there is an epimorphism

\[(4) \quad f : (e_i R)^{(A)} \to \overline{e_i R}.
\]

If \(\overline{e_i R}\) is countably generated, then from (4) we see that there is a countable subset \(B\) of \(A\) such that \(\overline{e_i R}\) is an epimorphic image of \((e_i R)^{(B)}\). Then by the same argument as in Lemma 3 we see that \(\overline{e_i R}\) is embeded in \(e_i R\), proving \(\overline{e_i R} = e_i R\), i.e., \(e_i R\) is quasi-injective. We are done in this case.

Assume that \(\overline{e_i R}\) is uncountably generated. Then in (4) \(A\) must be uncountable. There is an infinite-countably generated submodule \(M^*\) of \(\overline{e_i R}\) containing \(e_i R\). Hence it is easy to see that \(A\) contains a countable subset \(I\) with \(f(e_i R)^{(I)} \supseteq M^*\). Then using the epimorphism

\[f | (e_i R)^{(I)} : (e_i R)^{(I)} \to f((e_i R)^{(I)})\]

and the fact that \((e_i R)^{(I)}\) is CS we can show, by a similar argument as for Lemma 3, that \(f((e_i R)^{(I)})\) is embedded in \(e_i R\). Then by the \(e_i R\)-injectivity of \(\overline{e_i R}\), the inverse mapping of this embedding extends to a monomorphism \(g\) of \(e_i R\) into \(\overline{e_i R}\). Clearly, \(P^*_i := g(e_i R) \supseteq M^* \supseteq e_i R\) and \(P^*_i \neq e_i R\). Hence \(e_i R\) is embedded in itself and this produces an infinite strictly descending chain of projective uniform principal right ideals of \(R\) in \(e_i R\), a contradiction if we assume the DCC. Hence in the DCC case, \(\overline{e_i R}\) must be countably generated and then, as shown above, we have \(\overline{e_i R} = e_i R\), as desired. It remains to consider the ACC case.

Since \(P^*_i \neq e_i R\), we may repeat our argument and use an induction proof as in (i) to get an infinite strictly ascending chain of submodules \(P^*_k\) in \(\overline{e_i R}\):

\[P^*_1 \subset \cdots \subset P^*_k \subset \cdots\]

where each \(P^*_k\) is isomorphic to \(e_i R\). Let \(U^*\) be the union of these \(P^*_k\)'s. Then \(U^*_R\) is infinite-countably generated. By the above argument for considering \(M^*\), now applied to \(U^*\), we find a submodule \(P^*\) of \(\overline{e_i R}\) with \(P^* \cong e_i R\) and \(P^* \supseteq U^*\). This shows that in \(e_i R\) there is an infinite strictly ascending chain of projective uniform principal right ideals of \(R\), a contradiction. Thus \(\overline{e_i R}\) must be countably generated and so we can find a countable set \(A\) for which (4) holds.

Hence, as concluded above, we have \(\overline{e_i R} = e_i R\), proving the quasi-injectivity of \(e_i R\). \(\square\)

**Lemma 5.** Under assumptions of Lemma 4 (i), \(E = E(R_R)\) is \(\Sigma\)-injective.

**Proof.** By a result of Faith [5] mentioned in the introduction, it is enough to show that \(E\) is countably \(\Sigma\)-injective. For convenience we put

\[(5) \quad Q = E^{(N)} = \bigoplus_{i \in N} Q_i\]

where, by Lemma 4 (i), each \(Q_i\) is isomorphic to some injective \(e_j R\) in \(S(R)\). Moreover, because of this we easily verify that \(Q_R\) is a direct summand of a direct sum of countably many copies of \(R_R\). Hence \(Q_R\) is a CS module. First we show that (5) complements direct summands; i.e. if \(B\) is a direct summand of \(Q_R\), then there is a subset \(N'\) of \(N\) such that \(Q = B \oplus Q(N')\), where
here and below we denote by $Q(J)$ the direct sum of $Q_i$ in (5) with all $i \in J$ whenever $J$ is a subset of $\mathbb{N}$.

Thus we assume that $B$ is a direct summand of $Q$. By Zorn’s Lemma, there is a subset $H$ of $\mathbb{N}$ which is maximal with respect to $B \cap Q(H) = 0$. Since each $Q_i$ is uniform, it is clear that $C = B \oplus Q(H)$ is essential in $Q$. Put $K_1 = K_2 = Q$. Then like $Q$, $K_1 \oplus K_2$ is also a direct summand of a direct sum of countably many copies of $R_R$. Hence $K_1 \oplus K_2$ is a CS module, since $R$ is right countably $\Sigma$-CS. We have

$$K_1 \oplus K_2 = (B \oplus D) \oplus K_2$$

for some submodule $D$ of $K_1$. Since $K_2$ contains a direct summand $K'$ which is isomorphic to $Q(H)$, we see that $C = B \oplus Q(H)$ is isomorphic to $B \oplus K'$, a direct summand of $K_1 \oplus K_2$. Hence $C$ is a CS module. Moreover, if $V$ is a uniform direct summand of $C$, then $V$ is isomorphic to a uniform direct summand $V'$ of $B \oplus K'$ and hence one of $K_1 \oplus K_2$. Since each $Q_i$ is injective, uniform, so its endomorphism ring is local. Then we may use [9, 7.3.4] to see that $V'$ is isomorphic to some $Q_i$ in (5). In particular, $V'$ is injective and so is $V$.

Now assume that $C \neq Q$. Then there is a $Q_k$ in (5) with $Q_k \not\subseteq C$. Put $T = Q_k \cap C$. Then $T$ is a uniform submodule of $C$. Let $T^*$ be a maximal essential extension of $T$ in $C$. Then $T^*$ is a direct summand of $C$. Moreover, by the previous consideration, $T^*$ is injective. Hence $Q_k + T^* = T^* + T'$ for some submodule $T'$ of $Q_k + T^*$. If $T' = 0$, $T^* \subseteq Q_k$, implying $Q_k = T^* \subseteq C$, a contradiction. Hence $T' \neq 0$. By modularity we have $C \cap (Q_k + T^*) = (C \cap Q_k) + T^* = T + T^* = T^* = C \cap (T^* + T') = T^* + (C \cap T')$. Therefore $C \cap T' = 0$, a contradiction to the fact that $C$ is essential in $Q$. Thus $C = Q$, proving that (5) complements direct summands. From this we may use [11, Theorem 2.25] to obtain that local direct summands of $Q$ are direct summands. We apply this below to show that $Q$ is injective.

Let $U$ be a right ideal of $R$ and $f$ be an $R$-homomorphism of $U$ to $Q$. We may assume that $f \neq 0$ and $U$ is essential in $R_R$. Put $M = E \oplus Q$. Note that $U \subseteq R \subseteq E$. Since $Q = E(\mathbb{N})$, it is clear that $M \approx Q$ and so $M$ is CS and local direct summands of $M$ are direct summands. There exists a direct summand $V^*$ of $M$ such that the submodule $V = \{x - f(x); x \in U\}$ is essential in $V^*$. Put

$$M = V^* \oplus M^*$$

for some submodule $M^*$ of $M$. We have $V^* \cap Q = 0$ and moreover $V^* \oplus Q$ is essential in $M$. Let $\pi$ be the projection of $M$ onto $M^*$ given by (6). Then clearly, $\pi^* = (\pi(Q))$ is a monomorphism. It follows that $\{\pi^*(Q_i); i \in \mathbb{N}\}$ is an independent set of injective submodules of $M^*$ and so $W = \bigoplus_{i \in \mathbb{N}} \pi^*(Q_i)$ is a local direct summand of $M^*$ and also of $M$. Hence $W$ is a direct summand of $M^*$, say $M^* = W \oplus Y$. If $Y \neq 0$, there is $0 \neq x \in (V^* \oplus Q) \cap Y$ and so $x = u + v$ ($u \in V^*; u \in Q$). Hence $x = \pi(x) = \pi(u) + \pi(v) = \pi(v) \in W$, a contradiction. Hence $Y = 0$ and therefore $\pi(M) = W = \pi(Q)$, implying $M = V^* \oplus Q$.

Let $\pi'$ be the projection of $V^* \oplus Q$ onto $Q$. Then $\pi'|R_R$ is an extension of $f$ from $R_R$ to $Q_R$, proving the injectivity of $Q$. □
Proof of Theorem 1. (a) $\Rightarrow$ (b). Assume (a). Then by [12], $R$ is semiprimary. Hence $R$ has DCC and by [8] $R$ has ACC on principal right ideals. Thus (a) implies (b).

(b) $\Rightarrow$ (c). Assume (b). Clearly, it is enough to show that $R$ has finite right uniform dimension; this means there is no infinite direct sum of nonzero right ideals in $R$. Assume on the contrary that $R$ contains an infinite direct sum of nonzero right ideals. Since $R$ is right CS, by a standard argument we find an infinite set $\{f_i\}_{i=1}^{\infty}$ of nonzero orthogonal idempotents $f_i$ in $R$. Then the right ideals $R_i = f_iR \oplus \cdots \oplus f_iR, \ i = 1, 2, \ldots$, form an infinite strictly ascending chain of projective principal right ideals of $R$. This is a contradiction if we assume that $R$ has ACC on projective principal right ideals for (b). Hence $R$ must have finite right uniform dimension in this case.

For the DCC case of (b), put $f_i^* = f_i + \cdots + f_i, \ i = 1, 2, \ldots$. Then $R = f_i^*R \oplus (1 - f_i^*)R$. Clearly, each $(1 - f_i^*)R$ is nonzero and they form an infinite strictly descending chain of projective principal right ideals in $R$, a contradiction. Thus $R$ must have finite right uniform dimension, as desired.

(c) $\Rightarrow$ (a). Assume (c). Then by a standard argument we can show that $R_R$ has a direct decomposition of form (1). By Lemma 5, $E(R_R)$ is then $\Sigma$-injective. Hence by the mentioned result of Faith in the Introduction, and since $R_R^{(N)}$ is contained in $E^{(N)}$, $R$ has ACC on the annihilators of subsets from $R_R^{(N)}$. From this and since $R$ is right countably $\Sigma$-CS, every local direct summand of $R_R^{(N)}$ is a direct summand by [11, Proposition 2.18]. Moreover, by Lemma 4 (ii), $R_R^{(N)}$ is a direct sum of quasi-injective uniform modules (and therefore whose endomorphism rings are local). Hence we may use [11, Theorem 2.25] to see that this decomposition of $R_R^{(N)}$ complements direct summands. By [1, Theorem 28.14], $R$ is right perfect. Thus by [15, Theorem II], $R$ is right $\Sigma$-CS, proving (a).

(a) $\Rightarrow$ (d) is clear by [12].

(d) $\Rightarrow$ (a). Assume (d). Since $R$ is semiperfect and right CS, it is easy to see that $R_R$ has a direct decomposition of the form (1). Moreover, each $e_iR$ has a local endomorphism ring. Hence the module $R_R^{(N)}$ is a direct sum of uniform modules with local endomorphism rings. Therefore we may use the argument of proving (c) $\Rightarrow$ (a) to verify that $R$ is right $\Sigma$-CS, as desired. □

Proof of Corollary 2. One direction is clear (see [6, Theorem 24.20]). Assume conversely that $R$ is right countably $\Sigma$-CS, semiperfect and no nonzero projective right ideals are contained in $J(R)$. By Theorem 1, $R$ is right $\Sigma$-CS. In particular $R_R$ has a decomposition of form (1). Hence by Lemma 4 (i), each $E(e_iR)$ is isomorphic to some $e_kR$ in $S(R)$. If $E(e_iR) \neq e_iR$, then $e_iR$ is embedded in $J(e_kR) \subseteq J(R)$, a contradiction to our assumption. It follows that $e_iR = E(e_iR)$ for each $e_iR$; i.e. $R$ is right self-injective. Since $R$ is right $\Sigma$-CS, $R$ is QF by [12, Theorem 4.3]. □

From the considerations in this paper, especially from the proof of Lemma 4, we immediately obtain the following result:

Proposition 6. A right countably $\Sigma$-CS ring $R$ is right $\Sigma$-CS if and only if $R$ has finite right uniform dimension and each projective uniform principal right ideal of $R$ is not embedded properly in itself. □
It would be interesting to know whether Corollary 2 holds also for semilocal rings. After submitting this paper we received a preprint "Σ-Extending Modules" of J. Clark and R. Wisbauer, in which they showed, among others, that a right countably Σ-CS ring with ACC on right annihilators is right Σ-CS. This together with Theorem 1 yields a conclusion that in a right countably Σ-CS ring the ACC on right annihilators and the ACC on projective principal right ideals are equivalent. In general, these two kinds of ACC are quite different (see, for example, that the ring in Faith's Conjecture has ACC and DCC on principal right and principal left ideals, but if we could show that it has ACC on right annihilators, then (FC) would be established!).

We would like to ask the following questions:

(Q1) Is a right countably Σ-CS ring with finite right uniform dimension necessarily right Σ-CS?

(Q2) Is a right countably Σ-CS ring necessarily right Σ-CS if (i) $R$ has ACC on left annihilators or (ii) $R$ has ACC on (projective) principal left ideals?

From the results of Faith mentioned in the Introduction, a ring $R$ is QF iff $R$ is right countably Σ-injective iff $R$ is left countably Σ-injective. Of course, a similar result does not hold for right countably Σ-CS rings. However from the considerations in this note we easily verify that for a semiperfect right self-injective ring $R$ the following are equivalent: (i) $R$ is QF, (ii) $R$ is right countably Σ-CS, (iii) $R$ is left countably Σ-CS.

Acknowledgments

This paper was written during the author's stay at the Pusan National University (1992-1993). He wishes to thank the University for financial support and the members of the Department of Mathematics Education for their hospitality. He also wishes to thank his colleague J. K. Park for stimulating discussions during the preparation of this paper.

The author is very much indebted to the referee for useful suggestions.

References


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