CONVERGENCE OF DIAGONAL PADÉ APPROXIMANTS FOR FUNCTIONS ANALYTIC NEAR 0

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Abstract. For functions analytic in a neighbourhood of 0, we show that at least for a subsequence of the diagonal Padé approximants, the point 0 attracts a zero proportion of the poles. The same is true for every “sufficiently dense” diagonal subsequence. Consequently these subsequences have a convergence in capacity type property, which is possibly the correct analogue of the Nuttall-Pommerenke theorem in this setting.

1. Introduction

Recall that if \( f \) is analytic near 0, then for \( m, n \geq 0 \), the \( m, n \) Padé approximant to \( f \) is a rational function \( [m/n](z) = (P/Q)(z) \), where \( P, Q \) have degree \( \leq m, n \) respectively, \( Q \) is not identically zero, and

\[
(fQ - P)(z) = O(z^{m+n+1}), \quad z \to 0.
\]

For functions meromorphic in \( \mathbb{C} \), or even with singularities of capacity 0, it is known that the diagonal sequence \( \{[n/n]\}_{n=1}^{\infty} \) converges in capacity and in measure [11, 14]. Similar results are available in more general circumstances [3, 6, 16, 17, 19].

By contrast, for functions analytic only near 0, the full diagonal sequence of Padé approximants need not converge in capacity in any neighbourhood of zero [7, 8, 15], and moreover, at least for infinitely many \( n \), \( [n/n] \) may have at least \( n - \log n \) poles arbitrarily near 0 [18]. (We could replace \( \log n \) by any sequence increasing to \( \infty \).) The 1961 Baker-Gammel-Wills conjecture [1, 2] asserts that a subsequence of \( \{[n/n]\} \) converges uniformly near 0, but at present it is not even known if a subsequence converges in capacity.

In this paper we show that, at least for a subsequence of \( \{[n/n]\} \), the proportion of poles of \( [n/n] \) near 0 shrinks to 0, in a certain sense. This result also holds for subsequences of \( \{[n_j/n_j]\} \) provided \( n_{j+1}/n_j \to 1 \) as \( j \to \infty \). Then we deduce a convergence in capacity type property. Since by a variable scaling \( z \to rz \) any function analytic near 0 can be scaled to a function analytic in \( |z| < 1 \), the transformation properties of Padé approximants permit us to consider only the latter:

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Theorem 1.1. Let $f$ be analytic in $\{z : |z| < 1\}$. Let $\{n_j\}_{j=1}^\infty$ be an increasing sequence of positive integers with
\[\lim_{j \to \infty} n_{j+1}/n_j = 1.\]
Let $0 < \delta < 1$. Then there exists an infinite sequence of positive integers $S$ with the following property: For $j \in S$, the total multiplicity of poles of $[n_j/n_j]$ in $\{z : |z| \leq e^{-19/\delta}\}$ is at most $\delta n_j$.

Under a regularity assumption on the errors of best rational approximation, we can say the same for full sequences of Padé approximants: For $0 < \rho < 1$, we let
\[E_n(f; \rho) := \inf\{\|f - R\|_{L_\infty(|z| \leq \rho)} : R \text{ is a rational function of type } (n, n)\}.\]

Theorem 1.2. Assume that $f$ is analytic in $\{z : |z| < 1\}$, and that
\[0 < \limsup_{n \to \infty} E_n(f; \rho)^{1/n} \leq \kappa(\rho) \liminf_{n \to \infty} E_n(f; \rho)^{1/n},\]
where $\kappa(\rho)$ is finite for $\rho \in (0, 1)$. Then for large enough $n$, the total multiplicity of poles of $[n/n](z)$ in $\{z : |z| \leq \rho\}$ is at most $\delta n$, provided
\[\rho \kappa(\rho)^{1/\delta} \leq \exp(-19/\delta)\]
In particular, if $\lim_{n \to \infty} E_n(f; \rho)^{1/n}$ exists for $0 < \rho < 1$, then for $n$ large enough, the total multiplicity of poles of $[n/n](z)$ in $\{z : |z| \leq \exp(-19/\delta)\}$ is at most $\delta n$.

Remarks. (i) Similar results hold if we consider sectorial sequences of Padé approximants of the form $\{[m_j/n_j]\}$, where $\{m_j\}, \{n_j\}$ satisfy (1.1) and, for some fixed $\lambda$,
\[1/\lambda \leq m_j/n_j \leq \lambda, \quad j \geq 1.\]
The formulation will be more complicated and the proofs will be messier, but we hope to address this in a subsequent paper. (See [9], where similar results were proved for functions analytic in $\mathbb{C}$ except for singularities of capacity 0 and for general sectorial sequences $\{[m_j/n_j]\}$.)

(ii) Note that the size of the neighborhood in which there are at most $\delta n$ poles is a function of $\delta$ only, not of $f$. However, the factor $e^{-19}$ is not optimal.

(iii) Note that if
\[\liminf_{n \to \infty} E_n(f; \rho)^{1/n} = 0,\]
then it is easy to see that a subsequence of the $[n/n]$ Padé approximants actually converges in capacity on compact subsets of $\{z : |z| < \rho\}$. Note too that if
\[\lim_{n \to \infty} E_n(f; \rho)^{1/n} = 0,\]
then $f$ belongs to the Goncar-Walsh class [3, 10], the $[n/n]$ Padé approximants converge in capacity in $\{z : |z| < 1\}$ [3, 19], and for $j \in S$ all but $o(n_j)$ poles of $[n_j/n_j]$ leave every compact subset of $\{z : |z| < 1\}$ [9, 10].

Now we turn to convergence in capacity. Recall that, for a compact set $K$, the logarithmic capacity $\text{cap}(K)$ is defined by
\[\text{cap}(K) := \lim_{n \to \infty} \left(\min_{P_n} \|P_n\|_{L_\infty(K)}\right)^{1/n},\]
where the minimum is taken over all monic polynomials $P_n$ of degree $n$. For arbitrary $S$, the inner logarithmic capacity $\text{cap}(S)$ is defined by

$$\text{cap}(S) := \sup\{\text{cap}(K) : K \subset S, K \text{ compact}\}.$$ 

Convergence in capacity is essentially the same as convergence in measure. We say $f_n \to f$ in capacity in $\{z : |z| \leq r\}$ if $\forall \varepsilon > 0$,

$$\text{cap}\{z : |z| \leq r \text{ and } |f - f_n|(z) \leq \varepsilon\} \to 0, \quad n \to \infty.$$

The Nuttall-Pommerenke theorem [11, 14] and its extensions actually prove geometric convergence in capacity under suitable hypotheses on $f$:

$$\text{cap}\{z : |z| \leq r \text{ and } |f - [n/n]|(z) \geq \varepsilon^n\} \to 0, \quad n \to \infty.$$ 

Here we shall show that Theorem 1.1 implies a weak convergence in capacity property:

Given $0 < \Delta < \frac{1}{2}$, $A > 1$, there exists $\rho = \rho(\Delta) < 1$ (independent of $f$) such that, for all $n$ in a subsequence,

$$\text{cap}\{z : |z| \leq \rho : |f - [n/n]|(z) \geq \rho^{n\Delta}\} \leq \text{cap}\{z : |z| \leq \rho\}^A = \rho^A.$$ 

The same estimate holds if we replace cap by planar Lebesgue measure or one-dimensional Hausdorff content.

The point is that, in most of $\{z : |z| \leq \rho\}$, $[n/n]$ is geometrically close to $f$, and we have a weak convergence in capacity property: The capacity (or area or one-dimensional Hausdorff content) of the set on which $[n/n]$ does not approximate is an arbitrarily small proportion of the total capacity (or area or content). I believe that in the setting of the following theorem the conclusion of Theorem 1.3 may possibly be the correct analogue of the Nuttall-Pommerenke theorem: Nothing more can be said of subsequences of $\{[n_j/n_j]\}$, other than this weak convergence in capacity type property near 0. Of subsequences of the full diagonal sequence $\{[n/n]\}$, the Baker-Gammel-Wills conjecture may well be true.

**Theorem 1.3.** Let $f$ be analytic in $\{z : |z| < 1\}$. Let $0 < \Delta < \frac{1}{2}$, $A > 1$ and $\rho := \frac{1}{2} \exp(-19A/(\frac{1}{2} - \Delta))$. Let $\{n_j\}_{j=1}^{\infty}$ be an increasing sequence of positive integers satisfying (1.1). Then there exists an infinite sequence of positive integers $\mathcal{S}$ with the following property: For $j \in \mathcal{S}$,

$$\text{cap}\left\{z : |z| \leq \rho \text{ and } |f - [n_j/n_j]|(z) > \left(\frac{|z|}{\rho} \cdot \rho^\Delta\right)^{2n_j}\right\} \leq \text{cap}\{z : |z| \leq \rho\}^A.$$

**Remarks.** (i) The restriction $\Delta < \frac{1}{2}$ is related to the exponent $\frac{1}{2}$ in the right-hand side of

$$\limsup_{n \to \infty} E_{nn}(f; \rho)^{1/(2n)} \leq \rho^{1/2}.$$ 

It is now known [12] that

$$\liminf_{n \to \infty} E_{nn}(f; \rho)^{1/(2n)} \leq \rho.$$ 

Consequently, if we assume that $\lim_{n \to \infty} E_{nn}(f; \rho)^{1/(2n)}$ exists for $0 < \rho < 1$, then our proof allows us to replace $0 < \Delta < \frac{1}{2}$ by $0 < \Delta < 1$ and $\rho = \frac{1}{2} \exp(-19A/(\frac{1}{2} - \Delta))$ by $\rho = \frac{1}{2} \exp(-19A/(1 - \Delta))$ in the above result. In
that case also, the weak convergence in capacity will hold for the full diagonal sequence, and not just a subsequence.

(ii) The subsequence $\mathcal{S}$ in Theorem 1.3 is the same sequence as in Theorem 1.1, with a suitable choice of $\delta = \delta(A)$.

We prove Theorems 1.1 and 1.2 in §2 and Theorem 1.3 in §3.

2. Proof of Theorems 1.1 and 1.2

We shall use the notion of one-dimensional Hausdorff content:

$$m(E) := \inf \left\{ \sum_j \text{diam}(B_j) : E \subset \bigcup_j B_j \right\},$$

where the inf is taken over all countable collections of balls $\{B_j\}$ of diameters $\{\text{diam} B_j\}$ covering $E$. We first present four lemmas (at least two of which are standard), and then prove Theorems 1.2 and 1.1. Throughout $p_n$ denotes the polynomials of degree $< n$, and $C, C_1, C_2, \ldots$ denote constants independent of $n, p, \text{ and } z$. The same symbol does not necessarily denote the same constant in different occurrences. In the sequel, $[n/n] = p_n/q_n$.

Lemma 2.1. Let $U \in \mathcal{P} \setminus \{0\}$ and $0 < \epsilon < \rho$. Then there exists a set $\mathcal{E} \subset [0, \rho]$ such that $m(\mathcal{E}) < \epsilon$ and, for $\sigma \in [0, \rho] \setminus \mathcal{E}$,

\begin{equation}
\max\{|U(t)/U(z)| : |t| = \rho, |z| = \sigma\} \leq (12e\rho/\epsilon)\epsilon^\ell.
\end{equation}

Proof. Split $U = cVW$, where $c \neq 0$, and $V, W$ are monic polynomials of degree $\nu, \omega$, respectively, with zeros outside $|z| \leq 2\rho$, inside $|z| \leq 2\rho$, respectively. Now for $|a| \geq 2\rho$, $|t| = \rho$, $|z| \leq \rho$,

\begin{equation}
\left| \frac{t - a}{z - a} \right| \leq \frac{1 + |t/a|}{1 - |z/a|} \leq 3.
\end{equation}

We deduce that

\begin{equation}
|V(t)/V(z)| \leq 3^\nu, \quad |t| = \rho, \ |z| \leq \rho.
\end{equation}

Next, by Cartan’s lemma [1, p. 174],

$$|W(z)| \geq (\epsilon/4\epsilon)^\omega, \quad z \in \mathbb{C} \setminus \mathcal{F},$$

where $m(\mathcal{F}) \leq \epsilon$. Then using an easy covering argument, we see that $\mathcal{E} : = \{|z| : z \in \mathcal{F}\}$ also has $m(\mathcal{E}) \leq \epsilon$. Moreover for $|t| = \rho$,

$$|W(t)| \leq (3\rho)^\omega.$$

These last two inequalities and (2.2) give (2.1). □

Lemma 2.2. Let $f$ be analytic in $\{z : |z| < 1\}$. Let $0 < \epsilon < \rho < 1$. There exists $\mathcal{E}_n$ with $m(\mathcal{E}_n) \leq \epsilon$, such that, for $\sigma \in [0, \rho] \setminus \mathcal{E}_n$,

\begin{equation}
\max_{|z| = \sigma} |f - [n/n](z)| \leq E_n(f; \rho) \left( \frac{12e\sigma}{\epsilon} \right)^{2n} \frac{\sigma}{\rho - \sigma}.
\end{equation}

In particular, for some $\rho_1 \in [\frac{1}{3} \rho, \frac{2}{3} \rho]$,

\begin{equation}
\max_{|z| = \rho_1} |f - [n/n](z)| \leq 2E_n(f; \rho)(32e)^{2n}.
\end{equation}
Proof. Let \( r^*_n := p^*_n/q^*_n \) be the best approximant of type \((n, n)\) to \( f \) on \(|z| \leq \rho\). Then for \(|z| < \rho\),
\[
q^*_n(z)(f q_n - p_n)(z)/z^{n+1} = \frac{1}{2\pi i} \int_{|t| = \rho} \left[ q_n(t)(f q^*_n - p^*_n)(t)/t^{2n+1} \right] \frac{dt}{t - z}.
\]
This is an easy consequence of Cauchy’s integral formula and the fact that, for any \( \Pi \in \mathcal{P}_{2n} \),
\[
\frac{1}{2\pi i} \int_{|t| = \rho} \left[ \Pi(t)/t^{2n+1} \right] \frac{dt}{t - z} = 0.
\]
(We chose \( \Pi := p^*_n q_n - p_n q^*_n \)). We deduce that, for \( \sigma < \rho \),
\[
\max_{|z| = \sigma} |f - [n/n](z)| \leq \left( \frac{\sigma}{\rho} \right)^{2n+1} \max \left\{ \left| \frac{(q_n q^*_n)(t)}{(q_n q^*_n)(z)} \right| : |t| = \rho, |z| = \sigma \right\} \frac{\rho}{\rho - \sigma} E_{nn}(f; \rho)
\]
\[
\leq \left( \frac{12\rho \sigma}{\epsilon} \right)^{2n} \frac{\sigma}{\rho - \sigma} E_{nn}(f; \rho),
\]
by Lemma 2.1, provided \( \sigma \notin \mathcal{E} \), where \( \mathcal{M}(\mathcal{E}) \leq \epsilon \). In particular, if \( \epsilon = \rho/4 \), we can choose such a \( \rho_1 := \sigma \in [\frac{1}{2}\rho, \frac{3}{4}\rho] \setminus \mathcal{E} \). \( \square \)

We shall need a lemma of Gončar and Grigorjan:

Lemma 2.3. If \( g \) is analytic in \( \{z : |z| < \rho\} \) except for poles of total multiplicity \( m \), none lying on \( |z| = \rho \), and if \( \mathcal{A}_\rho(g) \) denotes the analytic part of \( g \) in \( \{z : |z| < \rho\} \) (that is, \( g \) minus its principal parts in \( |z| < \rho \)), then
\[
\|\mathcal{A}_\rho(g)\|_{L^\infty(|z| < \rho)} \leq 7m^2 \|g\|_{L^\infty(|z| = \rho)}.
\]

Proof. See [4]. For more precise results and references, see [5, 12, 13]. \( \square \)

Following is our main lemma:

Lemma 2.4. Let \( f \) be analytic in \( \{z : |z| < 1\} \), and \( 0 < \rho < 1, K > 1 \), with \( 3K\rho < 1 \). If \([n/n]\) has \( \tau = \tau(n) \) poles counting multiplicity in \( \{z : |z| \leq K\rho\} \), then, for large enough \( n \),
\[
E_{n-\tau, n-\tau}(f; K\rho) \leq [e^{16K}]^n E_{nn}(f; \rho)^{1+(\log 6K)/(\log 3K\rho)}.
\]

Proof. Let \( S_m \) be the \( m \)th partial sum of the Maclaurin series of \( f \). Let \( e := 3K\rho \). We have, for large enough \( m \),
\[
\|f - S_m\|_{L^\infty(|z| \leq 2K\rho)} \leq e^m.
\]
Let \( \lfloor x \rfloor \) denote the largest integer \( \leq x \). We let
\[
m := \left\lfloor \frac{\log E_{nn}(f; \rho)}{\log e} \right\rfloor + 1,
\]
so that \( e^m \leq E_{nn}(f; \rho) \). Let \( \rho_1 \in \left[\frac{1}{3}\rho, \frac{2}{3}\rho\right] \) be as in Lemma 2.2. We deduce from (2.4) and our choice of \( m \) that, for \( n \) large enough,
\[
\|S_m - [n/n]\|_{L^\infty(|z| = \rho_1)} \leq E_{nn}(f; \rho)\{1 + 2(32e)^{2n}\},
\]
so
\[
\|S_m q_n - p_n\|_{L^\infty(|z| = \rho_1)} \leq E_{nn}(f; \rho)(32e)^{2n}\|q_n\|_{L^\infty(|z| = \rho_1)}.
\]

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The Bernstein-Walsh lemma gives
\[ \|S_m q_n - p_n\|_{L^\infty(\{z: |z| \leq 2K\rho\})} \leq E_{nn}(f; \rho) 3(32\varepsilon)^{2n}\|q_n\|_{L^\infty(\{z: |z| = \rho_1\}) (6K)^{m+n}}. \]
We deduce that, for \( \sigma \leq 2K\rho \),
\[ \|f - [n/n]\|_{L^\infty(\{z: |z| = \sigma\})} \leq \|f - S_m\|_{L^\infty(\{z: |z| = \sigma\})} + \|S_m - [n/n]\|_{L^\infty(\{z: |z| = \sigma\})} \]
\[ \leq E_{nn}(f; \rho) \left[ 1 + 3(32\varepsilon)^{2n}(6K)^{m+n} \max \left\{ \frac{q_n(t)}{q_n(z)} : |t| = \rho_1, |z| = \sigma \right\} \right], \]
provided, of course, that the right-hand side is finite. By Lemma 2.1 (with \( \rho \) there replaced by \( 2K\rho \), and \( \varepsilon = \frac{3}{4}K\rho \)), we can choose \( \sigma_1 \in (K\rho, 2K\rho) \) such that
\[ \max \left\{ \frac{q_n(t)}{q_n(z)} : |t| = \rho_1, |z| = \sigma_1 \right\} \leq \max \left\{ \frac{q_n(t)}{q_n(z)} : |t| = 2K\rho, |z| = \sigma_1 \right\} \]
\[ \leq \left( \frac{12e \cdot 2K\rho}{3K\rho / 4} \right)^n = (32\varepsilon)^n. \]
Since \( f \) is analytic, \( \mathcal{A}_{\sigma_1}(f - [n/n]) = f - \mathcal{A}_{\sigma_1}([n/n]) \). Also \( \sigma_1 \geq K\rho \). Then Lemma 2.3 gives
\[ \|f - \mathcal{A}_{\sigma_1}([n/n])\|_{L^\infty(\{z: |z| \leq K\rho\})} \leq \|f - \mathcal{A}_{\sigma_1}([n/n])\|_{L^\infty(\{z: |z| = \sigma_1\})} \]
\[ \leq 7n^2 \|f - [n/n]\|_{L^\infty(\{z: |z| = \sigma_1\})} \leq 28n^2[6K(32\varepsilon)^3]n(6K)^mE_{nn}(f; \rho) \]
\[ \leq E_{nn}(f; \rho) 1 + \log 6K/\log 3K\rho (e^{16} K)^n, \]
for \( n \) large enough, by our choice of \( m \) and of \( \varepsilon = 3K\rho \). Since \( [n/n] \) has at least \( \tau \) poles in \( |z| \leq K\rho < \sigma_1 \), \( \mathcal{A}_{\sigma_1}([n/n]) \) is a rational function of type \( (n - \tau, n - \tau) \), and the result follows. \( \square \)

We turn to the proofs of the theorems. To indicate the ideas, we first prove the simpler Theorem 1.2. In the sequel, we let
\[ \Lambda(\rho) := \limsup_{n \to \infty} E_{nn}(f; \rho)^{1/n}. \]
Recall (as in the Introduction) that if \( \Lambda(\rho) = 0 \) for some \( \rho \in (0, 1) \), then, by a result of Gončar [3], \( \Lambda(\rho) = 0 \) for all \( 0 < \rho < 1 \), and then stronger results are available [9]. So we assume that \( \Lambda(\rho) > 0 \) for all \( \rho > 0 \) in the sequel.

**Proof of Theorem 1.2.** Assume that, for some \( \delta \in (0, 1) \), \( \rho \in (0, \frac{1}{2}) \), and for \( n \) belonging to some infinite sequence of integers \( \mathcal{N} \), \( [n/n] \) has poles of total multiplicity \( \geq \delta n \) in \( \{z : |z| \leq \rho\} \). We show that \( \rho \) cannot be too small assuming that \( \mathcal{N} \) is an infinite set. Applying Lemma 2.4 with \( K = 1 \) gives
\[ E_{(n(1 - \delta))}([n(1 - \delta)])(f; \rho) \leq e^{16n} E_{nn}(f; \rho) 1 + \log 6K/\log 3K\rho (e^{16} K)^n. \]
Taking \( n^{th} \) roots, letting \( n \to \infty \) through \( \mathcal{N} \), and using (1.2) gives
\[ (\kappa(\rho))^{-1} A(\rho)^{1 - \delta} \leq e^{16 A(\rho)} 1 + \log 6K/\log 3K\rho. \]
That is,
\[ A(\rho)^{-\delta - \log 6K/\log 3K\rho} \leq e^{16K(\rho)}. \]
(Recall that (1.2) forces \( \kappa(\rho) \geq 1 \).) The exponent of \( A(\rho) \) is negative if
\[ \rho \leq \frac{1}{3} \exp \left( -\frac{\log 6}{\delta} \right) \left( \leq \frac{1}{18} \right). \]
Now $A(\rho) \leq \rho$ by analyticity of $f$ in $|z| < 1$, so, for $\rho$ satisfying (2.7), it follows from (2.6) that

$$\delta |\log \rho| \leq 16 + \log \kappa(\rho) + \log 6 \left| \frac{\log \rho}{\log 3 \rho} \right|.$$  

Here for $\rho \leq \frac{1}{18}$,

$$\log 6 \left| \frac{\log \rho}{\log 3 \rho} \right| \leq \log 18 < 3.$$  

So we obtain

$$\rho \kappa(\rho)^{1/\delta} > \exp(-19/\delta).$$

Therefore for large enough $n$, $[n/n]$ can have no more than $\delta n$ poles in $\{z : |z| < \rho\}$ if $\rho \kappa(\rho)^{1/\delta} \leq \exp(-19/\delta)$. □

**Proof of Theorem 1.1.** The consequence of (1.1) that we shall use is

(2.8) \[ \limsup_{n \to \infty} E_{n,k,n}(f, \rho)^{1/n_k} = \limsup_{n \to \infty} E_{n,n}(f, \rho)^{1/n} = A(\rho). \]

This follows easily from the fact that $E_{n,n}(f, \rho)$ is decreasing in $n$. Let $0 < \eta < \delta < 1$. For large enough $k$, we define $j = j(k)$ to be the largest integer $j$ for which $n_k \geq n_j(1 - \eta)$, so that

$$n_j(k)(1 - \eta) \leq n_k < n_j(k) + 1(1 - \eta).$$

Let $\tau_k := n_j(k) - n_k$. We see from (1.1) and our choice of $j(k)$ that

(2.9) \[ \lim_{k \to \infty} \frac{\tau_k}{n_j(k)} = \eta; \quad \lim_{k \to \infty} \frac{n_k}{n_j(k)} = 1 - \eta. \]

Suppose that for some $0 < \rho < 1$ and for large enough $k$, $[n_k/n_k]$ has more than $\delta n_k$ poles in $\{z : |z| \leq \rho\}$. Then for large enough $k$, and $j = j(k)$, $[n_{j(k)}/n_{j(k)}]$ has $> \delta n_{j(k)} > \tau_k$ poles in $\{z : |z| \leq \rho\}$ (recall that $\eta < \delta$). As $n_j(k) - \tau_k = n_k$, Lemma 2.4 (with $P = 1$) gives

$$E_{n_k,n_k}(f, \rho) \leq e^{16n_{j(k)}} E_{n_{j(k)},n_{j(k)}}(f, \rho)^{1 + (\log 6)/(\log 3 \rho)}.$$  

Taking $n_{j(k)}$th roots in this last inequality, and then lim sups as $k \to \infty$, and using (2.8) and (2.9), give

$$A(\rho)^{1 - \eta} \leq e^{16} A(\rho)^{1 + (\log 6)/(\log 3 \rho)}.$$  

Since $\eta < \delta$ is arbitrary, we deduce that

$$A(\rho)^{-\delta} \leq e^{16} A(\rho)^{1 + (\log 6)/(\log 3 \rho)}.$$  

Then

$$A(\rho)^{-\delta - (\log 6)/(\log 3 \rho)} \leq e^{16}.$$  

This is the exact same relation as (2.6) with $\kappa(\rho) \equiv 1$. Proceeding exactly as in the previous proof with $\kappa(\rho) \equiv 1$, we obtain

$$\rho > \exp(-19/\delta).$$

□

3. **Proof of Theorem 1.3**

Let $0 < \rho < \frac{1}{2}$, $0 < \delta < 1$, $A > 1$, and assume that for $n$ belonging to some infinite sequence of integers $\mathcal{N}$, $[n/n] = p_n/q_n$ has no more than $\delta n$
poles, counting multiplicity, in \( \{ z : |z| \leq 2\rho \} \). Let \( r_n^* := p_n^*/q_n^* \) be a best approximation of type \((n, n)\) to \( f \) on \( \{ z : |z| \leq 2\rho \} \). We begin with the identity from Lemma 2.2: For \( |z| < 2\rho \),

\[
(f - [n/n])(z) = \frac{1}{2\pi i} \int_{|t|=2\rho} \left( \frac{z}{t} \right)^{2n+1} \frac{(q_n^* q_n)(t)}{(q_n^* q_n)(z)} \frac{(f - r_n^*)(t)}{t - z} dt.
\]

We deduce that, for \( |z| \leq \rho \),

\[
|f - [n/n]|(z) \leq 2 \left( \frac{|z|}{2\rho} \right)^{2n} E_{nn}(f; 2\rho) \max_{|t|=2\rho} \left| \frac{(q_n^* q_n)(t)}{(q_n^* q_n)(z)} \right|.
\]

Now for \( n \geq n_0(\rho) \),

\[
E_{nn}(f; 2\rho) \leq (3\rho)^n.
\]

Recall that \( q_n^* \) has all zeros outside \( |z| \leq 2\rho \). We split \( q_n = S_n U_n \), where \( S_n \) is monic of degree \( s_n \leq \delta n \) and has zeros in \( |z| \leq 2\rho \), and \( U_n \) has zeros in \( |z| > 2\rho \). Exactly as in the proof of Lemma 2.1, we see that, for \( |z| \leq \rho \),

\[
\max_{|t|=\rho} \left| \frac{(q_n^* U_n)(t)}{(q_n^* U_n)(z)} \right| \leq 3^{2n}.
\]

Next, as \( S_n \) is monic, the set \( \mathcal{E}_n := \{ z : |S_n(z)| \leq \rho^{A\delta} \} \) has \( \text{cap}(\mathcal{E}_n) = \rho^A \). Then as in Lemma 2.1, since \( S_n \) has all its zeros in \( |u| \leq 2\rho \), we have, for \( |z| \leq \rho \), \( z \notin \mathcal{E}_n \),

\[
\max_{|t|=\rho} \left| \frac{(S_n)(t)}{(S_n)(z)} \right| \leq \left( \frac{3\rho}{\rho^A} \right)^{s_n} \leq (3\rho^{-A})^{\delta n}.
\]

Combining (3.1)–(3.4), we have, for \( |z| \leq \rho \), \( z \notin \mathcal{E}_n \),

\[
|f - [n/n]|(z)^{1/(2n)} \leq 2^{1/2n} \left( \frac{|z|}{2\rho} \right)^{1/2} (3\rho)^{1/2} (3\rho^{-A})^{\delta/2} \leq \frac{|z|}{\rho} 8\rho^{(1-A\delta)/2} \leq \frac{|z|}{\rho} \rho^A,
\]

provided

\[
\rho^{1/2 - \Delta - A\delta/2} \leq \frac{1}{8}.
\]

In summary, we have shown that, for large enough \( n \in \mathcal{N} \),

\[
\text{cap} \left\{ z : |z| \leq \rho \text{ and } |f - [n/n]|(z)^{1/2n} > \frac{|z|}{\rho} \rho^A \right\} \leq \text{cap}(\mathcal{E}_n) = \rho^A = \text{cap} \{ z : |z| \leq \rho \}^A,
\]

provided (3.5) holds. Let us choose \( \delta \) and \( \rho \) by

\[
A\delta = \frac{1}{2} - \Delta, \quad 2\rho = \exp(-19/\delta) = \exp(-19A/(1/2 - \Delta)).
\]

Then

\[
\rho^{1/2 - \Delta - A\delta/2} = \rho^{(1/2 - \delta)/2} \leq \exp(-19A/2) \leq \exp(-19/2),
\]

as \( A \geq 1 \), so (3.5) is satisfied. Finally, with this choice of \( \delta \) and \( \rho \), Theorem 1.1 guarantees that, given \( \{ n_j \} \) satisfying (1.1), we can find infinitely many \( j \) such that for \( n = n_j \), \( j \in \mathcal{S} \); that is, \( \{ n_j/n_j \} \) has at most \( \delta n_j \) poles in \( \{ z : |z| \leq 2\rho \} \). \( \Box \)
We remark that when the limit

$$\lim_{n \to \infty} E_{nn}(f; \rho)^{1/n}$$

exists, then the aforementioned result of Parfenov guarantees that it is \( \leq \rho^2 \). Then we can replace (3.2) by

$$E_{nn}(f; 2\rho) \leq (3\rho^2)^n.$$  

Proceeding as before, we see that we can then choose \( \rho = \frac{1}{2} \exp(-19A/(1-\Delta)) \), for any \( 0 < \Delta < 1 \).

References


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