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# AN ANALOGUE OF THE JACOBSON-MOROZOV THEOREM FOR LIE ALGEBRAS OF REDUCTIVE GROUPS OF GOOD CHARACTERISTICS

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ABSTRACT. Let $\mathfrak{g}$ be the Lie algebra of a connected reductive group $G$ over an algebraically closed field of characteristic $p > 0$. Suppose that $G^{(1)}$ is simply connected and $p$ is good for the root system of $G$. Given a one-dimensional torus $\lambda \subset G$ let $\mathfrak{g}(\lambda, i)$ denote the weight component of $\mathrm{Ad}(\lambda)$ corresponding to weight $i \in X(\lambda) \cong \mathbb{Z}$. It is proved in the paper that, for any nonzero nilpotent element $e \in \mathfrak{g}$, there is a one-dimentional torus $\lambda_e \subset G$ such that $e \in \mathfrak{g}(\lambda_e, 2)$ and $\mathrm{Ker}\,\mathrm{ad}\, e \subseteq \bigoplus_{i \geq 0} \mathfrak{g}(\lambda_e, i)$.

## 1. INTRODUCTION

Let $G$ be a connected reductive group over an algebraically closed field $K$ of characteristic $p > 0$ and $\mathfrak{g} = \mathrm{Lie}(G)$. The group $G$ acts on $\mathfrak{g}$ via the adjoint representation Ad. Given a one-dimensional torus $\lambda \subset G$ denote by $\mathfrak{g}(\lambda, i)$ the weight component of $\mathrm{Ad}\,\lambda$ corresponding to weight $i \in X(\lambda) \cong \mathbb{Z}$. Throughout the paper we assume that $p = \mathrm{char}(K)$ is a good prime number for $G$ (see (2.1) for a precise definition). Note that if $p > 5$, then $p$ is good for any reductive group over $K$.

The Lie algebra $\mathfrak{g}$ has a canonical $[p]$-operation invariant under the adjoint action of $G$. An element $x \in \mathfrak{g}$ is said to be *nilpotent* or $[p]$-*nilpotent* (resp., *semisimple* or $[p]$-*semisimple*) if $x^{[p]^e} = 0$ for some $e \in \mathbb{Z}_+$ (resp., if $x$ lies in the $p$-envelope of $x^{[p]}$ in $\mathfrak{g}$). The group $G$ acts on the set of all nilpotent elements of $\mathfrak{g}$. The orbits of this action are classified by Bala-Carter under the assumption that $p \gg 0$ (see [1, 2]). Their results are extended by Pommerening to the case when $p$ is a good prime number for $G$ (see [13, 14]). Nothing seems to be published about nilpotent orbits of the Lie algebras of type $E_7$ and $E_8$ for $p \leq 5$ though it follows from [20] that the number of nilpotent orbits of $\mathfrak{g}$ is finite for any $p > 0$.

Let $p \gg 0$ and $e$ a nonzero nilpotent element of $\mathfrak{g}$. The Jacobson-Morozov theorem [21, III, 4.3] says that $\mathfrak{g}$ contains a subalgebra $\mathfrak{s}$ such that $\mathfrak{s} \cong sl(2)$, $e \in \mathfrak{s}$ and $\mathfrak{g}$ is a completely reducible $\mathfrak{s}$-module. Moreover, a standard Lie theory argument shows that there is a connected subgroup $S \subseteq G$ such that $\mathfrak{s} = \mathrm{Lie}(S)$ and any $\mathfrak{s}$-submodule of $\mathfrak{g}$ is $S$-stable. There exist a maximal unipotent subgroup $U_e \subset S$ and a one-dimensional torus $\lambda_e \subset S$ satisfying

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$\mathrm{Lie}(U_e) = Ke$, $\lambda_e(t)U_e\lambda_e(t)^{-1} = U_e$. Let $V$ be an irreducible $\mathfrak{s}$-submodule of $\mathfrak{g}$. As $V$ is $S$-stable and $\lambda_e U_e$ is a Borel subgroup of $S$, there exists $k \in \mathbb{Z}_+$ such that $V \cap \mathrm{Ker}\,\mathrm{ad}\,e \subseteq \mathfrak{g}(\lambda_e, k)$. As $\mathfrak{g}$ is a completely reducible $\mathfrak{s}$-module, we obtain that *for any nonzero nilpotent element* $e \in \mathfrak{g}$, *there exists a one-dimensional torus* $\lambda_e \subset G$ *such that* $e \in \mathfrak{g}(\lambda_e, 2)$ *and* $\mathrm{Ker}\,\mathrm{ad}\,e \subseteq \bigoplus_{i \geq 0} \mathfrak{g}(\lambda_e, i)$.

The purpose of this paper is to extend this result to the case of an arbitrary good $p$. Note that in this setting the result is known to be true provided all simple components of $G$ are groups of classical type (see [21, IV, §§1,2]). So in the sequel we mostly deal with the groups of exceptional types. Throughout the paper we assume that the derived subgroup of $G$ is simply connected. In proving our main theorem we crucially use Pommerening's classification of nilpotent elements of $\mathfrak{g}$ and the Kempf-Rousseau theory as exposed in [18].

The motive for this investigation originated in the representation theory of $\mathfrak{g}$. It is well known [6] that all irreducible representations of $\mathfrak{g}$ are of finite dimension. To each irreducible $\mathfrak{g}$-module $V$, one can assign in a canonical way a linear function $\chi \in \mathfrak{g}^*$ called the *p-character* of $V$. The ideal $I_\chi$ of the universal enveloping algebra $U(\mathfrak{g})$ generated by the central elements of the form $x^p - x^{[p]} - \chi(x)^p \cdot 1$, where $x \in \mathfrak{g}$, acts trivially on $V$. Given a restricted subalgebra $\mathfrak{a} \subseteq \mathfrak{g}$ denote by $u_\chi(\mathfrak{a})$ the associative subalgebra of $U(\mathfrak{g})/I_\chi$ generated by $\mathfrak{a}$. It follows from the PBW-theorem that $\dim u_x(\mathfrak{a}) = p^{\dim \mathfrak{a}}$.

In [23], Kac and Weisfeiler conjectured that if $G$ is simple and $\mathfrak{g}$ admits a nondegenerate $G$-invariant trace form, then any irreducible $\mathfrak{g}$-module with $p$-character $\chi$ has dimension divisible by $p^{(\dim \Omega(\chi))/2}$ where $\Omega(\chi)$ is the orbit of $\chi$ under the coadjoint action of $G$. As I recently observed (see [15]), for any $\chi \in \mathfrak{g}^*$, there exists a restricted nilpotent subalgebra $\tilde{\mathfrak{m}}_\chi$ of $\mathfrak{g}$ such that $\dim \tilde{\mathfrak{m}}_\chi = \frac{1}{2} \dim \Omega(\chi)$ and any irreducible, faithful $\mathfrak{g}$-module with $p$-character $\chi$ is free over $u_\chi(\tilde{\mathfrak{m}}_\chi)$. This result proves the Kac-Weisfeiler conjecture. In constructing the subalgebra $\tilde{\mathfrak{m}}_\chi$, I crucially use the main result of this paper.

Concluding the introduction, note that our main result is no longer true for some simple groups of adjoint type. Indeed, let $e$ be the image of a nilpotent Jordan block of order $p$ in $gl_p(K)/\mathfrak{z} = \mathrm{Lie}(G)$ where $G = PGL_p(K)$. It is easily seen that the preimage of $\mathrm{Ker}\,\mathrm{ad}\,e$ in $gl_p(K)$ acts irreducibly on the standard $gl_p(K)$-module of dimension $p$. It follows that $\mathrm{Ker}\,\mathrm{ad}\,e \not\subseteq \mathrm{Lie}(P)$ for any parabolic subgroup $P$ of $G$.

## 2. Dynkin tori for nilpotent elements

**2.1.** Let $G$ be a connected reductive algebraic group over an algebraically closed field $K$ of characteristic $p > 0$. We assume that $p$ is good for $G$; i.e. $p$ is greater than any coefficient of any positive root of the root system $R = R(G)$ relative to a basis of simple roots in $R$.

Given a maximal torus $T$ in $G$ decompose $\mathfrak{g}$ into weight spaces under the adjoint action of $T$ giving a Cartan decomposition

$$\mathfrak{g} = \mathfrak{t} \bigoplus \sum_{\alpha \in R} Ke_\alpha$$

where $\mathfrak{t} = \mathrm{Lie}(T)$. Let $B = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$ be a basis of simple roots in $R$, $R_+$ the corresponding system of positive roots, $\{\overline{\omega}_1, \overline{\omega}_2, \ldots, \overline{\omega}_l\}$ the corresponding system of fundamental weights in the lattice of the rational characters

of $T$. Everywhere below the indexing of the simple roots in $B$ corresponds to Bourbaki's tables [4, VI, Tables I–IX].

Given a subset $B_J \subset B$ one can define the standard parabolic subgroup $P_J$ of $G$ with Levi decomposition $P_J = U_J L_J$. Following Carter [5] define a function $\eta_J \colon R \to 2\mathbb{Z}$ by

$$\eta_J(\alpha) = \begin{cases} 0 & \text{if } \alpha \in B_J, \\ 2 & \text{if } \alpha \in B \backslash B_J \end{cases}$$

and extending to arbitrary root by linearity. Denote

$$\mathfrak{g}_J(i) = \begin{cases} \sum_{\eta_J(\alpha)=i} Ke_\alpha & \text{if } i \neq 0, \\ \mathfrak{t} \bigoplus \sum_{\eta_J(\alpha)=0} Ke_\alpha & \text{if } i = 0. \end{cases}$$

Then one has $\mathfrak{g} = \bigoplus_i \mathfrak{g}_J(i)$, $[\mathfrak{g}_J(i), \mathfrak{g}_J(k)] \subseteq \mathfrak{g}_J(i+k)$ and $\bigoplus_{i \geq 0} \mathfrak{g}_J(i) = \mathrm{Lie}(P_J)$. It is well known [5, p. 166] that $\dim \mathfrak{g}_J(0) = \dim L_J$, $\dim \mathfrak{g}_J(2) = \dim U_J/U_J^{(1)}$ and $\dim L_J \geq \dim U_J/U_J^{(1)}$.

For $G$ semisimple, a parabolic subgroup $P$ is called *distinguished* if $\dim P/U_P = \dim U_P/U_P^{(1)}$ where $U_P$ is the unipotent radical of $P$. Any parabolic subgroup of $G$ is conjugate in $G$ with precisely one of the standard parabolic subgroups. A standard parabolic subgroup $P_J$ is distinguished if and only if $\dim \mathfrak{g}_J(0) = \dim \mathfrak{g}_J(2)$.

**2.2.** Let $\mathfrak{p}$ denote the Lie algebra of a parabolic subgroup $P$ of $G$. Set $\mathfrak{n}_P = \mathrm{Lie}(U_P)$. An element $x \in \mathfrak{n}_P$ is called a *Richardson element* of $P$ if the orbit $(\mathrm{Ad}\, P) \cdot x$ is dense in $\mathfrak{n}_P$. Clearly, all Richardson elements of $\mathfrak{p}$ are conjugate with respect to the adjoint action of $P$.

If $P = P_J$ for some $J \subseteq \{1, 2, \dots, l\}$ we arrange $\mathfrak{p} = \mathfrak{p}_J$ and $\mathfrak{n} = \mathfrak{n}_J$. By [5, Proposition 5.8.5] any $U_J$-orbit containing a Richardson element of $\mathfrak{p}_J$ intersects with the graded subspace $\mathfrak{g}_J(2)$.

Given $x$ in $\mathfrak{g}$ denote by $Z_G(x)$ (resp., by $\mathfrak{z}_{\mathfrak{g}}(x)$) the centralizer of $x$ in $G$ (resp., in $\mathfrak{g}$). Clearly,

$$\mathrm{Lie}(Z_G(x)) = \mathrm{Lie}(Z_G(x)^\circ) \subseteq \mathfrak{z}_{\mathfrak{g}}(x)$$

(the symbol $H^\circ$ stands for the connected component of a Zariski closed subgroup $H \subseteq G$). By [21, I, §5], $\mathrm{Lie}(Z_G(x)) = \mathfrak{z}_{\mathfrak{g}}(x)$ provided $\mathfrak{g}$ admits a nondegenerate trace form associated with a rational representation of $G$. If $x$ is a Richardson element of a parabolic subalgebra $\mathfrak{p}$, then $Z_G(x)^\circ \subseteq P$ (see [5, Corollary 5.2.4]).

**2.3.** In the next three subsections we follow Slodowy's exposition [18].

Denote by $X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$ the group of all one-parameter subgroups of $T$ and by $X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ the group of the rational characters of $T$. As $T \cong (\mathbb{G}_m)^l$, one has $X_*(T) \cong \mathbb{Z}^l \cong X^*(T)$. The pairing $X_*(T) \times X^*(T) \to \mathbb{Z}$ given by

$$(\lambda, \omega) \mapsto \langle \lambda, \omega \rangle,$$

$\omega(\lambda(t)) = t^{\langle \lambda, \omega \rangle}$, is nondegenerate. The set $X_*(G)$ of all one-dimensional tori in $G$ is the union $\bigcup_H X_*(H)$ where $H$ runs over all maximal tori of $G$.

The Weyl group $W = N_G(T)/T$ acts on both $X_*(T)$ and $X^*(T)$. By fixing a $W$-invariant positively defined symmetric bilinear form $X_*(T) \times X_*(T) \to \mathbb{Z}$ one can identify the dual vector spaces $X^{\mathbb{R}} = X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ and $X_{\mathbb{R}} = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$.

To simplify notation we denote the scalar product on $X_{\mathbb{R}}$ by the above symbol $\langle\ ,\ \rangle$. Let $\|\ \|$ denote the corresponding norm mapping: $\|x\| = \sqrt{\langle x, x\rangle}$, $x \in X_{\mathbb{R}}$.

Using the $W$-invariance of $\langle\ ,\ \rangle$ and the fact that

$$X_*(G) = \bigcup_{g\in G} X_*(g^{-1}Tg)$$

one can extend the norm $\|\ \|$ up to a well-defined $G$-invariant mapping from $X_*(G)$ into $\mathbb{R}$. If $\lambda \in X_*(G)$ and $g \in G$ is such that $\mathrm{Int}(g)\circ\lambda \in X_*(T)$, then (by definition)

$$\|\lambda\| = \|\mathrm{Int}(g)\circ\lambda\|.$$

To each one-dimensional torus $\lambda \in X_*(G)$, one can assign a parabolic subgroup $P(\lambda)$ with Levi decomposition $P(\lambda) = U(\lambda)L(\lambda)$. If $\lambda \in X_*(T)$, then

$$\mathrm{Lie}(L(\lambda)) = \mathfrak{t}\bigoplus \sum_{\langle\lambda,\alpha\rangle=0} Ke_\alpha, \quad \mathrm{Lie}(U(\lambda)) = \sum_{\langle\lambda,\alpha\rangle>0} Ke_\alpha.$$

**2.4.** Let $\rho\colon G \to GL(V)$ be a finite-dimensional rational representation of $G$ in a vector space $V$ over $K$. If $\lambda \in X_*(G)$, then the induced action $\rho\circ\lambda\colon \mathbb{G}_m \to GL(V)$ turns $V$ into a $\mathbb{Z}$-graded vector space: $V = \bigoplus_{i\in\mathbb{Z}} V_i$ where

$$V_i = \{v \in V \mid \rho(\lambda(t))(v) = t^i v,\ \forall t \in \mathbb{G}_m\}.$$

If $\lambda \in X_*(T)$ and $V = \bigoplus_{\chi\in X^*(T)} V_\chi$ is the weight space decomposition of $V$ with respect to $T$, then

$$V_i = \bigoplus_{\langle\lambda,\chi\rangle=i} V_\chi.$$

It is easy to check that, for any $\lambda \in X_*(G)$, the parabolic subgroup $P(\lambda)$ defined in (2.3) preserves the subspaces $V_{(i)} = \bigoplus_{j\geq i} V_i$, $i \in \mathbb{Z}$.

Let $v \in V$ and $v = \sum_{i\in\mathbb{Z}} v_i$. Set

$$m(v, \lambda) = \max\{i \in \mathbb{Z} \mid v \in V_{(i)}\}$$

and

$$\mathrm{Supp}_T(v) = \{\chi \in X^*(T) \mid v_\chi \neq 0\}.$$

By the above, $m(v, \lambda) = \min_{\chi\in\mathrm{Supp}_T(v)}\langle\lambda, \chi\rangle$.

A vector $v \in V$ is called *instable* with respect to a closed subgroup $H \subset G$ (or $H$-instable) if the closure $\overline{H\cdot v}$ of the orbit $H\cdot v \subset V$ contains $0$. If $0 \notin \overline{H\cdot v}$, then $v$ is said to be *semistable* with respect to $H$ (or $H$-semistable). A one-dimensional torus $\lambda \in X_*(G)$ is called an *optimal* torus for a $G$-instable vector $v \in V$ if

$$\frac{m(v, \lambda)}{\|\lambda\|} \geq \frac{m(v, \mu)}{\|\mu\|}$$

for any nonzero $\mu \in X_*(G)$. An element $\lambda \in X_*(G)$ is called *primitive* if there is no $\mu \in X_*(G)$ with $\lambda = n\mu$, $n \in \mathbb{Z}$, $n \geq 2$.

Given a $G$-instable vector $v \in V$ define

$$\Lambda_v = \{\lambda \in X_*(G) \mid \lambda \text{ is primitive and optimal for } v\}.$$

**Theorem 2.1** (Kempf [10], Rousseau [16]). *Let $v \in V$ be $G$-instable. Then*

(i) *$\Lambda_v \neq \varnothing$ and there exists a parabolic subgroup $P(v) \subset G$ such that $P(v) = P(\lambda)$ for any $\lambda \in \Lambda_v$.*

(ii) *The stabilizer* $G_v = \{x \in G | \rho(x)(v) = v\}$ *is contained in* $P(v)$.

By (2.4), any $\lambda \in X_*(G)$ defines a $\mathbb{Z}$-grading of $V$. Being the centralizer of $\lambda$, the Levi subgroup $L(\lambda) \subset P(\lambda)$ preserves all graded components $V_i$, $i \in \mathbb{Z}$, of this $\mathbb{Z}$-grading. Denote

$$L_n(\lambda) = \{g \in L(\lambda) | \det(\rho(g)|_{V_n}) = 1\}.$$

**Proposition 2.2** (Kirwan [11], Ness [12]). *Let* $n > 0$, $v \in V_i$ *and* $v \neq 0$. *If* $v$ *is* $L_n(\lambda)$*-semistable, then* $\lambda$ *is an optimal torus for* $v$.

Note that any vector $v \in V_i$ must be $G$-instable if $i \neq 0$.

**2.5.** We will make use of Theorem 2.1 and Proposition 2.2 in the case $V = \mathfrak{g}$, $\rho = \mathrm{Ad}$. The adjoint action of $\lambda \in X_*(G)$ turns $\mathfrak{g}$ into a $\mathbb{Z}$-graded Lie algebra:

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i), \quad [\mathfrak{g}(i), \mathfrak{g}(j)] \subseteq \mathfrak{g}(i+j).$$

If $A \in \mathfrak{g}(2)$, then $A$ is a nilpotent element of $\mathfrak{g}$ and $(\mathrm{ad}\, A)^i$ maps $\mathfrak{g}(-i)$ into $\mathfrak{g}(i)$. Suppose that $\mathfrak{g}$ admits a nondegenerate trace form $b: \mathfrak{g} \times \mathfrak{g} \to K$ associated with a rational representation $\rho: G \to GL(V)$:

$$b(X, Y) = \mathrm{tr}\, d\rho(X) d\rho(Y) \qquad (X, Y \in \mathfrak{g})$$

where $d\rho$ denotes the differential of the rational representation $\rho$. Given $X \in \mathfrak{g}(2)$ define a bilinear form $b_X: \mathfrak{g}(-2) \times \mathfrak{g}(-2) \to K$ by setting

$$b_X(Y, Y') = b([X, Y], [X, Y']) = -b(Y, (\mathrm{ad}\, X)^2 \cdot Y').$$

Set $f(X) = \det(b_X)$.

**Lemma 2.3** (Kac [9], Slodowy [18]).
(i) *The polynomial function* $f: \mathfrak{g}(2) \to K$ *is* $L_2(\lambda)$*-invariant.*
(ii) *If the map* $(\mathrm{ad}\, A)^2: \mathfrak{g}(-2) \to \mathfrak{g}(2)$ *is surjective, then* $f(A) \neq 0$. *In particular,* $A$ *is semistable with respect to* $L_2(\lambda)$.

Lemma 2.3 together with Proposition 2.2 and Theorem 2.1 (ii) implies that if $(\mathrm{ad}\, A)^2: \mathfrak{g}(-2) \to \mathfrak{g}(2)$ is a surjective map, then $\lambda \in X_*(G)$ is an optimal torus for $A$ and so $Z_G(A) \subseteq P(\lambda)$. Since $\mathrm{Lie}(P(\lambda)) = \bigoplus_{i \geq 0} \mathfrak{g}(i)$, this forces $\mathfrak{z}_{\mathfrak{g}}(A) \subseteq \bigoplus_{i \geq 0} \mathfrak{g}(i)$ (see (2.2) for more detail).

**2.6.** From now on we suppose that $\mathrm{Lie}(G^{(1)})$ is isomorphic to the Lie algebra of a simply connected group isogeneous to $G^{(1)}$. Given $\lambda \in X_*(G)$ decompose $\mathfrak{g} = \mathrm{Lie}(G)$ into weight spaces under the adjoint action of $\lambda$:

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i).$$

By (2.3), $\mathrm{Lie}(P(\lambda)) = \bigoplus_{i \geq 0} \mathfrak{g}(i)$, $\mathrm{Lie}(U(\lambda)) = \bigoplus_{i > 0} \mathfrak{g}(i)$ and $\mathrm{Lie}(L(\lambda)) = \mathfrak{g}(0)$.

**Definition 2.4.** A one-dimensional torus $\lambda \in X_*(G)$ is called a *Dynkin torus* for a nilpotent element $e \in \mathfrak{g}$ if $e \in \mathfrak{g}(2)$ and $\mathfrak{z}_{\mathfrak{g}}(e) \subseteq \bigoplus_{i \geq 0} \mathfrak{g}(i)$.

The rest of the paper is devoted to proving the following

**Theorem 2.5.** *Any nonzero nilpotent element $e$ of $\mathfrak{g}$ has at least one Dynkin torus.*

Let $\mathfrak{g}' = \mathrm{Lie}(G^{(1)})$. Clearly, $\mathfrak{g} = \mathfrak{t}+\mathfrak{g}'$. It is well known that the canonical $[p]$-operation of $\mathfrak{g}$ is bijective on $\mathfrak{t}$. Since $\mathfrak{g}'$ is a restricted ideal of $\mathfrak{g}$, Jacobson's identity [8, V, §7] yields that any nilpotent element of $\mathfrak{g}$ lies in $\mathfrak{g}'$. Suppose that a nilpotent element $e \in \mathfrak{g}'$ has a Dynkin torus $\lambda \in X_*(G^{(1)})$. Decompose $\mathfrak{g}$ into weight spaces under the adjoint action of $\lambda$: $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(i)$. Let $T_1$ be a maximal torus of $G$ containing $\lambda$ and let $\mathfrak{t}_1 = \mathrm{Lie}(T_1)$. As $\mathfrak{t}_1 \subseteq \mathfrak{g}(0)$ and $\mathfrak{g} = \mathfrak{t}_1 + \mathfrak{g}'$, we have $\mathfrak{g} = \mathfrak{g}(0) + \mathfrak{g}'$. This yields that $\mathfrak{g}(i) \subset \mathfrak{g}'$ for each $i \neq 0$. As $\lambda$ is a Dynkin torus for $e \in \mathfrak{g}'$, $\mathfrak{z}_{\mathfrak{g}'}(e) \subseteq \sum_{i\geq 0} \mathfrak{g}(i)$ and $e \in \mathfrak{g}(2)$. Hence $\lambda$ preserves $\mathfrak{z}_{\mathfrak{g}}(e)$. But then

$$\mathfrak{z}_{\mathfrak{g}}(e) = \mathfrak{z}_{\mathfrak{g}}(e) \cap \mathfrak{g}(0) \bigoplus \sum_{i\neq 0} \mathfrak{z}_{\mathfrak{g}}(e) \cap \mathfrak{g}(i) \subseteq \mathfrak{g}(0) + \mathfrak{z}_{\mathfrak{g}}(e) \cap \mathfrak{g}' \subseteq \sum_{i\geq 0} \mathfrak{g}(i)$$

showing that $\lambda$ is a Dynkin torus for $e \in \mathfrak{g}$. Thus, we may suppose that $G$ is semisimple and simply connected.

Assuming that $G = G^{(1)}$, denote by $G_1, G_2, \dots, G_s$ the simple (and simply connected) normal subgroups of $G$. Let $\mathfrak{g}_i = \mathrm{Lie}(G_i)$, $1 \leq i \leq s$. Clearly, $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$ and $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$ if $i \neq j$. If $e = e_1 + \cdots + e_s$ where $e_i \in \mathfrak{g}_i$, then $\mathfrak{z}_{\mathfrak{g}}(e) = \mathfrak{z}_{\mathfrak{g}_1}(e_1) \oplus \cdots \oplus \mathfrak{z}_{\mathfrak{g}_s}(e_s)$. Suppose that each nonzero $e_i$ has a Dynkin torus $\lambda_i \in X_*(G_i)$. Let $\mathfrak{p}_i$ be the parabolic subalgebra of $\mathfrak{g}_i$ associated with $\lambda_i$. We may assume that $e_i \neq 0$ if $i \leq s_0 \leq s$ and $e_i = 0$ if $i > s_0$. Then $\mathfrak{p} = (\bigoplus_{i\leq s_0} \mathfrak{p}_i) \oplus (\bigoplus_{i>s_0} \mathfrak{g}_i)$ is the parabolic subalgebra of $\mathfrak{g}$ associated with $\lambda = \prod_i \lambda_i \in X_*(G)$. As $\mathfrak{z}_{\mathfrak{g}_i}(e_i) \subseteq \mathfrak{p}_i$ for all $1 \leq i \leq s_0$, then $\mathfrak{z}_{\mathfrak{g}}(e) \subseteq \mathfrak{p}$. Moreover,

$$(\mathrm{Ad}\,\lambda(t)) \cdot e = \sum_i (\mathrm{Ad}\,\lambda_i(t)) \cdot e_i = t^2 e.$$

Hence $\lambda$ is a Dynkin torus for $e$. Thus we may assume that $G$ is simple and simply connected.

If $R$ is of type $A_n$, $B_n$, $C_n$ or $D_n$, Theorem 2.5 follows immediately from the results of Springer and Steinberg (see [21, IV]). Indeed, if $G \cong GL_n(K)$ or $R$ is of type $B_n$, $C_n$ or $D_n$ and $p > 2$, then $G$ admits a nondegenerate trace form (by [21, I, Lemma 5.3]). Therefore, $\mathrm{Lie}(Z_G(e)) = \mathfrak{z}_{\mathfrak{g}}(e)$ by [21, I, Corollary 5.2] and one can apply [21, IV, §§1.7, 2.23]. If $G = SL_n(K)$, then $G = \widetilde{G}^{(1)}$ where $\widetilde{G} = GL_n(K)$. Any nilpotent element $e \in \mathfrak{g}$ can be regarded as an element of $\tilde{\mathfrak{g}} = \mathrm{Lie}(\widetilde{G}) = gl_n(K)$. By [21, IV, §1], one can find a Dynkin torus $\lambda \in X_*(\widetilde{G})$ for $e \in \tilde{\mathfrak{g}}$ contained in $G$. Let $\mathfrak{p}$ (resp., $\tilde{\mathfrak{p}}$) denote the parabolic subalgebra of $\mathfrak{g}$ (resp., $\tilde{\mathfrak{g}}$) associated with $\lambda \in X_*(G) \subseteq X_*(\widetilde{G})$. Clearly, $\mathfrak{p} = \tilde{\mathfrak{p}} \cap \mathfrak{g}$. But then $\mathfrak{z}_{\mathfrak{g}}(e) = \mathfrak{z}_{\tilde{\mathfrak{g}}}(e) \cap \mathfrak{g} \subseteq \tilde{\mathfrak{p}} \cap \mathfrak{g} = \mathfrak{p}$. Hence $\lambda \in X_*(G)$ is a Dynkin torus for $e \in \mathfrak{g}$.

**2.7.** Considering the remaining case of exceptional groups we will use some classification results due to Bala-Carter [5, V] and Pommerening [13, 14].

Recall that a nilpotent element $x$ in $\mathfrak{g}$ is said to be *distinguished* if it commutes with no nonzero semisimple element of $\mathfrak{g}$. Generalizing [1, 2], Pommerening proved (see [14, p. 377]) that any distinguished nilpotent element of $\mathfrak{g}$ is a Richardson element of a distinguished parabolic subalgebra of $\mathfrak{g}$. If $G$

is exceptional and $p$ is good for $G$, then the Killing form of $\mathfrak{g}$ is nondegenerate. Applying (2.2), one can now easily observe that any distinguished nilpotent element of $\mathfrak{g}$ has at least one Dynkin torus. Hence in proving Theorem 2.5 we may assume that $e \in \mathfrak{g}$ is not distinguished.

Since $\mathfrak{z}_{\mathfrak{g}}(e) = \mathrm{Lie}(Z_G(e)^\circ)$ and $e$ commutes with a nonzero semisimple element, the group $Z_G(e)^\circ$ contains a maximal torus $S$ of positive dimension. No generality is lost by assuming $S \subseteq T$.

Let $R_1$ denote the subsystem of roots vanishing on $S$. Set $\mathfrak{s} = \mathrm{Lie}(S)$. Combining [3, §9.2] and [5, §5.9] one can obtain that

$$\mathrm{Lie}(Z_G(S)) = \mathfrak{g}^S = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{s}) = \mathfrak{t} \bigoplus \sum_{\alpha \in R_1} K e_\alpha.$$

Moreover, by [17, p. 23], there exist a system of simple roots $\Delta \subset R$ and a subset $J$ with $\Delta_J \subseteq \Delta$ such that any $\gamma \in R_1$ is an integer linear combination of the elements from $\Delta_J$. Hence in what follows we may suppose that $R_1 = R_J = \{\gamma \in R \mid \gamma = \sum_{\alpha \in B_J} n_\alpha \alpha,\ n_\alpha \in \mathbb{Z}\}$ for some $J \subseteq \{1, 2, \dots, l\}$. Thus $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{s})$ coincides with the Levi subalgebra $\mathfrak{l}_J = \mathrm{Lie}(L_J)$ of the standard parabolic subalgebra $\mathfrak{p}_J$. It is immediate that the Killing form of $\mathfrak{g}$ is nondegenerate on $\mathfrak{l}_J$. By [21, II, §5], the semisimple group $L_J^{(1)}$ is simply connected. By Jacobson's identity [8, V, §7], $e \in \mathfrak{l}_J^{(1)} = \mathrm{Lie}(L_J^{(1)})$.

If $\mathfrak{z}(\mathfrak{l}_J^{(1)}) = 0$, then $\mathfrak{l}_J = \mathfrak{s} \oplus \mathfrak{l}_J^{(1)}$ and $[\mathfrak{s}, \mathfrak{l}_J] = 0$. The group $Z_{L_J^{(1)}}(e)^\circ$ is unipotent (otherwise $S$ would be properly contained in a bigger torus in $Z_G(e)^\circ$ contradicting the maximality of $S$). Since $\mathfrak{s} \perp \mathfrak{l}_J^{(1)}$, $\mathfrak{l}_J^{(1)}$ admits a nondegenerate trace form. Hence $\mathrm{Lie}(Z_{L_J^{(1)}}(e)^\circ) = \mathfrak{z}_{\mathfrak{l}_J^{(1)}}(e)$. This implies that $e$ is a distinguished nilpotent element of $\mathfrak{l}_J^{(1)}$.

If $\mathfrak{z}(\mathfrak{l}_J^{(1)}) \neq 0$, then either $p = 5$ and $R_J$ has a component of type $A_4$ or $p = 7$ and $R_J$ has a component of type $A_6$. As $R$ is exceptional, this yields that all components of $R_J$ have type $A$. Therefore, $\mathfrak{l}_J^{(1)}$ is a direct sum of commuting ideals $\mathfrak{l}_i$ isomorphic to $sl_{r_i}(K)$ for some $r_i \leq 6$. A standard argument used above shows that $Z_{L_J^{(1)}}(e)^\circ$ is unipotent. Together with [21, IV, §1] this yields that $e$ is a regular nilpotent element of $\mathfrak{l}_J^{(1)}$ (see [21, III]).

Combining [14, p. 377] with [21, IV, §1] we obtain now that in both cases $e$ is a Richardson element of a distinguished parabolic subalgebra of $\mathfrak{l}_J^{(1)} = \mathrm{Lie}(L_J^{(1)})$.

**2.8.** In what follows we may (and will) assume that there exists $I \subseteq J$ such that $e$ is a Richardson element of the standard parabolic subalgebra $\mathfrak{p}_I \cap \mathfrak{l}_J$ of the Levi subalgebra $\mathfrak{l}_J$. As $\mathfrak{l}_J = \mathfrak{g}_J(0)$, we have

$$\mathfrak{p}_I \cap \mathfrak{l}_J = \sum_{i \geq 0} \mathfrak{g}_J(0) \cap \mathfrak{g}_I(i).$$

By (2.2) we can also assume that $e \in \mathfrak{g}_J(0) \cap \mathfrak{g}_I(2)$.

We will use the $W$-invariant scalar product $(\ |\ )$ on $X^{\mathbb{R}} = \mathbb{R}\overline{\omega}_1 \oplus \cdots \oplus \mathbb{R}\overline{\omega}_l$ defined in [4, VI, Tables I–IX] via embedding $X^{\mathbb{R}}$ into a bigger Euclidean space.

Clearly, the $\mathbb{Q}$-span of $B_J$ in $X^{\mathbb{R}}$ has basis $\{\omega_i^J \mid i \in J\}$ satisfying

$$(\omega_i^J \mid \alpha_k) = \delta_{ik}$$

for all $i, k \in J$. This implies that

$$2\sum_{i \notin I} \omega_i^J = \sum_{k \in J} \frac{2}{(\alpha_k|\alpha_k)} m_k \alpha_k$$

for some $m_k \in \mathbb{Q}$. A direct computation based on the Bala-Carter classification of the distinguished parabolic subgroups (see [5, pp. 174–177]) shows that all $m_k$'s are positive integers (note that the classification of the distinguished parabolic subgroups given in [1, 2] remains true for any good $p$).

By Steinberg [22, §5], the maximal torus $T$ is generated by the one-parameter subgroups $h_\alpha(t)$, $\alpha \in R$, such that

$$(\operatorname{Ad} h_\alpha(t)) \cdot e_\beta = t^{\langle \beta, \alpha \rangle} e_\beta$$

for all $\alpha, \beta \in R$ (here $\langle \beta, \alpha \rangle = 2\frac{(\beta|\alpha)}{(\alpha|\alpha)}$). Put $h_i(t) = h_{\alpha_i}(t)$ for each $\alpha_i \in B$ and define $\lambda_e \in X_*(G)$ by setting

$$\lambda_e(t) = \prod_{k \in J} h_k(t^{m_k}) \qquad (t \in \mathbb{G}_m).$$

We intend to show that in most of the remaining cases $\lambda_e$ is an optimal torus for $e$ relative to the scalar product $(\ |\ )$. By construction,

$$(\operatorname{Ad} \lambda_e(t)) \cdot e_\alpha = \begin{cases} e_\alpha & \text{if } \alpha \in B_I, \\ t^2 e_\alpha & \text{if } \alpha \in B_J \backslash B_I. \end{cases}$$

Hence $\lambda_e(t)$ acts on $\mathfrak{g}_J(0) \cap \mathfrak{g}_I(i)$ by multiplying each vector by $t^i$, $i \in \mathbb{Z}$. In particular, $(\operatorname{Ad} \lambda_e(t)) \cdot e = t^2 e$. Since $\operatorname{Lie}(h_\alpha(t)) = K[e_\alpha, e_{-\alpha}]$ for any $\alpha \in R$, we have

$$\operatorname{Lie}(\lambda_e) \subset \sum_{k \in J} K[e_{\alpha_k}, e_{-\alpha_k}] \subseteq \mathfrak{t} \cap \mathfrak{l}_J^{(1)}.$$

Moreover, the Lie algebra $\operatorname{Lie}(\lambda_e)$ is spanned by $h \in \mathfrak{t} \cap \mathfrak{l}_J^{(1)}$ such that $[h, x] = ix$ for any $x \in \mathfrak{g}_J(0) \cap \mathfrak{g}_I(i)$, $i \in \mathbb{Z}$.

As $e$ is a Richardson element of $\mathfrak{p}_I \cap \mathfrak{l}_J^{(1)}$, a distinguished parabolic subalgebra of $\mathfrak{l}_J^{(1)} = \operatorname{Lie}(L_J^{(1)})$, the map

$$\operatorname{ad} e \colon \mathfrak{l}_J^{(1)} \cap \mathfrak{g}_I(-2) \to \mathfrak{l}_J^{(1)} \cap \mathfrak{g}_I(0)$$

is bijective (for $\dim \mathfrak{l}_J^{(1)} \cap \mathfrak{g}_I(-2) = \dim \mathfrak{l}_J^{(1)} \cap \mathfrak{g}_I(0)$ and $\mathfrak{z}_{\mathfrak{l}_J}(e) \subseteq \sum_{i \geq 0} \mathfrak{g}_J(0) \cap \mathfrak{g}_I(i)$ as the Killing form of $\mathfrak{g}$ is nondegenerate on $\mathfrak{l}_J$). This implies that there exists $f \in \mathfrak{l}_J^{(1)} \cap \mathfrak{g}_I(-2)$ such that $[e, f] = h$. Clearly, $\langle e, h, f \rangle$ is an $sl_2$-triple in $\mathfrak{l}_J^{(1)}$ (see [14]).

*Remark* 2.6. By construction, $h^{[p]} = h$ but it may happen for some small $p$ that $e^{[p]} \neq 0$ or $f^{[p]} \neq 0$.

**2.9.** Let $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ where

$$\mathfrak{g}_i = \{x \in \mathfrak{g} | (\operatorname{Ad} \lambda_e(t)) \cdot x = t^i x \text{ for all } t \in \mathbb{G}_m\}.$$

As $\lambda_e \subset L_J$, it preserves $\mathfrak{g}_J(k)$ for any $k \in \mathbb{Z}$. Set

$$M_J = \bigoplus_{i \neq 0} \mathfrak{g}_J(i), \quad M_J^i = M_J \cap \mathfrak{g}_i \quad \text{and} \quad \mathfrak{g}_J^i(k) = \mathfrak{g}_J(k) \cap \mathfrak{g}_i.$$

**Lemma 2.7.** *If $M_J^{2(p-1)} = 0$ and $B_J$ has no component of type $A_{p-1}$, then $\lambda_e$ is a Dynkin torus for $e$.*

*Proof.* Since $e \in \mathfrak{g}_2$, it suffices to show that $\lambda_e$ is an optimal torus for $e$ with respect to the scalar product $(\ |\ )$. Hence, in view of Lemma 2.3, it suffices to check that the map $(\operatorname{ad} e)^2 \colon \mathfrak{g}_{-2} \to \mathfrak{g}_2$ is surjective. We have

$$\mathfrak{g}_{-2} = \mathfrak{g}_J^{-2}(0) \oplus M_J^{-2}; \quad \mathfrak{g}_2 = \mathfrak{g}_J^2(0) \oplus M_J^2.$$

Moreover, by (2.8), $\mathfrak{g}_J^{\pm 2}(0) = \mathfrak{g}_J(0) \cap \mathfrak{g}_I(\pm 2) = \mathfrak{l}_J^{(1)} \cap \mathfrak{g}_I(\pm 2)$. Therefore, the map $\operatorname{ad} e \colon \mathfrak{g}_J^{-2}(0) \to \mathfrak{l}_J^{(1)} \cap \mathfrak{g}_I(0)$ is bijective (see (2.8)). If $[e, \mathfrak{l}_J^{(1)} \cap \mathfrak{g}_I(0)] \neq \mathfrak{g}_J^2(0)$, then a nonzero subspace $N \subset \mathfrak{g}_J^{-2}(0)$ is orthogonal to $[e, \mathfrak{l}_J^{(1)} \cap \mathfrak{g}_I(0)]$ with respect to the Killing form $k$ of $\mathfrak{g}$. But then $[e, N] \subset \mathfrak{l}_J^{(1)} \cap \mathfrak{g}_I(0)$ is orthogonal to $\mathfrak{l}_J^{(1)} \cap \mathfrak{g}_I(0)$. By our assumption, $\mathfrak{z}(\mathfrak{l}_J^{(1)}) = 0$. Thus $k$ remains nondegenerate if restricted to $\mathfrak{l}_J^{(1)}$ (see (2.7)). This implies that $k$ is nondegenerate on $\mathfrak{l}_J^{(1)} \cap \mathfrak{g}_I(0)$ forcing $[e, N] = 0$. Summarizing we obtain that $[e, \mathfrak{l}_J^{(1)} \cap \mathfrak{g}_I(0)] = \mathfrak{g}_J^2(0)$ and so

$$(\operatorname{ad} e)^2 \colon \mathfrak{g}_J^{-2}(0) \to \mathfrak{g}_J^2(0)$$

is one-to-one.

Since $e, h, f \in \mathfrak{l}_J$, $M_J$ is $\langle e, h, f \rangle$-stable. Our goal is to show that the map $(\operatorname{ad} e)^2 \colon M_J^{-2} \to M_J^2$ is bijective. We first check that $\operatorname{ad} e$ is injective on $M_J^{-2}$. Indeed, if $[e, v] = 0$ for some nonzero $v \in M_J^{-2}$, then

$$[e, (\operatorname{ad} f)^i(v)] = i(p-1-i)(\operatorname{ad} f)^{i-1}(v)$$

for any natural $i$. But then

$$(\operatorname{ad} e)^{p-2}(\operatorname{ad} f)^{p-2}(v) = v \neq 0$$

yielding $(\operatorname{ad} f)^{p-2}(v) \in M_J^{-2(p-1)} \backslash \{0\}$. Since $k$ induces a nondegenerate pairing between $M_J^k$ and $M_J^{-k}$ for all $k \in \mathbb{Z}$, this forces $M_J^{2(p-1)} \neq 0$ violating the assumption. Thus $\operatorname{ad} e$ is injective on $M_J^{-2}$.

Suppose that $[e, [e, x]] = 0$ for some nonzero $x \in M_J^{-2}$ and let $w = [e, x]$. Since

$$[e, (\operatorname{ad} f)^i(w)] = i(p+1-i)(\operatorname{ad} f)^{i-1}(w)$$

for any natural $i$, we have

$$(\operatorname{ad} e)^{p-2}(\operatorname{ad} f)^{p-1}(w) = [f, w].$$

As $(\operatorname{ad} f)^{p-1}(w) \in M_J^{-2(p-1)} = 0$, this yields $[f, w] = 0$. Using this fact it is easy to note that

$$(\operatorname{ad} e)^{p-2}(\operatorname{ad} f)^{p-2}(x) = x \neq 0.$$

But then $(\operatorname{ad} f)^{p-2}(x) \neq 0$ contradicting the equality $M_J^{-2(p-1)} = 0$. Therefore, the map $(\operatorname{ad} e)^2 \colon M_J^{-2} \to M_J^2$ is injective. To complete the proof of the lemma it remains to note that $\dim M_J^{-2} = \dim M_J^2$. $\square$

**2.10.** Denote by $m_i(e)$ the maximal weight of the $(\operatorname{Ad} \lambda_e)$-module $\mathfrak{g}_J(i)$ and set $m(e) = \max_{i \neq 0} m_i(e)$. It follows from the definition of $\lambda_e$ that the numbers $m_i(e)$ do not depend on the characteristic of the ground field. Thus in computing $m_i(e)$'s we may assume that $G$ and $\mathfrak{g}$ are both defined over $\mathbb{C}$.

We first consider the case when all roots in $R$ have the same length. In this case $(\alpha|\alpha) = 2$ for any $\alpha \in R$. This implies that

$$(\mathrm{Ad}\,\lambda_e(t)) \cdot e_\gamma = t^{(\lambda_{I,J}|\gamma)} \cdot e_\gamma \qquad (\gamma \in R)$$

where $\lambda_{I,J} = 2\sum_{i \notin I} \omega_i^J$. It is well known that, for any nonzero $i$, the subspace $\mathfrak{g}_J(i)$ is completely irreducible as an $L_J^{(1)}$-module. Moreover, any nontrivial irreducible $L_J^{(1)}$-submodule of $\mathfrak{g}_J(i)$ is generated by a highest weight vector that is a root element with respect to $T$ and corresponds to a minimal (minuscule) weight of the root system $R_J$ (indeed, as all roots of $R$ have the same length, it suffices to note that $\langle \gamma, \delta \rangle \in \{-1, 0, 1\}$ if $\gamma \in R_J$, $\delta \in R \backslash R_J$).

Since the linear function $(\lambda_{I,J}|\ )$ is nonpositive on $R_J \cap (-R_+)$, the $L_J^{(1)}$-module $\mathfrak{g}_J(i)$ contains a highest weight vector $e_{\gamma_i}$, $\gamma_i \in R$, such that $m_i(e) = (\lambda_{I,J}|\gamma_i)$. Clearly, $X^{\mathbb{R}} = X_J \oplus X_J^\perp$ where $X_J$ is the $\mathbb{R}$-span of $B_J$ and $X_J^\perp$ is its orthogonal complement relative to $(\ |\ )$. Let $\gamma_i = \gamma_i^J + \gamma_i^\perp$ where $\gamma_i^J \in X_J$, $\gamma_i^\perp \in X_J^\perp$. Then $(\lambda_{I,J}|\gamma_i) = (\lambda_{I,J}|\gamma_i^J)$ and $\gamma_i^J$ is a minimal weight of $R_J$. The vector $\gamma_i^J$ is a sum of minimal weights of irreducible components of $R_J$. These, in turn, lie in the set $\{\omega_i^J | i \in J\}$.

Let $\rho_J = \sum_{i \in J} \omega_i^J$ and $\rho = \sum_{i=1}^l \overline{\omega}_i$. As $(\omega_i^J|\omega_j^J) \geq 0$ for each $i, j \in J$ (see [4, VI, Tables I–IX]), we have

$$(\lambda_{I,J}|\gamma_i^J) \leq (2\rho_J|\gamma_i^J).$$

Computing $(2\rho_J|\gamma_i^J)$ can be reduced to the corresponding problem for the irreducible components of $R_J$. Using [4, VI,Tables I–IX], one can check that

$$(2\rho|\overline{\omega}_k) = k(l-k+1)$$

if $R$ is of type $A_l$, $1 \leq k \leq l$;

$$(2\rho|\overline{\omega}_1) = 2(l-1), \quad (2\rho|\overline{\omega}_{l-1}) = (2\rho|\overline{\omega}_l) = l(l-1)/2$$

if $R$ is of type $D_l$, $l \geq 4$.

These equations together with the above remarks yield that $m(e) < 2(p-1)$ if $e \in \mathfrak{l}_J$ and the root system $R_J$ has one of the following types:

$$A_1,\ A_1 \times A_1,\ A_1 \times A_1 \times A_1,\ A_2,\ A_2 \times A_1,\ A_2 \times A_1 \times A_1,\ A_2 \times A_2,$$
$$A_2 \times A_2 \times A_1,\ A_3,\ A_3 \times A_1,\ D_4$$

for $R \cong E_6, E_7$ or $E_8$, $p > 3$;

$$A_1 \times A_1 \times A_1 \times A_1,\ A_2 \times A_1 \times A_1 \times A_1,\ A_3 \times A_1 \times A_1,\ A_3 \times A_2,$$
$$A_3 \times A_2 \times A_1,\ D_4 \times A_1$$

for $R \cong E_7$ or $E_8$, $p > 3$;

$$A_2 \times A_2 \times A_1 \times A_1,\ A_3 \times A_3,\ A_4,\ A_4 \times A_1,\ A_4 \times A_1 \times A_1,\ A_4 \times A_2,$$
$$A_4 \times A_2 \times A_1,\ A_4 \times A_3,\ D_4 \times A_2,\ A_5,\ A_5 \times A_1,\ D_5,\ D_5 \times A_1$$

for $R \cong E_8$, $p > 5$.

By Lemma 2.7, if $R_J$ has one of the types listed above, then $\lambda_e$ is a Dynkin torus for $e \in \mathfrak{l}_J$.

**2.11.** Thus in what follows we may assume that $R_J$ has one of the following types:

$$A_4\,,\ A_4 \times A_1\,,\ A_5\,,\ D_5$$

for $R \cong E_6\,,\ p > 3\,;$

$$A_4\,,\ A_4 \times A_1\,,\ A_4 \times A_2\,,\ A_5\,,\ A_5 \times A_1\,,\ D_5\,,\ D_5 \times A_1\,,\ A_6\,,\ D_6\,,\ E_6$$

for $R \cong E_7\,,\ p > 3\,;$

$$D_5 \times A_2\,,\ A_6\,,\ A_6 \times A_1\,,\ E_6\,,\ E_6 \times A_1\,,\ D_6\,,\ A_7\,,\ D_7\,,\ E_7$$

for $R \cong E_8\,,\ p > 5\,.$

Note that if $e$ is a regular nilpotent element of $\mathfrak{l}_J$ we can always arrange that $e = \sum_{i \in J} e_{\alpha_i}$.

Denote $Q_+ = \{\sum_{i=1}^{l} n_i \alpha_i | n_i \in \mathbb{Z}_+\}$ and set $Q_+^J = Q_+ \cap X^J$. Let $w_J$ be the element of maximal length in the Weyl group $W(R_J) \subset W$. Given $\eta = \sum_{i=1}^{l} m_i \alpha_i$ in $Q_+$ and $k \le l$ define $\nu_k(\eta) = m_k$ and let $\mathrm{ht}(\eta) = \nu_1(\eta) + \nu_2(\eta) + \cdots + \nu_l(\eta)$. We call the number $\mathrm{ht}(\eta)$ the *height* of $\eta$. Set

$$Y(\eta) = \{\alpha_k \in B | \nu_k(\eta) \neq 0\}.$$

For $k \in \{1, 2, \ldots, l\} \backslash J$, define

$$\Gamma_J^k = \{\gamma \in R_+ | Y(\gamma) = B_J \cup \{\alpha_k\}\,,\ \nu_k(\gamma) = 1\}.$$

Let $\beta_J^k = w_J(\alpha_k)$. Clearly, $\beta_J^k \in \Gamma_J^k$.

**Lemma 2.8.** *$\beta_J^k$ is the only element of maximal height in $\Gamma_J^k$.*

*Proof.* Let $\delta \in \Gamma_J^k$. As $w_J$ acts on $\Gamma_J^k$, $\delta = w_J(\delta')$ for some $\delta' \in \Gamma_J^k$. We have $\delta' = \alpha_k + \sum_{i \in J} c_i e_{\alpha_i}$ where $c_i \in \mathbb{Z}_+$. As $-w_J$ acts on $B_J$,

$$\beta_J^k - \delta = w_J(\alpha_k - \delta') = \sum_{i \in J} c_i(-w_J \alpha_i)$$

yielding

$$\mathrm{ht}(\beta_J^k - \delta) = \sum_{i \in J} c_i \ge 0.$$

Clearly, $\sum_{i \in J} c_i = 0$ implies $\alpha_k = \delta'$ forcing $\beta_J^k = \delta$ as desired. □

**2.12.** Let $M_{J,+} = \bigoplus_{i>0} \mathfrak{g}_J(i)$ and $M_{J,-} = \bigoplus_{i<0} \mathfrak{g}_J(i)$. Obviously, the $L_J$-modules $M_{J,+}^*$ and $M_{J,-}$ are isomorphic. Let $M_{J,\pm}^{-2} = M_{J,\pm} \cap M_J^{-2}$ and $M_{J,\pm}^2 = M_{J,\pm} \cap M_J^2$. If the map

$$(\mathrm{ad}\, e)^2 \colon M_{J,+}^{-2} \to M_{J,+}^2$$

is bijective, then so is the induced map

$$((\mathrm{ad}\, e)^2|_{M_{J,+}})^* \colon (M_{J,+}^2)^* \to (M_{J,+}^{-2})^*$$

which can be identified with

$$(\mathrm{ad}\, e)^2 \colon M_{J,-}^{-2} \to M_{J,-}^2$$

via the above isomorphism.

Thus in order to show that $(\mathrm{ad}\, e)^2 \colon M_J^{-2} \to M_J^2$ is bijective, it suffices to prove that so is $(\mathrm{ad}\, e)^2 \colon M_{J,+}^{-2} \to M_{J,+}^2$. It is easy to check that $M_{J,+}^{-2}$ is spanned by $e_\gamma$ such that $\gamma \in R_+ \backslash R_J$ and $(\lambda_{I,J} | \gamma) = -2$.

**Lemma 2.9.** *Let* $b_J^k = \operatorname{ht}\beta_J^k + 1$, $k \in J$, *and* $\gamma = \sum_{i=1}^l m_i\alpha_i \in R_+$. *Then*

$$(2\rho_J|\gamma) = 2\operatorname{ht}\gamma - \sum_{i\notin J} m_i b_J^i.$$

*Proof.* We have $\rho - w_J\rho \in Q_+$ and $(\rho - w_J\rho|\alpha_i) = (\rho|\alpha_i - w_J\alpha_i) = 2$ for each $i \in J$. Since $2\rho_J \in X_J$ and $(2\rho_J|\alpha_i) = 2$ for each $i \in J$, we have $\rho - w_J\rho = 2\rho_J$. Hence

$$\begin{aligned}(2\rho_J|\gamma) &= (\rho - w_J\rho|\gamma) = (\rho|\gamma - w_J\gamma)\\ &= \left(\rho|\sum_{i\in J} m_i(\alpha_i - w_J\alpha_i) + \sum_{i\notin J} m_i(\alpha_i - \beta_J^i)\right)\\ &= 2\sum_{i\in J} m_i + \sum_{i\notin J} m_i(2 - b_J^i) = 2\operatorname{ht}\gamma - \sum_{i\notin J} m_i b_J^i\end{aligned}$$

as required. □

**2.13.** For any $k \in \mathbb{Z}_+$, $\dim \mathfrak{g}_J^{-2}(k) = \dim \mathfrak{g}_J^2(k)$. To observe this one can assume that $G$ and $\mathfrak{g}$ are defined over $\mathbb{C}$. In this case the statement follows from (2.8) and standard properties of the representations of $SL_2(\mathbb{C})$. If $m_k(e) < 2(p-1)$, then the argument used in proving Lemma 2.7 shows that $(\operatorname{ad} e)^2\colon \mathfrak{g}_J^{-2}(k) \to \mathfrak{g}_J^2(k)$ is one-to-one.

If $R_J$ had rank $l-1$, then $B\backslash B_J = \{\alpha_s\}$. Set

$$\Delta_s(a) = \{\alpha \in R_+ | \nu_s(\alpha) = a,\ (2\rho_J|\alpha) = -2\}.$$

Clearly, $\mathfrak{g}_J^{-2}(k)$ is spanned by $\{e_\gamma | \gamma \in \Delta_s(k)\}$.

Let $R_J$ be of type $D_5 \times A_2 \subset E_8$. Then $B_J = B\backslash\{\alpha_6\}$. It is straightforward that the $L_J^{(1)}$-modules $\mathfrak{g}_J(1)$, $\mathfrak{g}_J(2)$, $\mathfrak{g}_J(3)$ and $\mathfrak{g}_J(4)$ are irreducible and have highest weights $\omega_2^J + \omega_8^J$, $\omega_1^J + \omega_7^J$, $\omega_5^J$ and $\omega_8^J$ respectively. By (2.10), $(2\rho_J|\omega_2^J + \omega_8^J) = 12$, $(2\rho_J|\omega_1^J + \omega_7^J) = 10$, $(2\rho_J|\omega_5^J) = 10$ and $(2\rho_J|\omega_8^J) = 2$. If $e \in \mathfrak{l}_J$ is not regular, then $I \neq \varnothing$ and so $(\lambda_{I,J}|\omega_2^J + \omega_8^J) < 12 \le 2(p-1)$. Therefore, in this case $e$ satisfies the conditions of Lemma 2.7.

Suppose that $e$ is regular in $\mathfrak{l}_J^{(1)}$. By our previous remark the map $(\operatorname{ad} e)^2\colon \mathfrak{g}_J^{-2}(k) \to \mathfrak{g}_J^2(k)$ is bijective if $k > 1$ (if $p > 7$, it is bijective for all $k \ge 0$). To show that $(\operatorname{ad} e)^2\colon \mathfrak{g}_J^{-2}(1) \to \mathfrak{g}_J^2(1)$ is bijective, observe that $\beta_J^6 = {}^{1232111}_{2}$, $b_J^6 = 14$. Using Lemma 2.10 and [4, VI, Table VII] we obtain

$$\Delta_1(6) = \left\{ {}^{0011111}_{1}, {}^{0111110}_{1}, {}^{0111111}_{1}, {}^{1111110}_{0}, {}^{1111100}_{1}, {}^{0121100}_{1} \right\}.$$

Now it is not difficult to verify that $\operatorname{ad} e = \sum_{i\neq 6} \operatorname{ad} e_{\alpha_i}$ sends $\mathfrak{g}_J^{-2}(1)$ onto the subspace spanned by $e_\beta$ where

$$\beta \in \left\{ {}^{0111111}_{1}, {}^{1111111}_{0}, {}^{1111110}_{1}, {}^{0121110}_{1}, {}^{0122100}_{1}, {}^{1121100}_{1} \right\}.$$

This, in turn, is mapped by $\operatorname{ad} e$ onto the span of $e_\gamma$ where

$$\gamma \in \left\{ {}^{1111111}_{1}, {}^{0121111}_{1}, {}^{1121110}_{1}, {}^{0122110}_{1}, {}^{1122100}_{1}, {}^{1221100}_{1} \right\}$$

(note that this is true for any $p$).

Thus $(\operatorname{ad} e)^2\colon M_{J,+}^{-2} \to M_{J,+}^2$ is one-to-one and we can exclude the subsystem of type $D_5 \times A_2$ from our list.

**2.14.** Let $R_J$ be of type $A_7 \subset E_8$. Then $B_J = B\backslash\{\alpha_2\}$, $\beta_J^2 = {}^{1233321}_{1}$, $b_J^2 = 17$. Using [4, VI, Table VII] and Lemma 2.9 we get $\Delta_2(k) = \varnothing$ if $k \neq 2$ and $\Delta_2(2) = \{\gamma_1, \gamma_2, \gamma_3\}$ where

$$\gamma_1 = {}^{1233221}_{2}, \quad \gamma_2 = {}^{1243211}_{2}, \quad \gamma_3 = {}^{1343210}_{2}.$$

Set

$$\delta_1 = {}^{1343221}_{2}, \quad \delta_2 = {}^{1243321}_{2}, \quad \delta_3 = {}^{2343211}_{2}.$$

We may (and will) assume that the root elements $e_\gamma$, $\gamma \in R$, belong to a Chevalley basis of $\mathfrak{g}$:

$$[e_\alpha, e_\beta] = \pm e_{\alpha+\beta} \quad \text{if } \alpha, \beta, \alpha+\beta \in R.$$

We have

$$(\operatorname{ad} e)^2(e_{\gamma_1}) = [e_{\alpha_4}, [e_{\alpha_6}, e_{\gamma_1}]] + [e_{\alpha_3}, [e_{\alpha_4}, e_{\gamma_1}]] + [e_{\alpha_6}, [e_{\alpha_4}, e_{\gamma_1}]] = \pm e_{\delta_1} \pm 2e_{\delta_2}.$$

Similarly,

$$\begin{aligned}(\operatorname{ad} e)^2(e_{\gamma_2}) &= [e_{\alpha_3}, [e_{\alpha_7}, e_{\gamma_2}]] + [e_{\alpha_6}, [e_{\alpha_7}, e_{\gamma_2}]] + [e_{\alpha_7}, [e_{\alpha_3}, e_{\gamma_2}]] + [e_{\alpha_1}, [e_{\alpha_3}, e_{\gamma_2}]] \\ &= \pm 2e_{\delta_1} \pm e_{\delta_2} \pm e_{\delta_3}\end{aligned}$$

and

$$(\operatorname{ad} e)^2(e_{\gamma_3}) = [e_{\alpha_8}, [e_{\alpha_1}, e_{\gamma_3}]] + [e_{\alpha_1}, [e_{\alpha_8}, e_{\gamma_3}]] + [e_{\alpha_7}, [e_{\alpha_8}, e_{\gamma_3}]] = \pm e_{\delta_1} \pm 2e_{\delta_3}.$$

Since

$$\begin{vmatrix} \pm 1 & \pm 2 & \pm 1 \\ \pm 2 & \pm 1 & 0 \\ 0 & \pm 1 & \pm 2 \end{vmatrix} \neq 0$$

for $p > 3$, we conclude that $(\operatorname{ad} e)^2 \colon M_{J,+}^{-2} \to M_{J,+}^{2}$ is one-to-one. Since any distinguished parabolic subalgebra of $\mathfrak{l}_J$ is a Borel subalgebra (see [5, p. 174]), the subsystem of type $A_7$ can be excluded from our list.

**2.15.** Let $R_J \cong E_6 \times A_1 \subset E_8$. In this case $\mathfrak{g}_J(k) = 0$ if $k > 3$ and the $L_J^{(1)}$-modules $\mathfrak{g}_J(1)$, $\mathfrak{g}_J(2)$ and $\mathfrak{g}_J(3)$ are irreducible and have highest weights $\omega_1^J + \omega_8^J$, $\omega_6^J$ and $\omega_8^J$ respectively. The Lie algebra $\mathfrak{l}_J^{(1)}$ has three distinguished nilpotent conjugacy classes under the adjoint action of $L_J$ (Table 1):

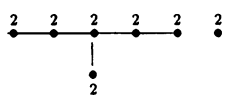
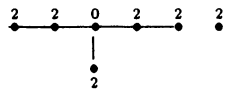
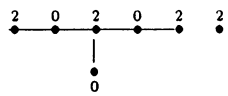
| Bala-Carter diagram | Type | Representative |
|---|---|---|
|   | $E_6 \times A_1$ | $e_{\alpha_1} + e_{\alpha_2} + e_{\alpha_3} + e_{\alpha_4} + e_{\alpha_5} + e_{\alpha_6} + e_{\alpha_8}$ |
|   | $E_6(a_1) \times A_1$ | $e_{\alpha_1} + e_{\alpha_3} + e_{\alpha_5} + e_{\alpha_6} + e_{\alpha_2+\alpha_4} + e_{\alpha_3+\alpha_4} + e_{\alpha_8}$ |
|   | $E_6(a_3) \times A_1$ | $e_{\alpha_1} + e_{\alpha_6} + e_{\alpha_1+\alpha_3} + e_{\alpha_2+\alpha_4} + e_{\alpha_5+\alpha_6} + e_{\alpha_3+\alpha_4+\alpha_5} + e_{\alpha_8}$ |

TABLE 1

(see, for example, [19]).

If $e$ is regular in $\mathfrak{l}_J^{(1)}$, then applying Lemma 2.9 yields $\Delta_7(1) = \Delta_7(3) = \varnothing$, $\Delta_7(2) = \{\gamma_1, \gamma_2\}$, where $\gamma_1 = {}^{1343221}_{2}$, $\gamma_2 = {}^{1243321}_{2}$ (one should take into account that $\beta_J^7 = {}^{2343211}_{2}$, $b_J^7 = 19$). Let $\delta_1 = {}^{1344321}_{2}$, $\delta_2 = {}^{2343321}_{2}$. Without loss of generality we may assume that $[e_{\alpha_3}, e_{\gamma_2}] = [e_{\alpha_6}, e_{\gamma_1}]$, $e_{\delta_1} = [e_{\alpha_1}, [e_{\alpha_6}, e_{\gamma_1}]]$, $e_{\delta_2} = [e_{\alpha_5}, [e_{\alpha_6}, e_{\gamma_1}]]$. Then

$$(\operatorname{ad} e)^2(e_{\gamma_1}) = [e_{\alpha_6}, [e_{\alpha_1}, e_{\gamma_1}]] + [e_{\alpha_1}, [e_{\alpha_6}, e_{\gamma_1}]] + [e_{\alpha_5}, [e_{\alpha_6}, e_{\gamma_1}]] = 2e_{\delta_1} + e_{\delta_2},$$

and

$$(\operatorname{ad} e)^2(e_{\gamma_2}) = [e_{\alpha_1}, [e_{\alpha_3}, e_{\gamma_2}]] + [e_{\alpha_5}, [e_{\alpha_3}, e_{\gamma_2}]] + [e_{\alpha_3}, [e_{\alpha_5}, e_{\gamma_2}]] = e_{\delta_1} + 2e_{\delta_2}.$$

Since

$$\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \neq 0$$

if $p > 3$, the map $(\operatorname{ad} e)^2 \colon M_{J,+}^{-2} \to M_{J,+}^{2}$ is one-to-one.

Suppose that $e$ has type $E_6(a_1) \times A_1$. Then $\lambda_{I,J} = 2\rho_J - 2\omega_4^J$. Since $\omega_4^J = {}^{2464200}_{3}$, we have $(\omega_4^J|\alpha_7) = -2$, $(\omega_4^J|\alpha_8) = 0$. Hence

$$(\lambda_{I,J}|\beta) = (2\rho_J|\beta) - (2\omega_4^J|\beta) = 2\operatorname{ht}\beta - \nu_7(\beta)b_J^7 - 2\nu_4(\beta) + 4\nu_7(\beta).$$

Since the number $b_J^7$ is odd, this implies that all weights of $\lambda_e$ on $\mathfrak{g}_J(1)$ and $\mathfrak{g}_J(3)$ are odd. Therefore, $M_{J,+}^{-2} = \mathfrak{g}_J^{-2}(2)$.

If $\nu_7(\beta) = 2$, then $(\lambda_{I,J}|\beta) = -2$ forces

$$\operatorname{ht}\beta = 14 + \nu_4(\beta).$$

Using [4, VI, Table VII] it is now easy to see that $\mathfrak{g}_J^{-2}(2)$ is spanned by $e_{\beta_1}$, $e_{\beta_2}$ and $e_{\beta_3}$ where

$$\beta_1 = {}^{1233321}_{2}, \quad \beta_2 = {}^{1343221}_{2}, \quad \beta_3 = {}^{1243321}_{2}.$$

Set

$$\eta_1 = {}^{1344321}_{2}, \quad \eta_2 = {}^{2343321}_{2}, \quad \eta_3 = {}^{1354321}_{2}.$$

Then

$$(\operatorname{ad} e)^2(e_{\beta_1}) = [e_{\alpha_1}, [e_{\alpha_3+\alpha_4}, e_{\beta_1}]] + [e_{\alpha_5}, [e_{\alpha_3+\alpha_4}, e_{\beta_1}]] = \pm e_{\eta_1} \pm e_{\eta_2}.$$

Similarly,

$$(\operatorname{ad} e)^2(e_{\beta_2}) = [e_{\alpha_6}, [e_{\alpha_1}, e_{\beta_2}]] + [e_{\alpha_1}, [e_{\alpha_6}, e_{\beta_2}]] + [e_{\alpha_5}, [e_{\alpha_6}, e_{\beta_2}]] = \pm e_{\eta_1} \pm 2e_{\eta_2}$$

and

$$\begin{aligned}(\operatorname{ad} e)^2(e_{\beta_3}) &= [e_{\alpha_1}, [e_{\alpha_3}, e_{\beta_3}]] + [e_{\alpha_5}, [e_{\alpha_3}, e_{\beta_3}]] \\ &\quad + [e_{\alpha_3}, [e_{\alpha_5}, e_{\beta_3}]] + [e_{\alpha_3+\alpha_4}[e_{\alpha_5}, e_{\beta_3}]] \\ &= \pm 2e_{\eta_1} \pm e_{\eta_2} \pm e_{\eta_3}.\end{aligned}$$

Since

$$\begin{vmatrix} \pm 1 & \pm 1 & \pm 2 \\ \pm 1 & \pm 2 & \pm 1 \\ 0 & 0 & \pm 1 \end{vmatrix} \neq 0$$

if $p \neq 3$, the map $(\operatorname{ad} e)^2 \colon M_{J,+}^{-2} \to M_{J,+}^{2}$ is one-to-one.

Let $e$ be of type $E_6(a_3) \times A_1$. Then $\lambda_{I,J} = 2\rho_J - 2(\omega_2^J + \omega_3^J + \omega_5^J)$. By [4, VI, Table V],

$$\omega_2^J + \omega_3^J + \omega_5^J = \begin{smallmatrix} 4 & 8 & 11 & 8 & 4 & 0 & 0 \\ & & 6 & & & & \end{smallmatrix}.$$

Hence

$$\begin{aligned}(\lambda_{I,J}|\beta) &= (2\rho_J|\beta) - 2(\nu_2(\beta) + \nu_3(\beta) + \nu_5(\beta)) + 8\nu_7(\beta) \\ &= 2\,\mathrm{ht}\,\beta - \nu_7(\beta)b_J^7 - 2(\nu_2(\beta) + \nu_3(\beta) + \nu_5(\beta)) + 8\nu_7(\beta) \\ &= 2(\nu_1(\beta) + \nu_4(\beta) + \nu_6(\beta) + \nu_8(\beta)) - 9\nu_7(\beta).\end{aligned}$$

This implies $M_{J,+}^{-2} = \mathfrak{g}_J^{-2}(2)$. If $\nu_7(\beta) = 2$, then $\nu_8(\beta) = 1$ and $(\lambda_{I,J}|\beta) = -2$ forces

$$\nu_1(\beta) + \nu_4(\beta) + \nu_6(\beta) = 7.$$

Using [4, VI, Table VII] one can find out that $\mathfrak{g}_J^{-2}(2)$ is spanned by $e_{\gamma_1}$, $e_{\gamma_2}$, $e_{\gamma_3}$ and $e_{\gamma_4}$ where

$$\gamma_1 = \begin{smallmatrix} 1&2&3&3&3&2&1 \\ &&1&&&& \end{smallmatrix}, \quad \gamma_2 = \begin{smallmatrix} 1&2&3&3&3&2&1 \\ &&2&&&& \end{smallmatrix}, \quad \gamma_3 = \begin{smallmatrix} 1&2&4&3&2&2&1 \\ &&2&&&& \end{smallmatrix}, \quad \gamma_4 = \begin{smallmatrix} 1&3&4&3&2&2&1 \\ &&2&&&& \end{smallmatrix}.$$

Let

$$\delta_1 = \begin{smallmatrix} 2&3&4&3&3&2&1 \\ &&2&&&& \end{smallmatrix}, \quad \delta_2 = \begin{smallmatrix} 1&3&5&4&3&2&1 \\ &&2&&&& \end{smallmatrix}, \quad \delta_3 = \begin{smallmatrix} 1&3&5&4&3&2&1 \\ &&3&&&& \end{smallmatrix}, \quad \delta_4 = \begin{smallmatrix} 2&3&4&4&3&2&1 \\ &&2&&&& \end{smallmatrix}.$$

Without loss of generality we may assume that

$$\begin{gathered} e_{\gamma_4} = [e_{\alpha_3}, e_{\gamma_3}], \quad [e_{\alpha_6}, e_{\gamma_3}] = [e_{\alpha_2+\alpha_4}, e_{\gamma_1}], \\ [e_{\alpha_3+\alpha_4+\alpha_5}, e_{\gamma_2}] = [e_{\alpha_5+\alpha_6}, e_{\gamma_4}], \quad [e_{\alpha_1}, e_{\alpha_3}] = e_{\alpha_1+\alpha_3}, \\ e_{\delta_1} = [e_{\alpha_1+\alpha_3}, [e_{\alpha_6}, e_{\gamma_3}]], \quad e_{\delta_2} = [e_{\alpha_3+\alpha_4+\alpha_5}, [e_{\alpha_2+\alpha_4}, e_{\gamma_1}]], \\ e_{\delta_3} = [e_{\alpha_2+\alpha_4}, [e_{\alpha_3+\alpha_4+\alpha_5}, e_{\gamma_2}]], \quad e_{\delta_4} = [e_{\alpha_1}, [e_{\alpha_5+\alpha_6}, e_{\gamma_4}]]. \end{gathered}$$

Direct computation shows that

$$\begin{aligned}(\operatorname{ad} e)^2(e_{\gamma_1}) &= [e_{\alpha_1+\alpha_3}, [e_{\alpha_2+\alpha_4}, e_{\gamma_1}]] + [e_{\alpha_3+\alpha_4+\alpha_5}, [e_{\alpha_2+\alpha_4}, e_{\gamma_1}]] \\ &= [e_{\alpha_1+\alpha_3}, [e_{\alpha_6}, e_{\gamma_3}]] + e_{\delta_2} = e_{\delta_1} + e_{\delta_2}\end{aligned}$$

and

$$\begin{aligned}(\operatorname{ad} e)^2(e_{\gamma_2}) &= [e_{\alpha_1}, [e_{\alpha_3+\alpha_4+\alpha_5}, e_{\gamma_2}]] + [e_{\alpha_2+\alpha_4}, [e_{\alpha_3+\alpha_4+\alpha_5}, e_{\gamma_2}]] \\ &= [e_{\alpha_1}, [e_{\alpha_5+\alpha_6}, e_{\gamma_4}]] + e_{\delta_3} = e_{\delta_3} + e_{\delta_4}.\end{aligned}$$

Also

$$\begin{aligned}(\operatorname{ad} e)^2(e_{\gamma_3}) &= [e_{\alpha_1+\alpha_3}, [e_{\alpha_6}, e_{\gamma_3}]] + [e_{\alpha_3+\alpha_4+\alpha_5}, [e_{\alpha_6}, e_{\gamma_3}]] + [e_{\alpha_6}, [e_{\alpha_1+\alpha_3}, e_{\gamma_3}]] \\ &\quad + [e_{\alpha_5+\alpha_6}, [e_{\alpha_1+\alpha_3}, e_{\gamma_3}]] + [e_{\alpha_1+\alpha_3}, [e_{\alpha_5+\alpha_6}, e_{\gamma_3}]] \\ &= 2e_{\delta_1} + [e_{\alpha_3+\alpha_4+\alpha_5}, [e_{\alpha_2+\alpha_4}, e_{\gamma_1}]] + 2[e_{\alpha_5+\alpha_6}, [e_{\alpha_1}, [e_{\alpha_3}, e_{\gamma_3}]]] \\ &= 2e_{\delta_1} + e_{\delta_2} + 2[e_{\alpha_1}, [e_{\alpha_5+\alpha_6}, e_{\gamma_4}]] = 2e_{\delta_1} + e_{\delta_2} + 2e_{\delta_4}\end{aligned}$$

and

$$\begin{aligned}(\operatorname{ad} e)^2(e_{\gamma_4}) &= [e_{\alpha_6}, [e_{\alpha_1}, e_{\gamma_4}]] + [e_{\alpha_5+\alpha_6}, [e_{\alpha_1}, e_{\gamma_4}]] + [e_{\alpha_1}, [e_{\alpha_6}, e_{\gamma_4}]] \\ &\quad + [e_{\alpha_1}, [e_{\alpha_5+\alpha_6}, e_{\gamma_4}]] + [e_{\alpha_2+\alpha_4}, [e_{\alpha_5+\alpha_6}, e_{\gamma_4}]] \\ &= 2[e_{\alpha_6}, [e_{\alpha_1}, [e_{\alpha_3}, e_{\gamma_3}]]] + 2e_{\delta_4} + [e_{\alpha_2+\alpha_4}, [e_{\alpha_3+\alpha_4+\alpha_5}, e_{\gamma_2}]] \\ &= 2e_{\delta_1} + e_{\delta_3} + 2e_{\delta_4}.\end{aligned}$$

Since

$$\begin{vmatrix} 1 & 0 & 2 & 2 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 2 \end{vmatrix} = -3,$$

the map $(\operatorname{ad} e)^2 \colon M_{J,+}^{-2} \to M_{J,+}^{2}$ is bijective.

**2.16.** Let $R_J \cong D_7 \subset E_8$. In this case $\mathfrak{g}_J(k) = 0$ for $k > 2$ and $\mathfrak{g}_J(1)$ and $\mathfrak{g}_J(2)$ are irreducible $L_J^{(1)}$-modules with highest weights $\omega_2^J$ and $\omega_8^J$ respectively (note that $B_J = B \backslash \{\alpha_1\}$).

We have $\beta_J^1 = {}^{1354321}_{3}$, $b_J^1 = 23$ and so

$$(2\rho_J|\gamma) = 2\operatorname{ht}\gamma - 23\nu_1(\gamma).$$

This yields $\Delta_1(1) = \varnothing$, $\Delta_1(2) = \{\gamma \in R | \nu_1(\gamma) = 2, \operatorname{ht}\gamma = 22\} = \{\beta\}$ where $\beta = {}^{2354321}_{2}$. If $e = \sum_{i=2}^{8} e_{\alpha_i}$, then

$$(\operatorname{ad} e)^2(e_\beta) = [e_{\alpha_2}, [e_{\alpha_3}, e_\beta]] + [e_{\alpha_3}, [e_{\alpha_2}, e_\beta]] = \pm 2 e_\eta$$

where $\eta = {}^{2454321}_{3}$. Hence $(\operatorname{ad} e)^2 \colon M_{J,+}^{-2} \to M_{J,+}^{2}$ is bijective.

The Lie algebra $\mathfrak{l}_J^{(1)}$ has two nonregular distinguished nilpotent classes under the adjoint action of $L_J$. Their Bala-Carter diagrams are given in Table 2.

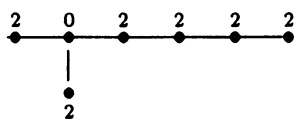
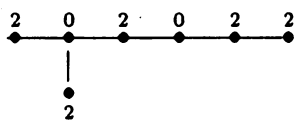
| Bala-Carter diagram | Type |
|---|---|
|   | $D_7(a_1)$ |
|   | $D_7(a_2)$ |

TABLE 2

If $e$ has type $D_7(a_2)$, then $\lambda_{I,J} = 2\rho_J - 2(\omega_4^J + \omega_6^J)$. Clearly,

$$2\omega_4^J = {}^{05108642}_{5}, \quad 2\omega_6^J = {}^{0366642}_{3}.$$

Hence

$$(\lambda_{I,J}|\gamma) = (2\rho_J|\gamma) - 2\nu_4(\gamma) - 2\nu_6(\gamma) + 8\nu_1(\gamma) = 2\operatorname{ht}\gamma - 2\nu_4(\gamma) - 2\nu_6(\gamma) - 15\nu_1(\gamma).$$

This implies that $\Delta_1(1) = \varnothing$ and $(\lambda_{I,J}|\tilde{\alpha}) = 8 < 2(p-1)$ where $\tilde{\alpha}$ is the highest root of $R_+$. As $(\lambda_{I,J}|\delta) \le (\lambda_{I,J}|\tilde{\alpha})$ for any $\delta \in R$ with $\nu_1(\delta) = 2$, we conclude that $m_2(e) < 2(p-1)$ and so $(\operatorname{ad} e)^2 \colon M_{J,+}^{-2} \to M_{J,+}^{2}$ is one-to-one (see our remark in (2.13)).

Let $e$ be of type $D_7(a_1)$. Direct verification based on the fact that $e$ is a Richardson element of $\mathfrak{p}_I \cap \mathfrak{l}_J$ shows that no generality is lost by assuming

$$e = e_{\alpha_3} + e_{\alpha_5} + e_{\alpha_6} + e_{\alpha_7} + e_{\alpha_8} + e_{\alpha_2+\alpha_4} + e_{\alpha_4+\alpha_5}.$$

As $\lambda_{I,J} = 2\rho_J - 2\omega_4^J$, one computes

$$(\lambda_{I,J}|\gamma) = 2\,\mathrm{ht}\,\gamma - 2\nu_4(\gamma) + 5\nu_1(\gamma) - b_J^1\nu_1(\gamma) = 2(\mathrm{ht}\,\gamma - \nu_4(\gamma) - 9\nu_1(\gamma)).$$

Hence $m_2(e) = (\lambda_{I,J}|\tilde{\alpha}) = 10 < 2(p-1)$. This implies that $(\mathrm{ad}\,e)^2\colon \mathfrak{g}_J^{-2}(2) \to \mathfrak{g}_J^2(2)$ is one-to-one. If $\nu_1(\gamma) = 1$, then $(\lambda_{I,J}|\gamma) = -2$ forces $\mathrm{ht}\,\gamma = \nu_4(\gamma) + 8$. Using [4, VI, Table VII] we obtain that $\mathfrak{g}_J^{-2}(1)$ is spanned by $e_{\gamma_i}$, $1 \le i \le 6$, where

$$\gamma_1 = {}^{1221111}_{\;\;1}, \quad \gamma_2 = {}^{1122111}_{\;\;1}, \quad \gamma_3 = {}^{1122210}_{\;\;1}$$
$$\gamma_4 = {}^{1222110}_{\;\;1}, \quad \gamma_5 = {}^{1232110}_{\;\;1}, \quad \gamma_6 = {}^{1232100}_{\;\;2}.$$

Let

$$\delta_1 = {}^{1122221}_{\;\;1}, \quad \delta_2 = {}^{1222211}_{\;\;1}, \quad \delta_3 = {}^{1232211}_{\;\;1},$$
$$\delta_4 = {}^{1233210}_{\;\;1}, \quad \delta_5 = {}^{1232111}_{\;\;2}, \quad \delta_6 = {}^{1232210}_{\;\;2}$$

and $\sigma = {}^{1122110}_{\;\;1}$. A suitable transformation of the form $e_\alpha \mapsto (-1)^{\varepsilon(\alpha)}e_\alpha$, $\alpha \in R$, allows one to assume that

$$e_{\alpha_4+\alpha_5} = [e_{\alpha_4}, e_{\alpha_5}], \quad e_{\gamma_2} = [e_{\alpha_8}, e_\sigma], \quad e_{\gamma_3} = [e_{\alpha_6}, e_\sigma], \quad e_{\gamma_4} = [e_{\alpha_3}, e_\sigma],$$
$$e_{\gamma_5} = [e_{\alpha_4}, e_{\gamma_4}], \quad [e_{\alpha_5}, e_{\gamma_1}] = [e_{\alpha_8}, e_{\gamma_4}], \quad [e_{\alpha_7}, e_{\gamma_6}] = [e_{\alpha_2+\alpha_4}, e_{\gamma_4}],$$
$$e_{\delta_1} = [e_{\alpha_7}, [e_{\alpha_6}, e_{\gamma_2}]], \quad e_{\delta_2} = [e_{\alpha_6}, [e_{\alpha_5}, e_{\gamma_1}]], \quad e_{\delta_3} = [e_{\alpha_6}, [e_{\alpha_4+\alpha_5}, e_{\gamma_1}]],$$
$$e_{\delta_4} = [e_{\alpha_4+\alpha_5}, [e_{\alpha_3}, e_{\gamma_3}]], \quad e_{\delta_5} = [e_{\alpha_2+\alpha_4}, [e_{\alpha_5}, e_{\gamma_1}]], \quad e_{\delta_6} = [e_{\alpha_2+\alpha_4}, [e_{\alpha_3}, e_{\gamma_3}]].$$

Computations show that

$$\begin{aligned}
(\mathrm{ad}\,e)^2(e_{\gamma_1}) &= [e_{\alpha_6}, [e_{\alpha_5}, e_{\gamma_1}]] + [e_{\alpha_2+\alpha_4}, [e_{\alpha_5}, e_{\gamma_1}]] + [e_{\alpha_6}, [e_{\alpha_4+\alpha_5}, e_{\gamma_1}]] \\
&= e_{\delta_2} + e_{\delta_3} + e_{\delta_5}, \\
(\mathrm{ad}\,e)^2(e_{\gamma_2}) &= [e_{\alpha_6}, [e_{\alpha_3}, e_{\gamma_2}]] + [e_{\alpha_2+\alpha_4}, [e_{\alpha_3}, e_{\gamma_2}]] \\
&\quad + [e_{\alpha_3}, [e_{\alpha_6}, e_{\gamma_2}]] + [e_{\alpha_7}, [e_{\alpha_6}, e_{\gamma_2}]] \\
&= e_{\delta_1} + 2[e_{\alpha_3}, [e_{\alpha_6}, [e_{\alpha_8}, e_\sigma]]] + [e_{\alpha_2+\alpha_4}, [e_{\alpha_3}, [e_{\alpha_8}, e_\sigma]]] \\
&= e_{\delta_1} + 2[e_{\alpha_6}, [e_{\alpha_8}, e_{\gamma_4}]] + [e_{\alpha_2+\alpha_4}, [e_{\alpha_8}, e_{\gamma_4}]] \\
&= e_{\delta_1} + 2[e_{\alpha_6}, [e_{\alpha_5}, e_{\gamma_1}]] + [e_{\alpha_2+\alpha_4}, [e_{\alpha_5}, e_{\gamma_1}]] = e_{\delta_1} + 2e_{\delta_2} + e_{\delta_5}, \\
(\mathrm{ad}\,e)^2(e_{\gamma_3}) &= [e_{\alpha_8}, [e_{\alpha_3}, e_{\gamma_3}]] + [e_{\alpha_2+\alpha_4}, [e_{\alpha_3}, e_{\gamma_3}]] + [e_{\alpha_4+\alpha_5}, [e_{\alpha_3}, e_{\gamma_3}]] \\
&\quad + [e_{\alpha_3}, [e_{\alpha_8}, e_{\gamma_3}]] + [e_{\alpha_7}, [e_{\alpha_8}, e_{\gamma_3}]] \\
&= [e_{\alpha_7}, [e_{\alpha_8}, [e_{\alpha_6}, e_\sigma]]] + 2[e_{\alpha_3}, [e_{\alpha_8}, [e_{\alpha_6}, e_\sigma]]] + e_{\delta_4} + e_{\delta_6} \\
&= e_{\delta_1} + 2[e_{\alpha_6}, [e_{\alpha_5}, e_{\gamma_1}]] + e_{\delta_4} + e_{\delta_6} = e_{\delta_1} + 2e_{\delta_2} + e_{\delta_4} + e_{\delta_6}, \\
(\mathrm{ad}\,e)^2(e_{\gamma_4}) &= [e_{\alpha_8}, [e_{\alpha_6}, e_{\gamma_4}]] + [e_{\alpha_2+\alpha_4}, [e_{\alpha_6}, e_{\gamma_4}]] + [e_{\alpha_4+\alpha_5}, [e_{\alpha_6}, e_{\gamma_4}]] \\
&\quad + [e_{\alpha_6}, [e_{\alpha_8}, e_{\gamma_4}]] \\
&\quad + [e_{\alpha_2+\alpha_4}, [e_{\alpha_8}, e_{\gamma_4}]] + [e_{\alpha_6}, [e_{\alpha_2+\alpha_4}, e_{\gamma_4}]] + [e_{\alpha_8}, [e_{\alpha_2+\alpha_4}, e_{\gamma_4}] \\
&= 2[e_{\alpha_6}, [e_{\alpha_5}, e_{\gamma_1}]] + 2[e_{\alpha_2+\alpha_4}, [e_{\alpha_6}, [e_{\alpha_3}, e_\sigma]]] + 2[e_{\alpha_2+\alpha_4}, [e_{\alpha_5}, e_{\gamma_1}]] \\
&\quad + [e_{\alpha_4+\alpha_5}, [e_{\alpha_6}, [e_{\alpha_3}, e_\sigma]]] \\
&= 2e_{\delta_2} + 2[e_{\alpha_2+\alpha_4}, [e_{\alpha_3}, e_{\gamma_3}]] + 2e_{\delta_5} + [e_{\alpha_4+\alpha_5}, [e_{\alpha_3}, e_{\gamma_3}]] \\
&= 2e_{\delta_2} + e_{\delta_4} + 2e_{\delta_5} + 2e_{\delta_6},
\end{aligned}$$

$$\begin{aligned}(\operatorname{ad} e)^2(e_{\gamma_5}) &= [e_{\alpha_5}, [e_{\alpha_6}, e_{\gamma_5}]] + [e_{\alpha_8}, [e_{\alpha_6}, e_{\gamma_5}]] + [e_{\alpha_6}, [e_{\alpha_8}, e_{\gamma_5}]] \\ &= 2[e_{\alpha_6}, [e_{\alpha_8}, [e_{\alpha_4}, e_{\gamma_4}]]] + [e_{\alpha_5}, [e_{\alpha_6}, [e_{\alpha_4}, e_{\gamma_4}]]] \\ &= 2[e_{\alpha_6}, [e_{\alpha_4}, [e_{\alpha_5}, e_{\gamma_1}]]] + [e_{\alpha_5}, [e_{\alpha_6}, [e_{\alpha_4}, [e_{\alpha_3}, e_{\sigma}]]]] \\ &= 2[e_{\alpha_6}, [[e_{\alpha_4}, e_{\alpha_5}], e_{\gamma_1}]] + [e_{\alpha_5}, [e_{\alpha_4}, [e_{\alpha_3}, e_{\gamma_3}]]] \\ &= 2e_{\delta_3} + [[e_{\alpha_5}, e_{\alpha_4}], [e_{\alpha_3}, e_{\gamma_3}]] = 2e_{\delta_3} - e_{\delta_4}, \\ (\operatorname{ad} e)^2(e_{\gamma_6}) &= [e_{\alpha_6}, [e_{\alpha_7}, e_{\gamma_6}]] + [e_{\alpha_8}, [e_{\alpha_7}, e_{\gamma_6}]] \\ &= [e_{\alpha_6}, [e_{\alpha_2+\alpha_4}, e_{\gamma_4}]] + [e_{\alpha_8}, [e_{\alpha_2+\alpha_4}, e_{\gamma_4}]] \\ &= [e_{\alpha_2+\alpha_4}, [e_{\alpha_6}, [e_{\alpha_3}, e_{\sigma}]]] + [e_{\alpha_2+\alpha_4}, [e_{\alpha_8}, e_{\gamma_4}]] \\ &= [e_{\alpha_2+\alpha_4}, [e_{\alpha_3}, e_{\gamma_3}]] + [e_{\alpha_2+\alpha_4}, [e_{\alpha_5}, e_{\gamma_1}]] = e_{\delta_5} + e_{\delta_6}.\end{aligned}$$

Since

$$\begin{vmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{vmatrix} = 4,$$

the map $(\operatorname{ad} e)^2 \colon M_{J,+}^{-2} \to M_{J,+}^{2}$ is bijective.

**2.17.** We now suppose that $R_J \cong E_7 \subset E_8$. Clearly, $B_J = B \backslash \{\alpha_8\}$. Therefore, $\mathfrak{g}_J(k) = 0$ if $k > 2$. Moreover, the $L_J^{(1)}$-module $\mathfrak{g}_J(2)$ is trivial and $\mathfrak{g}_J(1)$ is irreducible over $L_J^{(1)}$ and has highest weight $\omega_7^J$.

By [5, p. 176], any standard distinguished parabolic subalgebra $\mathfrak{p}_I \cap \mathfrak{l}_J^{(1)}$ of $\mathfrak{l}_J^{(1)}$ has the following property:

$$7 \notin I \text{ and either } \{2, 5\} \subset I \text{ or } \{2, 5\} \cap I = \varnothing.$$

Using [4, VI, Table VI] it is easy to note that $\omega_1^J, \omega_3^J, \omega_4^J, \omega_6^J, \omega_2^J + \omega_5^J \in Q_+^J$ and $\nu_7(2\omega_7^J) = 3$. This implies that, for any $\gamma \in R$ with $\nu_8(\gamma) = 1$,

$$(\lambda_{I,J}|\gamma) \equiv (\lambda_{I,J}|\alpha_8) \equiv (2\omega_7^J|\alpha_8) \equiv 1 \pmod 2.$$

Thus all $(\operatorname{Ad}\lambda_e)$-weights of $\mathfrak{g}_J(1)$ are odd. But then $M_J^{2(p-1)} = 0$ and Lemma 2.7 applies. Therefore, $\lambda_e$ is a Dynkin torus for $e \in \mathfrak{p}_I \cap \mathfrak{l}_J^{(1)}$.

**2.18.** We now deal with $R \cong E_7$. Let $R_J$ be of type $A_5 \times A_1$. Clearly, we may assume that $B_J = B \backslash \{\alpha_3\}$ and $e = \sum_{i \neq 3} e_{\alpha_i}$. It is immediate from [4, VI, Table VI] that $\mathfrak{g}_J(k) = 0$ if $k > 3$, $\beta_J^3 = {}^{112221}_{1}$, $b_J^3 = 11$. Hence $(2\rho_J|\gamma) = 2\operatorname{ht}\gamma - 11\nu_3(\gamma)$ (see Lemma 2.9). This implies that $\Delta_3(k) = \varnothing$ if $k \neq 2$ and $\Delta_3(2) = \{\beta_1, \beta_2\}$ where $\beta_1 = {}^{122211}_{1}$, $\beta_2 = {}^{123210}_{1}$. Set

$$\delta_1 = {}^{123221}_{1}, \quad \delta_2 = {}^{123211}_{2}, \quad \eta = {}^{122210}_{1}.$$

We may and do assume that

$$e_{\beta_1} = [e_{\alpha_7}, e_\eta], \quad e_{\beta_2} = [e_{\alpha_4}, e_\eta],$$
$$e_{\delta_1} = [e_{\alpha_6}, [e_{\alpha_4}, e_{\beta_1}]], \quad e_{\delta_2} = [e_{\alpha_2}, [e_{\alpha_7}, e_{\beta_2}]].$$

We have

$$\begin{aligned}(\operatorname{ad} e)^2(e_{\beta_1}) &= [e_{\alpha_2}, [e_{\alpha_4}, e_{\beta_1}]] + [e_{\alpha_6}, [e_{\alpha_4}, e_{\beta_1}]] + [e_{\alpha_4}, [e_{\alpha_6}, e_{\beta_1}]] \\ &= 2e_{\delta_1} + [e_{\alpha_2}, [e_{\alpha_4}, [e_{\alpha_7}, e_\eta]]] = 2e_{\delta_1} + [e_{\alpha_2}, [e_{\alpha_7}, e_{\beta_2}]] = 2e_{\delta_1} + e_{\delta_2}\end{aligned}$$

and

$$\begin{aligned}(\operatorname{ad} e)^2(e_{\beta_2}) &= [e_{\alpha_7}, [e_{\alpha_2}, e_{\beta_2}]] + [e_{\alpha_2}, [e_{\alpha_7}, e_{\beta_2}]] + [e_{\alpha_6}, [e_{\alpha_7}, e_{\beta_2}]] \\ &= 2e_{\delta_2} + [e_{\alpha_6}, [e_{\alpha_7}, [e_{\alpha_4}, e_\eta]]] = 2e_{\delta_2} + [e_{\alpha_6}, [e_{\alpha_4}, e_{\beta_1}]] = e_{\delta_1} + 2e_{\delta_2}.\end{aligned}$$

Since

$$\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \neq 0$$

if $p > 3$, the map $(\operatorname{ad} e)^2 \colon M_{J,+}^{-2} \to M_{J,+}^{2}$ is bijective.

**2.19.** Let $R_J$ be of type $D_5 \times A_1$. In this case $B_J = B \backslash \{\alpha_6\}$. First suppose that $e$ is regular in $\mathfrak{l}_J^{(1)}$. By [4, VI, Table VI], $\mathfrak{g}_J(k) = 0$ for $k > 2$, $\beta_J^6 = {}^{123211}_{2}$ and $b_J^6 = 13$. Hence $(2\rho_J|\gamma) = 2\operatorname{ht}\gamma - 13\nu_6(\gamma)$ yielding $\Delta_6(1) = \varnothing$, $\Delta_6(2) = \{\gamma\}$ where $\gamma = {}^{123221}_{1}$.

As

$$\begin{aligned}(\operatorname{ad} e)^2(e_\gamma) &= \left(\sum_{i \neq 6} e_{\alpha_i}\right)^2 (e_\gamma) = [e_{\alpha_5}, [e_{\alpha_2}, e_\gamma]] + [e_{\alpha_2}, [e_{\alpha_5}, e_\gamma]] \\ &= 2[e_{\alpha_5}, [e_{\alpha_2}, e_\gamma]] \neq 0,\end{aligned}$$

we conclude that $(\operatorname{ad} e)^2 \colon M_{J,+}^{-2} \to M_{J,+}^{2}$ is one-to-one.

Now we suppose that $e$ is not regular. Then $e$ has the data given in Table 3:

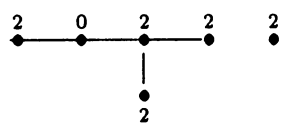
| Bala-Carter diagram | Type |
|---|---|
|   | $D_5(a_1) \times A_1$ |

TABLE 3

A simple checking shows that no generality is lost by assuming

$$e = e_{\alpha_1} + e_{\alpha_3} + e_{\alpha_5} + e_{\alpha_7} + e_{\alpha_2+\alpha_4} + e_{\alpha_4+\alpha_5}.$$

Since $\lambda_{I,J} = 2\rho_J - 2\omega_4^J$ and $2\omega_4^J = {}^{246300}_{3}$, we have

$$(\lambda_{I,J}|\gamma) = 2\operatorname{ht}\gamma - 13\nu_6(\gamma) + 3\nu_6(\gamma) - 2\nu_4(\gamma) = 2(\operatorname{ht}\gamma - \nu_4(\gamma) - 5\nu_6(\gamma)).$$

In particular, $(\lambda_{I,J}|\tilde{\alpha}) = 6 < 2(p-1)$. Since $(\lambda_{I,J}|\beta) \leq (\lambda_{I,J}|\tilde{\alpha})$ for any $\beta \in R$ with $\nu_6(\beta) = 2$, we conclude that $m_2(e) < 2(p-1)$. By our remark in (2.13), it follows that $(\operatorname{ad} e)^2 \colon \mathfrak{g}_J^{-2}(2) \to \mathfrak{g}_J^{2}(2)$ is bijective.

Using [4, VI, Table VI] one can check that $\mathfrak{g}_J^{-2}(1)$ is spanned by $e_{\beta_i}$, $1 \leq i \leq 5$, where

$$\beta_1 = {}^{111110}_{0}, \quad \beta_2 = {}^{011110}_{1}, \quad \beta_3 = {}^{001111}_{1},$$
$$\beta_4 = {}^{011111}_{0}, \quad \beta_5 = {}^{012110}_{1}.$$

Set

$$\delta_1 = {}^{111111}_{1}, \quad \delta_2 = {}^{112111}_{1}, \quad \delta_3 = {}^{112210}_{1},$$
$$\delta_4 = {}^{122110}_{1}, \quad \delta_5 = {}^{012211}_{1}, \quad \eta = {}^{011110}_{0}.$$

We may assume that

$$[e_{\alpha_5}, e_{\alpha_5}] = e_{\alpha_4+\alpha_5}, \quad e_{\beta_1} = [e_{\alpha_1}, e_\eta], \quad e_{\beta_4} = [e_{\alpha_7}, e_\eta],$$
$$e_{\beta_5} = [e_{\alpha_2+\alpha_4}, e_\eta], \quad [e_{\alpha_4}, e_{\beta_2}] = e_{\beta_5}, \quad [e_{\alpha_7}, e_{\beta_2}] = [e_{\alpha_3}, e_{\beta_3}],$$
$$e_{\delta_1} = [e_{\alpha_1}, [e_{\alpha_7}, e_{\beta_2}]], \quad e_{\delta_2} = [e_{\alpha_7}, [e_{\alpha_2+\alpha_4}, e_{\beta_1}]], \quad e_{\delta_3} = [e_{\alpha_5}, [e_{\alpha_2+\alpha_4}, e_{\beta_1}]],$$
$$e_{\delta_4} = [e_{\alpha_3}, [e_{\alpha_2+\alpha_4}, e_{\beta_1}]], \quad e_{\delta_5} = [e_{\alpha_4+\alpha_5}, [e_{\alpha_7}, e_{\beta_2}]].$$

Then

$$\begin{aligned}
(\operatorname{ad} e)^2(e_{\beta_1}) &= [e_{\alpha_3}, [e_{\alpha_2+\alpha_4}, e_{\beta_1}]] + [e_{\alpha_5}, [e_{\alpha_2+\alpha_4}, e_{\beta_1}]] + [e_{\alpha_7}, [e_{\alpha_2+\alpha_4}, e_{\beta_1}]] \\
&\quad + [e_{\alpha_2+\alpha_4}, [e_{\alpha_7}, e_{\beta_1}]] \\
&= 2e_{\delta_2} + e_{\delta_3} + e_{\delta_4}, \\
(\operatorname{ad} e)^2(e_{\beta_2}) &= [e_{\alpha_4+\alpha_5}, [e_{\alpha_1}, e_{\beta_2}]] + [e_{\alpha_7}, [e_{\alpha_1}, e_{\beta_2}]] + [e_{\alpha_1}, [e_{\alpha_4+\alpha_5}, e_{\beta_2}]] \\
&\quad + [e_{\alpha_7}, [e_{\alpha_4+\alpha_5}, e_{\beta_2}]] + [e_{\alpha_1}, [e_{\alpha_7}, e_{\beta_2}]] + [e_{\alpha_4+\alpha_5}, [e_{\alpha_7}, e_{\beta_2}]] \\
&= 2e_{\delta_1} + 2e_{\delta_5} - 2[e_{\alpha_1}, [e_{\alpha_5}, [e_{\alpha_4}, e_{\beta_2}]]] \\
&= 2e_{\delta_1} + 2e_{\delta_5} - 2[e_{\alpha_1}, [e_{\alpha_5}, e_{\beta_5}]] \\
&= 2e_{\delta_1} + 2e_{\delta_5} - 2[e_{\alpha_5}, [e_{\alpha_1}, [e_{\alpha_2+\alpha_4}, e_\eta]]] \\
&= 2e_{\delta_1} + 2e_{\delta_5} - 2[e_{\alpha_5}, [e_{\alpha_2+\alpha_4}, e_{\beta_1}]] = 2(e_{\delta_1} - e_{\delta_3} + e_{\delta_5}), \\
(\operatorname{ad} e)^2(e_{\beta_3}) &= [e_{\alpha_1}, [e_{\alpha_3}, e_{\beta_3}]] + [e_{\alpha_4+\alpha_5}, [e_{\alpha_3}, e_{\beta_3}]] \\
&= [e_{\alpha_1}, [e_{\alpha_7}, e_{\beta_2}]] + [e_{\alpha_4+\alpha_5}, [e_{\alpha_7}, e_{\beta_2}]] = e_{\delta_1} + e_{\delta_5}, \\
(\operatorname{ad} e)^2(e_{\beta_4}) &= [e_{\alpha_2+\alpha_4}, [e_{\alpha_1}, e_{\beta_4}]] + [e_{\alpha_1}, [e_{\alpha_2+\alpha_4}, e_{\beta_4}]] + [e_{\alpha_5}, [e_{\alpha_2+\alpha_4}, e_{\beta_4}]] \\
&= 2[e_{\alpha_2+\alpha_4}, [e_{\alpha_1}, [e_{\alpha_7}, e_\eta]]] + [e_{\alpha_5}, [e_{\alpha_2+\alpha_4}, [e_{\alpha_7}, e_\eta]]] \\
&= 2[e_{\alpha_2+\alpha_4}, [e_{\alpha_7}, e_{\beta_1}]] + [e_{\alpha_5}, [e_{\alpha_7}, e_{\beta_5}]] \\
&= 2e_{\delta_2} + [e_{\alpha_7}, [e_{\alpha_5}, [e_{\alpha_4}, e_{\beta_2}]]] = 2e_{\delta_2} + [e_{\alpha_7}, [[e_{\alpha_5}, e_{\alpha_4}], e_{\beta_2}]] \\
&= 2e_{\delta_2} - e_{\delta_5}, \\
(\operatorname{ad} e)^2(e_{\beta_5}) &= [e_{\alpha_3}, [e_{\alpha_1}, e_{\beta_5}]] + [e_{\alpha_5}, [e_{\alpha_1}, e_{\beta_5}]] \\
&\quad + [e_{\alpha_7}, [e_{\alpha_1}, e_{\beta_5}]] + [e_{\alpha_1}, [e_{\alpha_5}, e_{\beta_5}]] \\
&\quad + [e_{\alpha_7}, [e_{\alpha_5}, e_{\beta_5}]] + [e_{\alpha_1}, [e_{\alpha_7}, e_{\beta_5}]] + [e_{\alpha_5}, [e_{\alpha_7}, e_{\beta_5}]] \\
&= 2[e_{\alpha_7}, [e_{\alpha_1}, [e_{\alpha_2+\alpha_4}, e_\eta]]] + 2[e_{\alpha_1}, [e_{\alpha_5}, [e_{\alpha_2+\alpha_4}, e_\eta]]] \\
&\quad + [e_{\alpha_3}, [e_{\alpha_1}, [e_{\alpha_2+\alpha_4}, e_\eta]]] + 2[e_{\alpha_7}, [e_{\alpha_5}, [e_{\alpha_4}, e_{\beta_2}]]] \\
&= 2[e_{\alpha_7}, [e_{\alpha_2+\alpha_4}, e_{\beta_1}]] + 2[e_{\alpha_5}, [e_{\alpha_2+\alpha_4}, e_{\beta_1}]] + [e_{\alpha_3}, [e_{\alpha_2+\alpha_4}, e_{\beta_1}]] \\
&\quad + 2[e_{\alpha_7}, [[e_{\alpha_5}, e_{\alpha_4}], e_{\beta_2}]] \\
&= 2e_{\delta_2} + 2e_{\delta_3} + e_{\delta_4} - 2e_{\delta_5}.
\end{aligned}$$

Since

$$\begin{vmatrix} 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & 0 & 2 & 2 \\ 1 & -2 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & -1 & -2 \end{vmatrix} = -8,$$

we can exclude $D_5 \times A_1$ from our list.

**2.20.** Let $R_J \cong A_6 \subset E_7$. Then $B_J = B\backslash\{\alpha_2\}$, $\beta_J^2 = {}^{123321}_{1}$, $b_J^2 = 14$ and $e = \sum_{i\neq 2} e_{\alpha_i}$. Clearly, $\mathfrak{g}_J(k) = 0$ if $k > 2$. By using [4, VI, Table VI] and Lemma 2.9 one gets

$$\Delta_2(1) = \{\beta_1, \beta_2, \beta_3, \beta_4\}, \quad \Delta_2(2) = \{\gamma\},$$

where

$$\beta_1 = {}^{112100}_{1}, \quad \beta_2 = {}^{111110}_{1}, \quad \beta_3 = {}^{012110}_{1},$$
$$\beta_4 = {}^{011111}_{1}, \quad \gamma = {}^{123221}_{2}.$$

It is straightforward that $(\operatorname{ad} e)^2(e_\gamma) = \pm e_\eta$ where $\eta = {}^{124321}_{2}$. Denote

$$\delta_1 = {}^{122110}_{1}, \quad \delta_2 = {}^{112210}_{1}, \quad \delta_3 = {}^{112111}_{1}, \quad \delta_4 = {}^{012211}_{1}.$$

It can be easily seen that we may suppose that

$$[e_{\alpha_6}, e_{\beta_1}] = [e_{\alpha_4}, e_{\beta_2}] = [e_{\alpha_1}, e_{\beta_3}],$$

$$[e_{\alpha_4}, e_{\beta_4}] = [e_{\alpha_7}, e_{\beta_3}], \quad e_{\delta_1} = [e_{\alpha_3}, [e_{\alpha_6}, e_{\beta_1}]], \quad e_{\delta_2} = [e_{\alpha_5}, [e_{\alpha_6}, e_{\beta_1}]],$$
$$e_{\delta_3} = [e_{\alpha_7}, [e_{\alpha_6}, e_{\beta_1}]], \quad e_{\delta_4} = [e_{\alpha_5}, [e_{\alpha_7}, e_{\beta_3}]].$$

Calculations show that

$$\begin{aligned}
(\operatorname{ad} e)^2(e_{\beta_1}) &= [e_{\alpha_6}, [e_{\alpha_3}, e_{\beta_1}]] + [e_{\alpha_3}, [e_{\alpha_6}, e_{\beta_1}]] \\
&\quad + [e_{\alpha_5}, [e_{\alpha_6}, e_{\beta_1}]] + [e_{\alpha_7}, [e_{\alpha_6}, e_{\beta_1}]] \\
&= 2e_{\delta_1} + e_{\delta_2} + e_{\delta_3}, \\
(\operatorname{ad} e)^2(e_{\beta_2}) &= [e_{\alpha_3}, [e_{\alpha_4}, e_{\beta_2}]] + [e_{\alpha_5}, [e_{\alpha_4}, e_{\beta_2}]] \\
&\quad + [e_{\alpha_7}, [e_{\alpha_4}, e_{\beta_2}]] + [e_{\alpha_4}, [e_{\alpha_7}, e_{\beta_2}]] \\
&= e_{\delta_1} + e_{\delta_2} + 2e_{\delta_3}, \\
(\operatorname{ad} e)^2(e_{\beta_3}) &= [e_{\alpha_3}, [e_{\alpha_1}, e_{\beta_3}]] + [e_{\alpha_5}, [e_{\alpha_1}, e_{\beta_3}]] \\
&\quad + [e_{\alpha_7}, [e_{\alpha_1}, e_{\beta_3}]] + [e_{\alpha_1}, [e_{\alpha_5}, e_{\beta_3}]] + [e_{\alpha_7}, [e_{\alpha_5}, e_{\beta_3}]] \\
&\quad + [e_{\alpha_1}, [e_{\alpha_7}, e_{\beta_3}]] + [e_{\alpha_5}, [e_{\alpha_7}, e_{\beta_3}]] \\
&= e_{\delta_1} + 2e_{\delta_2} + 2e_{\delta_3} + 2e_{\delta_4}, \\
(\operatorname{ad} e)^2(e_{\beta_4}) &= [e_{\alpha_4}, [e_{\alpha_1}, e_{\beta_4}]] + [e_{\alpha_7}, [e_{\alpha_4}, e_{\beta_4}]] + [e_{\alpha_5}, [e_{\alpha_4}, e_{\beta_4}]] \\
&= 2[e_{\alpha_1}, [e_{\alpha_7}, e_{\beta_3}]] + [e_{\alpha_5}, [e_{\alpha_7}, e_{\beta_3}]] \\
&= 2[e_{\alpha_7}, [e_{\alpha_6}, e_{\beta_1}]] + e_{\delta_4} = 2e_{\delta_3} + e_{\delta_4}.
\end{aligned}$$

Since

$$\begin{vmatrix} 2 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 1 \end{vmatrix} = -7,$$

the map $(\operatorname{ad} e)^2 \colon M_{J,+}^{-2} \to M_{J,+}^{2}$ is one-to-one if $p \neq 7$. Note that if $p = 7$, then $R_J$ has type $A_{p-1}$. We shall handle this case later.

**2.21.** Let $R_J$ be of type $D_6 \subset E_7$. Then $B_J = B\backslash\{\alpha_1\}$, $\beta_J^1 = {}^{134321}_{2}$, $b_J^1 = 17$ and $\mathfrak{g}_J(k) = 0$ for $k > 2$. By [5, p. 175], there are two nonempty subsets $I \subset J$ for which $\mathfrak{p}_I \cap \mathfrak{l}_J^{(1)}$ is distinguished in $\mathfrak{l}_J^{(1)}$, namely, $\{4\}$ and $\{4, 6\}$. As

$$2\omega_4^J = {}^{048642}_{4}, \quad 2\omega_6^J = {}^{024442}_{2},$$

we have $(\lambda_{I,J}|\gamma) \equiv (2\rho_J|\gamma) \pmod 2$ for any $\gamma \in R$. But $(2\rho_J|\gamma) = 2\,\mathrm{ht}\,\gamma - 17$ is odd provided $\nu_1(\gamma) = 1$. It follows that all weights of $\lambda_e$ on $\mathfrak{g}_J(1)$ are odd. Since $\mathfrak{g}_J(2)$ is a trivial $L_J^{(1)}$-module, we derive that $M_J^{2(p-1)} = 0$. Thus $D_6 \subset E_7$ can be excluded by Lemma 2.7.

**2.22.** Now let $R_J$ be of type $E_6 \subset E_7$. In this case $B_J = B\backslash\{\alpha_1\}$ and $M_{J,+} = \mathfrak{g}_J(1)$ is an irreducible $L_J^{(1)}$-module with highest weight $\omega_1^J$.

Let $R_{\tilde{J}}$ be of type $E_6 \times A_1$ in $E_8$ (see 2.15). We suppose that $R_J \subset R_{\tilde{J}}$, $\mathfrak{l}_J^{(1)} \subset \mathfrak{l}_{\tilde{J}}^{(1)}$ and $L_J^{(1)} \subset L_{\tilde{J}}^{(1)}$. Denote by $\tilde{\mathfrak{g}}$ a Lie algebra of type $E_8$ that contains $\mathfrak{l}_{\tilde{J}}^{(1)}$. By (2.15), $\tilde{\mathfrak{g}}_{\tilde{J}}(2)$ is irreducible over $L_{\tilde{J}}^{(1)}$ and has highest weight $\omega_6^{\tilde{J}}$. Let $\tilde{e}$ denote a nilpotent element from Table 1. Then $\tilde{e} = e + e_{\alpha_8}$ where $e \in \mathfrak{l}_J^{(1)}$. Clearly $\lambda_{\tilde{e}}(t) = \lambda_e(t)h_{\alpha_8}(t)$ for each $t \in \mathbb{G}_m$. The $L_J^{(1)}$-modules $\tilde{\mathfrak{g}}_{\tilde{J}}(2)$ and $\mathfrak{g}_J(1)$ are dual to each other and $e_{\alpha_8}$ and $h_{\alpha_8}(t)$ both act trivially on $\tilde{\mathfrak{g}}_{\tilde{J}}(2)$. Therefore, we can apply a computation presented in (2.15) to conclude that $(\mathrm{ad}\,e)^2\colon \mathfrak{g}_J^{-2}(-1) \to \mathfrak{g}_J^2(-1)$ is bijective if $p \neq 3$ (recall that $\mathfrak{g}_J(-1) \cong \mathfrak{g}_J(1)^*$). By our remark in (2.12), it follows that

$$(\mathrm{ad}\,e)^2\colon M_{J,+}^{-2} \to M_{J,+}^2$$

is one-to-one.

**2.23.** Let $R_J$ be of type $A_5$ in $R \cong E_6$. Then $e$ is regular in $\mathfrak{l}_J^{(1)}$, $B_J = B\backslash\{\alpha_2\}$, $\beta_J^2 = {}^{12321}_{1}$ and $b_J^2 = 11$. Using this information and [4, VI, Table V] it is now easy to observe that $\Delta_2(k) = \varnothing$ for all $k > 0$. Therefore, this case can be excluded by applying Lemma 2.7.

If $R_J$ is of type $D_5$ in $R \cong E_6$, then $M_{J,+} = \mathfrak{g}_J(1)$. By conjugating $R_J$ by $w_0 \in W$ if necessary we obtain $B_J = B\backslash\{\alpha_1\}$. We have $\beta_J^1 = \tilde{\alpha}$, $b_J^1 = 12$, $(2\rho_J|\tilde{\alpha}) = 10$. If $e$ is not regular in $\mathfrak{l}_J^{(1)}$, then $e \in \mathfrak{p}_I \cap \mathfrak{l}_J^{(1)}$ where $I = \{4\}$ (see [5, p. 175]). Since $2\omega_4^J = {}^{03642}_{3}$, then $m_1(e) = (\lambda_{I,J}|\tilde{\alpha}) = 7 < 2(p-1)$ and Lemma 2.7 applies. Thus, we can suppose that $e = \sum_{i>1} e_{\alpha_i}$.

Using [4, VI, Table V] we get $\Delta_1(1) = \{\gamma_1, \gamma_2\}$ where $\gamma_1 = {}^{11111}_{0}$, $\gamma_2 = {}^{11110}_{1}$. Set $\delta_1 = {}^{11211}_{1}$, $\delta_2 = {}^{12210}_{1}$. Then

$$(\mathrm{ad}\,e)^2(e_{\gamma_1}) = [e_{\alpha_4}, [e_{\alpha_2}, e_{\gamma_1}]] = \pm e_{\delta_1},$$
$$(\mathrm{ad}\,e)^2(e_{\gamma_2}) = [e_{\alpha_3}, [e_{\alpha_4}, e_{\gamma_2}]] + [e_{\alpha_6}[e_{\alpha_4}, e_{\gamma_2}]] + [e_{\alpha_4}, [e_{\alpha_6}, e_{\gamma_2}]] = \pm 2e_{\delta_1} \pm e_{\delta_2}.$$

It follows that $(\mathrm{ad}\,e)^2\colon M_{J,+}^{-2} \to M_{J,+}^2$ is bijective for any prime $p$.

**2.24.** We now turn to the case when $R_J$ has rank $l-2$ in $R$. If $B_J = B\backslash\{\alpha_i, \alpha_j\}$ and $k > 0$, then $\mathfrak{g}_J(k)$ is spanned by all $e_\gamma$ such that $\nu_i(\gamma)+\nu_j(\gamma) = k$. Given $a, b \in \mathbb{Z}_+$ denote

$$\Delta_{i,j}(a, b) = \{\gamma \in R_+ | \nu_i(\gamma) = a, \nu_j(\gamma) = b, (2\rho_J|\gamma) = -2\}.$$

Let $R_J$ be of type $A_5$ in $R \cong E_7$. By conjugating $R_J$ by a suitable $w \in W$ one obtains $B_J = B\backslash\{\alpha_2, \alpha_7\}$. Using [4, VI, Table VI] we get $\beta_J^2 = {}^{123210}_{1}$, $\beta_J^7 = {}^{111111}_{0}$, $b_J^2 = 11$, $b_J^7 = 7$. Applying Lemma 2.9 yields $\Delta_{2,7}(1, 0) = \Delta_{2,7}(2, 1) = \Delta_{2,7}(2, 0) = \varnothing$ and $\Delta_{2,7}(1, 1) = \{\beta_1, \beta_2\}$ where

$$\beta_1 = {}^{112111}_{1}, \quad \beta_2 = {}^{012211}_{1}.$$

Put $\delta_1 = {}^{12221}_{1}$, $\delta_2 = {}^{112221}_{1}$, $e_{\delta_1} = [e_{\alpha_3}, [e_{\alpha_5}, e_{\beta_1}]]$, $e_{\delta_2} = [e_{\alpha_6}, [e_{\alpha_5}, e_{\beta_1}]]$. Without loss of generality we may assume that $e = \sum_{i \neq 2,7} e_{\alpha_i}$ and $[e_{\alpha_5}, e_{\beta_1}] = [e_{\alpha_1}, e_{\beta_2}]$. Then

$$(\operatorname{ad} e)^2(e_{\beta_1}) = [e_{\alpha_5}, [e_{\alpha_3}, e_{\beta_1}]] + [e_{\alpha_3}, [e_{\alpha_5}, e_{\beta_1}]] + [e_{\alpha_6}, [e_{\alpha_5}, e_{\beta_1}]] = 2e_{\delta_1} + e_{\delta_2},$$
$$(\operatorname{ad} e)^2(e_{\beta_2}) = [e_{\alpha_3}, [e_{\alpha_1}, e_{\beta_2}]] + [e_{\alpha_6}, [e_{\alpha_1}, e_{\beta_2}]] + [e_{\alpha_1}, [e_{\alpha_6}, e_{\beta_2}]] = e_{\delta_1} + 2e_{\delta_2}.$$

Since

$$\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \neq 0$$

if $p \neq 3$, we conclude that $(\operatorname{ad} e)^2 \colon M_{J,+}^{-2} \to M_{J,+}^{2}$ is bijective.

Now let $R_J$ be of type $D_5$ in $R \cong E_7$. By conjugating $R_J$ by a suitable $w \in W$ we get $B_J = B\backslash\{\alpha_6, \alpha_7\}$. Let $\bar{\mathfrak{g}}$ denote the subalgebra of type $E_6$ generated by $e_{\pm\alpha_i}$, $i \leq 6$. Then $\mathfrak{g}_J(1) = \bar{\mathfrak{g}}_J(1) \oplus Ke_{\alpha_7}$. Clearly, $Ke_{\alpha_7}$ is a trivial $L_J^{(1)}$-module. Using [4, VI, Table VI] it is easy to check that the $L_J^{(1)}$-modules $\bar{\mathfrak{g}}_J(1)$ and $\mathfrak{g}_J(2)$ are isomorphic. Applying the second part of (2.22) we obtain now that

$$(\operatorname{ad} e)^2 \colon \mathfrak{g}_J^{-2}(k) \to \mathfrak{g}_J^{2}(k)$$

is bijective for $k = 1, 2$. Clearly, $\beta_J^6 = {}^{123210}_{2}$, $\beta_J^7 = \alpha_7$, $b_J^6 = 12$, $b_J^7 = 2$.

The subspace $\mathfrak{g}_J(3)$ is spanned by all $e_\gamma$ with $\nu_6(\gamma) = 2$ and $\nu_1(\gamma) = 1$. By Lemma 2.9, $(2\rho_J|\tilde{\alpha}) = 34 - 2 \cdot 12 - 2 = 8 \leq 2(p-1)$. It follows that $m_3(e) = (\lambda_{I,J}|\tilde{\alpha}) < 2(p-1)$ provided $I \neq \varnothing$. Hence we may assume that $e = \sum_{i \leq 6} e_{\alpha_i}$. By Lemma 2.9, $\Delta_{6,7}(2, 1) = \{\gamma\}$ where $\gamma = {}^{123221}_{1}$. Since $(\operatorname{ad} e)^2(e_\gamma) = 2[e_{\alpha_2}, [e_{\alpha_5}, e_\gamma]] \neq 0$, we can exclude $D_5$ from our list.

Let $R_J$ be of type $D_6$ in $R \cong E_8$. In this case $B_J = B\backslash\{\alpha_1, \alpha_8\}$. For any $k > 0$, the $L_J^{(1)}$-module $\mathfrak{g}_J(k)$ is completely reducible. Moreover, the highest weights of the irreducible submodules of $\mathfrak{g}_J(k)$ lie in the set $\{0, \omega_2^J, \omega_3^J, \omega_7^J\}$ (see (2.10)). By (2.10), $(2\rho_J|\omega_2^J) = (2\rho_J|\omega_3^J) = 15$, $(2\rho_J|\omega_7^J) = 10 < 2(p-1)$. Reasoning as in (2.21) it can now be easily seen that $(\lambda_{I,J}|\omega_2^J) = (\lambda_{I,J}|\omega_3^J)$ is odd for any $I \subset J$ such that $\mathfrak{p}_I \cap \mathfrak{l}_J^{(1)}$ is distinguished in $\mathfrak{l}_J^{(1)}$. Summarizing we obtain that each $(\operatorname{Ad}\lambda_e)$-weight of $M_{J,+}$ is either odd or less than $2(p-1)$. But then $M_J^{2(p-1)} = 0$ and, by Lemma 2.7, $\lambda_e$ is a Dynkin torus for $e \in \mathfrak{p}_I \cap \mathfrak{l}_J^{(1)}$.

If $R_J$ has type $E_6$ in $R \cong E_8$, then $B_J = B\backslash\{\alpha_7, \alpha_8\}$. Clearly, $M_J$ is a completely reducible $L_J^{(1)}$-module. Let $E_J(\omega)$ denote the (unique) irreducible $L_J^{(1)}$-module with highest weight $\omega = \sum_{i \in J} a_i \omega_i^J$ where $a_i \in \mathbb{Z}_+$. Let $E_J(\omega)_s$ be the weight space of $E_J(\omega)$ corresponding to weight $s \in X(\lambda_e) \cong \mathbb{Z}$ under the action of $\lambda_e \subset L_J^{(1)}$ on $E_J(\omega)$. The Lie algebra $\mathfrak{l}_J^{(1)} = \operatorname{Lie}(L_J^{(1)})$ acts on $E_J(\omega)$ via the differential $d\rho$ of a rational representation $\rho \colon L_J^{(1)} \to GL(E_J(\omega))$.

If $V$ is a nontrivial irreducible $L_J^{(1)}$-submodule of $M_J$, then either $V \cong E_J(\omega_1^J)$ or $V \cong E_J(\omega_6^J)$ (see 2.10)). Combining (2.22) with the computation

in (2.15) one easily sees that the map $(d\rho(e))^2\colon E_J(\omega_i^J)_{-2} \to E_J(\omega_i^J)_2$ is a bijection if $i = 1, 6$, $p \neq 3$ (note that $E_J(\omega_1^J)$ is contragradient to $E_J(\omega_6^J)$). From this it is immediate that $(\operatorname{ad} e)^2\colon M_J^{-2} \to M_J^2$ is bijective.

**2.25.** An argument employed in proving Lemma 2.7 shows that, if $R_J$ has no components of type $A_{p-1}$ and $(\operatorname{ad} e)^2\colon M_J^{-2} \to M_J^2$ is a bijection, then so is $(\operatorname{ad} e)^2\colon \mathfrak{g}_{-2} \to \mathfrak{g}_2$. Applying Lemma 2.3 shows now that in all examined cases $\lambda_e$ is a Dynkin torus for $e$.

It remains to consider the following subsystems $R_J \subset R$:

$$A_4,\ A_4 \times A_1,\ A_4 \times A_2 \text{ for } R \cong E_6 \text{ or } E_7, \qquad p \geq 5;$$
$$A_6 \text{ for } R \cong E_7 \text{ or } E_8,\ A_6 \times A_1 \text{ for } R \cong E_8, \qquad p \geq 7.$$

In all these cases we can suppose that $e = \sum_{i \in J} e_{\alpha_i}$. If $p > 5$ (resp., $p > 7$) and $R_J$ is from the first line (resp., from the second line), then $M_J^{2(p-1)} = 0$ (to obtain this one can argue as in (2.10)). Since in this case $R_J$ has no components of type $A_{p-1}$, $\lambda_e$ is a Dynkin torus for $e$ by Lemma 2.7.

Thus, in what follows we may assume that $p = 5$ (resp., $p = 7$) for the subsystems from the first line (resp., from the second line). Note that, in any event, $e^{[p]} = f^{[p]} = 0$.

**2.26.** Let $H = \operatorname{diag}(t, t^{-1})$ be the standard Cartan subgroup of the algebraic group $SL_2$ over $K$. Let $F$ denote the ideal of $H$ in the algebra $A$ of all regular functions on $SL_2$. The infinitesimal neighborhood of $H$ in $SL_2$ is defined as the group scheme $(SL_2)_{(H)}$ corresponding to the algebra $A/F^p$.

The structure of an $(SL_2)_{(H)}$-module in a finite-dimensional vector space $V$ is given by a triple $(\theta, X, Y)$ where $\theta$ is a rational representation of $\mathbb{G}_m$ in $V$ and $X, Y$ are endomorphisms of $V$ such that

$$\begin{aligned} &X^p = Y^p = 0, \\ &\theta(t)X\theta(t)^{-1} = t^2 X, \\ &\theta(t)Y\theta(t)^{-1} = t^{-2} Y, \\ &[X, Y] = d\theta, \end{aligned} \tag{1}$$

where $d\theta$ is the differential of $\theta$.

It is well known (see, for example, [7]) that for any $n = 0, 1, \dots, p-1$ and any $k \in \mathbb{Z}$ there exists a unique irreducible $(SL_2)_{(H)}$-module $V_{n,k}$ with highest weight $n + kp$. Moreover, $V_{n,k} \cong V_{n,0} \otimes \Pi^k$ where $\Pi$ is the one-dimensional $(SL_2)_{(H)}$-module corresponding to the triple $(t^p, 0, 0)$. Any simple $(SL_2)_{(H)}$-module is isomorphic to one of $V_{n,k}$. Since the action of $(SL_2)_{(H)}$ on $V_{n,0}$ is induced by the $n$th symmetric power of the standard representation of $SL_2(K)$, the weights of $V_{n,k}$ are $n+kp, n-2+kp, \dots, -n+2+kp, -n+kp$.

For any $k \in \mathbb{Z}$, the module $V_{p-1,k}$ is projective. For any $n = 0, 1, \dots, p-2$ and any $k \in \mathbb{Z}$ there exists a $2p$-dimensional projective indecomposable $(SL_2)_{(H)}$-module $P_{n,k}$ whose socle and cosocle are both isomorphic to $V_{n,k}$. The highest (resp., lowest) weight of $P_{n,k}$ is equal to $(k+1)p + p - n - 2$ (resp., $(k-1)p - (p-n-2)$). Any projective $(SL_2)_{(H)}$-module is isomorphic to a direct sum of indecomposable projective modules listed above (see [7] for more detail).

Given an $(SL_2)_{(H)}$-module $M$ denote by $X(M)$ the set of weights of $M$ relative to $\theta(\mathbb{G}_m)$. Let $M_s$ denote the weight component of $M$ corresponding to weight $s \in X(M)$.

**Lemma 2.10.** *Suppose that $M$ is a projective $(SL_2)_{(H)}$-module such that $X(M) = -X(M)$ and $s < 2p-1$ for each $s \in X(M)$. Then* $\operatorname{Ker} X \subset \sum_{s \geq 0} M_s$.

*Proof*. Since $M$ is projective, it is isomorphic to a direct sum of some of $V_{p-1,k}$'s and $P_{n,k}$'s (with multiplicities). If $V_{p-1,r}$ or $P_{n,r}$ with $r > 0$ (resp., $r < 0$) has nonzero multiplicity in $M$, then there is $d \in X(M)$ with $d \geq 2p-1$ (resp., $d \leq 1-2p$). As $X(M)$ is symmetric, this violates the assumption that $s < 2p-1$ for each $s \in X(M)$. Therefore, any indecomposable direct summand of $M$ is isomorphic either to $V_{p-1,0}$ or to $P_{m,0}$ where $0 \leq m \leq p-2$.

Clearly, $V_{p-1,0} \cap \operatorname{Ker} X = (V_{p-1,0})_{p-1}$. Using [7, p. 600] one sees that $P_{m,0} \cap \operatorname{Ker} X \subset (P_{m,0})_{2p-m-2} \oplus (P_{m,0})_m$. This implies that

$$\operatorname{Ker} X \subset \sum_{s \geq 0} M_s$$

as desired. □

**2.27.** Since the triple $(\lambda_e, \operatorname{ad} e, \operatorname{ad} f)$ restricted to $M_{J,+}$ satisfies the conditions (2.26 (1)), we may regard $M_{J,+}$ as an $(SL_2)_{(H)}$-module. By (2.13), $X(M_{J,+}) = -X(M_{J,+})$.

If $R_J \cong A_6 \times A_1$, then $B_J = B \backslash \{\alpha_3\}$. Hence $M_{J,+} = \mathfrak{g}_J(1) \oplus \mathfrak{g}_J(2) \oplus \mathfrak{g}_J(3) \oplus \mathfrak{g}_J(4)$. Looking over [4, VI, Table VII] one obtains that $\mathfrak{g}_J(1)$, $\mathfrak{g}_J(2)$, $\mathfrak{g}_J(3)$ and $\mathfrak{g}_J(4)$ are irreducible over $L_J^{(1)}$ and have highest weights $\omega_1^J + \omega_7^J$, $\omega_5^J$, $\omega_1^J + \omega_2^J$ and $\omega_8^J$ respectively.

Let $N_r$ denote the standard $SL_r(K)$-module of dimension $r$. As $L_J^{(1)} \cong SL_2(K) \times SL_7(K)$, one has the following module isomorphisms:

$$\mathfrak{g}_J(1) \cong N_2 \otimes (\overset{2}{\wedge} N_7)^*, \quad \mathfrak{g}_J(2) \cong \overset{3}{\wedge} N_7, \quad \mathfrak{g}_J(3) \cong N_2 \otimes N_7, \quad \mathfrak{g}_J(4) \cong N_7^*.$$

One can view $N_2$ (resp., $N_7$) as a natural $L_J^{(1)}$-module via the trivial action of the second (resp., the first) component of $L_J^{(1)} \cong SL_2(K) \times SL_7(K)$. Let $\sigma_2$ (resp., $\sigma_7$) denote the corresponding representation of $L_J^{(1)}$. The differential $d\sigma_2$ (resp, $d\sigma_7$) restricted to the principal $sl_2$-triple $Ke \oplus Kh \oplus Kf \subset \operatorname{Lie}(L_J^{(1)})$ together with the rational representation $\sigma_2 \circ \lambda_e$ (resp., $\sigma_7 \circ \lambda_e$) of $\mathbb{G}_m$ defines a representation $\theta_2 = (\sigma_2 \circ \lambda_e, d\sigma_2(e), d\sigma_2(f))$ (resp., $\theta_7 = (\sigma_7 \circ \lambda_e, d\sigma_7(e), d\sigma_7(f))$) of the group scheme $(SL_2)_{(H)}$ in the vector space $N_2$ (resp., $N_7$). It is immediate from the above remarks that

$$(\lambda_e, \operatorname{ad} e, \operatorname{ad} f)|_{M_{J,+}} \cong \theta_2 \otimes (\overset{2}{\wedge} \theta_7)^* \oplus \overset{3}{\wedge} \theta_7 \oplus \theta_2 \otimes \theta_7 \oplus \theta_7^*.$$

Since $N_7$ is an irreducible $(SL_2)_{(H)}$-module and $\dim N_7 = p$, we conclude that $N_7 \cong V_{p-1,k}$ for some $k \in \mathbb{Z}$ (see (2.26)). But then $N_7$ and $N_7^*$ are both projective as $(SL_2)_{(H)}$-modules. This implies that $M_{J,+}$ is projective over $(SL_2)_{(H)}$ (bear in mind that $\overset{2}{\wedge} \theta_7$ and $\overset{3}{\wedge} \theta_7$ are direct summands of $\theta_7 \otimes \theta_7$ and $\theta_7 \otimes \theta_7 \otimes \theta_7$ respectively).

An easy calculation based on our remarks in (2.10) shows that $m_1(e) = 11$, $m_2(e) = 12$, $m_3(e) = 7$ and $m_4(e) = 6$. Hence $s < 2p-1 = 13$ for any

$s \in X(M_{J,+})$. Applying Lemma 2.10 we get

$$\mathfrak{z}_{\mathfrak{g}}(e) \cap M_{J,+} \subset \sum_{i \geq 0} \mathfrak{g}_i.$$

Since $M_{J,-}$ is contragradient to $M_{J,+}$ in the category of finite-dimensional $(SL_2)_{(H)}$-modules, Lemma 2.10 applies to $M_{J,-}$ as well yielding

$$\mathfrak{z}_{\mathfrak{g}}(e) \cap M_J \subset \sum_{i \geq 0} \mathfrak{g}_i.$$

By construction, $\mathfrak{l}_J \cap \sum_{i \geq 0} \mathfrak{g}_i = \mathfrak{p}_I \cap \mathfrak{l}_J$. Since $e$ is a Richardson element of $\mathfrak{p}_I \cap \mathfrak{l}_J$, then $\mathfrak{z}_{\mathfrak{g}}(e) \cap \mathfrak{l}_J \subset \sum_{i \geq 0} \mathfrak{g}_i$ in view of (2.2) (recall that $\mathfrak{l}_J$ admits a nondegenerate trace form). Therefore, $\mathfrak{z}_{\mathfrak{g}}(e) \subset \sum_{i \geq 0} \mathfrak{g}_i$ and so $\lambda_e$ is a Dynkin torus for $e$.

**2.28.** One can analyze the remaining four cases repeating almost verbatim the argument from (2.27). Details are left to the reader.

If $R$ is of type $G_2$, then $B_J = \{\alpha_i\}$ where $i \in \{1, 2\}$. In this case $\lambda_e(t) = h_i(t)$ for each $t \in \mathbb{G}_m$. As $p > 3$,

$$\mathfrak{z}_{\mathfrak{g}}(e_{\alpha_i}) \subseteq \mathfrak{t} \bigoplus \sum_{(\gamma|\alpha_i) \geq 0} K e_\gamma = \sum_{i \geq 0} \mathfrak{g}_i$$

whence $\lambda_e$ is a Dynkin torus for $e = e_{\alpha_i}$.

If $R$ is of type $F_4$, then $G \cong \mathrm{Aut}(\mathfrak{g})$. We regard $\mathfrak{g}$ as a subalgebra of a Lie algebra $\tilde{\mathfrak{g}}$ of type $E_6$. Let $\sigma$ denote the outer automorphism of $\tilde{\mathfrak{g}}$ defined by extending

$$\sigma(e_{\pm\alpha_1}) = e_{\pm\alpha_6}, \quad \sigma(e_{\pm\alpha_6}) = e_{\pm\alpha_1}, \quad \sigma = (e_{\pm\alpha_3}) = e_{\pm\alpha_5},$$
$$\sigma(e_{\pm\alpha_5}) = e_{\pm\alpha_3}, \quad \sigma(e_{\pm\alpha_2}) = e_{\pm\alpha_2}, \quad \sigma(e_{\pm\alpha_4}) = e_{\pm\alpha_4}.$$

It is well known that $\mathfrak{g}$ is isomorphic to the subalgebra $\tilde{\mathfrak{g}}^\sigma = \{x \in \tilde{\mathfrak{g}} | x^\sigma = x\}$. Moreover, the elements $e_1 = e_{\alpha_2}$, $e_2 = e_{\alpha_4}$, $e_3 = e_{\alpha_3} + e_{\alpha_5}$ and $e_4 = e_{\alpha_1} + e_{\alpha_6}$ can be viewed as root elements corresponding to the simple roots of $R$ in Bourbaki's indexing.

If $e$ is a regular nilpotent element of $\mathfrak{l}_J^{(1)} \subset \mathfrak{g}$ where $J \subset \{1, 2, 3, 4\}$, then, up to conjugacy in $G$, $e = \sum_{i \in J} e_i$. It is clear from the above that there exists $\widetilde{J} \subset \{1, 2, \ldots, 6\}$ such that $e = \sum_{i \in \widetilde{J}} e_{\alpha_i}$. Therefore, $e$ is a regular nilpotent element of the standard Levi subalgebra of $\tilde{\mathfrak{g}}$ associated to the subset $\widetilde{J}$.

We may assume that $\tilde{\mathfrak{g}} = \mathrm{Lie}(\widetilde{G})$ where $\widetilde{G}$ is a simply connected group of type $E_6$. It has already been proved that $\lambda_e \subset X_*(\widetilde{G})$ is a Dynkin torus for $e \in \tilde{\mathfrak{g}}$. The automorphism $\sigma$ is induced by the nontrivial symmetry of the Dynkin diagram of type $E_6$. Clearly, the subset $B_{\widetilde{J}} = \{\alpha_i | i \in \widetilde{J}\}$ is $\sigma$-stable. Since the scalar product $(\,|\,)$ is $\sigma$-stable as well, $\sigma$ acts on the set $\{\omega_i^{\widetilde{J}} | i \in \widetilde{J}\}$. It follows that $\sigma(\rho_{\widetilde{J}}) = \rho_{\widetilde{J}}$. As

$$(\mathrm{Ad}\,\lambda_e(t)) \cdot e_\gamma = t^{(2\rho_{\widetilde{J}}|\gamma)} e_\gamma$$

for each $\gamma \in \{\pm\alpha_i | 1 \leq i \leq 6\}$ we conclude that

$$\sigma(\mathrm{Ad}\,\lambda_e(t))\sigma^{-1} = \mathrm{Ad}\,\lambda_e(t)$$

for each $t \in \mathbb{G}_m$. Therefore, $\operatorname{Ad}\lambda_e$ acts on $\tilde{\mathfrak{g}}^\sigma = \mathfrak{g}$. Let $\bar{\lambda}_e \colon \mathbb{G}_m \to \operatorname{Aut}\mathfrak{g}$ denote the homomorphism induced by restricting $\operatorname{Ad}\lambda_e$ to $\tilde{\mathfrak{g}}^\sigma$. Clearly, $\bar{\lambda}_e \in X_*(G)$ and $\bar{\lambda}_e(t)\cdot e = t^2 e$ for any $t \in \mathbb{G}_m$. Let $\tilde{\mathfrak{g}}_i$ (resp., $\mathfrak{g}_i$) be the weight component of $\operatorname{Ad}\lambda_e$ (resp., $\operatorname{Ad}\bar{\lambda}_e$) corresponding to weight $i \in \mathbb{Z}$. Obviously, $\mathfrak{g}_i = \tilde{\mathfrak{g}}_i \cap \mathfrak{g}$. But then

$$\mathfrak{z}_{\mathfrak{g}}(e) = \mathfrak{z}_{\tilde{\mathfrak{g}}}(e) \cap \mathfrak{g} \subseteq \left(\sum_{i\geq 0} \tilde{\mathfrak{g}}_i\right) \cap \mathfrak{g} = \sum_{i\geq 0} \mathfrak{g}_i.$$

Therefore, $\bar{\lambda}_e$ is a Dynkin torus for $e \in \mathfrak{g}$.

By [5, pp. 174, 175], any distinguished parabolic subalgebra of a Lie algebra of type $A_1$, $A_1 \times A_1$, $A_2 \times A_1$, $B_2$ or $B_3$ is a Borel subalgebra. Looking over [4, VI, Table VIII] we conclude now that the Levi subalgebra $\mathfrak{l}_{J_0}$ corresponding to the subset $J_0 = \{2, 3, 4\}$ is the only standard Levi subalgebra of $\mathfrak{g}$ that contains a nonregular distinguished nilpotent element. Let $e$ be such an element. By (2.2) and [5, p. 174], we may assume that $e \in \mathfrak{l}_{J_0} \cap \mathfrak{g}_{I_0}(2)$ where $I_0 = \{3\}$.

Using [4, VI, Table VIII] we get $\alpha_2 = \varepsilon_3 - \varepsilon_4$, $\alpha_3 = \varepsilon_4$, $\alpha_4 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)$. Computation shows that $\omega_2^J = \frac{3}{2}\alpha_2 + 2\alpha_3 + \alpha_4$ and $\omega_4^J = \alpha_2 + 2\alpha_3 + 2\alpha_4$. Therefore, $\lambda_{I_0, J_0} = 2(\omega_2^J + \omega_4^J) = 5\alpha_2 + 8\alpha_3 + 6\alpha_4 = m_2\alpha_2 + 2m_3\alpha_3 + 2m_4\alpha_4$ (as $(\alpha_2|\alpha_2) = 2$ and $(\alpha_3|\alpha_3) = (\alpha_4|\alpha_4) = 1$). This implies

$$\lambda_e(t) = h_2(t^{m_2})h_3(t^{m_3})h_4(t^{m_4}) = h_2(t^5)h_3(t^4)h_4(t^3).$$

As $\nu_1(\gamma) \leq 2$ for each $\gamma \in R_+$, $M_{J_0,+} = \mathfrak{g}_{J_0}(1) \oplus \mathfrak{g}_{J_0}(2)$. It is immediate from [4, VI, Table VIII] that $\mathfrak{g}_{J_0}(2)$ is trivial over $L_{J_0}^{(1)}$ and the $L_{J_0}^{(1)}$-module $\mathfrak{g}_{J_0}(1)$ is generated by the highest weight vector $e_\gamma$ where $\gamma = 1\,3\,4\,2$.

Since $(\gamma|\alpha_3) = (\gamma|\alpha_4) = 0$ and $\langle \gamma, \alpha_2 \rangle = 1$, we obtain

$$(\operatorname{Ad}\lambda_e(t)) \cdot e_\gamma = t^5 e_\gamma.$$

Consequently, $m_1(e) = 5$, $m_2(e) = 0$. But then $M_{J_0}^{2(p-1)} = 0$. Applying Lemma 2.7 we obtain that $\lambda_e$ is a Dynkin torus for $e \in \mathfrak{l}_{J_0}$.

The proof of Theorem 2.5 is now complete.

## References


1. P. Bala and R. W. Carter, *Classes of unipotent elements in simple algebraic groups.* I, Math. Proc. Cambridge Philos. Soc. **79** (1976), 401–425.
2. ——, *Classes of unipotent elements in simple algebraic groups.* II, Math. Proc. Cambridge Philos. Soc. **80** (1976), 1–18.
3. A. Borel, *Linear algebraic groups*, Benjamin, New York, 1969.
4. N. Bourbaki, *Groupes et algèbres de Lie*, Chapitres IV–VI, Hermann, Paris, 1968.
5. R. W. Carter, *Finite groups of Lie type, conjugacy classes and complex characters*, Wiley, Chichester and New York, 1985.
6. C. W. Curtis, *Noncommutative extensions of Hilbert rings*, Proc. Amer. Math. Soc. **4** (1953), 945–955.
7. S. Gelfand and D. Kazhdan, *Examples of tensor categories*, Invent. Math. **109** (1992), 595–617.
8. N. Jacobson, *Lie algebras*, Interscience, New York, 1962.
9. V. G. Kac, *Some remarks on nilpotent orbits*, J. Algebra **64** (1980), 190–213.
10. G. R. Kempf, *Instability in invariant theory*, Ann. of Math. (2) **108** (1978), 299–316.

11. F. C. Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, Math. Notes, no. 31, Princeton Univ. Press, Princeton, NJ, 1984.

12. L. Ness, *A stratification of the null-cone via the moment map*, Amer. J. Math. **106** (1984), 1281–1329.

13. K. Pommerening, *Über die unipotenten Klassen reduktiver Gruppen*, J. Algebra **49** (1977), 525–536.

14. ——, *Über die unipotenten Klassen reduktiver Gruppen.* II, J. Algebra **65** (1980), 373–398.

15. A. Premet, *Irreducible representations of Lie algebras of reductive groups and the Kac-Weisfeiler conjecture*, Invent. Math. (to appear).

16. G. Rousseau, *Immeubles sphériques et théorie des invariants*, C. R. Acad. Sci. Paris **286** (1978), 247–250.

17. P. Slodowy, *Simple singularities and simple algebraic groups*, Lecture Notes in Math., vol. 815, Springer, Berlin and Heidelberg, 1980.

18. ——, *Die Theorie der optimalen Einparametergruppen für instabile Vektoren*, Algebraic Transformation Groups and Invariant Theory, Birkhäuser, Basel, Boston, and Berlin, pp. 115–131.

19. N. Spaltenstein, *On unipotent and nilpotent elements of groups of type* $E_6$, J. London Math. Soc. (2) **27** (1983), 413–420.

20. T. A. Springer, *The Steinberg function of a finite Lie algebra*, Invent. Math. **58** (1980), 211–215.

21. T. A. Springer and R. Steinberg, *Conjugacy classes*, Seminar on Algebraic Groups and Related Topics, Lecture Notes in Math., vol. 131, Springer, Berlin, Heidelberg, and New York, 1970, pp. 167–266.

22. R. Steinberg, *Lectures on Chevalley groups*, Yale Univ., New Haven, Conn., 1968.

23. B. Ju. Weisfeiler and V. G. Kac, *Irreducible representations of Lie p-algebras*, Funct. Anal. Appl. **5** (1971), 28–36.

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