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THE INDEX OF DETERMINACY FOR MEASURES AND THE ℓ^2-NORM OF ORTHONORMAL POLYNOMIALS

CHRISTIAN BERG AND ANTONIO J. DURAN

ABSTRACT. For determinate measures μ having moments of every order we define and study an index of determinacy which checks the stability of determinacy under multiplication by even powers of $|t-z|$ for z a complex number. Using this index of determinacy, we solve the problem of determining for which $z \in \mathbb{C}$ the sequence $(p_n^{(m)}(z))_n$ ($m \in \mathbb{N}$) belongs to ℓ^2, where $(p_n)_n$ is the sequence of orthonormal polynomials associated with the measure μ.

1. INTRODUCTION

By $\mathscr{M}^*$ we denote the set of positive measures μ on $\mathbb{R}$ having moments of every order and infinite support. A measure $\mu \in \mathscr{M}^*$ is determinate if no other measure has the same moments as those of μ, otherwise μ is indeterminate. With $\mu \in \mathscr{M}^*$ we can associate the sequence $(p_n)_n$ of orthonormal polynomials. We always assume that p_n is of degree n with positive leading coefficient, and this condition together with orthonormality determines $(p_n)_n$ uniquely from μ. We stress that $(p_n)_n$ depends only on the class of all the measures having the same moments as μ. If p is a non-negative polynomial, we denote by $p\mu$ the measure with density p with respect to μ.

It is well known that if a measure μ is indeterminate, then for all $z \in \mathbb{C}$ the sequence $(p_n(z))_n$ belongs to ℓ^2 and that if μ is determinate, the sequence $(p_n(z))_n$ belongs to ℓ^2 only when $\mu(\{z\}) > 0$. Moreover, it is not hard to prove (see (2.2) below) that if μ is indeterminate, then for all $m \in \mathbb{N}$ and $z \in \mathbb{C}$ the sequence $(p_n^{(m)}(z))_n$ belongs to ℓ^2. However, it does not seem to be known when $(p_n^{(m)}(z))_n \in \ell^2$ for $m \geq 1$ and μ a determinate measure. In this paper we shall solve this problem. To do that, we need to define and study an index of determinacy with respect to a point $z \in \mathbb{C}$ for determinate measures $\mu \in \mathscr{M}^*$. It is defined as

$$\operatorname{ind}_z(\mu) = \sup\{k \in \mathbb{N} : |t-z|^{2k}\mu \quad \text{is determinate}\} \tag{1.1}$$

where $\mathbb{N} = \{0, 1, 2, \cdots\}$. For a similar index of determinacy for measures in $\mathscr{M}^*$ supported by the positive half-line and related to determinacy in the

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sense of Stieltjes, see [BT1], [BT2]. The two indices are compared at the end of Section 3.

Concerning this determinacy index, we shall prove (Section 3) that $\operatorname{ind}_z(\mu)$ is constant $k \in \mathbb{N} \cup \{\infty\}$ for z in the complement of the support of μ and constant equal to $k+1$ for z in the support of μ. If μ is a non-discrete determinate measure it turns out that $\operatorname{ind}_z(\mu) = \infty$ for all $z \in \mathbb{C}$. Also, we shall prove a characterization theorem for those measures μ satisfying $\operatorname{ind}_z(\mu) < \infty$: Indeed, for $\mu \in \mathscr{M}^*$, we can consider the set of polynomials with complex coefficients $\mathbb{C}[t]$ as a subspace of $L^2(\mu)$. We recall the theorem of M. Riesz (cf. [A, p. 43]): If μ is determinate, then $\mathbb{C}[t]$ is dense in $L^2(\mu)$ and if μ is indeterminate, then $\mathbb{C}[t]$ is dense in $L^2(\mu)$ if and only if μ is Nevanlinna extremal (N-extremal in short). The N-extremal measures are discrete, and if μ is obtained from an N-extremal by removing the mass of $k+1$ points in the support of this N-extremal measure, then μ is determinate with

$$\operatorname{ind}_z(\mu) = \begin{cases} k & \text{for } z \notin \operatorname{supp}(\mu), \\ k+1 & \text{for } z \in \operatorname{supp}(\mu) \end{cases} \tag{1.2}$$

(cf. Theorem 3.6 below). We shall furthermore show that this is the only possibility for a measure with finite index of determinacy (cf. Theorem 3.9).

Finally, concerning the problem of determining for which $z \in \mathbb{C}$ the sequence $(p_n^{(m)}(z))_n$ belongs to ℓ^2, we shall prove that if for some $z \in \mathbb{C}$ the index of determinacy of μ is not finite, then for all $m \geq 1$ and $z \in \mathbb{C}$ the sequence $(p_n^{(m)}(z))_n$ does not belong to ℓ^2. Otherwise, if the index of determinacy of μ satisfies (1.2), then for $k \geq 1$ once again $(p_n^{(m)}(z))_n$ does not belong to ℓ^2 for all $m \geq 1$ and $z \in \mathbb{C}$. However, for $k = 0$ we find that there exist infinitely many numbers z satisfying $(p_n^{(m)}(z))_n \in \ell^2$ and these numbers are real. Moreover, $(p_n^{(m)}(z))_n \in \ell^2$ if and only if $F_\mu^{(m)}(z) = 0$ where F_μ is the entire function

$$F_\mu(w) = e^{-\left(\sum_n \frac{1}{x_n} w\right)} \prod_{n=0}^{\infty} \left(1 - \frac{w}{x_n}\right) e^{\frac{w}{x_n}}$$

and $\{x_n : n \in \mathbb{N}\}$ is the support of μ. This function F_μ is the uniquely determined entire function of minimal exponential type having $\operatorname{supp}(\mu)$ as its set of zeros and satisfying $F_\mu(0) = 1$ (cf. Theorem 4.4 below). In the above formulation we tacitly assume $0 \notin \operatorname{supp}(\mu)$. If, however, $0 \in \operatorname{supp}(\mu)$, the above expression for F_μ shall be multiplied with w and $\{x_n : n \in \mathbb{N}\} = \operatorname{supp}(\mu) \setminus \{0\}$.

2. Preliminaries

Let $(s_n)_n$ be an indeterminate Hamburger moment sequence, V the set of measures $\mu \in \mathscr{M}^*$ having $(s_n)_n$ as sequence of moments, $(p_n)_n$ the corresponding orthonormal polynomials and $(q_n)_n$ the associated polynomials of the second kind, i.e.

$$q_n(z) = \int \frac{p_n(z) - p_n(t)}{z - t}\, d\mu(t), \quad z \in \mathbb{C},\ \mu \in V. \tag{2.1}$$

It is well known that $(p_n(z))_n \in \ell^2$, $(q_n(z))_n \in \ell^2$ for all $z \in \mathbb{C}$, but more is known: the series $\sum_n |p_n(z)|^2$, $\sum_n |q_n(z)|^2$ converge uniformly on compact

subsets of $\mathbb{C}$. By the following lemma we see that the series

$$\sum_n |p_n^{(m)}(z)|^2, \quad \sum_n |q_n^{(m)}(z)|^2 \tag{2.2}$$

also converge uniformly on compact subsets of $\mathbb{C}$.

Lemma 2.1. *Let $f_n : \Omega \to \mathbb{C}$ be a sequence of holomorphic functions on a domain $\Omega \subset \mathbb{C}$ such that $\sum_n |f_n(z)|^2$ converges uniformly on compact subsets of $\mathbb{C}$. Then the vector-valued function $F : \Omega \to \ell^2$ defined by $F(z) = (f_n(z))_n$ is holomorphic with $F^{(m)}(z) = (f_n^{(m)}(z))_n$, $m \geq 0$.*

Proof. By assumption $\|F\|$ is continuous, and F is weakly holomorphic since for $a \in \ell^2$

$$\lim_n \sum_{k=0}^{n} f_k(z)\overline{a_k} = \langle F(z), a\rangle$$

uniformly on compact subsets of Ω. By a standard result F is automatically holomorphic (quoting K. Hoffman "*any two reasonable-sounding definitions of a holomorphic function with values in a Banach space are equivalent*").

The convergence of the series (2.2) implies that the m-th derivative of a function $f \in L^2(\sigma)$ (σ being an N-extremal measure) can be defined as:

Proposition 2.2. *Let σ be an N-extremal indeterminate measure. For $m \in \mathbb{N}$, $z \in \mathbb{C}$ the mapping $p \to p^{(m)}(z)$ extends from $\mathbb{C}[t]$ to a continuous linear functional $\delta_z^{(m)} : L^2(\sigma) \to \mathbb{C}$ given by*

$$\delta_z^{(m)}(f) = \sum_{n=0}^{\infty} p_n^{(m)}(z) \int f(t)p_n(t)\,d\sigma(t) \quad \text{for } f \in L^2(\sigma), \tag{2.3}$$

where $(p_n)_n$ are the orthonormal polynomials corresponding to σ.

Proof. The orthonormal polynomials $(p_n)_n$ form an orthonormal basis for the Hilbert space $L^2(\sigma)$ when σ is N-extremal. By (2.2) it is clear that the expression (2.3) defines a continuous linear functional on $L^2(\sigma)$. For $f = p_n$ we find $\delta_z^{(m)}(p_n) = p_n^{(m)}(z)$, so by linearity $\delta_z^{(m)}(p) = p^{(m)}(z)$ for all $p \in \mathbb{C}[t]$.

In the next proposition we will calculate a suitable expresion for $\delta_u^{(m)}(\frac{1}{t-u})$, $u \notin \operatorname{supp}(\sigma)$. For this we shall make use of the entire functions on $\mathbb{C} \times \mathbb{C}$ defined by

$$\begin{cases} A(u, v) = (u - v)\displaystyle\sum_{k=0}^{\infty} q_k(u)q_k(v), \\ B(u, v) = -1 + (u - v)\displaystyle\sum_{k=0}^{\infty} p_k(u)q_k(v), \\ D(u, v) = (u - v)\displaystyle\sum_{k=0}^{\infty} p_k(u)p_k(v) \end{cases} \tag{2.4}$$

satisfying

$$A(u, v)D(u, v) + B(u, v)B(v, u) = 1. \tag{2.5}$$

Except for a change of sign these functions occur in [BuCa]. Setting $A(u) = A(u, 0)$, $B(u) = B(u, 0)$, $C(u) = -B(0, u)$ and $D(u) = D(u, 0)$, we obtain the Nevanlinna matrix

$$\begin{pmatrix} A & C \\ B & D \end{pmatrix} \tag{2.6}$$

of entire functions associated with the indeterminate moment problem. The determinant of (2.6) is identically one. The N-extremal solutions to the moment problem for $(s_n)_n$ are given via the Nevanlinna matrix as the measures $(\sigma_t)_{t\in\mathbb{R}\cup\{\infty\}}$ determined by the formula

$$\int \frac{d\sigma_t(x)}{u-x} = \frac{A(u)t - C(u)}{B(u)t - D(u)} \tag{2.7}$$

which holds for all $u \in \mathbb{C} \setminus \operatorname{supp}(\sigma_t)$, and $\operatorname{supp}(\sigma_t)$ is the discrete set of zeros of the entire function $B(u)t - D(u)$ (for $t = \infty$ this shall be interpreted by $B(u)$). We recall from [BuCa] (or from [A, p. 123]) that

$$B(u, v) = B(u)C(v) - A(v)D(u), \tag{2.8}$$

$$D(u, v) = B(u)D(v) - B(v)D(u), \tag{2.9}$$

from which we easily get ($m \geq 0$)

$$(m+1)\sum_{k=0}^{\infty} p_k^{(m)}(u)q_k(u) = B^{(m+1)}(u)C(u) - D^{(m+1)}(u)A(u), \tag{2.10}$$

$$(m+1)\sum_{k=0}^{\infty} p_k^{(m)}(u)p_k(u) = B^{(m+1)}(u)D(u) - D^{(m+1)}(u)B(u). \tag{2.11}$$

For the N-extremal measure σ_t, $t \in \mathbb{R} \cup \{\infty\}$ and $m \in \mathbb{N}$, (2.7) gives

$$\int \frac{p_k(x)d\sigma_t(x)}{u-x} = p_k(u)\frac{A(u)t - C(u)}{B(u)t - D(u)} - q_k(u),$$

then, it follows easily from (2.10) and (2.11) that

Proposition 2.3. *For the N-extremal measure σ_t, $t \in \mathbb{R} \cup \{\infty\}$ and $m \in \mathbb{N}$, we have*

$$(m+1)\sum_{k=0}^{\infty} p_k^{(m)}(u)\int \frac{p_k(x)\,d\sigma_t(x)}{u-x} = \frac{B^{(m+1)}(u)t - D^{(m+1)}(u)}{B(u)t - D(u)}$$

$$\textit{for} \quad u \in \mathbb{C} \setminus \operatorname{supp}(\sigma_t).$$

In the next result we omit the parameter t in the notation and put $\sigma = \sigma_t$, $F(u) = B(u)t - D(u)$. Let $\delta_z^{(m)}$ be the corresponding functional on $L^2(\sigma)$ given by (2.3).

Proposition 2.4. *Let $m \in \mathbb{N}$ and $z \in \mathbb{C} \setminus \operatorname{supp}(\sigma)$. For each $n \geq 1$ the following conditions are equivalent.*

(i) $\langle \delta_z^{(m)}, \frac{1}{(t-z)^j} \rangle = 0$ *for* $j = 1, \cdots, n$.

(ii) $F^{(j)}(z) = 0$ *for* $j = m+1, \cdots, m+n$.

Proof. By Proposition 2.3 we have

$$(j+1)\sum_{k=0}^{\infty} p_k^{(j)}(u)\int_{\mathbb{R}} \frac{p_k(t)}{u-t}\,d\sigma(t) = \frac{F^{(j+1)}(u)}{F(u)}, \qquad j \geq 0,\ u \in \mathbb{C}\setminus \operatorname{supp}(\sigma). \tag{2.12}$$

We proceed by induction, and the equivalence follows for $n = 1$ from (2.12) with $j = m$ and from (2.3).

Suppose next that the equivalence holds for some $n \geq 1$, and let us prove the equivalence for $n+1$. Put $j = m+i$ in (2.12) and differentiate $n-i$ times with respect to u. For $i = n, n-1, \cdots, 0$ we get

(2.13)

$$\begin{cases} (m+n+1)\displaystyle\sum_{k=0}^{\infty} p_k^{(m+n)}(u)\int_{\mathbb{R}} \frac{p_k(t)d\,\sigma(t)}{u-t} = \frac{F^{(n+m+1)}(u)}{F(u)}, \\ (m+n)\displaystyle\sum_{k=0}^{\infty}\left(p_k^{(m+n)}(u)\int_{\mathbb{R}} \frac{p_k(t)d\,\sigma(t)}{u-t}\right. \\ \qquad\qquad \left. -p_k^{(m+n-1)}(u)\displaystyle\int_{\mathbb{R}} \frac{p_k(t)d\,\sigma(t)}{(u-t)^2}\right) = \frac{d}{du}\left(\frac{F^{(n+m)}(u)}{F(u)}\right), \\ \qquad\qquad\qquad\qquad \vdots \\ (m+1)\displaystyle\sum_{k=0}^{\infty}\left(p_k^{(m+n)}(u)\int_{\mathbb{R}} \frac{p_k(t)d\,\sigma(t)}{u-t}\right. \\ \qquad\qquad - np_k^{(m+n-1)}(u)\displaystyle\int_{\mathbb{R}} \frac{p_k(t)d\,\sigma(t)}{(u-t)^2} + \cdots \\ \qquad\qquad\qquad \left. +(-1)^n n! p_k^{(m)}(u)\displaystyle\int_{\mathbb{R}} \frac{p_k(t)d\,\sigma(t)}{(u-t)^{n+1}}\right) = \frac{d^n}{du^n}\left(\frac{F^{(m+1)}(u)}{F(u)}\right). \end{cases}$$

Assume first that $F^{(j)}(z) = 0$ for $j = m+1, \cdots, m+n+1$. By the induction hypothesis it suffices to prove that

$$\left\langle \delta_z^{(m)}, \frac{1}{(t-z)^{n+1}}\right\rangle = 0. \tag{2.14}$$

By assumption the right-hand sides of (2.13) vanish for $u = z$, and we conclude equation by equation that all series vanish for $u = z$ ending with

$$(-1)^n n! \sum_{k=0}^{\infty} p_k^{(m)}(z)\int_{\mathbb{R}} \frac{p_k(t)d\,\sigma(t)}{(z-t)^{n+1}} = 0,$$

which proves (2.14).

Assume next that $\langle \delta_z^{(m)}, \frac{1}{(t-z)^j}\rangle = 0$ for $j = 1, \cdots, n+1$. Again by the induction hypothesis $F^{(j)}(z) = 0$ for $j = m+1, \cdots, m+n$ and we shall establish $F^{(m+n+1)}(z) = 0$. The two first equations in (2.13) can be transformed

for $u = z$ to the following

$$\sum_{k=0}^{\infty} p_k^{(m+n)}(z) \int_{\mathbb{R}} \frac{p_k(t)d\,\sigma(t)}{z-t} = \frac{1}{m+n+1} \frac{F^{(n+m+1)}(z)}{F(z)},$$

$$\sum_{k=0}^{\infty} p_k^{(m+n-1)}(z) \int_{\mathbb{R}} \frac{p_k(t)d\,\sigma(t)}{(z-t)^2} = \frac{-1}{(m+n+1)(m+n)} \frac{F^{(n+m+1)}(z)}{F(z)}.$$

Inserting this in the third equation and so forth we find successively

$$\sum_{k=0}^{\infty} p_k^{(m+n-j)}(z) \int_{\mathbb{R}} \frac{p_k(t)d\,\sigma(t)}{(z-t)^{j+1}}$$
$$= \frac{(-1)^j}{(m+n+1)(m+n)\cdots(m+n+1-j)} \frac{F^{(n+m+1)}(z)}{F(z)}, \quad j = 0, 1, \cdots, n,$$

and for $j = n$ this gives

$$0 = \left\langle \delta_z^{(m)}, \frac{1}{(t-z)^{n+1}} \right\rangle = \frac{(-1)^n}{(m+n+1)(m+n)\cdots(m+1)} \frac{F^{(m+n+1)}(z)}{F(z)}.$$

Each of the four functions which appear in the matrix (2.6) are of minimal exponential type, i.e. they satisfy an inequality of the following type:

$$\forall \epsilon > 0\ \exists C_\epsilon > 0 : |f(u)| \le C_\epsilon e^{\epsilon|u|} \quad \text{for } u \in \mathbb{C}, \tag{2.15}$$

showing that their order is less than or equal to 1 and that if the order is one, then their type is 0. In [BP], it was proved that all four functions have the same order and the same type.

The entire functions of minimal exponential type have the important property that they are characterized up to a multiplicative constant by their sequence of zeros. Indeed, let f be an entire function of minimal exponential type with zeros (x_n) numbered so that $|x_1| \le |x_2| \le \cdots$, and let us for the sake of simplicity assume $x_1 \ne 0$. We have two posibilities: either the genus of the zeros is 0, i.e. $\sum_n \frac{1}{|x_n|} < \infty$, and by the Hadamard factorization theorem f is proportional to the canonical product

$$\prod_{n=1}^{\infty} \left(1 - \frac{u}{x_n}\right),$$

or the genus is 1, i.e. $\sum_n \frac{1}{|x_n|} = \infty$, and by the Hadamard factorization theorem and Lindelöf's theorem (cf. [B]) f is proportional to

$$e^{\alpha u} \prod_{n=1}^{\infty} \left(1 - \frac{u}{x_n}\right) e^{\frac{u}{x_n}},$$

where

$$\lim_{r\to\infty} \sum_{|x_n|\le r} \frac{1}{x_n} = -\alpha.$$

If also $x_0 = 0$ is a zero of order k of f, then the factor u^k should appear in the above canonical products. These results show that two functions of minimal exponential type having the same zeros are proportional.

Finally, from Laguerre's Theorem (cf. [B, p. 23]) we get the following property of entire functions of minimal exponential type which we will later need.

Proposition 2.5. *Let f be an entire function of minimal exponential type, with only real and simple zeros. Then the function f' has the same properties.*

3. The index of determinacy

In this section, we shall give a complete description of the index of determinacy defined by (1.1) and prove the properties which we mentioned in the introduction.

To begin with, we shall use some basic results about determinacy which we state for easy reference.

Lemma A (cf. [R]). *Let $\mu \in \mathscr{M}^*$, $a \in \mathbb{C}$.*

(1) *If μ is determinate, then the polynomials are dense in the space $L^2(|t-a|^2\mu)$.*
(2) *If $\mu(\{a\}) = 0$, in particular if $a \in \mathbb{C} \setminus \mathbb{R}$, then the converse holds.*
(3) *If the polynomials are dense in the space $L^2((1+t^2)\mu)$, then μ is determinate.*

Lemma B (cf. [A, p. 115], [BC, p. 111]). *If μ is N-extremal, hence of the form*

$$\mu = \sum_{\lambda \in \Lambda} a_\lambda \delta_\lambda \tag{3.1}$$

where $\Lambda \subset \mathbb{R}$ is an infinite discrete set and $a_\lambda > 0$ for $\lambda \in \Lambda$, then $\mu - a_\lambda \delta_\lambda$ is determinate.

Lemma C. *Let $\mu \in \mathscr{M}^*$, $a \in \mathbb{R}$ and $\alpha > 0$. Then*

(1) *If μ is determinate, then the polynomials are dense in $L^2(\mu + \alpha\delta_a)$.*
(2) *If $\mu(\{a\}) = 0$ the converse holds.*

Proof. In (1), it is no restriction to assume that $\mu(\{a\}) = 0$, and then the assertion is proved in [BC, p. 113].

Suppose next that the polynomials are dense in $L^2(\mu + \alpha\delta_a)$ and $\mu(\{a\}) = 0$. If $\mu + \alpha\delta_a$ is determinate, so is the smaller measure μ. If $\mu + \alpha\delta_a$ is indeterminate and hence N-extremal, then μ is determinate by the previous lemma.

Lemma D (cf. [BC, p. 113]). *If μ is N-extremal and given by* (3.1), *then the measure $a\delta_{\lambda_0} + (\mu - a_\lambda\delta_\lambda)$ is again N-extremal if $a > 0$ and $\lambda_0 \notin \Lambda \setminus \{\lambda\}$.*

The following result, which seems to be new, can be viewed as a stability result for the class of N-extremal measures.

Proposition 3.1. *Let μ be an N-extremal measure given by* (3.1), *and let $\Lambda_0 \subset \Lambda$ be a finite set. Then the measure $\tilde{\mu} = (\prod_{\lambda \in \Lambda_0}(t-\lambda)^2)\mu$ is again N-extremal.*

Proof. It suffices to prove the result for $\Lambda_0 = \{\lambda_0\} \subset \Lambda$. The measure $\tilde{\mu} = (t-\lambda_0)^2\mu$ is clearly indeterminate, and $\sigma = \mu - a_{\lambda_0}\delta_{\lambda_0}$ is determinate by Lemma B. It follows by Lemma A that the polynomials are dense in $L^2((t-\lambda_0)^2\sigma)$, and we finally have $\tilde{\mu} = (t-\lambda_0)^2\mu = (t-\lambda_0)^2\sigma$.

If we multiply an indeterminate measure μ with a non-negative polynomial with zeros outside the support of μ the situation is different from Proposition 3.1.

Proposition 3.2. *Let $\mu \in \mathscr{M}^*$ be indeterminate.*

(1) *If p is a non-negative polynomial of degree greater than 1 with no zeros in $\operatorname{supp}(\mu)$, then the polynomials are not dense in $L^2(p\mu)$.*
(2) *The polynomials are not dense in $L^2((t-x_0)^{2k}\mu)$ for $x_0 \in \mathbb{R}$ and $k \in \mathbb{N}$, $k \geq 2$.*

Proof. (1) If we assume the polynomials to be dense in $L^2(p\mu)$, then the measure $p(t)(1+t^2)^{-1}\mu$ is determinate by Lemma A. By assumption there exists $C > 0$ such that $p(t) \geq C(1+t^2)$ for $t \in \operatorname{supp}(\mu)$; this implies that $C\mu$ is determinate, which is a contradiction.

(2) The measure $\sigma = (t-x_0)^{2k-2}\mu$ is indeterminate with $\sigma(\{x_0\}) = 0$, so the result follows from Lemma A.

The first result about the index of determinacy will be a straightforward consequence of

Theorem 3.3. *For $\mu \in \mathscr{M}^*$ and $a \in \mathbb{C}$ the following conditions are equivalent.*

(1) *The polynomials are dense in $L^2(|t-a|^{2k}\mu)$ for all $k \in \mathbb{N}$.*
(2) *$|t-a|^{2k}\mu$ is determinate for all $k \in \mathbb{N}$.*
(3) *The polynomials are dense in $L^2(\phi\mu)$ for any Borel function $\phi : \mathbb{R} \to [0, \infty)$ which is bounded by some polynomial.*

If μ is non-discrete, then (1)–(3) are equivalent with μ being determinate.

Proof. (1) $\to$ (2) follows from Proposition 3.2.

(2) $\to$ (3). There exist $C > 0$ and $k \in \mathbb{N}$ such that $0 \leq \phi(t) \leq C(1+t^2)^k$, so it suffices to prove that the polynomials are dense in $L^2((1+t^2)^k\mu)$ for all $k \in \mathbb{N}$. If this was not the case there would exist an integer $k_0 \geq 1$ such that the polynomials are dense in $L^2((1+t^2)^{k_0}\mu)$ but not dense in $L^2((1+t^2)^{k_0+1}\mu)$. Then $\sigma = (1+t^2)^{k_0}\mu$ is N-extremal implying that μ is discrete. Defining $\nu = \mu - \mu(\{a\})\delta_a$ we get by Proposition 3.2 that $\mathbb{C}[t]$ is not dense in L^2 with respect to $|t-a|^4\sigma = |t-a|^4(1+t^2)^{k_0}\nu$.

On the other hand, by assumption (2) $|t-a|^{2k_0+4}\mu$ is determinate and in particular $\mathbb{C}[t]$ is dense in L^2 with respect to $|t-a|^{2k_0+4}\mu = |t-a|^{2k_0+4}\nu$. This gives a contradiction because there exist constants $0 < c < C < \infty$ so that

$$c \leq \frac{|t-a|^4(1+t^2)^{k_0}}{|t-a|^{2k_0+4}} \leq C \quad \text{for } t \in \operatorname{supp}(\nu).$$

The implication (3) $\to$ (1) is obvious.

If μ is determinate and non-discrete, then (2) holds because if $|t-a|^{2k}\mu$ is determinate and $|t-a|^{2k+2}\mu$ is indeterminate, then the latter is N-extremal, and therefore μ is discrete.

In terms of the index of determinacy, we have

Corollary 3.4. *Let $\mu \in \mathscr{M}^*$ be determinate.*

(1) *If $\operatorname{ind}_{z_0}(\mu) = \infty$ for some $z_0 \in \mathbb{C}$, then $\operatorname{ind}_z(\mu) = \infty$ for all $z \in \mathbb{C}$.*
(2) *If μ is non-discrete, then $\operatorname{ind}_z(\mu) = \infty$ for all $z \in \mathbb{C}$.*

The rest of this section is devoted to proving that if $\operatorname{ind}_{z_0}(\mu)$ is finite for some $z_0 \in \mathbb{C}$, then necessarily $\operatorname{ind}_z(\mu)$ is constant k for z in the complement of the support of μ and constant equal to $k+1$ in the support of μ. Moreover,

in this case μ is obtained from an N-extremal measure by removing the mass of $k+1$ points in the support of this N-extremal measure. We first prove

Lemma 3.5. *Let $\mu \in \mathscr{M}^*$ be determinate. Then* $\operatorname{ind}_z(\mu)$ *is constant on* $\mathbb{C} \setminus \operatorname{supp}(\mu)$.

Proof. By Corollary 3.4, we can assume that μ is discrete and that $\operatorname{ind}_z(\mu) = k < \infty$ for some $z \notin \operatorname{supp}(\mu)$. Then $|t-z|^{2k}\mu$ is determinate and $|t-z|^{2k+2}\mu$ is indeterminate. By Lemma A the polynomials are dense in L^2 with respect to $|t-z|^{2k+2}\mu$ and also with respect to $|t-w|^{2k+2}\mu$ if $w \notin \operatorname{supp}(\mu)$ since $(\frac{|t-w|}{|t-z|})^{2k+2}$ is bounded on $\operatorname{supp}(\mu)$. It follows by Lemma A that $|t-w|^{2k}\mu$ is determinate. In the same way we see that $|t-w|^{2k+2}\mu$ is indeterminate.

The next theorem establishes a half of our result.

Theorem 3.6. *Let $\sigma = \sum_{\lambda\in\Lambda} a_\lambda \delta_\lambda$ be N-extremal; let $\Lambda_0 \subset \Lambda$ be a subset with $k+1$ elements, $k \geq 0$; and put*

$$\mu = \sigma - \sum_{\lambda\in\Lambda_0} a_\lambda\delta_\lambda = \sum_{\lambda\in\Lambda\setminus\Lambda_0} a_\lambda\delta_\lambda. \tag{3.2}$$

Then μ is determinate and

$$\operatorname{ind}_z(\mu) = \begin{cases} k & \text{for } z \notin \operatorname{supp}(\mu), \\ k+1 & \text{for } z \in \operatorname{supp}(\mu). \end{cases} \tag{3.3}$$

Proof. The measure μ is determinate by Lemma B. We proceed by induction after k.

For $k=0$, we put $\Lambda_0 = \{\lambda_0\} \subset \Lambda$ and $\mu = \sigma - a_{\lambda_0}\delta_{\lambda_0}$. We then have

$$(t-\lambda_0)^2\mu = (t-\lambda_0)^2\sigma,$$

and the last expression is N-extremal by Proposition 3.1, showing that $\operatorname{ind}_{\lambda_0}(\mu) = 0$. By Proposition 3.5 we have $\operatorname{ind}_z(\mu) = 0$ for $z \notin \operatorname{supp}(\mu)$.

We shall finally prove that $\operatorname{ind}_z(\mu) = 1$ for $z \in \operatorname{supp}(\mu)$. We have

$$|t-z|^2\mu = |t-z|^2\sigma - |\lambda_0 - z|^2 a_{\lambda_0}\delta_{\lambda_0},$$

which is determinate by Proposition 3.1 and Lemma B. Since $|t-z|^2$ does not vanish on $\operatorname{supp}(|t-z|^2\sigma)$, we know by Proposition 3.2 (1) that $\mathbb{C}[t]$ is not dense in L^2 with respect to

$$|t-z|^4\sigma = |t-z|^4\mu + |\lambda_0 - z|^4 a_{\lambda_0}\delta_{\lambda_0}.$$

By Lemma C (1) we conclude that $|t-z|^4\mu$ is indeterminate and hence that $\operatorname{ind}_z(\mu) = 1$.

Assume now that (3.3) holds if Λ_0 has k elements, and let us prove (3.3) under the assumption that Λ_0 has $k+1$ elements ($k \geq 1$).

Choose $\lambda_0 \in \Lambda_0$ and define $\tilde{\Lambda} = \Lambda_0 \setminus \{\lambda_0\}$, which has k elements. Put $\tilde{\mu} = \sigma - \sum_{\lambda\in\tilde{\Lambda}} a_\lambda\delta_\lambda$ so that $\mu = \tilde{\mu} - a_{\lambda_0}\delta_{\lambda_0}$. The measure

$$\tau = (t-\lambda_0)^2\mu = (t-\lambda_0)^2\sigma - \sum_{\lambda\in\tilde{\Lambda}} a_\lambda(\lambda-\lambda_0)^2\delta_\lambda$$

is obtained from the N-extremal measure $(t-\lambda_0)^2\sigma$ by removing k masses, so the induction hypothesis gives

$$\operatorname{ind}_z(\tau) = \begin{cases} k-1 & \text{for } z \notin \operatorname{supp}(\mu), \\ k & \text{for } z \in \operatorname{supp}(\mu), \end{cases} \tag{3.4}$$

in particular $\operatorname{ind}_{\lambda_0}(\tau) = \operatorname{ind}_{\lambda_0}((t-\lambda_0)^2\mu) = k-1$, i.e. $\operatorname{ind}_{\lambda_0}(\mu) = k$, and by Lemma 3.5 we then have $\operatorname{ind}_z(\mu) = k$ for $z \notin \operatorname{supp}(\mu)$.

For $z \in \operatorname{supp}(\mu) = \operatorname{supp}(\tau)$ the measure $|t-z|^{2k}(t-\lambda_0)^2\mu$ is determinate by (3.4), and so is $|t-z|^{2k+2}\mu$ by Lemma A since $\lambda_0 \notin \operatorname{supp}(\mu)$. Furthermore, $|t-z|^{2k+2}(t-\lambda_0)^2\mu$ is indeterminate by (3.4) and so is $|t-z|^{2k+4}\mu$, again by Lemma A. This shows that $\operatorname{ind}_z(\mu) = k+1$.

To prove the converse of Theorem 3.6, we need the following lemma which establishes this result for $k = 0$.

Lemma 3.7. *Let $\mu \in \mathscr{M}^*$ be determinate and assume that $\operatorname{ind}_z(\mu) = 0$ for some $z \in \mathbb{C}$. Then $z \notin \operatorname{supp}(\mu)$ and $\mu + \alpha\delta_x$ is N-extremal for any $\alpha > 0$ and $x \in \mathbb{R} \setminus \operatorname{supp}(\mu)$.*

Proof. By assumption $|t-z|^2\mu$ is indeterminate and so is $\tau = (1+|t-z|^2)\mu$. The polynomials are dense in $L^2(\tau)$ by Lemma A, so τ is N-extremal and in particular μ is discrete. If we assume $\mu(\{z\}) > 0$ then $\tau - \tau(\{z\})\delta_z$ is determinate by Lemma B and so is the smaller measure $|t-z|^2\mu$, which contradicts $\operatorname{ind}_z(\mu) = 0$.

For $\alpha > 0$ and $x \in \mathbb{R} \setminus \operatorname{supp}(\mu)$ if the measure $\mu + \alpha\delta_x$ is determinate, then $\operatorname{ind}_x(\mu + \alpha\delta_x) \geq 1$ by the first part of this lemma. Therefore $(t-x)^2(\mu+\alpha\delta_x) = (t-x)^2\mu$ is determinate and hence $\operatorname{ind}_z(\mu) = \operatorname{ind}_x(\mu) \geq 1$ by Lemma 3.5. Hence $\mu + \alpha\delta_x$ is indeterminate, and now Lemma C says that $\mu + \alpha\delta_x$ is N-extremal.

Remark 3.8. The above result is a counterpart for the Hamburger moment problem to the results 5.4 and 5.5 in [BT1].

Finally, we prove that the measures μ from Theorem 3.6 are the only measures with finite index of determinacy.

Theorem 3.9. *Let $\mu \in \mathscr{M}^*$ be determinate, and assume that $\operatorname{ind}_z(\mu) = k$ $(1 \leq k < \infty)$ for some $z \in \mathbb{C}$.*

(1) *If $z \notin \operatorname{supp}(\mu)$, then μ has the form* (3.2) *for an N-extremal measure σ and Λ_0 containing $k+1$ points.*
(2) *If $z \in \operatorname{supp}(\mu)$, then μ has the form* (3.2) *for an N-extremal measure σ and Λ_0 containing k points.*

Proof. (1) We choose $\Lambda_0 = \{\lambda_0, \lambda_1, \cdots, \lambda_k\}$ consisting of $k+1$ different points in $\mathbb{R} \setminus \operatorname{supp}(\mu)$, which is certainly possible by Corollary 3.4. We shall prove that the measure $\sigma = \mu + \sum_{i=0}^{k} \delta_{\lambda_i}$ is N-extremal.

To prove that $\mathbb{C}[t]$ is dense in $L^2(\sigma)$ we proceed as follows. The polynomials are dense in $L^2(\mu + \delta_{\lambda_0})$ by Lemma C, and $\mu + \delta_{\lambda_0}$ is also determinate for otherwise it would be N-extremal and hence $\operatorname{ind}_{\lambda_0}(\mu) = 0$ by Theorem 3.6. This contradicts the assumption $\operatorname{ind}_z(\mu) = k \geq 1$ since $\operatorname{ind}_z(\mu) = \operatorname{ind}_{\lambda_0}(\mu)$ by Lemma 3.5.

Having established that $\mu + \delta_{\lambda_0} + \cdots + \delta_{\lambda_i}$ is determinate for $0 \leq i < k$ we again conclude from Lemma C that $\mathbb{C}[t]$ is dense in L^2 with respect to $\mu + \delta_{\lambda_0} + \cdots + \delta_{\lambda_{i+1}}$ and the latter is determinate if $i+1 < k$ since otherwise

$$\operatorname{ind}_{\lambda_{i+1}}(\mu) = \operatorname{ind}_z(\mu) = i+1 < k.$$

This shows that $\mathbb{C}[t]$ is dense in $L^2(\sigma)$.

We shall next establish that σ is indeterminate and hence N-extremal. Assume that σ is determinate. Since σ has mass at λ_k, it follows by Lemma 3.7 that $(t-\lambda_k)^2\sigma$ is determinate; so $\mathbb{C}[t]$ is dense in $L^2(\nu)$, where

$$\nu = (t-\lambda_k)^2\sigma + \delta_{\lambda_k} = (t-\lambda_k)^2\left(\mu + \sum_{i=0}^{k-1}\delta_{\lambda_i}\right) + \delta_{\lambda_k}.$$

We shall now see that the two alternatives: (a) ν is determinate, (b) ν is indeterminate, lead to the same conclusion

$$(t-\lambda_k)^4(\mu + \sum_{i=0}^{k-2}\delta_{\lambda_i}) \quad \text{is determinate.} \tag{3.5}$$

In the case (a), since ν has mass at λ_k, also $(t-\lambda_k)^2\nu$ is determinate and so is the smaller measure in (3.5). In the case (b) ν is N-extremal, so by Lemma B

$$\tau = (t-\lambda_k)^2(\mu + \sum_{i=0}^{k-2}\delta_{\lambda_i}) + \delta_{\lambda_k} \quad \text{is determinate.}$$

Since τ has mass at λ_k, also $(t-\lambda_k)^2\tau$ is determinate, and (3.5) holds.

Repeating this argument $k-1$ times we conclude that $(t-\lambda_k)^{2k+2}\mu$ is determinate, i.e. $\operatorname{ind}_{\lambda_k}(\mu) \geq k+1$, which is a contradiction since $\operatorname{ind}_{\lambda_k}(\mu) = \operatorname{ind}_z(\mu) = k$.

(2) The measure $\nu = \mu - \mu(\{z\})\delta_z$ is determinate and we have $\operatorname{ind}_z(\mu) = \operatorname{ind}_z(\nu) = k$. By part (1) ν is of the form (3.2) with an N-extremal measure σ and Λ_0 containing $k+1$ points. By the proof of (1) we can choose Λ_0 as any set of $k+1$ points from $\mathbb{R}\setminus\operatorname{supp}(\nu)$, so we can assume $z \in \Lambda_0$. By Lemma D we may also assume that $\mu(\{z\}) = \sigma(\{z\})$, and hence μ has the form (3.2), where Λ_0 consists of k points.

Let us compare the above index of determinacy with the index of determinacy in the sense of Stieltjes introduced in [BT1].

By $\mathscr{M}^*([0,\infty))$ we denote the set of measures $\sigma \in \mathscr{M}^*$ for which $\operatorname{supp}(\sigma) \subset [0,\infty)$. Such a measure is called determinate in the sense of Stieltjes (det(S) in short) if there is no other measure in $\mathscr{M}^*([0,\infty))$ with the same moments as σ. If $\sigma \in \mathscr{M}^*([0,\infty))$ is det(S) we put

$$\operatorname{ind}(\sigma) = \sup\{k \in \mathbb{N} : t^k\sigma \text{ is det(S)}\}. \tag{3.6}$$

It is possible that $\sigma \in \mathscr{M}^*([0,\infty))$ is det(S) but indeterminate (cf. [BT1]) and in this case the index of this paper is not defined. However, if $\sigma \in \mathscr{M}^*([0,\infty))$ is determinate the following holds.

Proposition 3.10. *Suppose $\sigma \in \mathscr{M}^*([0,\infty))$ is determinate. Then*

$$\operatorname{ind}_0(\sigma) = \left[\frac{1}{2}\operatorname{ind}(\sigma)\right], \tag{3.7}$$

where $[x]$ is the integral part of x in case $x < \infty$ and $[\infty] = \infty$.

Proof. Suppose first $\operatorname{ind}_0(\sigma) = k < \infty$. Then $t^{2k}\sigma$ is determinate and hence det(S), and $t^{2k+2}\sigma$ is indeterminate. However, since $t^{2k+2}\sigma$ has no mass at zero, it cannot be det(S) (see Proposition 1.1 in [BT1]). It follows that $\operatorname{ind}(\sigma)$

is either $2k$ or $2k+1$ and (3.7) follows. (It is easy to give examples showing that both possibilities can occur.) If $\mathrm{ind}_0(\sigma) = \infty$, then clearly $\mathrm{ind}(\sigma) = \infty$.

If $\sigma \in \mathscr{M}^*([0, \infty))$ there exists a unique symmetric measure $\mu \in \mathscr{M}^*$ such that $\mu^\psi = \sigma$, where μ^ψ is the image measure of μ under the mapping $\psi : \mathbb{R} \to [0, \infty)$, $\psi(x) = x^2$. It is well known that μ is determinate if and only if σ is det(S) (cf. [BT2]). With this notation we clearly have

$$\mathrm{ind}_0(\mu) = \mathrm{ind}(\sigma), \tag{3.8}$$

since $(t^{2k}\mu)^\psi = t^k\sigma$.

4. The ℓ^2-norm of the derivatives of orthonormal polynomials

Let $\mu \in \mathscr{M}^*$, and let $(p_n)_n$ be the corresponding sequence of orthonormal polynomials. In this section, we deal with the problem of determining for which complex numbers z, the sequence $(p_n^{(m)}(z))_n$ ($m \in \mathbb{N}$) belongs to ℓ^2 and give estimates on its ℓ^2-norm. For $m = 0$ the solution of this problem is well known: if μ is indeterminate, then for all $z \in \mathbb{C}$ $(p_n(z))_n \in \ell^2$ and if μ is determinate, then the sequence $(p_n(z))_n$ belongs to ℓ^2 only when $\mu(\{z\}) > 0$. In both cases, we have

$$\|p_n(z)\|_{\ell^2}^{-2} = \sup\{\nu(\{z\}) : \nu \in \mathscr{M}^* \text{ having the same moments as } \mu\}.$$

From Section 2, we know that if μ is indeterminate, then for all $m \in \mathbb{N}$ and $z \in \mathbb{C}$, the sequence $(p_n^{(m)}(z))_n$ belongs to ℓ^2. Formulas for the ℓ^2-norm of these sequences have recently been found by the second author in [D]. In these formulas the orthogonality of the polynomials $(p_n)_n$ with respect to positive definite matrices of measures plays a fundamental role.

However, it does not seem to be known when $(p_n^{(m)}(z))_n \in \ell^2$ for $m \geq 1$ and μ a determinate measure. In this section, we shall solve this problem.

First of all, we are going to show the link between this problem and the index of determinacy defined in the previous section. Our starting point is the following theorem which extends a well-known extremality property of the orthonormal polynomials (cf. [A, p. 60]). (For a subset A of a normed vector space E we denote by $\langle A \rangle$ the algebraic span of A, and for $x \in E$ we denote by $d_E(x, A)$ the distance between x and A: $d_E(x, A) = \inf\{\|x-a\| : a \in A\}$.)

Theorem 4.1. *Let $\mu \in \mathscr{M}^*$, and let $(p_n)_n$ be the corresponding sequence of orthonormal polynomials. Then for $m \in \mathbb{N}$*

$$\|p_n^{(m)}(z)\|_{\ell^2}^{-2} = d^2_{L^2(\mu)}\left(\frac{(t-z)^m}{m!}, \langle (t-z)^n : n \neq m\rangle\right).$$

Proof. It is straightforward that

$$\begin{aligned} &d^2_{L^2(\mu)}\left(\frac{(t-z)^m}{m!}, \langle (t-z)^n : n \neq m\rangle\right) \\ &\qquad = \inf\{\|p\|^2_{L^2(\mu)} : p \in \mathbb{C}[t] \text{ with } p^{(m)}(z) = 1\}. \end{aligned}$$

If we expand the polynomial p in terms of the polynomials $(p_n)_n$, $p(t) = \sum_n a_n p_n(t)$, the condition $p^{(m)}(z) = 1$ is equivalent to $\sum_n a_n p_n^{(m)}(z) = 1$.

Since $\|p\|^2_{L^2(\mu)} = \sum_n |a_n|^2$, we have to find $\inf\{\sum_n |a_n|^2 : \sum_n a_n p_n^{(m)}(z) = 1\}$ which clearly is $1/\sum_n |p_n^{(m)}(z)|^2$.

Taking into account this theorem, we get

$$\begin{aligned} \|p_n^{(m)}(z)\|_{\ell^2}^{-2} &= d^2_{L^2(\mu)}\left(\frac{(t-z)^m}{m!}, \langle (t-z)^n : n \neq m\rangle\right) \\ &\leq d^2_{L^2(\mu)}\left(\frac{(t-z)^m}{m!}, \langle (t-z)^n : n \geq m+1\rangle\right) \\ &= \frac{1}{m!^2} d^2_{L^2(|t-z|^{2m}\mu)}(1, \langle (t-z)^n : n \geq 1\rangle) \\ &= \frac{1}{m!^2}\|p_{n,m,z}(z)\|_{\ell^2}^{-2} \end{aligned} \tag{4.1}$$

where $(p_{n,m,z})_n$ is the sequence of orthonormal polynomials with respect to the measure $|t-z|^{2m}\mu$. Hence, we have proved the following lemma

Lemma 4.2. *Let μ and $(p_n)_n$ be a measure in $\mathscr{M}^*$ and its sequence of orthonormal polynomials. If $\operatorname{ind}_z(\mu) = k$ $(k \in \mathbb{N} \cup \{\infty\})$, we have for $m \in \mathbb{N}$*

$$d^2_{L^2(\mu)}\left(\frac{(t-z)^m}{m!}, \langle (t-z)^n : n \geq m+1\rangle\right) \begin{cases} = 0 & \text{for } 1 \leq m \leq k, \\ > 0 & \text{for } k < m. \end{cases}$$

Thus, the following corollary follows directly from Corollary 3.4 and Lemma 4.2.

Corollary 4.3. *If there exists a complex number z_0 such that $\operatorname{ind}_{z_0}(\mu) = \infty$, then for all $z \in \mathbb{C}$ the sequence $(p_n^{(m)}(z))_n$ is never in ℓ^2 for $m \geq 1$.*

So, we have reduced the problem to the case of a measure μ for which $\operatorname{ind}_z(\mu) < \infty$ for all $z \in \mathbb{C}$. In this case, according to Theorems 3.6 and 3.9, there exists a non-negative integer k such that $\operatorname{ind}_z(\mu)$ equals k if $z \notin \operatorname{supp}(\mu)$ and $k+1$ if $z \in \operatorname{supp}(\mu)$. For each $z \notin \operatorname{supp}(\mu)$, the measure $\sigma_z = |t-z|^{2k+2}\mu$ is N-extremal by Lemma A in Section 3. The four entire functions from the Nevanlinna matrix associated to σ_z (cf. Section 2) are denoted by A_z, B_z, C_z, D_z, and by (2.7) there exists $t_z \in \mathbb{R} \cup \{\infty\}$ such that

$$\int \frac{d\sigma_z(x)}{w-x} = \frac{A_z(w)t_z - C_z(w)}{B_z(w)t_z - D_z(w)}, \qquad w \in \mathbb{C} \setminus \operatorname{supp}(\mu). \tag{4.2}$$

We write F_z for the entire function

$$F_z(w) = B_z(w)t_z - D_z(w), \tag{4.3}$$

which like B_z, D_z is of minimal exponential type by Section 2; since all the functions F_z, $z \in \mathbb{C} \setminus \operatorname{supp}(\mu)$, have the same zeros equal to the points of $\operatorname{supp}(\mu)$, they are proportional. Let F_μ denote the canonical product

$$F_\mu(w) = e^{-\left(\sum_n \frac{1}{x_n} w\right)} \prod_{n=0}^{\infty}\left(1 - \frac{w}{x_n}\right) e^{\frac{w}{x_n}},$$

where $\{x_n : n \in \mathbb{N}\}$ is the support of μ arranged so that $|x_0| \leq |x_1| \leq \cdots$; cf. Section 2. For any $z \in \mathbb{C} \setminus \operatorname{supp}(\mu)$ there exists a constant c_z such that

$$F_z(w) = c_z F_\mu(w), \qquad w \in \mathbb{C}. \tag{4.4}$$

Since the zeros of the function F_μ are the points in $\operatorname{supp}(\mu)$, we have that $(p_n(z))_n \in \ell^2$ if and only if $F_\mu(z) = 0$, i.e. the zeros of the function F_μ determine when the sequence $(p_n(z))_n$ belongs to ℓ^2. As the main result in this section, we prove that the zeros of the derivatives of this function F_μ determine when the sequence $(p_n^{(m)}(z))_n$ belongs to ℓ^2.

Theorem 4.4. *Let k be a non-negative integer and μ a determinate measure with index of determinacy $\operatorname{ind}_z(\mu)$ equal to k if $z \notin \operatorname{supp}(\mu)$ and $k+1$ if $z \in \operatorname{supp}(\mu)$. Let F_μ be the entire function associated with μ as before. Then for a complex number z we have*

(i) *The sequence $(p_n(z))_n$ belongs to ℓ^2 if and only if $F_\mu(z) = 0$.*

(ii) *For $k, m \geq 1$ the sequence $(p_n^{(m)}(z))_n \notin \ell^2$.*

(iii) *For $k = 0$ the sequence $(p_n^{(m)}(z))_n$ belongs to ℓ^2 if and only if $F_\mu^{(m)}(z) = 0$.*

Moreover, for $k = 0$ there exist infinitely many numbers z satisfying $(p_n^{(m)}(z))_n \in \ell^2$, and these numbers are real.

Proof. We have already proved (i).

To prove (ii) and (iii) we first prove the following lemma.

Lemma 4.5. *For $k < m$, the following conditions are equivalent.*

(a) *The sequence $(p_n^{(m)}(z))_n$ belongs to ℓ^2.*

(b) *One of the following conditions hold*:

(1) *$z \notin \operatorname{supp}(\mu)$ and $d_{L^2(\mu)}\Big((t-z)^l, \langle (t-z)^n : n \geq k+1, n \neq m\rangle\Big) = 0$ for $l = 0, \cdots, k$;*

(2) *$z \in \operatorname{supp}(\mu)$, $k+1 < m$ and*

$$d_{L^2(\mu)}\Big((t-z)^l, \langle (t-z)^n : n \geq k+2, n \neq m\rangle\Big) = 0 \quad \text{for } l = 1, \cdots, k+1.$$

(c) *$F_\mu^{(m-l)}(z) = 0$ for $l = 0, \cdots, k$.*

Proof of Lemma 4.5. (a) $\Rightarrow$ (b) Suppose that the sequence $(p_n^{(m)}(z))_n$ belongs to ℓ^2. Theorem 4.1 gives that

$$d_{L^2(\mu)}\left((t-z)^m, \langle (t-z)^n : n \neq m\rangle\right) > 0. \tag{4.5}$$

For $z \notin \operatorname{supp}(\mu)$ we have $\operatorname{ind}_z(\mu) = k$, and so for $l = 0, \cdots, k$ the measure $|t-z|^{2l}\mu$ is determinate. It follows from Lemma 4.2 that

$$d_{L^2(\mu)}\Big((t-z)^l, \langle (t-z)^n : n \geq l+1\rangle\Big) = 0 \quad \text{for } l = 0, \cdots, k.$$

Hence, we have

$$d_{L^2(\mu)}\Big((t-z)^l, \langle (t-z)^n : n \geq k+1\rangle\Big) = 0 \quad \text{for } l = 0, \cdots, k. \tag{4.6}$$

For $z \in \operatorname{supp}(\mu)$ we have $\operatorname{ind}_z(\mu) = k+1$, and again from Lemma 4.2 we get

$$d_{L^2(\mu)}\Big((t-z)^l, \langle (t-z)^n : n \geq k+2\rangle\Big) = 0 \quad \text{for } l = 1, \cdots, k+1. \tag{4.7}$$

Putting $l = k+1$ in (4.7), (4.5) proves that in this case $m \neq k+1$. Suppose that there exists l, $0 \leq l \leq k$, if z is outside of the support of μ or $1 \leq l \leq k+1$ if z is in the support of μ, for which

$$d_{L^2(\mu)}\Big((t-z)^l : \langle (t-z)^n : n \geq k+j, n \neq m\rangle\Big) > 0, \tag{4.8}$$

where $j = 2$ or 1, if z is in or outside of the support of μ respectively. Then, (4.6) or (4.7) give a sequence of complex numbers $(a_n)_n$ and a sequence of polynomials $(r_n)_n$ such that $r_n \in \langle (t-z)^n : n \geq k+j,\, n \neq m \rangle$ and

$$\lim_{n\to\infty} \|(t-z)^l - a_n(t-z)^m - r_n(t)\|_{L^2(\mu)} = 0.$$

The inequality (4.8) gives $\liminf_n |a_n| > 0$; hence

$$\lim_{n\to\infty} \left\| \frac{(t-z)^l}{a_n} - (t-z)^m - \frac{r_n(t)}{a_n} \right\|_{L^2(\mu)} = 0,$$

and we deduce that $d_{L^2(\mu)}((t-z)^m, \langle (t-z)^n : n \neq m \rangle) = 0$, which contradicts (4.5). So, we have proved (b).

(b.1) $\Leftrightarrow$ (c) Since z is outside of the support of μ, Lemma A in Section 3 gives the representation $\mu = \frac{\sigma_z}{|t-z|^{2k+2}}$, where σ_z is an N-extremal measure. Then it is clear that (b.1) is equivalent to the following condition: *For* $l = 0, \cdots, k$ *there exists a sequence of polynomials* $(r_{n,l})_n$ *for which* $r_{n,l}^{(m-k-1)}(z) = 0$ $(n \in \mathbb{N})$ *and such that*

$$\lim_{n\to\infty} \left(\frac{1}{(t-z)^{k+1-l}} - r_{n,l}(t) \right) = 0 \tag{4.9}$$

in $L^2(\sigma_z)$. Using the continuous linear functional $\delta_z^{(m-k-1)}$ on $L^2(\sigma_z)$ given by (2.3), we claim that (4.9) is equivalent to

$$\langle \delta_z^{(m-k-1)}(t), \frac{1}{(t-z)^{k+1-l}} \rangle = 0 \quad \text{for } l = 0, \cdots, k. \tag{4.10}$$

Condition (4.10) clearly follows from (4.9) by continuity. To see that (4.10) implies (4.9) we choose a sequence $(s_{n,l})_n$ of polynomials such that

$$\lim_{n\to\infty} s_{n,l} = \frac{1}{(t-z)^{k+1-l}} \quad \text{in } L^2(\sigma_z),$$

and hence $\lim_{n\to\infty} \delta_z^{(m-k-1)}(t)(s_{n,l}) = 0$. Therefore

$$r_{n,l}(t) = s_{n,l}(t) - \frac{(t-z)^{m-k-1}}{(m-k-1)!} s_{n,l}^{(m-k-1)}(z)$$

satisfies (4.9). Proposition 2.4, (4.3) and (4.4) show that (4.10) is equivalent to (c).

(b.2) $\Leftrightarrow$ (c) First of all we see that if $z \in \operatorname{supp}(\mu)$ and (c) holds, then $k+1 < m$. In fact if $k+1 = m$, then $F^{(j)}(z) = 0$ for $j = 0, \cdots, k+1$, but this contradicts that the zeros of F_μ are simple.

We write $\mu_1 = |t-z|^2\mu$. Then, $z \notin \operatorname{supp}(\mu_1)$, $\operatorname{ind}_z \mu_1 = k$ and $\operatorname{supp}(\mu) = \operatorname{supp}(\mu_1) \cup \{z\}$. We put F_μ and F_{μ_1} for the entire functions associated to μ and μ_1 respectively. Since the zeros of the functions F_μ and F_{μ_1} are the points in the support of μ and μ_1, respectively, and F_μ, F_{μ_1} are entire functions of minimal exponential type, we have the following link between these functions:

$$F_\mu(w) = \alpha F_{\mu_1}(w)(w-z)$$

for a certain real constant α.

Then, we have that $F_{\mu_1}^{(m-1-l)}(z) = 0$ for $l = 0, \cdots, k$ if and only if $F_{\mu}^{(m-l)}(z) = 0$ for $l = 0, \cdots, k$. If we apply the equivalence between (b.1) and (c) to the measure μ_1 ($k+1 < m$), we get that $F_{\mu_1}^{(m-1-l)}(z) = 0$ for $l = 0, \cdots, k$ if and only if

$$d_{L^2(|t-z|^2\mu)}\Big((t-z)^l, \langle (t-z)^n : n \geq k+1, n \neq m-1\rangle\Big) = 0 \qquad \text{for } l = 0, \cdots, k,$$

which gives the equivalence between (b.2) and (c).

(b.1) $\Rightarrow$ (a) If the sequence $(p_n^{(m)}(z))_n$ does not belong to ℓ^2, Theorem 4.1 gives

$$d_{L^2(\mu)}((t-z)^m, \langle (t-z)^n : n \neq m\rangle) = 0,$$

and from (b.1) we deduce

$$d_{L^2(\mu)}((t-z)^m, \langle (t-z)^n : n \geq k+1, n \neq m\rangle) = 0.$$

So, there exist a sequence of complex numbers $(a_n)_n$ and a sequence of polynomials $(r_n)_n$ such that $r_n \in \langle (t-z)^n : n \geq k+2, n \neq m\rangle$ and

$$\lim_{n\to\infty} \|(t-z)^m - a_n(t-z)^{k+1} - r_n(t)\|_{L^2(\mu)} = 0. \tag{4.11}$$

Since $\operatorname{ind}_z(\mu) = k$, we get from Lemma 4.2 that

$$d_{L^2(\mu)}\Big((t-z)^{k+1}, \langle (t-z)^n : n \geq k+2\rangle\Big) > 0,$$

so from (4.11) we deduce that $\lim_n a_n = 0$, and hence

$$d_{L^2(\mu)}((t-z)^m, \langle (t-z)^n : n \geq k+2, n \neq m\rangle) = 0.$$

If we proceed in the same way, we end by getting

$$d_{L^2(\mu)}((t-z)^m, \langle (t-z)^n : n \geq m+1\rangle) = 0,$$

but since $\operatorname{ind}_z(\mu) = k < m$, Lemma 4.2 provides a contradiction.

(b.2) $\Rightarrow$ (a) If $(p_n^{(m)}(z))_n \notin \ell^2$, then by Theorem 4.1

$$d_{L^2(\mu)}((t-z)^m, \langle (t-z)^n : n \neq m\rangle) = 0.$$

Hence, there exist a sequence of complex numbers $(a_n)_n$ and a sequence of polynomials $(r_n)_n$ such that $r_n \in \langle (t-z)^n : n \geq 1, n \neq m\rangle$ and

$$\lim_{n\to\infty} \|(t-z)^m - a_n - r_n(t)\|_{L^2(\mu)} = 0.$$

Since $z \in \operatorname{supp}(\mu)$, we have $d_{L^2(\mu)}(1, \langle (t-z)^n : n \geq 1\rangle) > 0$; then we deduce that $\lim_{n\to\infty} a_n = 0$, so

$$d_{L^2(\mu)}((t-z)^m, \langle (t-z)^n : n \neq 0, m\rangle) = 0.$$

Now, from (b.2), we deduce that

$$d_{L^2(\mu)}((t-z)^m, \langle (t-z)^n : n \geq k+2, n \neq m\rangle) = 0.$$

Proceeding as in the case $z \notin \operatorname{supp}(\mu)$, we get

$$d_{L^2(\mu)}((t-z)^m, \langle (t-z)^n : n \geq m+1\rangle) = 0.$$

Then, since $\operatorname{ind}_z \mu = k+1 < m$, Lemma 4.2 provides a contradiction.

Finally, we return to the proof of Theorem 4.4. Part (iii) is just the equivalence between (a) and (c) in Lemma 4.5 for $k = 0$. Part (ii) will follow from Lemma 4.5 if we prove that the zeros of the derivatives of the function F_μ always have multiplicity 1. But this is a consequence of Proposition 2.5.

Finally, we give some estimates on the ℓ^2-norm of the sequence $(p_n^{(m)}(z))_n$.

Corollary 4.6. *Let μ be a determinate measure with index of determinacy* $\operatorname{ind}_z(\mu)$ *equal to 0 if $z \notin \operatorname{supp}(\mu)$ and 1 if $z \in \operatorname{supp}(\mu)$, and let z be a complex number for which the sequence $(p_n^{(m)}(z))_n$ belongs to ℓ^2. Then for $m \geq 2$,*

$$\|p_n^{(m)}(z)\|_{\ell^2}^{-2} \leq \frac{1}{m!^2} \sup\{\nu(\{z\}) : \nu \text{ and } |t-z|^{2m}\mu \text{ have the same moments}\}$$

and for $m = 1$

$$\|p_n'(z)\|_{\ell^2}^{-2} = \sup\{\nu(\{z\}) : \nu \text{ and } |t-z|^2\mu \text{ have the same moments}\}.$$

Proof. The first estimate follows simply from (4.1). For the second estimate: We have proved that $(p_n'(z))_n \in \ell^2$ if and only if $F_\mu'(z) = 0$. Since the zeros of the function F_μ are simple, we get $z \notin \operatorname{supp}(\mu)$. Then, the formula follows from (4.1) and Lemma 4.5 (b.1).

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Matematisk Institut, Københavns Universitet, Universitetsparken 5, DK-2100 København, Denmark

E-mail address: berg@math.ku.dk

Departamento de Análisis Matemático, Universidad de Sevilla, Apdo. 1160. 41080 Sevilla, Spain

E-mail address: duran@cica.es