ALL FINITE GENERALIZED TRIANGLE GROUPS

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Abstract. We complete the classification of those generalized triangle groups that are finite.

Introduction

A generalized triangle group is one given by a presentation \( \langle a, b | a^p = b^q = R^m = 1 \rangle \), where \( p, q, m \) are integers greater than 1, and \( R \) is a word of the form \( a^{\alpha_1} b^{\beta_1} \cdots a^{\alpha_k} b^{\beta_k} \), \( k \geq 1 \), \( 1 < \alpha_i < p \), \( 1 < \beta_i < q \) for \( i = 1, \ldots, k \), which is not a proper power. Building on previous results of Baumslag, Morgan and Shalen [1], Conder [2] and Fine, Levin and Rosenberger [3], [4], [6] and [7], Howie, Metaftsis and Thomas [5] were able to give an almost complete list of those generalized triangle groups that are finite.

There are precisely two groups for which they could not decide whether or not they are finite. The two groups in question were

\[ G_1 = \langle a, b | a^2 = b^3 = (ababab^2abab^2ab^2)^2 = 1 \rangle \]

and

\[ G_2 = \langle a, b | a^2 = b^3 = (ababab^2abab^2ab^2)^2 = 1 \rangle. \]

Here we show that \( G_1 \) is infinite and \( G_2 \) is finite of order \( 2^{20} \cdot 3^4 \cdot 5 = 424673280 \).

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2. Theorem. (a) \( G_1 = \langle a, b | a^2 = b^3 = (ababab^2abab^2ab^2)^2 = 1 \rangle \) is infinite.

(b) \( G_2 = \langle a, b | a^2 = b^3 = (ababab^2abab^2ab^2)^2 = 1 \rangle \) is finite of order \( 2^{20} \cdot 3^4 \cdot 5 \).

Proof. (a) The commutator subgroup \( G'_1 \) of \( G_1 \) is generated by \( x = b \) and \( y = aba \), and, in these generators, it has a presentation

\[ G'_1 = \langle x, y | x^3 = y^3 = xyyx^{-1}yx^{-1}yx^{-1}yx^{-1}yx^{-1}yx^{-1}yx^{-1}yx^{-1} = 1 \rangle. \]

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Let $F \supset \mathbb{Q}$ be an algebraic number field containing an element $t$ with $t^6 - 3t^3 + 1 = 0$. The map

$$x \mapsto X = \begin{pmatrix} A & B & C \\ 0 & 0 & 1 \\ D & E & -A \end{pmatrix},$$

$$y \mapsto Y = \begin{pmatrix} D & E & -A \\ 3(Bt - D) & -D & -Ct \\ 1 & 0 & 0 \end{pmatrix}$$

where $A = -3t^4 + 8t$, $B = -4t^4 + 11t$, $C = 2t^3 - 6$, $D = -5t^5 + 14t^2$ and $E = -7t^5 + 19t^2$ extends to a representation of $G_1$ over $F$.

$(XY)^2$ has trace 3 which implies that $XY$ has infinite order. Indeed, if $XY$ would be of finite order then $(XY)^2$, as a matrix of finite order, would be diagonalizable and, hence, having trace 3, it would be the identity matrix $I$ which is not the case. Hence, in particular, $G_1$ is infinite and $ab$ has infinite order in $G_1$.

(b) For the computations for the second group $G_2$ we used the computer system GAP developed at Lehrstuhl D für Mathematik (J. Neubüser) at the RWTH Aachen. $G_2$ was found to contain a subgroup $H$ of index 24 and generated by $a$, $bab^{-1}$, $b^{-1}abab^{-1}ab$, $b^{-1}ab^{-1}abab^{-1}abab^{-1}abab$, $b^{-1}ab^{-1}abab^{-1}abab^{-1}abab^{-1}abab$ and

$$b^{-1}ab^{-1}abab^{-1}abab^{-1}ab^{-1}abab^{-1}abab.$$

The core $N$ of $H$ has index 13824. To get a presentation for $N$ one constructs the coset table of $N$ using the regular representation of $G_2$ on $H$. One then feeds the resulting coset table in the reduced Reidemeister-Schreier algorithm to get a presentation for $N$ in 1208 generators and 5745 relators. This presentation can be reduced via Tietze transformations to a presentation with 7 generators and 148 relators of total length 1584. The latter presentation is manageable with the Todd-Coxeter algorithm to get the order of $G_2$ which is $2^{20} \cdot 3^4 \cdot 5$. In fact, $G_2$ is a normal product of $SL(2, 5)$ and a solvable group with amalgamated central subgroup of order 2. □

Remarks. (1) The group $G_2$ is a counterexample to the conjecture at the end of Chapter I in [6]. Besides this group the conjecture seems to hold.

(2) There is a certain difference between $G_1$ and $G_2$ that is quite suggestive. In the case of $G_1$ there are certain subgroups for which the $p$-quotient algorithm revealed larger and larger $p$-factors. On the other hand if one takes $G_2$ and looks at various subgroups one does not get nilpotent factors of large class, all lower central series stop very soon. This fact was one of the reasons for suspecting that $G_2$ might be finite.

References


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