ON THE COHOMOLOGY OF $\Gamma_p$

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Abstract. Let $\Gamma_g$ denote the mapping class group of genus $g$. In this paper, we calculate $p$-torsion of Farrell cohomology $\tilde{H}^*(\Gamma_p)$ for any odd prime $p$.

Introduction

The mapping class group $\Gamma_g$ of a connected oriented surface $F_g^s$ of genus $g$ with $s$ punctures is defined as the group of connected components of the group of orientation-preserving diffeomorphisms of $F_g^s$ which possibly permute $s$ punctures. We will also denote $\Gamma_0^g$ simply by $\Gamma_g$. The cohomology $H^*(\Gamma_g)$ is one of the central topics in contemporary mathematics since it is closely related to algebraic topology, algebraic geometry, the theory of Riemann surfaces, the theory of three-dimensional manifolds, the theory of combinatorial groups and physics. It is well known that $\Gamma_1$ is the special linear group $SL_2(\mathbb{Z})$ and the cohomology $\tilde{H}^*(\Gamma_1; \mathbb{Z}) = H^*(SL(2, \mathbb{Z}); \mathbb{Z}) = \mathbb{Z}[u]/(12u)$, where $u$ is a generator of degree 2. The cohomology $H^*(\Gamma_2; \mathbb{Z})$ was completely calculated by Benson and Cohen in [BC]. Recently, Looijenga obtained $H^*(\Gamma_3; \mathbb{Q})$ with rational coefficient [L]. Recall that Farrell and ordinary cohomologies of $\Gamma_3$ coincide above the vcd($\Gamma_3$) = 7 (see [Br]). It is easy to see that the Farrell cohomology $\tilde{H}^*(\Gamma_3; \mathbb{Z})$ contains only 2, 3 and 7 torsion since $\Gamma_3$ does. The 7-component $\tilde{H}^*(\Gamma_3; \mathbb{Z})(7)$ is included in a general result of $\tilde{H}^*(\Gamma(p-1)/2; \mathbb{Z})(p)$ by the author in [XI]. The 2-component $\tilde{H}^*(\Gamma_3; \mathbb{Z})(2)$ is more difficult to calculate and remains open. In this note, we give the 3-component $\tilde{H}^*(\Gamma_3; \mathbb{Z})(3)$.

Let $\pi_1$ and $\pi_2$ denote representatives of the two different conjugacy classes of order 3 subgroups of $\Gamma_3$. We describe explicitly the quotients $N(\pi_1)/\pi_1$ and $N(\pi_2)/\pi_2$ as finite index subgroups of $\Gamma_1^2$ and $\Gamma_0^3$, where $N(-)$ stands for the normalizer. The cohomology $\tilde{H}^*(\Gamma_1^2)$ is completely calculated. The Shapiro lemma and a result of Cohen about $H^*(\Gamma_0^3; \mathbb{Z})$ as $\Sigma_3$-module are employed for computing $\tilde{H}^*(N(\pi_1); \mathbb{Z})(3)$ and $\tilde{H}^*(N(\pi_2); \mathbb{Z})(3)$ respectively. Then, the Farrell cohomology $\tilde{H}^*(\Gamma_3; \mathbb{Z})(3)$ follows immediately because $\Gamma_3$ is 3-periodic. It is generally believed that $\tilde{H}^*(\Gamma_g)$ (and $H^*(\Gamma_g)$) might be calculated inductively via $H^*(\Gamma_h)^s \ (h < g)$, the mapping class groups of lower genus with punctures.

For a fixed prime $p > 2$, the first two genera $g$'s such that $\Gamma_g$ contains a cyclic subgroup of order $p$ are $(p-1)/2$ and $p-1$. We have completed the
calculations of the \( p \)-component of \( \tilde{H}^*(\Gamma_{(p-1)/2} ; \mathbb{Z}) \) and \( \tilde{H}^*(\Gamma_p ; \mathbb{Z}) \) in our previous papers [X1] and [X2] respectively. Next, the third genus \( g \) such that \( \Gamma_g \) contains a cyclic subgroup of order \( p \) is \( p \). As one more successful example along these basic lines, we finish by calculating the \( p \)-component of \( \tilde{H}^*(\Gamma_p ; \mathbb{Z}) \) for any prime \( p \geq 3 \) (not only \( p = 3 \)) in this note. Note that the 2-component of \( H^*(\Gamma_2 ; \mathbb{Z}) \) is given in [BC].

The main results of this note are as follows.

**Theorem 5.4.**

\[
\tilde{H}^n(\Gamma_3 ; \mathbb{Z})_{(3)} = \mathbb{Z}/3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/9
\]

for \( n \equiv 0 \pmod{4} \);

\[
\tilde{H}^n(\Gamma_3 ; \mathbb{Z})_{(3)} = \mathbb{Z}/3
\]

for \( n \) odd;

\[
\tilde{H}^n(\Gamma_3 ; \mathbb{Z})_{(3)} = \mathbb{Z}/3 \oplus \mathbb{Z}/9
\]

for \( n \equiv 2 \pmod{4} \).

It is easy to see a dihedral subgroup \( D_{2p} \) of order \( 2p \) sitting in \( \Gamma_p \) for any prime \( p > 2 \).

**Theorem 6.5.** For any prime \( p > 3 \), the restriction map

\[
R : \tilde{H}^n(\Gamma_p ; \mathbb{Z})_{(p)} \to \tilde{H}^n(D_{2p} ; \mathbb{Z})_{(p)}
\]

is an isomorphism for any \( n \). Namely,

\[
\tilde{H}^n(\Gamma_p ; \mathbb{Z})_{(p)} = \mathbb{Z}/p
\]

for \( n \equiv 0 \pmod{4} \);

\[
\tilde{H}^n(\Gamma_p ; \mathbb{Z})_{(p)} = 0
\]

for other \( n \)’s.

The organization of the rest of this note is as follows. In section 1, we exactly describe two quotients \( N(\pi_1)/\pi_1 \) and \( N(\pi_2)/\pi_2 \) as finite index subgroups of \( \Gamma_3 \) and \( \Gamma_5 \). In section 2, we calculate \( H^*(\Gamma_3^2) \). In sections 3 and 4, we compute \( H^*(N(\pi_1)/\pi_1) \) and \( H^*(N(\pi_2)/\pi_2) \) respectively. In section 5, we obtain \( H^*(N(\pi_1)) \), \( H^*(N(\pi_2)) \) and prove the main result, Theorem 5.4. In last section, we finish the proof of Theorem 6.2.

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1. **The \( \mathbb{Z}/3 \)/\( \mathbb{Z}/3 \)’s of \( \Gamma_3 \)**

Recall that for \( x \) an orientation-preserving periodic diffeomorphism of a closed orientable surface \( F_g \) of prime period \( p \), the fixed point data of \( x \) are a set (unordered) \( \delta(x) = (\beta_1, \beta_2, \ldots, \beta_q) \), where \( q \) is the number of fixed points of \( x \) and \( \beta_i \) is the integer (mod \( p \)) such that \( x^{\beta_i} \) acts as multiplication by \( e^{2\pi i/p} \) in the local invariant complex structure at the \( i \)th fixed point. The fixed point data are well defined for an element \( x \in \Gamma_g \) of period \( p \) too. According to a classical theorem of Nielsen, the conjugacy classes of elements of \( \Gamma_g \) of period \( p \) are exactly given by all possible fixed point data. It is easy to check that there are exactly two conjugacy classes of order 3 subgroups of \( \Gamma_3 \).
the one with the fixed point data of a generator \( \langle 1, 2 \rangle \) is denoted as \( \pi_1 \) and the
other with the fixed point data of a generator \( \langle 1, 1, 1, 1, 2 \rangle \) is denoted as \( \pi_2 \).

The structure of quotients \( N(\pi_1)/\pi_1 \) and \( N(\pi_2)/\pi_2 \) are described as follows.

A result of MacLachlan and Harvey [MH] states that for a finite subgroup
\( G \subset \Gamma_g \) the quotient \( N(G)/G \) maps injectively into the mapping class group
\( \Gamma_h \), where \( h \) is the genus of orbit space \( F_g/G \), and \( q \) the number of singular
points. It is clear in our cases that the quotients \( N(\pi_1)/\pi_1 \) and \( N(\pi_2)/\pi_2 \) are
isomorphic to subgroups of mapping class groups \( \Gamma_1 \) and \( \Gamma_0 \) respectively. We
give a more precise description now.

Consider a natural homomorphism

\[
\lambda : \Gamma_h \rightarrow GL(n - 1 + 2h, \mathbb{Z})
\]

that is given by mapping a diffeomorphism \( f \in \text{Diff}_+(F_h; \{n\}) \) to its action
on \( H_1(F_h - \{n\}; \mathbb{Z}) \) with a base \( \{x_1, x_2, \ldots, x_{n-1}, a_1, \ldots, a_h, b_1, \ldots, b_h\} \)
in the obvious notation. The map \( \lambda \) is clearly not a surjection. An element of
\( \text{Im}(\lambda) \) must be in the form

\[
\begin{pmatrix}
A & \quad B
\end{pmatrix}
\begin{pmatrix}
0 & \quad D
\end{pmatrix}
\]

where \( A \in G \) (\( \cong \Sigma_n \), the symmetric group of \( n \) letters), \( D \in Sp(2g, \mathbb{Z}) \), the
symplectic group. Reducing the group \( GL(n - 1 + 2h, \mathbb{Z}) \) to a finite group
\( GL(n - 1 + 2h, \mathbb{Z}/p) \) with coefficient in the field \( \mathbb{Z}/p \), one gets a map \( \hat{\lambda} : \Gamma_h^n \rightarrow
GL(n - 1 + 2h, \mathbb{Z}/p) \). Actually, for any elementary abelian \( p \) subgroup \( E \subset \Gamma_g \),
the quotient \( N(E)/E \) is isomorphic to a finite index subgroup of \( \Gamma_h^n \), which is
a preimage of a subgroup \( K_E \subset GL(n - 1 + 2h, \mathbb{Z}/p) \) under the map \( \hat{\lambda} \). The
group \( K_E \) is specifically determined by some geometric data, for example, the
fixed point data of \( E \). The details of this general result will appear somewhere
else. Here, only special cases of the quotients \( N(\pi_1)/\pi_1 \) and \( N(\pi_2)/\pi_2 \) are
illustrated for the purpose of the calculation of \( H^*(\Gamma_3; \mathbb{Z}) \).

Consider the natural map

\[
\hat{\lambda} : \Gamma_1^2 \rightarrow GL(3, \mathbb{Z}/3)
\]

defined as above for \( h = 1 \) and \( n = 2 \). Let \( K_1 \) denote a subgroup of \( \text{Im}(\hat{\lambda}) \)
consisting of all elements of \( GL(3, \mathbb{Z}/3) \) in the form of

\[
\begin{pmatrix}
A & \quad 0
\end{pmatrix}
\begin{pmatrix}
0 & \quad D
\end{pmatrix}
\]

with \( A \in \{1, -1\} \), and \( D \in SL(2, \mathbb{Z}/3) \).

**Proposition 1.1.** The quotient \( N(\pi_1)/\pi_1 \) is isomorphic to \( \hat{\lambda}^{-1}(K_1) \subset \Gamma_1^2 \).

The following well-known lemma is needed in the proof of Proposition 1.1
above.

**Lemma 1.2.** Let \( p : F_g \rightarrow F_h \) be a \( p \)-sheeted branched covering map with \( n \)
ramification points. Then a diffeomorphism \( w \in \text{Diff}_+(F_h; \{n\}) \) lifts to a diffeomorphism
\( w \in \text{Diff}_+(F_g; \{n\}) \) if and only if every closed curve which lifts to
a closed curve maps (via \( w \)) to a closed curve which lifts to a closed curve.

**Proof** (of Proposition 1.1). Let \( p : F_3 \rightarrow F_1 \) be the 3-sheeted branched cov-
ering map with ramification points \( x_1 \) and \( x_2 \) induced by a generator of \( \pi_1 \)
(strictly speaking, some lift of \( \pi_1 \) to \( \text{Diff}_+(F_3, \{2\}) \)). We show that \( w \in \text{Diff}_+(F_1, \{2\}) \) lifts if and only if \( \lambda(w) \in K_1 \) (we abuse the notation \( w \) here). Let \( f : \pi_1(F_1 - \{x_1, x_2\}) \to \pi_1 \) be the surjective map determined by the map \( p \). Up to conjugation of \( \pi_1 \), one could choose 

\[
f : \pi_1(F_1 - \{x_1, x_2\}) = \langle a, b, x_1, x_2 \mid [a; b] x_1 x_2 = 1 \rangle \to \pi_1 = \langle y \rangle
\]

as \( f(a) = f(b) = 1 \), \( f(x_1) = y \) and \( f(x_2) = y^2 \). The basic covering space theory says that a closed curve \( \gamma \in F_1 - \{x_1, x_2\} \) lifts to a closed curve \( \gamma' \in F_2 - \{x_1, x_2\} \) if and only if \( f([\gamma]) = 1 \), where \([\cdot]\) stands for homotopy class here. Note that the set of surjective homomorphisms \( \text{epi}(\pi_1(F_1 - \{x_1, x_2\}) \to \pi_1) \) is in one-to-one correspondence to the set of surjective homomorphisms \( \text{epi}(H_1(F_1 - \{x_1, x_2\}; \mathbb{Z}) \to \pi_1) \) since the group \( \pi_1 \) is abelian. Let \( \gamma \in H_1(F_1 - \{x_1, x_2\}; \mathbb{Z}) \) be the homology class of \( \gamma \). Suppose \( \gamma = x_1^a x_2^b \) and \( \tilde{f} : H_1(F_1 - \{x_1, x_2\}; \mathbb{Z}) \to \pi_1 \) is induced by \( f : \pi_1(F_1 - \{x_1, x_2\}) \to \pi_1 \). It is easy to see that \( \tilde{f}(\gamma) = 1 \) is equivalent to \( m = 0 \) (mod 3). Let \( \lambda(w) \) be denoted by 

\[
\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}
\]

Then \( \tilde{f}(w \gamma) = 1 \) is equivalent to \( Am + BL = 0 \) (mod 3), where \( \begin{pmatrix} m \\ L \end{pmatrix} \) is a 3-vector of \( H_1(F_1 - \{x_1, x_2\}; \mathbb{Z}) \) with 

\[
L = \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}.
\]

Lemma 1.2 above says that \( w \) lifts is equivalent to the statement \( \tilde{f}(w \gamma) = 1 \) if \( f(\gamma) = 1 \); i.e., \( B = 0 \) (mod 3) because \( L \) could be an arbitrary two vector. We complete the proof.

Consider, for any \( n \), the well-known map \( \mu : \Gamma_0^n \to \Sigma_n \) defined via the permutation of \( f \in \text{Diff}_+(S^2, \{n\}) \) on \( n \) punctures. Recall that the quotient \( N(\pi_2)/\pi_2 \) is isomorphic to a subgroup of \( \Gamma_0^3 \). Then, one has

**Proposition 1.3.** The quotient \( N(\pi_2)/\pi_2 \) is isomorphic to \( \mu^{-1}(\Sigma_4) \subset \Gamma_0^3 \).

This proposition is a special case of Lemma 1.1 of [X2].

### 2. Cohomology of \( \Gamma_1^3 \)

Let \( P\Gamma_0^n \) denote the pure mapping class group of genus \( g \) with \( n \) punctures, i.e., the group of path components of orientation-preserving diffeomorphisms of a connected oriented surface \( F_g^n \) with \( n \) punctures which fix \( n \) punctures. Consider the group extension (see [Bi])

\[
1 \to F(2) = \pi_1(F_1 - \{x_1\}) \to P\Gamma_1^2 \to P\Gamma_1^2 = SL(2, \mathbb{Z}) \to 1
\]

given by forgetting one puncture, where \( F(2) \) is the free group of 2 generators. The Lyndon-Hochschild-Serre spectral sequence (LHS\(^3\)) for the extension above is given by 

\[
E_2^{p,q} = H^p(SL(2, \mathbb{Z}); H^q(F(2); \mathbb{Z})) \Rightarrow H^{p+q}(P\Gamma_1^2; \mathbb{Z})
\]

where \( H^0(F(2); \mathbb{Z}) = \mathbb{Z} \) as a trivial \( SL(2, \mathbb{Z}) \) module; \( H^1(F(2); \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \) as the \( SL(2, \mathbb{Z}) \) module is obtained by the usual \( SL(2, \mathbb{Z}) \) action on \( \mathbb{Z} \oplus \mathbb{Z} \).
It is well known that there is an amalgamated product decomposition $SL(2, \mathbb{Z}) = \mathbb{Z}/6 \ast_{\mathbb{Z}/2} \mathbb{Z}/4$. Choose generators $x \in \mathbb{Z}/6$, $y \in \mathbb{Z}/4$ and $z \in \mathbb{Z}/2$ as

$$x = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$$

and

$$z = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

A direct calculation gives

$$H^1(F(2); \mathbb{Z})^\mathbb{Z}/6 = 0, \quad H^1(F(2); \mathbb{Z})^\mathbb{Z}/6 = 0, \quad H^1(F(2); \mathbb{Z})^\mathbb{Z}/4 = 0,$$

$$H^1(F(2); \mathbb{Z})^\mathbb{Z}/4 = H^1(F(2); \mathbb{Z})/M_4 = \mathbb{Z}/2$$

where $M_4$ is a submodule consisting of all elements $(-2b, a - 2b)^T$ ($a$ and $b$ are integers);

$$H^1(F(2); \mathbb{Z})^{Z/2} = 0$$

and

$$H^1(F(2); \mathbb{Z})^{Z/2} = H^1(F(2); \mathbb{Z})/M_2 = \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

where $M_2$ is a submodule consisting of all elements $(-2a, -2b)^T$. This implies

$$H^n(\mathbb{Z}/6; H^1(F(2); \mathbb{Z})) = 0$$

for any $n$;

$$H^{odd}(\mathbb{Z}/4; H^1(F(2); \mathbb{Z})) = \mathbb{Z}/2,$$

$$H^{even}(\mathbb{Z}/4; H^1(F(2); \mathbb{Z})) = 0$$

and

$$H^{odd}(\mathbb{Z}/2; H^1(F(2); \mathbb{Z})) = \mathbb{Z}/2 \oplus \mathbb{Z}/2,$$

$$H^{even}(\mathbb{Z}/2; H^1(F(2); \mathbb{Z})) = 0.$$

Applying the M-V sequence to the group $SL(2, \mathbb{Z})$ with module $H^1(F(2); \mathbb{Z})$, one gets a long exact sequence

$$\rightarrow H^n(\mathbb{Z}/4; H^1(F(2); \mathbb{Z})) \oplus H^n(\mathbb{Z}/6; H^1(F(2); \mathbb{Z}))$$

$$\rightarrow H^n(\mathbb{Z}/2; H^1(F(2); \mathbb{Z})) \rightarrow H^{n+1}(SL(2, \mathbb{Z}); H^1(F(2); \mathbb{Z}))$$

$$\rightarrow H^{n+1}(\mathbb{Z}/4; H^1(F(2); \mathbb{Z})) \oplus H^{n+1}(\mathbb{Z}/6; H^1(F(2); \mathbb{Z}))$$

$$\rightarrow H^{n+1}(\mathbb{Z}/2; H^1(F(2); \mathbb{Z})) \rightarrow.$$ 

Note that the restriction map

$$H^n(\mathbb{Z}/4; H^1(F(2); \mathbb{Z})) \rightarrow H^n(\mathbb{Z}/2; H^1(F(2); \mathbb{Z}))$$

is an injection. It follows that

$$H^n(SL(2, \mathbb{Z}); H^1(F(2); \mathbb{Z})) = 0$$

if $n = 0$ or odd; and

$$H^n(SL(2, \mathbb{Z}); H^1(F(2); \mathbb{Z})) = \mathbb{Z}/2$$

if $n > 0$ even. Recall $H^*(SL(2, \mathbb{Z}); \mathbb{Z}) = \mathbb{Z}[u]/(12u)$. One claims that the LHS for (1) collapses by dimension reason. We conclude now
Proposition 2.1.

\[ H^0(\mathbb{P} \Gamma_1^2 ; \mathbb{Z}) = \mathbb{Z}, \quad H^{\text{odd}}(\mathbb{P} \Gamma_1^2 ; \mathbb{Z}) = \mathbb{Z}/2, \quad H^{\text{even}}(\mathbb{P} \Gamma_1^2 ; \mathbb{Z}) = \mathbb{Z}/12. \]

It is a routine to construct a \( \mathbb{Z}/3 \) action on Torus \( F_1 \) with three fixed points. This gives an order 3 subgroup \( \pi \subset \mathbb{P} \Gamma_1^2 \subset \Gamma_1^2 \). Proposition 2.1 tells that the restriction map \( H^*(\mathbb{P} \Gamma_1^2 ; \mathbb{Z})_{(3)} \rightarrow H^*(\pi ; \mathbb{Z})_{(3)} \) is an isomorphism. Furthermore, the universal coefficient theorem implies that the restriction map \( H^*(\mathbb{P} \Gamma_1^2 ; M)_{(3)} \rightarrow H^*(\pi ; M)_{(3)} \) is an isomorphism for any trivial \( \mathbb{P} \Gamma_1^2 \)-module \( M \). Note \( H^*(\Gamma_1^2 ; \mathbb{Z})_{(3)} = H^*(\mathbb{P} \Gamma_1^2 ; \mathbb{Z})_{(3)} \). In order to show the restriction map \( H^*(\Gamma_1^2 ; \mathbb{Z})_{(3)} \rightarrow H^*(\pi ; \mathbb{Z})_{(3)} \) is an isomorphism too, we only need to show the 3-period of \( \Gamma_1^2 \) is 2. The general form of the 3-period of a group is \( \text{LCM}\{2|N(n)/C(n)\}\alpha \) (see [GMX] for details). We know that \( \alpha = 0 \) above from Proposition 2.1. Therefore, we only need to see the order \( |N_{\Gamma_1^2}(\pi)/C_{\Gamma_1^2}(\pi)| = 1 \) in this case. Let \( x \in \text{Diff}_+(F_1, \{2\}) \) denote a period 3 element with three fixed points. It is obvious that \( x \) is not conjugate to \( x^2 \) because they are not conjugate even mapping to \( SL(2, \mathbb{Z}) \). In summary, one obtains

**Theorem 2.2.** The restriction map

\[ R : H^*(\Gamma_1^2 ; M)_{(3)} \rightarrow H^*(\mathbb{P} \Gamma_1^2 ; M)_{(3)} \rightarrow H^*(\pi ; M)_{(3)} \]

is an isomorphism for any trivial \( \Gamma_1^2 \)-module \( M \).

3. Cohomology of \( N(\pi_1)/\pi_1 \)

Recall that we defined the map \( \tilde{\lambda} : \Gamma_1^2 \rightarrow GL(3, \mathbb{Z}/3) \) and a subgroup \( K_1 \subset GL(3, \mathbb{Z}/3) \) in section 1. Proposition 1.1 says the quotient \( N(\pi_1)/\pi_1 \) is isomorphic to \( \tilde{\lambda}^{-1}(K_1) \). Let \( G \) denote the image of \( \tilde{\lambda} \). Recall that any element of \( G \) must be in the form

\[
\begin{pmatrix}
A & B \\
0 & D 
\end{pmatrix}
\]

(see section 1 for details). We remark here that in our case \( G \) is exactly the group consisting of all such matrices. In fact, one can see from geometry that \( \tilde{\lambda}(F(2)) \) contains matrices

\[
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1 
\end{pmatrix}.
\]

Thus, the index of \( K_1 \) in \( G \) is 9 and \( F(2) \) acts on \( G/K_1 \) via the map \( \tilde{\lambda} \) transitively. It is clear that \( \Gamma_1^2/N(\pi_1)/\pi_1 \) is in one-one correspondence to \( G/K_1 \) as cosets. By the well-known Shapiro lemma, one has \( H^*(N(\pi_1)/\pi_1 ; \mathbb{Z}) = H^*(\Gamma_1^2 ; \mathbb{Z}[G/K_1]) \), where \( \Gamma_1^2 \) acts on the permutation module \( \mathbb{Z}[G/K_1] \) via the map \( \tilde{\lambda} \).

We have seen that \( \Gamma_1^2 \) contains a subgroup \( \pi \) of order 3 in section 2. However, one can show

**Proposition 3.1.** The group \( N(\pi_1)/\pi_1 \) does not contain any subgroup of order 3.

**Proof.** It is obvious from the Riemann-Hurwitz formula that \( \Gamma_3 \) does not contain \( \mathbb{Z}/3 \times \mathbb{Z}/3 \). We only need to show that the third power 3 of any order 9
diffeomorphism of $F_3$ has five fixed points, not two fixed points like a lift of $\pi_1$. This again follows directly from the Riemann-Hurwitz formula.

Proposition 3.1 above implies that the permutation module $\mathbb{Z}[G/K_1]$ is not the trivial module $\mathbb{Z}$ and $\pi_1$ acts on $\mathbb{Z}[G/K_1]$ (by multiplication) nontrivially. It is elementary to observe that

**Lemma 3.2.** The group $\pi_1$ acts on the coset $G/K_1$ freely.

**Proof.** If not, assume that $x \in \pi_1$ fixes $g \in G/K_1$; i.e., $xgk = gk'$, or $g^{-1}xg = k'k^{-1} \in K_1$. This contradicts Proposition 3.1.

Therefore, one has the invariant $\mathbb{Z}[G/K_1]^{n_1} = \bigoplus \mathbb{Z}\langle \bar{n}_i \rangle$, where $\bar{n}_i = \bar{g}_i + x\bar{g}_i + x^2\bar{g}_i$ for some $\bar{g}_i$ ($1 \leq i \leq 3$) in this case. The co-invariant $\mathbb{Z}[G/K_1]_{n_1} = \mathbb{Z}[G/K_1]/M_1 = \bigoplus \mathbb{Z}$ spanned by $\bar{g}_i$'s. A direct computation implies the normal map

$$N: \mathbb{Z}[G/K_1]_{n_1} \to \mathbb{Z}[G/K_1]^{n_1}$$

is an isomorphism. So, one gets

**Proposition 3.3.** $H^n(\pi_1; \mathbb{Z}[G/K_1]) = 0$ for $n > 0$.

Consider the LHS$^3$ given by

$$E^2_{p,q} = H^p(SL(2, \mathbb{Z}); H^q(F(2); \mathbb{Z}[G/K_1])) \Rightarrow H^{p+q}(\Gamma_2^5; \mathbb{Z}[G/K_1])$$

for the extension (1) with coefficient $\mathbb{Z}[G/K_1]$.

It is immediate from Proposition 3.3 and the M-V sequence that

**Proposition 3.4.** $H^n(SL(2, \mathbb{Z}); \mathbb{Z}[G/K_1]^{F(2)})_{(3)} = 0$ for $n > 0$.

Note that the $SL(2, \mathbb{Z})$ acts on

$$H^1(F(2); \mathbb{Z}[G/K_1]) = H^1(\mathbb{Z}; \mathbb{Z}[G/K_1]) \oplus H^1(\mathbb{Z}; \mathbb{Z}[G/K_1])$$

as matrix multiplications given in Section 2. One obtains

**Proposition 3.5.** $H^n(SL(2, \mathbb{Z}); H^1(F(2); \mathbb{Z}[G/K_1]))_{(3)} = 0$ for $n > 0$.

Combining Propositions 3.4 and 3.5, one concludes

**Proposition 3.6.** $H^n(N(\pi_1)/\pi_1; \mathbb{Z})_{(3)} = 0$ for any $n \geq 0$.

Repeating the argument above with $\mathbb{Z}/3$ coefficient, one gets

**Proposition 3.7.** $H^n(N(\pi_1)/\pi_1; \mathbb{Z}/3) = 0$ for $n > 0$.

A similar proof of Proposition 2.1 and the Shapiro lemma give

**Proposition 3.8.** $H^n(N(\pi_1)/\pi_1; \mathbb{Z})$ does not contain any copy of $\mathbb{Z}$ for $n > 0$.

## 4. Cohomology of $N(\pi_2)/\pi_2$

Consider the group extension

$$1 \to \Gamma_0^5 \to N(\pi_2)/\pi_2 \to \Sigma_4 \to 1$$

described in Proposition 1.3. The LHS$^3$ for the extension above is given by

$$E^2_{p,q} = H^p(\Sigma_4; H^q(\Gamma_0^5; \mathbb{Z}/3)) \Rightarrow H^{p+q}(N(\pi_2)/\pi_2; \mathbb{Z}/3)$$

where $\Sigma_4$ acts on $H^q(\Gamma_0^5; \mathbb{Z}/3)$ as shown in work of Cohen (the $\Gamma_0^5$ is denoted by $K_5$ in [BC]). Recall that $H^*(\Gamma_0^5; \mathbb{Z}/3)$ is generated by one-dimen-
sional elements $B_{42}, B_{43}, B_{52}, B_{53}$ and $B_{54}$ subject to some relations specifically given in [BC]. Let $x = (123) \in \Sigma_4$ be a generator of a Sylow 3-subgroup. It is a routine to have

$$H^1(PT_0^5; \mathbb{Z}/3)^{(x)} = \mathbb{Z}/3 \oplus \mathbb{Z}/3 = (2B_{42} + B_{43}, B_{52} + 2B_{53})$$

and

$$H^1(PT_0^5; \mathbb{Z}/3)^{(x)} = H^1(PT_0^5; \mathbb{Z}/3)/M_5$$

where the submodule $M_5$ consists of all elements in the form

$$(m_1 + m_2 + m_3)B_{42} + (2m_2 - m_1)B_{43} + (m_3 + m_4 - m_5)B_{52} + (2m_4 - m_3 - m_5)B_{53}$$

with $m_i \in \mathbb{Z}/3$. Let $b_1 = m_1 + m_2 + m_3$, $b_2 = 2m_2 - m_1$, $b_3 = m_3 + m_4 - m_5$ and $b_4 = 2m_4 - m_3 - m_5$. Elementary linear algebra implies $3m_1 = 2b_1 - b_2 - 2m_5 = 0$, $3m_2 = b_1 + b_2 - m_5 = 0$, $3m_3 = 2b_3 - b_4 + m_5$ and $3m_4 = b_3 + b_4 + 2m_5 = 0$. Thus, the equation $b_1 + b_2 + 2b_3 + 2b_4 = 0$ holds. This amounts to showing

$$H^1(PT_0^5; \mathbb{Z}/3)^{(x)} = \mathbb{Z}/3 \oplus \mathbb{Z}/3$$

generated by $(B_{54}, B_{42})$. It is easy to check that the normal map

$$N : H^1(PT_0^5; \mathbb{Z})^{(x)} \to H^1(PT_0^5; \mathbb{Z})$$

is given by $N(B_{54}) = B_{42} + 2B_{43} + B_{52} + 2B_{53}$ and $N(B_{42}) = 0$. Thus, one obtains

**Lemma 4.1.**

$$H^0((x); H^1(PT_0^5; \mathbb{Z}/3)^{(x)}) = \mathbb{Z}/3 \oplus \mathbb{Z}/3,$$

$$H^{\text{odd}}((x); H^1(PT_0^5; \mathbb{Z}/3)^{(x)}) = \mathbb{Z}/3,$$

$$H^{\text{even}}((x); H^1(PT_0^5; \mathbb{Z}/3)^{(x)}) = \mathbb{Z}/3.$$
given by
\[ N(\overline{B}_{42}B_{52}) = 2B_{42}B_{52} + B_{42}B_{53} + B_{43}B_{52} + 2B_{43}B_{52} \]
and
\[ N(\overline{B}_{43}B_{54}) = -B_{42}B_{52} - B_{43}B_{52} - B_{43}B_{53} \]
is an isomorphism. This implies

**Lemma 4.2.** \( H^0((x); H^2(\Gamma_0^5; \mathbb{Z}/3)) = \mathbb{Z}/3 \oplus \mathbb{Z}/3 \) and \( H^n((x); H^2(\Gamma_0^5; \mathbb{Z}/3)) = 0 \) for \( n > 0 \).

Recall that \( H^1(\Gamma_0^5; \mathbb{Z}/3)_x \) is generated by \( \overline{B}_{54} \) and \( \overline{B}_{42} \). We can check directly that \( (12) \in \Sigma_4 \) permutes \( \overline{B}_{54} \) to \( \overline{B}_{54} - B_{42} \) and \( \overline{B}_{42} \) to \( -\overline{B}_{42} \); that is, \( H^1(\Gamma_0^5; \mathbb{Z}/3)_{(12)} = 0 \). It is also straightforward to verify \( (12) \in \Sigma_4 \) acts on generators \( 2B_{42} + B_{43} \) and \( B_{52} + 2B_{53} \) of \( H^1(\Gamma_0^5; \mathbb{Z}/3)_x \) trivially and acts on the one-dimensional space generated by
\[ 2B_{42}B_{52} + B_{43}B_{52} + B_{42}B_{53} + 2B_{43}B_{53} \in H^2(\Gamma_0^5; \mathbb{Z}/3)_x \]
trivially. These calculations imply

**Lemma 4.3.**
\[ H^0(\Sigma_4; H^1(\Gamma_0^5; \mathbb{Z}/3)) = \mathbb{Z}/3 \oplus \mathbb{Z}/3, \]
\[ H^0(\Sigma_4; H^2(\Gamma_0^5; \mathbb{Z})) = \mathbb{Z}/3, \]
\[ H^n(\Sigma_4; H^1(\Gamma_0^5; \mathbb{Z}/3)) = \mathbb{Z}/3 \]
for \( n \equiv 0, 1 \) (mod 4);
\[ H^n(\Sigma_4; H^1(\Gamma_0^5; \mathbb{Z}/3)) = 0 \]
for \( n \equiv 2, 3 \) (mod 4).

It is easy to see a \( \mathbb{Z}/3 \subset N(\pi_2)/\pi_2 \subset \Gamma_0^5 \) by constructing a \( \mathbb{Z}/3 \) action on \( S^2 \) with two fixed points and permuting three points. The following lemma is needed for the study of \( \text{LHS}^3 \) associated to the extension (2) in the beginning of this section.

**Lemma 4.4.** The group \( N(\pi_2)/\pi_2 \) has the \( \mathbb{Z}/3 \) as a retract.

**Proof.** Recall the group \( N(\pi_2)/\pi_2 \) is an extension of \( \Gamma_0^5 \) over \( \Sigma_4 \). There is a surjective map by forgetting the fifth puncture from \( N(\pi_2)/\pi_2 \) to \( \Gamma_0^5 \), therefore, to \( H_1(\Gamma_4; \mathbb{Z}) = \mathbb{Z}/2 \oplus \mathbb{Z}/3 \), to \( \mathbb{Z}/3 \). Note the \( \mathbb{Z}/3 \subset N(\pi_2)/\pi_2 \) is compatible with the \( \mathbb{Z}/3 \subset \Gamma_0^5 \). The lemma follows since \( \Gamma_0^4 \) has the \( \mathbb{Z}/3 \) as a retract.

Now, one can conclude the \( \text{LHS}^3 \) collapses by Lemma 4.4 and

**Proposition 4.5.**
\[ H^0(N(\pi_2)/\pi_2; \mathbb{Z}/3) = \mathbb{Z}/3, \]
\[ H^n(N(\pi_2)/\pi_2; \mathbb{Z}/3) = \mathbb{Z}/3 \oplus \mathbb{Z}/3 \]
if \( n = 1, 2 \);
\[ H^n(N(\pi_2)/\pi_2; \mathbb{Z}/3) = \mathbb{Z}/3 \]
if \( n \geq 3 \).

Repeat the calculation in this section above with coefficient \( \mathbb{Z} \) and consider \( \text{LHS}^3 \) for the extension (2) with coefficient \( \mathbb{Z} \); one gets
Proposition 4.6. The restriction map
\[ R : H^n(N(\pi_2)/\pi_2 ; \mathbb{Z}) \rightarrow H^n(\mathbb{Z}/3 ; \mathbb{Z})(3) \]
induces an isomorphism; the group \( H^n(N(\pi_2)/\pi_2 ; \mathbb{Z}) \) contains exactly one copy of \( \mathbb{Z} \) for \( n = 0, 1, 2 \) and contains no copy of \( \mathbb{Z} \) for \( n \geq 3 \).

5. Farrell cohomology of \( \Gamma_3 \)

We actually calculate not only the 3-components of
\[ H^*(N(\pi_1) ; \mathbb{Z}) \quad \text{and} \quad H^*(N(\pi_2) ; \mathbb{Z}), \]
but also their free parts. Consider the group extensions
\[ 1 \rightarrow \pi_1 \rightarrow N(\pi_1) \rightarrow N(\pi_1)/\pi_1 \rightarrow 1 \]
and
\[ 1 \rightarrow \pi_2 \rightarrow N(\pi_2) \rightarrow N(\pi_2)/\pi_2 \rightarrow 1. \]
One has the LHS\(^3\) for the extensions above giving as
\[ E_p^{q,q} = H^p(N(\pi_1)/\pi_1 ; H^q(\pi_1 ; \mathbb{Z})) \Rightarrow H^{p+q}(N(\pi_1) ; \mathbb{Z}). \]
Note that the group \( N(\pi_1) \) acts on \( \pi_1 \) nontrivially and the group \( N(\pi_2) \) acts on \( \pi_2 \) trivially from the observation of the fixed point data of generators of \( \pi_1 \) and \( \pi_2 \).

It is easy to see a dihedral subgroup \( D_6 \subset \Gamma_3 \) of order 6 containing the \( \pi_1 \) by realizing a \( D_6 \) action on \( F_3 \) with four singular points of order 2 and one singular point of order 3 in the orbit space \( F_3/D_6 = S^2 \) (2 sphere). The following proposition is immediate.

Proposition 5.1.
(1) The restriction map
\[ R : H^n(N(\pi_1) ; \mathbb{Z}) \rightarrow H^n(D_6 ; \mathbb{Z})(3) \]
is an isomorphism for any \( n \geq 0 \).
(2) \( H^n(N(\pi_1) ; \mathbb{Z}) \) does not contain any \( \mathbb{Z} \) for \( n > 0 \).

Again, it is clear that the \( \pi_2 \) is contained in a \( \mathbb{Z}/9 \subset \Gamma_3 \) if one notices that there is a \( \mathbb{Z}/9 \) action on \( F_3 \) with two singular points of order 9 and one singular point on the orbit space \( F_3/\mathbb{Z}/9 = S^2 \) (2 sphere). Comparing the LHS\(^3\) for the extension
\[ 1 \rightarrow \pi_2 \rightarrow \mathbb{Z}/9 \rightarrow \mathbb{Z}/3 \rightarrow 1 \]
with Proposition 4.5, one obtains

Proposition 5.2.
\[ H^n(N(\pi_2) ; \mathbb{Z})(3) = 0 \]
for \( n = 0, 1; \)
\[ H^2(N(\pi_2) ; \mathbb{Z})(3) = \mathbb{Z}/9, \quad H^n(N(\pi_2) ; \mathbb{Z})(3) = \mathbb{Z}/3 \]
for \( n \geq 3 \) odd;
\[ H^n(N(\pi_2) ; \mathbb{Z})(3) = \mathbb{Z}/3 \oplus \mathbb{Z}/9 \]
for \( n \geq 4 \) even.
Proposition 5.3. $H^n(N(\pi_2); \mathbb{Z})$ contains exactly one copy of $\mathbb{Z}$ for $n = 0, 1, 2$ and contains no $\mathbb{Z}$ for $n \geq 3$.

The main result about Farrell cohomology now follows readily since $\Gamma_3$ is 3-periodic.

Theorem 5.4.

$\tilde{H}^n(\Gamma_3; \mathbb{Z}_{(3)}) = \mathbb{Z}/3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/9$
for $n \equiv 0 \pmod{4}$;

$\tilde{H}^n(\Gamma_3; \mathbb{Z}_{(3)}) = \mathbb{Z}/3$
for $n$ odd;

$H^n(\Gamma_3; \mathbb{Z}_{(3)}) = \mathbb{Z}/3 \oplus \mathbb{Z}/9$
for $n \equiv 2 \pmod{4}$.

6. The $p$-component of Farrell cohomology of $\Gamma_p$ for $p > 3$

For any prime $p > 3$, it is easy to check from possible fixed point data that there is one and only one conjugacy class of order $p$ subgroup of $\Gamma_p$, denoted as $\pi \subset \Gamma_p$. The fixed point data of a generator of $\pi$ is $(1, p - 1)$. Thus, the cyclic group $N(\pi)/C(\pi)$ is $\mathbb{Z}/2$. Actually, it is not difficult to observe a dihedral subgroup $D_{2p} \subset \Gamma_p$ by constructing a surjective map from $\pi_1(F_1 - \{x_1, x_2\})$ onto $D_{2p}$.

Let $K_1$ denote a subgroup of $\text{Im}(\tilde{\lambda})$ consisting of all elements of $GL(3, \mathbb{Z}/p)$ in the form of

$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$

with $A \in \{1, -1\}$ and $D \in SL(2, \mathbb{Z}/p)$, where

$\tilde{\lambda}: \Gamma_1^2 \to GL(3, \mathbb{Z}/p)$

is defined as in section 1.

Proposition 6.1. The quotient $N(\pi)/\pi$ is isomorphic to $\tilde{\lambda}^{-1}(K_1) \subset \Gamma_1^2$.

The proof is the same as Proposition 1.1.

Proposition 6.2. $H^n(\Gamma_1^2; M)_{(p)} = 0$ for any prime $p > 3$, $n > 0$ and $\mathbb{Z}\Gamma_1^2$-module $M$.

Repeat the argument in section 2 with any coefficient $\mathbb{Z}\Gamma_1^2$-module $M$; the proof follows immediately.

By using the Shapiro lemma again, one gets

Proposition 6.3. $H^n(N(\pi)/\pi; \mathbb{Z})_{(p)} = 0$ for any $p > 3$ and $n > 0$.

Finally, comparing two short exact sequences

$1 \to \pi \to N(\pi) \to N(\pi)/\pi \to 1$,

$1 \to \pi \to D_{2p} \to \mathbb{Z}/2 \to 1$

and considering two LHS$^3$ associated to them, one obtains
Proposition 6.4. The restriction map
\[ R: H^n(N(n); \mathbb{Z}_p) \to H^n(D_{2p}; \mathbb{Z}_p) \]
is an isomorphism for any \( p > 3 \) and \( n \geq 0 \).

Theorem 6.5. For a prime \( p > 3 \), the restriction map
\[ R: \hat{H}^n(\Gamma_p; \mathbb{Z}_p) \to \hat{H}^n(D_{2p}; \mathbb{Z}_p) \]
is an isomorphism for any \( n \). Namely,
\[ \hat{H}^n(\Gamma_p; \mathbb{Z}_p) = \mathbb{Z}_p \]
for any \( n \equiv 0 \pmod{4} \);
\[ \hat{H}^n(\Gamma_p; \mathbb{Z}_p) = 0 \]
for any \( n \not\equiv 0 \pmod{4} \).

References


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