ON THE COHOMOLOGY OF $\Gamma_p$

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Abstract. Let $\Gamma_g$ denote the mapping class group of genus $g$. In this paper, we calculate $p$-torsion of Farrell cohomology $\tilde{H}^*(\Gamma_p)$ for any odd prime $p$.

Introduction

The mapping class group $\Gamma_g$ of a connected oriented surface $F_g$ of genus $g$ with $s$ punctures is defined as the group of connected components of the group of orientation-preserving diffeomorphisms of $F_g$ which possibly permute $s$ punctures. We will also denote $\Gamma^0_g$ simply by $\Gamma_g$. The cohomology $H^*(\Gamma_g)$ is one of the central topics in contemporary mathematics since it is closely related to algebraic topology, algebraic geometry, the theory of Riemann surfaces, the theory of three-dimensional manifolds, the theory of combinatorial groups and physics. It is well known that $\Gamma_1$ is the special linear group $SL_2(\mathbb{Z})$ and the cohomology $H^*(\Gamma_1; \mathbb{Z}) = H^*(SL(2, \mathbb{Z}); \mathbb{Z}) = \mathbb{Z}[u]/(12u)$, where $u$ is a generator of degree 2. The cohomology $H^*(\Gamma_2; \mathbb{Z})$ was completely calculated by Benson and Cohen in [BC]. Recently, Looijenga obtained $H^*(\Gamma_3; \mathbb{Q})$ with rational coefficient [L]. Recall that Farrell and ordinary cohomologies of $\Gamma_3$ coincide above the vcd($\Gamma_3$) = 7 (see [Br]). It is easy to see that the Farrell cohomology $\tilde{H}^*(\Gamma_3; \mathbb{Z})$ contains only 2, 3 and 7 torsion since $\Gamma_3$ does. The 7-component $\tilde{H}^*(\Gamma_3; \mathbb{Z})_7$ is included in a general result of $\tilde{H}^*(\Gamma_{(p-1)/2}; \mathbb{Z})_7$ by the author in [XI]. The 2-component $\tilde{H}^*(\Gamma_3; \mathbb{Z})_2$ is more difficult to calculate and remains open. In this note, we give the 3-component $\tilde{H}^*(\Gamma_3; \mathbb{Z})_3$.

Let $\pi_1$ and $\pi_2$ denote representatives of the two different conjugacy classes of order 3 subgroups of $\Gamma_3$. We describe explicitly the quotients $N(\pi_1)/\pi_1$ and $N(\pi_2)/\pi_2$ as finite index subgroups of $\Gamma_1^3$ and $\Gamma_0^3$, where $N(-)$ stands for the normalizer. The cohomology $H^*(\Gamma_3^3)$ is completely calculated. The Shapiro lemma and a result of Cohen about $H^*(\Gamma_3^3; \mathbb{Z})$ as $\Sigma_3$-module are employed for computing $H^*(N(\pi_1))(3)$ and $H^*(N(\pi_2))(3)$ respectively. Then, the Farrell cohomology $\tilde{H}^*(\Gamma_3; \mathbb{Z})_3$ follows immediately because $\Gamma_3$ is 3-periodic. It is generally believed that $\tilde{H}^*(\Gamma_g)$ (and $H^*(\Gamma_g)$) might be calculated inductively via $H^*(\Gamma_h)^*(h < g)$, the mapping class groups of lower genus with punctures. For a fixed prime $p > 2$, the first two genera $g$'s such that $\Gamma_g$ contains a cyclic subgroup of order $p$ are $(p-1)/2$ and $p-1$. We have completed the
calculations of the $p$-component of $\hat{H}^*(\Gamma_{p-1}/2; \mathbb{Z})$ and $\hat{H}^*(\Gamma_p; \mathbb{Z})$ in our previous papers [X1] and [X2] respectively. Next, the third genus $g$ such that $\Gamma_g$ contains a cyclic subgroup of order $p$ is $p$. As one more successful example along these basic lines, we finish by calculating the $p$-component of $\hat{H}^*(\Gamma_p; \mathbb{Z})$ for any prime $p \geq 3$ (not only $p = 3$) in this note. Note that the 2-component of $H^*(\Gamma_2; \mathbb{Z})$ is given in [BC].

The main results of this note are as follows.

**Theorem 5.4.**

\[ \hat{H}^n(\Gamma_3; \mathbb{Z})(3) = \mathbb{Z}/3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/9 \]

for $n \equiv 0 \pmod{4}$;

\[ \hat{H}^n(\Gamma_3; \mathbb{Z})(3) = \mathbb{Z}/3 \]

for $n$ odd;

\[ \hat{H}^n(\Gamma_3; \mathbb{Z})(3) = \mathbb{Z}/3 \oplus \mathbb{Z}/9 \]

for $n \equiv 2 \pmod{4}$.

It is easy to see a dihedral subgroup $D_{2p}$ of order $2p$ sitting in $\Gamma_p$ for any prime $p > 2$.

**Theorem 6.5.** For any prime $p > 3$, the restriction map

\[ R: \hat{H}^n(\Gamma_p; \mathbb{Z})(p) \rightarrow \hat{H}^n(D_{2p}; \mathbb{Z})(p) \]

is an isomorphism for any $n$. Namely,

\[ \hat{H}^n(\Gamma_p; \mathbb{Z})(p) = \mathbb{Z}/p \]

for $n \equiv 0 \pmod{4}$;

\[ \hat{H}^n(\Gamma_p; \mathbb{Z})(p) = 0 \]

for other $n$'s.

The organization of the rest of this note is as follows. In section 1, we exactly describe two quotients $N(\pi_1)/\pi_1$ and $N(\pi_2)/\pi_2$ as finite index subgroups of $\Gamma_2^2$ and $\Gamma_0^3$. In section 2, we calculate $H^*(\Gamma_2^2)$. In sections 3 and 4, we compute $H^*(N(\pi_1)/\pi_1)$ and $H^*(N(\pi_2)/\pi_2)$ respectively. In section 5, we obtain $H^*(N(\pi_1)), H^*(N(\pi_2))$ and prove the main result, Theorem 5.4. In last section, we finish the proof of Theorem 6.2.

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1. The $N(\mathbb{Z}/3)/\mathbb{Z}/3$'s of $\Gamma_3$

Recall that for $x$ an orientation-preserving periodic diffeomorphism of a closed orientable surface $F_g$ of prime period $p$, the fixed point data of $x$ are a set (unordered) $\delta(x) = (\beta_1, \beta_2, \ldots, \beta_q)$, where $q$ is the number of fixed points of $x$ and $\beta_i$ is the integer (mod $p$) such that $x^{\beta_i}$ acts as multiplication by $e^{2\pi i / p}$ in the local invariant complex structure at the $i$th fixed point. The fixed point data are well defined for an element $x \in \Gamma_g$ of period $p$ too. According to a classical theorem of Nielsen, the conjugacy classes of elements of $\Gamma_g$ of period $p$ are exactly given by all possible fixed point data. It is easy to check that there are exactly two conjugacy classes of order 3 subgroups of $\Gamma_3$. 
the one with the fixed point data of a generator \( (1, 2) \) is denoted as \( \pi_1 \) and the other with the fixed point data of a generator \( (1, 1, 1, 1, 2) \) is denoted as \( \pi_2 \). The structure of quotients \( N(\pi_1)/\pi_1 \) and \( N(\pi_2)/\pi_2 \) are described as follows.

A result of MacLachlan and Harvey [MH] states that for a finite subgroup \( G \subset \Gamma_g \) the quotient \( N(G)/G \) maps injectively into the mapping class group \( \Gamma_h^q \), where \( h \) is the genus of orbit space \( F_g/G \), and \( q \) the number of singular points. It is clear in our cases that the quotients \( N(\pi_1)/\pi_1 \) and \( N(\pi_2)/\pi_2 \) are isomorphic to subgroups of mapping class groups \( \Gamma_1^1 \) and \( \Gamma_0^1 \) respectively. We give a more precise description now.

Consider a natural homomorphism

\[
\lambda : \Gamma_h^q \rightarrow GL(n-1+2h, \mathbb{Z})
\]

that is given by mapping a diffeomorphism \( f \in Diff_+(F_h; \{n\}) \) to its action on \( H_1(F_h-\{n\}; \mathbb{Z}) \) with a base \( \langle x_1, x_2, \ldots, x_{n-1}, a_1, \ldots, a_h, b_1, \ldots, b_h \rangle \) in the obvious notation. The map \( \lambda \) is clearly not a surjection. An element of \( \text{Im}(\lambda) \) must be in the form

\[
\begin{pmatrix}
A & B \\
0 & D
\end{pmatrix}
\]

where \( A \in G \) (\( \cong \Sigma_n \), the symmetric group of \( n \) letters), \( D \in Sp(2g, \mathbb{Z}) \), the symplectic group. Reducing the group \( GL(n-1+2h, \mathbb{Z}) \) to a finite group \( GL(n-1+2h, \mathbb{Z}/p) \) with coefficient in the field \( \mathbb{Z}/p \), one gets a map \( \tilde{\lambda} : \Gamma_h^q \rightarrow GL(n-1+2h, \mathbb{Z}/p) \). Actually, for any elementary abelian \( p \) subgroup \( E \subset \Gamma_g \), the quotient \( N(E)/E \) is isomorphic to a finite index subgroup of \( \Gamma_h^q \), which is a preimage of a subgroup \( K_E \subset GL(n-1+2h, \mathbb{Z}/p) \) under the map \( \tilde{\lambda} \). The group \( K_E \) is specifically determined by some geometric data, for example, the fixed point data of \( E \). The details of this general result will appear somewhere else. Here, only special cases of the quotients \( N(\pi_1)/\pi_1 \) and \( N(\pi_2)/\pi_2 \) are illustrated for the purpose of the calculation of \( \tilde{H}^*(\Gamma_3; \mathbb{Z}) \).

Consider the natural map

\[
\lambda : \Gamma_2^2 \rightarrow GL(3, \mathbb{Z}/3)
\]

defined as above for \( h = 1 \) and \( n = 2 \). Let \( K_1 \) denote a subgroup of \( \text{Im}(\lambda) \) consisting of all elements of \( GL(3, \mathbb{Z}/3) \) in the form of

\[
\begin{pmatrix}
A & 0 \\
0 & D
\end{pmatrix}
\]

with \( A \in \{1, -1\} \), and \( D \in SL(2, \mathbb{Z}/3) \).

**Proposition 1.1.** The quotient \( N(\pi_1)/\pi_1 \) is isomorphic to \( \lambda^{-1}(K_1) \subset \Gamma_2^2 \).

The following well-known lemma is needed in the proof of Proposition 1.1 above.

**Lemma 1.2.** Let \( p : F_h \rightarrow F_g \) be a \( p \)-sheeted branched covering map with \( n \) ramification points. Then a diffeomorphism \( w \in Diff_+(F_h, \{n\}) \) lifts to a diffeomorphism \( w \in Diff_+(F_g, \{n\}) \) if and only if every closed curve which lifts to a closed curve maps (via \( w \)) to a closed curve which lifts to a closed curve.

**Proof** (of Proposition 1.1). Let \( p : F_3 \rightarrow F_1 \) be the 3-sheeted branched covering map with ramification points \( x_1 \) and \( x_2 \) induced by a generator of \( \pi_1 \).
(strictly speaking, some lift of $\pi_1$ to $\text{Diff}_+(F_3, \{2\})$). We show that $w \in \text{Diff}_+(F_1, \{2\})$ lifts if and only if $\tilde{\lambda}(w) \in K_1$ (we abuse the notation $w$ here). Let $f : \pi_1(F_1 - \{x_1, x_2\}) \to \pi_1$ be the surjective map determined by the map $p$. Up to conjugation of $\pi_1$, one could choose

$$f : \pi_1(F_1 - \{x_1, x_2\}) \to \langle a, b, x_1, x_2 | [a; b]x_1x_2 = 1 \rangle \to \pi_1 = \langle y \rangle$$

as $f(a) = f(b) = 1$, $f(x_1) = y$ and $f(x_2) = y^2$. The basic covering space theory says that a closed curve $\gamma \in F_1 - \{x_1, x_2\}$ lifts to a closed curve $\gamma' \in F_2 - \{x_1, x_2\}$ if and only if $f([\gamma]) = 1$, where $[\cdot]$ stands for homotopy class here. Note that the set of surjective homomorphisms $\text{epi}(\pi_1(F_1 - \{x_1, x_2\}) \to \pi_1)$ is in one-to-one correspondence to the set of surjective homomorphisms $\text{epi}(H_1(F_1 - \{x_1, x_2\}; \mathbb{Z}) \to \pi_1)$ since the group $\pi_1$ is abelian. Let $\gamma \in H_1(F_1 - \{x_1, x_2\}; \mathbb{Z})$ be the homology class of $\gamma$. Suppose $\tilde{\gamma} = x_1^m a^h b^k$ and $\tilde{f} : H_1(F_1 - \{x_1, x_2\}; \mathbb{Z}) \to \pi_1$ is induced by $f : \pi_1(F_1 - \{x_1, x_2\}) \to \pi_1$. It is easy to see that $\tilde{f}(\gamma) = 1$ is equivalent to $m \equiv 0 \pmod{3}$. Let $\tilde{\lambda}(w)$ be denoted by

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}.$$ 

Then $\tilde{f}(w \gamma) = 1$ is equivalent to $Am + BL = 0 \pmod{3}$, where $\begin{pmatrix} m \\ L \end{pmatrix}$ is a 3-vector of $H_1(F_1 - \{x_1, x_2\}; \mathbb{Z})$ with

$$L = \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}.$$ 

Lemma 1.2 above says that $w$ lifts is equivalent to the statement $\tilde{f}(w \gamma) = 1$ if $\tilde{f}(\gamma) = 1$; i.e., $B = 0 \pmod{3}$ because $L$ could be an arbitrary two vector. We complete the proof.

Consider, for any $n$, the well-known map $\mu : \Gamma_0^n \to \Sigma_n$ defined via the permutation of $f \in \text{Diff}_+(S^2, \{n\})$ on $n$ punctures. Recall that the quotient $N(\pi_2)/\pi_2$ is isomorphic to a subgroup of $\Gamma_0^n$. Then, one has

**Proposition 1.3.** The quotient $N(\pi_2)/\pi_2$ is isomorphic to $\mu^{-1}(\Sigma_4) \subset \Gamma_0^5$.

This proposition is a special case of Lemma 1.1 of [X2].

## 2. Cohomology of $\Gamma_1^2$

Let $P\Gamma_2^n$ denote the pure mapping class group of genus $g$ with $n$ punctures, i.e., the group of path components of orientation-preserving diffeomorphisms of a connected oriented surface $F^n_g$ with $n$ punctures which fix $n$ punctures. Consider the group extension (see [Bi])

$$1 \to F(2) = \pi_1(F_1 - \{x_1\}) \to P\Gamma_1^2 \to P\Gamma_1^1 = SL(2, \mathbb{Z}) \to 1$$

given by forgetting one puncture, where $F(2)$ is the free group of 2 generators. The Lyndon-Hochschild-Serre spectral sequence ($\text{LHS}^3$) for the extension above is given by

$$E_2^{p,q} = H^p(SL(2, \mathbb{Z}); H^q(F(2); \mathbb{Z})) \Rightarrow H^{p+q}(P\Gamma_1^2; \mathbb{Z})$$

where $H^0(F(2); \mathbb{Z}) = \mathbb{Z}$ as a trivial $SL(2, \mathbb{Z})$ module; $H^1(F(2); \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$ as the $SL(2, \mathbb{Z})$ module is obtained by the usual $SL(2, \mathbb{Z})$ action on $\mathbb{Z} \oplus \mathbb{Z}$. 

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It is well known that there is an amalgamated product decomposition $SL(2, \mathbb{Z}) = \mathbb{Z}/6 \ast_{\mathbb{Z}/2} \mathbb{Z}/4$. Choose generators $x \in \mathbb{Z}/6$, $y \in \mathbb{Z}/4$ and $z \in \mathbb{Z}/2$ as

$$x = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$$

and

$$z = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

A direct calculation gives

$$H^1(F(2); \mathbb{Z})_{\mathbb{Z}/6} = 0, \quad H^1(F(2); \mathbb{Z})_{\mathbb{Z}/4} = 0, \quad H^1(F(2); \mathbb{Z})_{\mathbb{Z}/2} = 0$$

$$H^2(F(2); \mathbb{Z})_{\mathbb{Z}/6} = H^1(F(2); \mathbb{Z})_{\mathbb{Z}/4} = H^1(F(2); \mathbb{Z})_{\mathbb{Z}/2} = \mathbb{Z}/2$$

where $M_4$ is a submodule consisting of all elements $(-2b, a - 2b)^T$ ($a$ and $b$ are integers);

$$H^1(F(2); \mathbb{Z})_{\mathbb{Z}/2} = H^1(F(2); \mathbb{Z})_{\mathbb{Z}/4} = 0,$$

and

$$H^1(F(2); \mathbb{Z})_{\mathbb{Z}/2} = H^1(F(2); \mathbb{Z})_{\mathbb{Z}/4} = \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

where $M_2$ is a submodule consisting of all elements $(-2a, -2b)^T$. This implies

$$H^n(\mathbb{Z}/6; H^1(F(2); \mathbb{Z})) = 0$$

for any $n$;

$$H^{odd}(\mathbb{Z}/4; H^1(F(2); \mathbb{Z})) = \mathbb{Z}/2,$$

$$H^{even}(\mathbb{Z}/4; H^1(F(2); \mathbb{Z})) = 0$$

and

$$H^{odd}(\mathbb{Z}/2; H^1(F(2); \mathbb{Z})) = \mathbb{Z}/2 \oplus \mathbb{Z}/2,$$

$$H^{even}(\mathbb{Z}/2; H^1(F(2); \mathbb{Z})) = 0.$$

Applying the M-V sequence to the group $SL(2, \mathbb{Z})$ with module $H^1(F(2); \mathbb{Z})$, one gets a long exact sequence

$$\cdots \rightarrow H^n(\mathbb{Z}/4; H^1(F(2); \mathbb{Z})) \oplus H^n(\mathbb{Z}/6; H^1(F(2); \mathbb{Z}))$$

$$\rightarrow H^n(\mathbb{Z}/2; H^1(F(2); \mathbb{Z})) \rightarrow H^{n+1}(SL(2, \mathbb{Z}); H^1(F(2); \mathbb{Z}))$$

$$\rightarrow H^{n+1}(\mathbb{Z}/4; H^1(F(2); \mathbb{Z})) \oplus H^{n+1}(\mathbb{Z}/6; H^1(F(2); \mathbb{Z}))$$

$$\rightarrow H^{n+1}(\mathbb{Z}/2; H^1(F(2); \mathbb{Z})) \rightarrow \cdots.$$

Note that the restriction map

$$H^n(\mathbb{Z}/4; H^1(F(2); \mathbb{Z})) \rightarrow H^n(\mathbb{Z}/2; H^1(F(2); \mathbb{Z}))$$

is an injection. It follows that

$$H^n(SL(2, \mathbb{Z}); H^1(F(2); \mathbb{Z})) = 0$$

if $n = 0$ or odd; and

$$H^n(SL(2, \mathbb{Z}); H^1(F(2); \mathbb{Z})) = \mathbb{Z}/2$$

if $n > 0$ even. Recall $H^*(SL(2, \mathbb{Z}); \mathbb{Z}) = \mathbb{Z}[u]/(12u)$. One claims that the LHS$^3$ for (1) collapses by dimension reason. We conclude now
Proposition 2.1.

\[ H^0(\Gamma_1^2; \mathbb{Z}) = \mathbb{Z}, \quad H^{\text{odd}}(\Gamma_1^2; \mathbb{Z}) = \mathbb{Z}/2, \quad H^{\text{even}}(\Gamma_1^2; \mathbb{Z}) = \mathbb{Z}/12. \]

It is a routine to construct a \( \mathbb{Z}/3 \) action on Torus \( F_1 \) with three fixed points. This gives an order 3 subgroup \( \pi \subset \Gamma_1^2 \subset \Gamma_2^2 \). Proposition 2.1 tells that the restriction map \( H^*(\Gamma_1^2; \mathbb{Z})_{(3)} \rightarrow H^*(\pi; \mathbb{Z})_{(3)} \) is an isomorphism. Furthermore, the universal coefficient theorem implies that the restriction map \( H^*(\Gamma_1^2; M)_{(3)} \rightarrow H^*(\pi; M)_{(3)} \) is an isomorphism for any trivial \( \Gamma_1^2 \)-module \( M \). Note \( H^*(\Gamma_1^2; \mathbb{Z})_{(3)} = H^*(\Gamma_2^2; \mathbb{Z})_{(3)} \). In order to show the restriction map \( H^*(\Gamma_1^2; \mathbb{Z})_{(3)} \rightarrow H^*(\pi; \mathbb{Z})_{(3)} \) is an isomorphism too, we only need to show the 3-period of \( \Gamma_1^2 \) is 2. The general form of the 3-period of a group is \( \text{LCM}\{2 | N(n)/C(n)\}p^\alpha \) (see [GMX] for details). We know that \( \alpha = 0 \) above from Proposition 2.1. Therefore, we only need to see the order \( |N_{\Gamma_1^2}(\pi)/C_{\Gamma_1^2}(\pi)| = 1 \) in this case. Let \( x \in \text{Diff}_+(F_1, \{2\}) \) denote a period 3 element with three fixed points. It is obvious that \( x \) is not conjugate to \( x^2 \) because they are not conjugate even mapping to \( SL(2, \mathbb{Z}) \). In summary, one obtains

**Theorem 2.2.** The restriction map

\[ R : H^*(F_1; M)_{(3)} \rightarrow H^*(\Gamma_1^2; M)_{(3)} \rightarrow H^*(\pi; M)_{(3)} \]

is an isomorphism for any trivial \( \Gamma_1^2 \)-module \( M \).

3. Cohomology of \( N(\pi_1)/\pi_1 \)

Recall that we defined the map \( \bar{\lambda} : \Gamma_1^2 \rightarrow GL(3, \mathbb{Z}/3) \) and a subgroup \( K_1 \subset GL(3, \mathbb{Z}/3) \) in section 1. Proposition 1.1 says the quotient \( N(\pi_1)/\pi_1 \) is isomorphic to \( \bar{\lambda}^{-1}(K_1) \). Let \( G \) denote the image of \( \bar{\lambda} \). Recall that any element of \( G \) must be in the form

\[
\begin{pmatrix}
A & B \\
0 & D
\end{pmatrix}
\]

(see section 1 for details). We remark here that in our case \( G \) is exactly the group consisting of all such matrices. In fact, one can see from geometry that \( \bar{\lambda}(F(2)) \) contains matrices

\[
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Thus, the index of \( K_1 \) in \( G \) is 9 and \( F(2) \) acts on \( G/K_1 \) via the map \( \bar{\lambda} \) transitively. It is clear that \( \Gamma_1^2/N(\pi_1)/\pi_1 \) is in one-one correspondence to \( G/K_1 \) as cosets. By the well-known Shapiro lemma, one has \( H^*(N(\pi_1)/\pi_1 ; \mathbb{Z}) = H^*(\Gamma_1^2; \mathbb{Z}[G/K_1]) \), where \( \Gamma_1^2 \) acts on the permutation module \( \mathbb{Z}[G/K_1] \) via the map \( \bar{\lambda} \).

We have seen that \( \Gamma_1^2 \) contains a subgroup \( \pi \) of order 3 in section 2. However, one can show

**Proposition 3.1.** The group \( N(\pi_1)/\pi_1 \) does not contain any subgroup of order 3.

**Proof.** It is obvious from the Riemann-Hurwitz formula that \( \Gamma_3 \) does not contain \( \mathbb{Z}/3 \times \mathbb{Z}/3 \). We only need to show that the third power 3 of any order 9
of $F_3$ has five fixed points, not two fixed points like a lift of $\pi_1$. This again follows directly from the Riemann-Hurwitz formula.

Proposition 3.1 above implies that the permutation module $Z[G/K_1]$ is not the trivial module $Z$ and $\pi_1$ acts on $Z[G/K_1]$ (by multiplication) nontrivially. It is elementary to observe that

**Lemma 3.2.** The group $\pi_1$ acts on the coset $G/K_1$ freely.

**Proof.** If not, assume that $x \in \pi_1$ fixes $g \in G/K_1$; i.e., $xgk = gk'$, or $g^{-1}xg = k'k^{-1} \in K_1$. This contradicts Proposition 3.1.

Therefore, one has the invariant $Z[G/K_1]^{\pi_1} = \bigoplus Z\langle n_i \rangle$, where $n_i = g_i + xg_i + x^2g_i$ for some $g_i \ (1 \leq i \leq 3)$ in this case. The co-invariant $Z[G/K_1]_{\pi_1} = Z[G/K_1]/M_1 = \bigoplus Z$ spanned by $g_i$'s. A direct computation implies the normal map

$$N : Z[G/K_1]_{\pi_1} \rightarrow Z[G/K_1]^{\pi_1}$$

is an isomorphism. So, one gets

**Proposition 3.3.** $H^n(\pi_1; Z[G/K_1]) = 0$ for $n > 0$.

Consider the LHS$^3$ given by

$$E_2^{p,q} = H^p(SL(2, \mathbb{Z}); H^q(F(2); Z[G/K_1])) \Rightarrow H^{p+q}(P\Gamma^5_2; Z[G/K_1])$$

for the extension (1) with coefficient $Z[G/K_1]$.

It is immediate from Proposition 3.3 and the M-V sequence that

**Proposition 3.4.** $H^n(SL(2, \mathbb{Z}); Z[G/K_1]^{F(2)})(3) = 0$ for $n > 0$.

Note that the $SL(2, \mathbb{Z})$ acts on $H^1(F(2); Z[G/K_1]) = H^1(Z; Z[G/K_1]) \oplus H^1(Z; Z[G/K_1])$ as matrix multiplications given in Section 2. One obtains

**Proposition 3.5.** $H^n(SL(2, \mathbb{Z}); H^1(F(2); Z[G/K_1])(3) = 0$ for $n > 0$.

Combining Propositions 3.4 and 3.5, one concludes

**Proposition 3.6.** $H^n(N(\pi_1)/\pi_1; Z)(3) = 0$ for any $n \geq 0$.

Repeating the argument above with $Z/3$ coefficient, one gets

**Proposition 3.7.** $H^n(N(\pi_1)/\pi_1; Z/3) = 0$ for $n > 0$.

A similar proof of Proposition 2.1 and the Shapiro lemma give

**Proposition 3.8.** $H^n(N(\pi_1)/\pi_1; Z)$ does not contain any copy of $Z$ for $n > 0$.

4. COHOMOLOGY OF $N(\pi_2)/\pi_2$

Consider the group extension

$$1 \rightarrow P\Gamma_0^5 \rightarrow N(\pi_2)/\pi_2 \rightarrow \Sigma_4 \rightarrow 1$$

described in Proposition 1.3. The LHS$^3$ for the extension above is given by

$$E_2^{p,q} = H^p(\Sigma_4; H^q(P\Gamma_0^5; Z/3)) \Rightarrow H^{p+q}(N(\pi_2)/\pi_2; Z/3)$$

where $\Sigma_4$ acts on $H^q(P\Gamma_0^5; Z/3)$ as shown in work of Cohen (the $P\Gamma_0^5$ is denoted by $K_3$ in [BC]). Recall that $H^*(P\Gamma_0^5; Z/3)$ is generated by one-dimen-
sional elements $B_{42}$, $B_{43}$, $B_{52}$, $B_{53}$ and $B_{54}$ subject to some relations specifically given in [BC]. Let $x = (123) \in \Sigma_4$ be a generator of a Sylow 3-subgroup. It is a routine to have

$$H^1(PT^3_0; \mathbb{Z}/3)^{(x)} = \mathbb{Z}/3 \oplus \mathbb{Z}/3 = (2B_{42} + B_{43}, B_{52} + 2B_{53})$$

and

$$H^1(PT^3_0; \mathbb{Z}/3)^{(x)} = H^1(PT^3_0; \mathbb{Z}/3)/M_5$$

where the submodule $M_5$ consists of all elements in the form

$$(m_1 + m_2 + m_5)B_{42} + (2m_2 - m_1)B_{43} + (m_3 + m_4 - m_5)B_{52} + (2m_4 - m_3 - m_5)B_{53}$$

with $m_i \in \mathbb{Z}/3$. Let $b_1 = m_1 + m_2 + m_5$, $b_2 = 2m_2 - m_1$, $b_3 = m_3 + m_4 - m_5$ and $b_4 = 2m_4 - m_3 - m_5$. Elementary linear algebra implies $3m_1 = 2b_1 - b_2 - 2m_5 = 0$, $3m_2 = b_1 + b_2 - m_5 = 0$, $3m_3 = 2b_3 - b_4 + m_5$ and $3m_4 = b_3 + b_4 + 2m_5 = 0$. Thus, the equation $b_1 + b_2 + 2b_3 + b_4 = 0$ holds. This amounts to showing

$$H^1(PT^3_0; \mathbb{Z}/3)^{(x)} = \mathbb{Z}/3 \oplus \mathbb{Z}/3$$

generated by $\langle B_{54}, B_{42} \rangle$. It is easy to check that the normal map

$$N : H^1(PT^3_0; \mathbb{Z})^{(x)} \to H^1(PT^3_0; \mathbb{Z})$$

is given by $N(B_{54}) = B_{42} + 2B_{43} + B_{52} + 2B_{53}$ and $N(B_{42}) = 0$. Thus, one obtains

**Lemma 4.1.**

$$H^0((x); H^1(PT^3_0; \mathbb{Z}/3)) = \mathbb{Z}/3 \oplus \mathbb{Z}/3,$$

$$H^{odd}(x); H^1(PT^3_0; \mathbb{Z}/3)) = \mathbb{Z}/3,$$

$$H^{even}(x); H^1(PT^3_0; \mathbb{Z}/3)) = \mathbb{Z}/3.$$  

Consider the $x$ action on $H^2(PT^3_0; \mathbb{Z}/3)$; one gets the invariant

$$H^2(PT^3_0; \mathbb{Z}/3)^{(x)} = \mathbb{Z}/3 \oplus \mathbb{Z}/3 = \langle B_{42}B_{53} + 2B_{43}B_{52}, B_{42}B_{52} + B_{43}B_{52} + B_{43}B_{53} \rangle$$

and the co-invariant

$$H^2(PT^3_0; \mathbb{Z})^{(x)} = H^2(PT^3_0; \mathbb{Z})/M_5$$

where the submodule $M_5$ consists of all elements in the form

$$(m_1 - m_5 + m_6)B_{42}B_{52} + (m_2 + m_4 - m_5 + m_6)B_{42}B_{53}$$

$$+ (m_3 + m_6)B_{42}B_{54} + (m_2 - m_3 + m_4 - m_5 + m_6)B_{43}B_{52}$$

$$+ (m_2 - m_1 - m_3 + m_4 + m_6)B_{43}B_{53} + (-m_3 + 2m_6)B_{43}B_{54}$$

with $m_i \in \mathbb{Z}/3$. Let $b_1 = m_1 - m_2 + m_6$, $b_2 = m_2 + m_4 - m_5 + m_6$, $b_3 = m_3 + m_6$, $b_4 = m_2 - m_3 + m_4 - m_5 + m_6$, $b_5 = -m_1 + m_2 - m_3 + m_4 + m_6$ and $b_6 = -m_3 + 2m_6$. It is easy to have from linear algebra that $-2b_3 + b_6 = 0$, $b_1 + 2b_2 - b_3 - 2b_4 + b_5 = 0$, $m_1 = b_1 + b_2 - b_3 - b_4 + m_5$, $m_2 = 2b_2 - b_3 - b_4 - m_5 + m_4$, $m_3 = b_2 - b_4$ and $m_6 = -b_2 + b_3 + b_4$. Thus, one gets

$$H^2(PT^3_0; \mathbb{Z}/3)^{(x)} = \mathbb{Z}/3 \oplus \mathbb{Z}/3$$

generated by $\langle B_{42}B_{52}, B_{43}B_{54} \rangle$. Also, it is straightforward to check that the normal map

$$N : H^2(PT^3_0; \mathbb{Z}/3)^{(x)} \to H^2(PT^3_0; \mathbb{Z}/3)^{(x)}$$

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given by
\[ N(\overline{B}_{42}B_{52}) = 2B_{42}B_{52} + B_{42}B_{53} + B_{43}B_{52} + 2B_{43}B_{52} \]
and
\[ N(\overline{B}_{43}B_{54}) = -B_{42}B_{52} - B_{43}B_{52} - B_{43}B_{53} \]
is an isomorphism. This implies

Lemma 4.2. \( H^0((x) ; H^2(\Gamma_5^5 ; \mathbb{Z}/3)) = \mathbb{Z}/3 \oplus \mathbb{Z}/3 \) and \( H^n((x) ; H^2(\Gamma_0^5 ; \mathbb{Z}/3)) = 0 \) for \( n > 0 \).

Recall that \( H^1(\Gamma_5^5 ; \mathbb{Z}/3) \) is generated by \( \overline{B}_{54} \) and \( \overline{B}_{42} \). We can check directly that \( (12) \in \Sigma_4 \) permutes \( \overline{B}_{54} \) to \( \overline{B}_{54} - \overline{B}_{42} \) and \( \overline{B}_{42} \) to \( -\overline{B}_{42} \); that is, \( H^1(\Gamma_0^5 ; \mathbb{Z}/3)(12) = 0 \). It is also straightforward to verify \( (12) \in \Sigma_4 \) acts on generators \( 2B_{42} + B_{43} \) and \( B_{52} + 2B_{53} \) of \( H^1(\Gamma_5^5 ; \mathbb{Z}/3) \) trivially and acts on the one-dimensional space generated by
\[ 2B_{42}B_{52} + B_{43}B_{52} + B_{42}B_{53} + 2B_{43}B_{53} \in H^2(\Gamma_0^5 ; \mathbb{Z}/3)(x) \]
trivially. These calculations imply

Lemma 4.3.
\[
\begin{align*}
H^0(\Sigma_4 ; H^1(\Gamma_5^5 ; \mathbb{Z}/3)) &= \mathbb{Z}/3 \oplus \mathbb{Z}/3, \\
H^0(\Sigma_4 ; H^2(\Gamma_0^5 ; \mathbb{Z})) &= \mathbb{Z}/3, \\
H^n(\Sigma_4 ; H^1(\Gamma_0^5 ; \mathbb{Z}/3)) &= \mathbb{Z}/3
\end{align*}
\]
for \( n \equiv 0, 1 \pmod{4} \);
\[
H^n(\Sigma_4 ; H^1(\Gamma_5^5 ; \mathbb{Z}/3)) = 0
\]
for \( n \equiv 2, 3 \pmod{4} \).

It is easy to see a \( \mathbb{Z}/3 \subset N(\pi_2)/\pi_2 \subset \Gamma_0^5 \) by constructing a \( \mathbb{Z}/3 \) action on \( S^2 \) with two fixed points and permuting three points. The following lemma is needed for the study of \( \text{LHS}^3 \) associated to the extension (2) in the beginning of this section.

Lemma 4.4. The group \( N(\pi_2)/\pi_2 \) has the \( \mathbb{Z}/3 \) as a retract.

Proof. Recall the group \( N(\pi_2)/\pi_2 \) is an extension of \( \Gamma_0^5 \) over \( \Sigma_4 \). There is a surjective map by forgetting the fifth puncture from \( N(\pi_2)/\pi_2 \) to \( \Gamma_0^5 \), therefore, to \( H_1(\Gamma_4 ; \mathbb{Z}) = \mathbb{Z}/2 \oplus \mathbb{Z}/3 \), to \( \mathbb{Z}/3 \). Note the \( \mathbb{Z}/3 \subset N(\pi_2)/\pi_2 \) is compatible with the \( \mathbb{Z}/3 \subset \Gamma_0^5 \). The lemma follows since \( \Gamma_0^5 \) has the \( \mathbb{Z}/3 \) as a retract.

Now, one can conclude the \( \text{LHS}^3 \) collapses by Lemma 4.4 and

Proposition 4.5.
\[
\begin{align*}
H^0(N(\pi_2)/\pi_2 ; \mathbb{Z}/3) &= \mathbb{Z}/3, \\
H^n(N(\pi_2)/\pi_2 ; \mathbb{Z}/3) &= \mathbb{Z}/3 \oplus \mathbb{Z}/3
\end{align*}
\]
if \( n = 1, 2 \);
\[
H^n(N(\pi_2)/\pi_2 ; \mathbb{Z}/3) = \mathbb{Z}/3
\]
if \( n \geq 3 \).

Repeat the calculation in this section above with coefficient \( \mathbb{Z} \) and consider \( \text{LHS}^3 \) for the extension (2) with coefficient \( \mathbb{Z} \); one gets
Proposition 4.6. The restriction map

\[ R : H^n(N(\pi_2) / \pi_2 ; \mathbb{Z}) \rightarrow H^n(\mathbb{Z} / 3 ; \mathbb{Z})(3) \]

induces an isomorphism; the group \( H^n(N(\pi_2) / \pi_2 ; \mathbb{Z}) \) contains exactly one copy of \( \mathbb{Z} \) for \( n = 0, 1, 2 \) and contains no copy of \( \mathbb{Z} \) for \( n \geq 3 \).

5. Farrell cohomology of \( \Gamma_3 \)

We actually calculate not only the 3-components of

\[ H^*(N(\pi_1) ; \mathbb{Z}) \quad \text{and} \quad H^*(N(\pi_2) ; \mathbb{Z}), \]

but also their free parts. Consider the group extensions

\[ \begin{align*}
1 & \to \pi_1 \to N(\pi_1) \to N(\pi_1) / \pi_1 \to 1 \\
1 & \to \pi_2 \to N(\pi_2) \to N(\pi_2) / \pi_2 \to 1.
\end{align*} \]

One has the \( \text{LHS}^3 \) for the extensions above giving as

\[ E_{p+q}^2 = H^p(N(\pi_1) / \pi_1 ; H^q(\pi_1 ; \mathbb{Z})) \Rightarrow H^{p+q}(N(\pi_1) ; \mathbb{Z}). \]

Note that the group \( N(\pi_1) \) acts on \( \pi_1 \) nontrivially and the group \( N(\pi_2) \) acts on \( \pi_2 \) trivially from the observation of the fixed point data of generators of \( \pi_1 \) and \( \pi_2 \).

It is easy to see a dihedral subgroup \( D_6 \subset \Gamma_3 \) of order 6 containing the \( \pi_1 \) by realizing a \( D_6 \) action on \( F_3 \) with four singular points of order 2 and one singular point of order 3 in the orbit space \( F_3 / D_6 = S^2 \) (2 sphere). The following proposition is immediate.

Proposition 5.1.

(1) The restriction map

\[ R : H^n(N(\pi_1) ; \mathbb{Z})(3) \rightarrow H^n(D_6 ; \mathbb{Z})(3) \]

is an isomorphism for any \( n \geq 0 \).

(2) \( H^n(N(\pi_1) ; \mathbb{Z}) \) does not contain any \( \mathbb{Z} \) for \( n > 0 \).

Again, it is clear that the \( \pi_2 \) is contained in a \( \mathbb{Z}/9 \subset \Gamma_3 \) if one notices that there is a \( \mathbb{Z}/9 \) action on \( F_3 \) with two singular points of order 9 and one singular point on the orbit space \( F_3 / \mathbb{Z}/9 = S^2 \) (2 sphere). Comparing the \( \text{LHS}^3 \) for the extension

\[ 1 \to \pi_2 \to \mathbb{Z}/9 \to \mathbb{Z}/3 \to 1 \]

with Proposition 4.5, one obtains

Proposition 5.2.

\[ H^n(N(\pi_2) ; \mathbb{Z})(3) = 0 \]

for \( n = 0, 1 \);

\[ H^2(N(\pi_2) ; \mathbb{Z})(3) = \mathbb{Z}/9, \quad H^n(N(\pi_2) ; \mathbb{Z})(3) = \mathbb{Z}/3 \]

for \( n \geq 3 \) odd;

\[ H^n(N(\pi_2) ; \mathbb{Z})(3) = \mathbb{Z}/3 \oplus \mathbb{Z}/9 \]

for \( n \geq 4 \) even.
Proposition 5.3. \( H^n(N(n_2) ; \mathbb{Z}) \) contains exactly one copy of \( \mathbb{Z} \) for \( n = 0, 1, 2 \) and contains no \( \mathbb{Z} \) for \( n \geq 3 \).

The main result about Farrell cohomology now follows readily since \( \Gamma_3 \) is 3-periodic.

Theorem 5.4.

\[
\tilde{H}^n(\Gamma_3 ; \mathbb{Z})_{(3)} = \mathbb{Z}/3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/9
\]

for \( n \equiv 0 \pmod{4} \);

\[
\tilde{H}^n(\Gamma_3 ; \mathbb{Z})_{(3)} = \mathbb{Z}/3
\]

for \( n \) odd;

\[
H^n(\Gamma_3 ; \mathbb{Z})_{(3)} = \mathbb{Z}/3 \oplus \mathbb{Z}/9
\]

for \( n \equiv 2 \pmod{4} \).

6. The \( p \)-component of Farrell cohomology of \( \Gamma_p \) for \( p > 3 \)

For any prime \( p > 3 \), it is easy to check from possible fixed point data that there is one and only one conjugacy class of order \( p \) subgroup of \( \Gamma_p \), denoted as \( \pi \subset \Gamma_p \). The fixed point data of a generator of \( \pi \) is \( (1, p-1) \). Thus, the cyclic group \( N(\pi)/C(\pi) \) is \( \mathbb{Z}/2 \). Actually, it is not difficult to observe a dihedral subgroup \( D_{2p} \subset \Gamma_p \) by constructing a surjective map from \( \pi_1(F_1 - \{x_1, x_2\}) \) onto \( D_{2p} \).

Let \( K_1 \) denote a subgroup of \( \text{Im}(\tilde{\lambda}) \) consisting of all elements of \( GL(3, \mathbb{Z}/p) \) in the form of

\[
\begin{pmatrix}
A & 0 \\
0 & D
\end{pmatrix}
\]

with \( A \in \{1, -1\} \) and \( D \in SL(2, \mathbb{Z}/p) \), where

\[
\tilde{\lambda} : \Gamma^2_1 \to GL(3, \mathbb{Z}/p)
\]

is defined as in section 1.

Proposition 6.1. The quotient \( N(\pi)/\pi \) is isomorphic to \( \tilde{\lambda}^{-1}(K_1) \subset \Gamma^2_1 \).

The proof is the same as Proposition 1.1.

Proposition 6.2. \( H^n(\Gamma^2_1 ; M)_{(p)} = 0 \) for any prime \( p > 3 \), \( n > 0 \) and \( \mathbb{Z}\Gamma^2_1 \)-module \( M \).

Repeat the argument in section 2 with any coefficient \( \mathbb{Z}\Gamma^2_1 \)-module \( M \); the proof follows immediately.

By using the Shapiro lemma again, one gets

Proposition 6.3. \( H^n(N(\pi)/\pi ; \mathbb{Z})_{(p)} = 0 \) for any \( p > 3 \) and \( n > 0 \).

Finally, comparing two short exact sequences

\[
1 \to \pi \to N(\pi) \to N(\pi)/\pi \to 1,
\]

\[
1 \to \pi \to D_{2p} \to \mathbb{Z}/2 \to 1
\]

and considering two \( LHS^3 \) associated to them, one obtains
Proposition 6.4. The restriction map
\[ R: H^n(N(p); \mathbb{Z}) \to H^n(D_{2p}; \mathbb{Z}) \]
is an isomorphism for any \( p > 3 \) and \( n \geq 0 \).

Theorem 6.5. For a prime \( p > 3 \), the restriction map
\[ R: \tilde{H}^n(\Gamma_p; \mathbb{Z}) \to \tilde{H}^n(D_{2p}; \mathbb{Z}) \]
is an isomorphism for any \( n \). Namely,
\[ \tilde{H}^n(\Gamma_p; \mathbb{Z}) = \mathbb{Z}/p \]
for any \( n \equiv 0 \) (mod 4);
\[ \tilde{H}^n(\Gamma_p; \mathbb{Z}) = 0 \]
for any \( n \not\equiv 0 \) (mod 4).

References


[X2] ———, The \( p \)-torsion of the Farrell-Tate cohomology of the mapping class group \( \Gamma_{p-1} \), J. Pure Appl. Algebra 78 (1992), 319–334.