THE ZERO-SETS OF THE RADIAL-LIMIT FUNCTIONS
OF INNER FUNCTIONS

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Abstract. A set $E$ on the unit circle is the zero-set of the radial-limit function of some inner function if and only if $E$ is a countable intersection of $F_0$-sets of measure 0.

1. Introduction

By Fatou's theorem [F] (or [CL, Theorem 2.1, p. 17]), the points $\zeta$ on the unit circle $C$ where the radial limit $f^*(\zeta)$ of a bounded holomorphic function $f$ in the unit disk $D$ exists constitute a set of Lebesgue measure $2\pi$. If in addition $f \neq w_0$, then the set $E(f, w_0)$ of points $\zeta$ on $C$ where $f^*(\zeta) = w_0$ has measure 0, by a theorem of F. Riesz and M. Riesz [RR, p. 41] (or [CL, Theorem 2.5, p. 22]). On the other hand, to each set $E$ of measure 0 on $C$ correspond nonconstant, bounded holomorphic functions $f$ in $D$ such that the ordinary limit of $f$ (and therefore the radial limit of $f$) is 0 everywhere on $E$ (see [P, p. 214] or [Z, Vol. I, end of p. 276]).

G. T. Cargo has pointed out that not every set of measure 0 on $C$ is contained in the set $E(f, 0)$ of some inner function $f$, that is, of some bounded holomorphic function satisfying almost everywhere on $C$ the condition $|f^*(\zeta)| = 1$. (We refer to [CL] for background concerning the class of inner functions and the subclasses of singular inner functions and Blaschke products that will be considered in the sequel.) Cargo showed [C, Theorems 1 and 4] that for each complex number $w_0$ and each nonconstant inner function $f$ the set $E(f, w_0)$ is meagre, in other words, it is a set of first category.

In [B], the second author considered the problem of characterizing the sets $E(f, 0)$ for inner functions $f$. He showed that if $f$ is an inner function, then the set of points where $f$ has the radial limit 0 is the intersection of countably many $F_0$-sets of measure 0. Theorem 2.4 in [B] gives some sufficient conditions for a set $E$ on $C$ to be the zero-set of the radial-limit function of an inner function. Example: if $E$ is the intersection of an $F_0$-set of measure 0 with a $G_\delta$-set, then the zero-set of the radial-limit function of some Blaschke product is equal to $E$.

The present paper completes the characterization.
Theorem. In order that a set on the unit circle $C$ be the set where some inner function has the radial limit 0, it is necessary and sufficient that it be a countable intersection of $F_\alpha$-sets of measure 0.

Our proof of the sufficiency in the theorem requires a construction involving certain inner functions. In Section 2, we develop our elementary building blocks; in Section 3, we perform the final synthesis. Section 4 is devoted to some concluding remarks.

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2. Singular inner functions

We use functions of the form

$$f(z) = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \frac{z + e^{it}}{z - e^{it}} \, d\mu(t)\right),$$

where the symbol $\mu$ in the Stieltjes integral denotes a nondecreasing function that is defined on the real line and satisfies at each point $t$ the three conditions

- $\mu(0) = 0$ (initial standardization),
- $\mu(t + 2\pi) - \mu(t) = \mu(2\pi) = \|\mu\|$ (2\pi-periodicity of the distribution $d\mu$),
- $\mu(t^+) + \mu(t^-) = 2\mu(t)$ (normalization at discontinuities).

At each point $z = re^{i\theta}$ in $D$, the real part of $\log|1/f(z)|$ is

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{(1 - r)^2 + 4r \sin^2[(t-\theta)/2]} \, d\mu(t).$$

For each fixed point $z = re^{i\theta}$ ($r > 0$) in $D$, the integrand in (2) is a decreasing function of $|t - \theta|$ ($0 \leq |t - \theta| \leq \pi$). This implies that if we interpret $d\mu$ as a distribution on the unit circle $C$ and shift all (or a portion) of the mass in $d\mu$ to a point nearer to $e^{i\theta}$, the change does not decrease the value of $u(z)$.

Suppose, for example, that $\mu$ is constant on an arc

$$A = \{ \zeta : \zeta = e^{it}, \quad t_1 < t < t_2 \}$$

and that the point $z$ lies in the sector $S$ of $D$ bounded by $A$ and by the two line segments joining the origin to the endpoints of $A$. To be definite, suppose that $|z - e^{it_1}| \leq |z - e^{it_2}|$. Shifting the entire mass of $d\mu$ to $e^{it_1}$, we obtain a new function

$$\mu^*(t) = \begin{cases} 0, & t_1 - 2\pi < t < t_1, \\ \|\mu\|, & t_1 < t < t_1 + 2\pi, \end{cases}$$

and we deduce that

$$\log \frac{1}{|f(z)|} \leq \frac{\|\mu\|}{2\pi} \text{Re} \frac{e^{it_1} + z}{e^{it_1} - z}.$$

The mapping $w(z) = (e^{it_1} + z)/(e^{it_1} - z)$ carries each circle $\Gamma$ internally tangent to $C$ at $e^{it_1}$ onto a vertical line; moreover, if $\delta$ denotes the diameter of $\Gamma$, the image of $\Gamma$ passes through the point $w = (2 - \delta)/\delta = 2/\delta - 1$. Dealing similarly with points $z$ in the half of $S$ whose boundary passes through $e^{it_1}$, we see that if $\mu$ is constant on $A$, the inequality

$$\log \frac{1}{|f(z)|} \leq \frac{\|\mu\|}{2\pi} \left(\frac{2}{\delta} - 1\right)$$

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holds at each point $z$ in $S$ lying outside the union of the two disks of diameter $\delta$ that are internally tangent to $C$ at $e^{it_1}$ and $e^{it_2}$, respectively.

In connection with (1), we also recall that if the function $\mu$ in (1) has an infinite derivative at a point $t = \theta$, then, by virtue of a standard proof of Fatou’s radial-limit theorem, the function $f$ has the radial limit $0$ at $e^{i\theta}$.

In the sequel, we shall use a lemma that is proved in passing (but not stated) in the upper half of page 9 of [LP].

**Lemma LP.** If $E$ is a closed subset of $C$ of measure $0$, then there exists a function $\mu$ that meets the specifications stated in connection with the formula (1) and has the property

$$\mu'(t) = \begin{cases} +\infty, & e^{it} \in E, \\ 0, & e^{it} \in C \setminus E. \end{cases}$$

To prove the lemma, we point out that if $E$ is a closed set of measure $0$ in $C$, then each point of $E$ is either a bilateral limit point of $E$ or the endpoint of a component of $C \setminus E$. In the first case, the function $\mu$ constructed in the second paragraph of page 9 of [LP] has infinite right- and left-hand derivatives at $t$, and therefore $\mu'(t) = +\infty$. In the second case, $\mu$ has a saltus at $t$, and therefore $\mu'(t) = \infty$ by virtue of the normalization.

Corresponding to each number $\delta$ in the interval $(0,2)$, let $T = T(E, \delta)$ denote the union of all circular disks that have diameter $\delta$ and are internally tangent to $C$ at some point of $E$. We note that the number of components of $T$ is less than $2n_\delta$. In our application of Lemma LP to a function $f$ of the form (1), it will be useful to consider the auxiliary mental picture consisting of the graph $W$ of the real-valued function $|f|$ over the unit disk $D$ in $zh$-space ($h \geq 0$). The following informal proposition may be visualized in terms of the topography encountered by an imaginary traveler on the surface $W$.

**Proposition.** Let $f$ be a function given by (1), and denote by $W$ the surface described above. Then, for each point $z$ in $D \setminus T$, the point $(z, |f(z)|)$ on $W$ lies above the plane $h = \exp(-\|\mu\|/\delta \pi)$. If the function $\mu$ has an infinite derivative at the point $t$, then the closure of $D$ is (in a reasonable sense) tangent at $(e^{it}, 0)$ to the image on $W$ of each triangular domain in $D$ that has a vertex at $e^{i\theta}$.

Consequently, any traveler on $W$ who strays onto the image of the territory $T$ incurs the risk of sliding to the bottom of a steep-walled pit.

A subset $E$ of $C$ satisfies the condition in our theorem if and only if it has a representation

$$E = \bigcap_{j=1}^{\infty} E_j,$$

where each set $E_j$ is the union of closed sets $E_{jk}$ ($k = 1, 2, \ldots$) of measure $0$. Because every closed set of measure $0$ on $C$ is nowhere dense, we can assume that for each index $j$ the sets $E_{jk}$ ($k = 1, 2, \ldots$) are pairwise disjoint ([S], see also [B, Theorem 3.2 and Corollary 3.4]). Also, since a point $e^{it}$ lies in $E$ if and only if it lies in each of the sets $E_j$ ($j = 1, 2, \ldots$), and since, moreover, the intersection of two sets of type $F_\delta$ is again of type $F_\delta$, we can assume that the relation $E_j \supset E_{j+1}$ holds for each index $j$. Finally, to eliminate certain
trivialities from our exposition, we assume that $E$ is nonempty and if one of the sets $E_{jk}$ is empty, then $E_{mn}$ is empty whenever $m = j$ and $n > k$.

Corresponding to each of our sets $E_j$, we shall construct a singular inner function $f_j$ such that $E(f_j, 0) = E_j$. The construction of $f_j$ will be strongly affected by a certain sequence $(\delta_{jk})$ of positive numbers; our choice of each number $\delta_{jk}$ will depend on the geometric relations among the sets $E_{im}$ ($m \leq k$); if $j > 1$, the choice of $\delta_{jk}$ will reflect also the relations between the sets $E_{j-1}$ and $E_j$.

For the sake of notational consistency with what will follow, we introduce the alternate symbol $F_{jk}$ for the set $E_{jk} (k = 1, 2, \ldots)$. The intersections $E_{2m} \cap F_{1n} (m, n = 1, 2, \ldots)$ form a countable collection of disjoint closed sets whose union is the set $E_2$. We order the nonempty elements of this collection into a simple sequence $(F_{2k}) (k = 1, 2, \ldots)$. More generally, once the sequence $(F_{j-1,k})_{k=1}^\infty$ is defined, we order the family of all nonempty closed sets $E_{jm} \cap F_{j-1,n} (m, n = 1, 2, \ldots)$ into a simple sequence $(F_{jk})_{k=1}^\infty$ of disjoint closed sets.

For $\delta_{11}$ we choose any number in the interval $(0, 1)$, and we construct the corresponding territory $T_{11} = T(F_{11}, \delta_{11})$. After choosing constants $\delta_{1m}$ and constructing the corresponding territories $T_{1m} = T(F_{1m}, \delta_{1m}) (m = 1, \ldots, j - 1)$, we choose a number $\delta_{1j}$ lying in $(0, \delta_{11})$ and small enough so that the territory $T_{1j} = T(F_{1j}, \delta_{1j})$ lies at a positive distance from each of the territories $T_{1m} \cap F_{1m} (m = 1, \ldots, j - 1)$.

Suppose we have chosen the numbers $\delta_{mv}$ for $m = 1, \ldots, j - 1$ and $v = 1, 2, \ldots$ and for $m = j$ and $v < k$. Let $n$ be the unique positive integer for which the set $F_{jk}$ is a subset of $F_{j-1,n} (n = 1, \ldots, j - 1)$. Clearly, we can choose $\delta_{jk}$ small enough so that the territory $T_{jk} = T(F_{jk}, \delta_{jk})$ lies at a positive distance from each of the territories $T_{j-1,n} \cap F_{j-1,n}$ and lies at a positive distance from each of the territories $T_{jh} (h = 1, 2, \ldots, k - 1)$.

Corresponding to each index $j$, we denote the union of the territories $T_{jk} (k = 1, 2, \ldots)$ by $T_j$. It is easy to see that if $e^{it} \in C \setminus E_j$ and the radius $R_t = |0, e^{it} |$ meets a component of a territory $T_{jk}$, then the endpoints of the segment $R_t \cap T_{jk}$ lie in none of the territories $T_{jh} (h \neq k)$ and in none of the sets $T_{j+1}, T_{j+2}, \ldots$.

If $c_{jk}$ is a positive constant and $\mu_{jk}^*$ is a function satisfying with regard to the set $F_{jk}$ the condition described in Lemma LP, then the function $c_{jk} \mu_{jk}^* = \mu_{jk}$ also satisfies that condition. By virtue of the inequality (3), we can choose the constant $c_{jk}$ small enough so that the function $f_{jk}$ generated by $\mu_{jk}$ in accordance with the formula (1) has at each point of $F_{jk}$ the radial limit 0 and satisfies everywhere in $D \setminus T_{jk}$ the inequality

$$\log \frac{1}{|f_{jk}(z)|} < \frac{1}{2j+k}.$$ 

If $e^{it} \in F_{jk}$, then $f_{jk}$ has at $e^{it}$ the radial limit 0; if $e^{it} \in C \setminus F_{jk}$, then the function $\mu_{jk}$ is constant on some open segment with the midpoint $t$, and therefore $f_{jk}$ is holomorphic at $e^{it}$ and $|f_{jk}(e^{it})| = 1$. Clearly, the function $\mu_j = \sum_k \mu_{jk}$ has finite norm, and by a classical differentiation theorem of
Fubini, $\mu_j' = 0$ a.e. Also, the product

$$f_j = \prod_{k=1}^{\infty} f_{jk}$$

converges uniformly on compact sets, satisfies everywhere in $D \setminus T_j$ the inequality

$$\log \frac{1}{|f_j(z)|} < \frac{1}{2j},$$

and has at each point of $E_j$ the radial limit 0.

Suppose now that $e^{it} \in C \setminus E_j$. Because $\delta_{jk} \to 0$ as $k \to \infty$ and

$$\text{dist}(T_{jh}, T_{jk}) > 0$$

whenever $h \neq k$, each of the radial segments

$$\{z : z = re^{it}, 1 - 1/m \leq r < 1\}$$

contains a point $z_m$ in $D \setminus T_j$. The inequality (4) implies that the relation

$$\prod_{k \geq K} |f_{jk}(z_m)| > \exp(-2^{-j-K})$$

holds for all positive integers $K$ and $m$. Since the right-hand side tends to 1 as $K \to \infty$, and since each of the $K$ factors $f_{j1}, \ldots, f_{jK}$ is holomorphic and has modulus 1 at $e^{it}$, it follows that the cluster set at $e^{it}$ of $f_j$ contains at least one point of modulus 1.

3. Celebration after the grubby work

It remains only to combine the functions $f_j$ into a function $f$ whose radial limit at each point of the set $E = \bigcap E_j$ is 0 and whose radial cluster set at each of the points of $C \setminus E$ includes a point other than 0.

Corresponding to each index $j$, let

$$g_j = \frac{1/2 + f_j}{1 + f_j/2}.$$

Clearly, $g_j$ is an inner function. At each point where the radial limit of $f_j$ is 0, the radial limit of $g_j$ is 1/2, and at each point where the radial cluster set of $f_j$ includes a point of $C$, the radial cluster set of $g_j$ also includes a point of $C$. Moreover, because $-1/2$ is not the radial limit of $f_j$ at any point of $C$, the function $g_j$ does not have the radial limit 0 at any point, and therefore $g_j$ has no singular factor; in other words, $g_j$ is a Blaschke product, that is, a function of the form

$$B_j(z) = e^{i\gamma_j} z^{\nu_j} \prod_{n=1}^{\infty} |a_{jn}| \frac{1-z/a_{jn}}{1-\overline{a_{jn}} z},$$

where $\gamma_j$ and $\nu_j$ represent a real constant and a nonnegative integer, respectively, and where the points $a_{jn}$ lie in $D \setminus \{0\}$ and satisfy the condition

$$\sum_n (1 - |a_{jn}|) < \infty.$$  

Except in the trivial case where $E = \emptyset$, formula (1) guarantees that the value of $\log 1/f_j(0)$ has a positive value, and therefore $B_j(0)$ is positive. This in turn
implies that the factors \( e^{i\nu_j} \) and \( \nu_j \) in our formula for \( B_j(z) \) have the values 1 and 0, respectively.

The final stage of our synthesis is the creation of the formal product \( f = \prod g_j \). That the symbol \( f \) represents a genuine Blaschke product is equivalent to the convergence of the double series

\[
\sum_{j=1}^{\infty} \sum_{n=1}^{\infty} (1 - |a_{jn}|).
\]

This follows from the inequality \( \log r < r - 1 \) (\( r > 0 \)), the equality \( 1/B_j(0) = \prod_{n=1}^{\infty} |a_{jn}|^{-1} \), and the absolute convergence of the series \( \sum_{j=1}^{\infty} \log 1/B_j(0) \).

To see that the Blaschke product \( f \) has the desired property, suppose first that \( e^{it} \in \mathcal{E} \). Then \( e^{it} \in E_j \) \( (j = 1, 2, \ldots) \), and because each of the functions \( g_j \) has at \( e^{it} \) the radial limit 1/2, the radial limit of \( f \) is 0, at the point \( e^{it} \).

Next, suppose that \( e^{it} \in \mathcal{C} \setminus \mathcal{E} \), and let \( J \) denote the least nonnegative integer such that \( e^{it} \in \mathcal{C} \setminus E_{j+1} \). At the point \( e^{it} \), the product of the first \( J \) factors \( g_j \) has the radial limit \( 2^{-J} \). As we pointed out in Section 2, the radius \( (0, e^{it}) \) supports a sequence \( \{z_m\} \) that converges to \( e^{it} \) and lies in \( D \setminus T_{j+1} \). Because the sequence of sets \( T_j \) decreases, the sequence \( \{z_m\} \) lies in each of the sets \( D \setminus T_j \) \( (j = J + 1, J + 2, \ldots) \). To study the behavior of the corresponding factors \( g_j \) \( (j = J+1, J+2, \ldots) \) on the sequence of points \( z_m \) \( (m = 1, 2, \ldots) \), we point out that if \( |f_j(z_m)| = \rho \), then, by virtue of formula (5), the function \( g_j \) carries the point \( z_m \) to a point on the circle having a diameter with the endpoints

\[
\frac{1-2\rho}{2-\rho} \quad \text{and} \quad \frac{1+2\rho}{2+\rho}
\]

on the real line. Because the left-hand endpoint is nearer to the origin than the one on the right,

\[
|g_j(z_m)| \geq \left| \frac{1-2\rho}{2-\rho} \right|.
\]

The inequality \( |f_j(z_m)| > 1/2 \) resolves the ambiguity of sign, and we see that

\[
1 - |g_j(z_m)| \leq 1 - \frac{2\rho - 1}{2 - \rho} = \frac{3(1 - \rho)}{2 - \rho} < 3(1 - |f_j(z_m)|).
\]

Since the values of \( g_j(z_m) \) are bounded away from zero, there exists a constant \( c \) such that

\[
-\log |g_j(z_m)| < c(1 - |g_j(z_m)|) < 3c(1 - |f_j(z_m)|).
\]

It follows that

\[
\log \prod_{j > J} \frac{1}{|g_j(z_m)|} < 3c \sum_{j > J} (1 - |f_j(z_m)|).
\]

The inequality (4) implies that to each positive number \( \epsilon \) there corresponds an integer \( N \) \( (N \geq J) \) such that the inequality

\[
\sum_{j > J} (1 - |f_j(z_m)|) < \epsilon
\]

holds for each index \( m \). The argument at the end of Section 2 shows that for all sufficiently large values of the index \( m \), the number \( \epsilon \) is also an upper bound for the sum of the analogous terms with the indices \( j = J + 1, \ldots, N \).
It follows that the radial cluster set at $e^{it}$ of the product $\prod_{j \geq J} g_j(z_m)$ contains at least one point of $C$, and we see immediately that the radial cluster set at $e^{it}$ of the function $f$ contains at least one point of the circle $|w| = 2^{-J}$.

This concludes the proof that every subset of $C$ that is a countable intersection of $F_\alpha$-sets of measure 0 is the set where an inner function has radial limit 0.

We point out that if some open subset $A$ of $C$ contains no points of the set $E$, then each component of $A$ lies in a domain (cuspidate at both ends of the component) that contains no zeros of any of the Blaschke products $g_j$. Consequently, the Blaschke product $f$ is holomorphic and has absolute value 1 on each component of $A$. Moreover, corresponding to each positive $\epsilon$ and each compact subset $K$ of $D \cup A$, we can choose our set $\{\delta_{jk}\}$ ($j, k = 1, 2, \ldots$) so that the inequality $|f(z) - 1| < \epsilon$ holds everywhere in $K$.

4. THE CASES WHERE $w \neq 0$

The following extension of our main theorem holds. If $w \in D \setminus \{0\}$, then a set $E$ on $C$ is the set $E(f_w, w)$ for some Blaschke product $f_w$ if and only if $E$ is a countable intersection of $F_\beta$-sets of measure 0.

The necessity of the condition is obvious. To prove sufficiency, suppose first that $|w| \neq 2^{-1}, 2^{-2}, 2^{-3}, \ldots$. We construct the function $f$ as in Sections 2 and 3 and define $f_w$ by the formula

$$f_w = \frac{w + f}{1 + \overline{w}f}.$$

Then $f_w$ has the radial limit $w$ at all points $e^{it}$ where $f$ has the radial limit 0. Moreover, since $-w$ is not a radial limit value for $f$, the set $E(f_w, 0)$ is empty, and therefore $f_w$ is a Blaschke product.

If $|w|$ has one of the values $2^{-m}$ ($m = 1, 2, \ldots$), we proceed in the same way, except that we now define the Blaschke products $B_j$ by the formula

$$B_j = \frac{1/3 + f_j}{1 + f_j/3}.$$

In the case where $f$ is an inner function and $|w| = 1$, the set $E(f, w)$ is still of type $F_\alpha$; also, the theorems of F. and M. Riesz and of Cargo mentioned in our introduction are still applicable. But it has been shown (see [BN]) that $E(f, w)$ need not be contained in an $F_\alpha$-set of measure 0. In other words, the characterization of the sets $E(f, w)$ for inner functions depends on whether $w \in D$ or $w \in C$.

An interesting question—perhaps easier than the problem of complete characterization of the sets $E(f, 1)$—is whether every meagre $F_\alpha$-set of measure 0 is contained in the set $E(f, 1)$ for some nonconstant inner function $f$.

Another natural extension of the radial-limit zero-set problem concerns the possibility of radial-limit interpolation. We refer to [BN] for results in this direction and a discussion of the literature.

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