THE ZERO-SETS OF THE RADIAL-LIMIT FUNCTIONS
OF INNER FUNCTIONS

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Abstract. A set \( E \) on the unit circle is the zero-set of the radial-limit function of some inner function if and only if \( E \) is a countable intersection of \( F_\alpha \)-sets of measure 0.

1. Introduction

By Fatou's theorem [F] (or [CL, Theorem 2.1, p. 17]), the points \( \zeta \) on the unit circle \( C \) where the radial limit \( f^*(\zeta) \) of a bounded holomorphic function \( f \) in the unit disk \( D \) exists constitute a set of Lebesgue measure \( 2\pi \). If in addition \( f \neq w_0 \), then the set \( E(f, w_0) \) of points \( \zeta \) on \( C \) where \( f^*(\zeta) = w_0 \) has measure 0, by a theorem of F. Riesz and M. Riesz [RR, p. 41] (or [CL, Theorem 2.5, p. 22]). On the other hand, to each set \( E \) of measure 0 on \( C \) correspond nonconstant, bounded holomorphic functions \( f \) in \( D \) such that the ordinary limit of \( f \) (and therefore the radial limit of \( f \)) is 0 everywhere on \( E \) (see [P, p. 214] or [Z, Vol. I, end of p. 276]).

G. T. Cargo has pointed out that not every set of measure 0 on \( C \) is contained in the set \( E(f, 0) \) of some inner function \( f \), that is, of some bounded holomorphic function satisfying almost everywhere on \( C \) the condition \( |f^*(\zeta)| = 1 \). (We refer to [CL] for background concerning the class of inner functions and the subclasses of singular inner functions and Blaschke products that will be considered in the sequel.) Cargo showed [C, Theorems 1 and 4] that for each complex number \( w_0 \) and each nonconstant inner function \( f \) the set \( E(f, w_0) \) is meagre; in other words, it is a set of first category.

In [B], the second author considered the problem of characterizing the sets \( E(f, 0) \) for inner functions \( f \). He showed that if \( f \) is an inner function, then the set of points where \( f \) has the radial limit 0 is the intersection of countably many \( F_\alpha \)-sets of measure 0. Theorem 2.4 in [B] gives some sufficient conditions for a set \( E \) on \( C \) to be the zero-set of the radial-limit function of an inner function. Example: if \( E \) is the intersection of an \( F_\alpha \)-set of measure 0 with a \( G_\delta \)-set, then the zero-set of the radial-limit function of some Blaschke product is equal to \( E \).

The present paper completes the characterization.
Theorem. In order that a set on the unit circle \( C \) be the set where some inner function has the radial limit 0, it is necessary and sufficient that it be a countable intersection of \( F_o \)-sets of measure 0.

Our proof of the sufficiency in the theorem requires a construction involving certain inner functions. In Section 2, we develop our elementary building blocks; in Section 3, we perform the final synthesis. Section 4 is devoted to some concluding remarks.

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2. Singular inner functions

We use functions of the form

\[
 f(z) = \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{z + e^{it}}{z - e^{it}} d\mu(t) \right),
\]

where the symbol \( \mu \) in the Stieltjes integral denotes a nondecreasing function that is defined on the real line and satisfies at each point \( t \) the three conditions

\[
 \begin{align*}
 \mu(0) &= 0 \quad \text{(initial standardization)}, \\
 \mu(t + 2\pi) - \mu(t) &= \mu(2\pi) = \|\mu\| \quad \text{(2\pi-periodicity of the distribution \( d\mu \))}, \\
 \mu(t^+) + \mu(t^-) &= 2\mu(t) \quad \text{(normalization at discontinuities)}.
\end{align*}
\]

At each point \( z = re^{i\theta} \) in \( D \), the real part of \( \log[1/f(z)] \) is

\[
 u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{(1 - r)^2 + 4r \sin^2[(\theta - \phi)/2]} d\mu(t).
\]

For each fixed point \( z = re^{i\theta} \) \((r > 0)\) in \( D \), the integrand in (2) is a decreasing function of \( |t - \theta| \) \((0 \leq |t - \theta| \leq \pi)\). This implies that if we interpret \( d\mu \) as a distribution on the unit circle \( C \) and shift all (or a portion) of the mass in \( d\mu \) to a point nearer to \( e^{i\theta} \), the change does not decrease the value of \( u(z) \). Suppose, for example, that \( \mu \) is constant on an arc

\[
 A = \{ \zeta : \zeta = e^{it}, t_1 < t < t_2 \}
\]

and that the point \( z \) lies in the sector \( S \) of \( D \) bounded by \( A \) and by the two line segments joining the origin to the endpoints of \( A \). To be definite, suppose that \( |z - e^{i\theta_1}| \leq |z - e^{i\theta_2}| \). Shifting the entire mass of \( d\mu \) to \( e^{i\theta_1} \), we obtain a new function

\[
 \mu^*(t) = \begin{cases} 
 0, & t_1 - 2\pi < t < t_1, \\
 \|\mu\|, & t_1 < t < t_1 + 2\pi,
\end{cases}
\]

and we deduce that

\[
 \log \frac{1}{|f(z)|} \leq \frac{\|\mu\|}{2\pi} \Re \frac{e^{i\theta_1} + z}{e^{i\theta_1} - z}.
\]

The mapping \( w(z) = (e^{i\theta_1} + z)/(e^{i\theta_1} - z) \) carries each circle \( \Gamma \) internally tangent to \( C \) at \( e^{i\theta_1} \) onto a vertical line; moreover, if \( \delta \) denotes the diameter of \( \Gamma \), the image of \( \Gamma \) passes through the point \( w = (2 - \delta)/\delta = 2/\delta - 1 \). Dealing similarly with points \( z \) in the half of \( S \) whose boundary passes through \( e^{i\theta_1} \), we see that if \( \mu \) is constant on \( A \), the inequality

\[
 \log \frac{1}{|f(z)|} \leq \frac{\|\mu\|}{2\pi} \left( \frac{2}{\delta} - 1 \right)
\]

is satisfied.
holds at each point \( z \) in \( S \) lying outside the union of the two disks of diameter \( \delta \) that are internally tangent to \( C \) at \( e^{it_1} \) and \( e^{it_2} \), respectively.

In connection with (1), we also recall that if the function \( \mu \) in (1) has an infinite derivative at a point \( t = \theta \), then, by virtue of a standard proof of Fatou's radial-limit theorem, the function \( f \) has the radial limit 0 at \( e^{i\theta} \).

In the sequel, we shall use a lemma that is proved in passing (but not stated) in the upper half of page 9 of [LP].

**Lemma LP.** If \( E \) is a closed subset of \( C \) of measure 0, then there exists a function \( \mu \) that meets the specifications stated in connection with the formula (1) and has the property

\[
\mu'(t) = \begin{cases} 
+\infty, & e^{it} \in E, \\
0, & e^{it} \in C \setminus E.
\end{cases}
\]

To prove the lemma, we point out that if \( E \) is a closed set of measure 0 in \( C \), then each point of \( E \) is either a bilateral limit point of \( E \) or the endpoint of a component of \( C \setminus E \). In the first case, the function \( \mu \) constructed in the second paragraph of page 9 of [LP] has infinite right- and left-hand derivatives at \( t \), and therefore \( \mu'(t) = \infty \). In the second case, \( \mu \) has a saltus at \( t \), and therefore \( \mu'(t) = \infty \) by virtue of the normalization.

Corresponding to each number \( \delta \) in the interval \((0,2)\), let \( T = T(E, \delta) \) denote the union of all circular disks that have diameter \( \delta \) and are internally tangent to \( C \) at some point of \( E \). We note that the number of components of \( T \) is less than \( 2\pi/\delta \). In our application of Lemma LP to a function \( f \) of the form (1), it will be useful to consider the auxiliary mental picture consisting of the graph \( W \) of the real-valued function \(|f|\) over the unit disk \( D \) in \( z/h \)-space \((h \geq 0)\). The following informal proposition may be visualized in terms of the topography encountered by an imaginary traveler on the surface \( W \).

**Proposition.** Let \( f \) be a function given by (1), and denote by \( W \) the surface described above. Then, for each point \( z \) in \( D \setminus T \), the point \((z, |f(z)|)\) on \( W \) lies above the plane \( h = \exp(-||f||/\delta \pi) \). If the function \( \mu \) has an infinite derivative at the point \( t \), then the closure of \( D \) is (in a reasonable sense) tangent at \((e^{it}, 0)\) to the image on \( W \) of each triangular domain in \( D \) that has a vertex at \( e^{it} \).

Consequently, any traveler on \( W \) who strays onto the image of the territory \( T \) incurs the risk of sliding to the bottom of a deep-walled pit.

A subset \( E \) of \( C \) satisfies the condition in our theorem if and only if it has a representation

\[
E = \bigcap_{j=1}^{\infty} E_j,
\]

where each set \( E_j \) is the union of closed sets \( E_{jk} \) \((k = 1, 2, \ldots)\) of measure 0. Because every closed set of measure 0 on \( C \) is nowhere dense, we can assume that for each index \( j \) the sets \( E_{jk} \) \((k = 1, 2, \ldots)\) are pairwise disjoint ([S], see also [B, Theorem 3.2 and Corollary 3.4]). Also, since a point \( e^{it} \) lies in \( E \) if and only if it lies in each of the sets \( E_j \) \((j = 1, 2, \ldots)\), and since, moreover, the intersection of two sets of type \( F_\alpha \) is again of type \( F_\alpha \), we can assume that the relation \( E_j \supset E_{j+1} \) holds for each index \( j \). Finally, to eliminate certain
Corresponding to each of our sets $E_j$, we shall construct a singular inner function $f_j$ such that $E(f_j, 0) = E_j$. The construction of $f_j$ will be strongly affected by a certain sequence $(\delta_{jk})$ of positive numbers; our choice of each number $\delta_{jk}$ will depend on the geometric relations among the sets $E_{lm}$ ($m \leq k$); if $j > 1$, the choice of $\delta_{jk}$ will reflect also the relations between the sets $E_{j-1}$ and $E_j$. For the sake of notational consistency with what will follow, we introduce the alternate symbol $F_{jk}$ for the set $E_{jk}$ ($k = 1, 2, \ldots$). The intersections $E_{2m} \cap F_{1n}$ ($m, n = 1, 2, \ldots$) form a countable collection of disjoint closed sets whose union is the set $E_2$. We order the nonempty elements of this collection into a simple sequence $(F_{2k})$ ($k = 1, 2, \ldots$). More generally, once the sequence $(F_{j-1,k})_{k=1}^{\infty}$ is defined, we order the family of all nonempty closed sets $E_{jm} \cap F_{j-1,n}$ ($m, n = 1, 2, \ldots$) into a simple sequence $(F_{jk})_{k=1}^{\infty}$ of disjoint closed sets.

For $\delta_{11}$ we choose any number in the interval $(0, 1)$, and we construct the corresponding territory $T_{11} = T(F_{11}, \delta_{11})$. After choosing constants $\delta_{1m}$ and constructing the corresponding territories $T_{1m} = T(F_{1m}, \delta_{1m})$ ($m = 1, \ldots, j-1$), we choose a number $\delta_{1j}$ lying in $(0, \delta_{11})$ and small enough so that the territory $T_{1j} = T(F_{1j}, \delta_{1j})$ lies at a positive distance from each of the territories $T_{1m} \cap F_{1m}$ ($m < j$).

Suppose we have chosen the numbers $\delta_{mv}$ for $m = 1, \ldots, j-1$, and $\nu = 1, 2, \ldots$ and for $m = j$ and $\nu < k$. Let $n$ be the unique positive integer for which the set $F_{jk}$ is a subset of $F_{j-1,n}$ ($n = 1, 2, \ldots$). Clearly, we can choose $\delta_{jk}$ small enough so that the territory $T_{jk} = T(F_{jk}, \delta_{jk})$ is a subset of $T_{j-1,n}$ and lies at a positive distance from each of the territories $T_{jh}$ ($h = 1, 2, \ldots, k-1$).

Corresponding to each index $j$, we denote the union of the territories $T_{jk}$ ($k = 1, 2, \ldots$) by $T_j$. It is easy to see that if $e^{it} \in C \setminus E_j$ and the radius $R_i = [0, e^{it})$ meets a component of a territory $T_{jk}$, then the endpoints of the segment $R_i \cap T_{jk}$ lie in none of the territories $T_{jh}$ ($h \neq k$) and in none of the sets $T_{j+1}, T_{j+2}, \ldots$.

If $c_{jk}$ is a positive constant and $\mu_{jk}^*$ is a function satisfying with regard to the set $F_{jk}$ the condition described in Lemma LP, then the function $c_{jk} \mu_{jk}^* = \mu_{jk}$ also satisfies that condition. By virtue of the inequality (3), we can choose the constant $c_{jk}$ small enough so that the function $f_{jk}$ generated by $\mu_{jk}$ in accordance with the formula (1) has at each point of $F_{jk}$ the radial limit 0 and satisfies everywhere in $D \setminus T_{jk}$ the inequality

$$\log \frac{1}{|f_{jk}(z)|} \leq \frac{1}{2j+k}.$$  

If $e^{it} \in F_{jk}$, then $f_{jk}$ has at $e^{it}$ the radial limit 0; if $e^{it} \in C \setminus F_{jk}$, then the function $\mu_{jk}$ is constant on some open segment with the midpoint $t$, and therefore $f_{jk}$ is holomorphic at $e^{it}$ and $|f_{jk}(e^{it})| = 1$. Clearly, the function $\mu_j = \sum_k \mu_{jk}$ has finite norm, and by a classical differentiation theorem of
Fubini, \( \mu'_j = 0 \) a.e. Also, the product

\[
f_j = \prod_{k=1}^{\infty} f_{jk}
\]

converges uniformly on compact sets, satisfies everywhere in \( D \setminus T_j \) the inequality

\[
\log \frac{1}{|f_j(z)|} < \frac{1}{2^j},
\]

and has at each point of \( E_j \) the radial limit 0.

Suppose now that \( e^{it} \in C \setminus E_j \). Because \( \delta_{jk} \to 0 \) as \( k \to \infty \) and

\[
dist(T_{jh}, T_{jk}) > 0
\]

whenever \( h \neq k \), each of the radial segments

\[
\{ z : z = re^{it}, 1 - \frac{1}{m} \leq r < 1 \}
\]

contains a point \( z_m \) in \( D \setminus T_j \). The inequality (4) implies that the relation

\[
\prod_{k>K} |f_{jk}(z_m)| > \exp(-2^{-j-K})
\]

holds for all positive integers \( K \) and \( m \). Since the right-hand side tends to 1 as \( K \to \infty \), and since each of the \( K \) factors \( f_{j1}, \ldots, f_{jK} \) is holomorphic and has modulus 1 at \( e^{it} \), it follows that the cluster set at \( e^{it} \) of \( f_j \) contains at least one point of modulus 1.

3. Celebration after the grubby work

It remains only to combine the functions \( f_j \) into a function \( f \) whose radial limit at each point of the set \( E = \bigcap E_j \) is 0 and whose radial cluster set at each of the points of \( C \setminus E \) includes a point other than 0.

Corresponding to each index \( j \), let

\[
g_j = \frac{1/2 + f_j}{1 + f_j/2}.
\]

Clearly, \( g_j \) is an inner function. At each point where the radial limit of \( f_j \) is 0, the radial limit of \( g_j \) is 1/2, and at each point where the radial cluster set of \( f_j \) includes a point of \( C \), the radial cluster set of \( g_j \) also includes a point of \( C \). Moreover, because \(-1/2\) is not the radial limit of \( f_j \) at any point of \( C \), the function \( g_j \) does not have the radial limit 0 at any point, and therefore \( g_j \) has no singular factor; in other words, \( g_j \) is a Blaschke product, that is, a function of the form

\[
B_j(z) = e^{iz_j} z^\nu_j \prod_{n=1}^{\infty} \frac{1 - z/a_{jn}}{1 - a_{jn} z},
\]

where \( \gamma_j \) and \( \nu_j \) represent a real constant and a nonnegative integer, respectively, and where the points \( a_{jn} \) lie in \( D \setminus \{0\} \) and satisfy the condition

\[
\sum_{n=1}^{\infty} (1 - |a_{jn}|) < \infty.
\]

Except in the trivial case where \( E = \varnothing \), formula (1) guarantees that the value of \( \log 1/f_j(0) \) has a positive value, and therefore \( B_j(0) \) is positive. This in turn
implies that the factors $e^{iy_j}$ and $\nu_j$ in our formula for $B_j(z)$ have the values 1 and 0, respectively.

The final stage of our synthesis is the creation of the formal product $f = \prod g_j$. That the symbol $f$ represents a genuine Blaschke product is equivalent to the convergence of the double series

$$\sum_{j=1}^{\infty} \sum_{n=1}^{\infty} (1 - |a_{jn}|).$$

This follows from the inequality $\log r < r - 1$ ($r > 0$), the equality $1/B_j(0) = \prod_{n=1}^{\infty} |a_{jn}|^{-1}$, and the absolute convergence of the series $\sum_{j=1}^{\infty} \log 1/B_j(0)$.

To see that the Blaschke product $f$ has the desired property, suppose first that $e^{it} \in E$. Then $e^{it} \in E_j$ ($j = 1, 2, \ldots$), and because each of the functions $g_j$ has at $e^{it}$ the radial limit 1/2, the radial limit of $f$ is 0, at the point $e^{it}$.

Next, suppose that $e^{it} \in C \setminus E$, and let $J$ denote the least nonnegative integer such that $e^{it} \in C \setminus E_{J+1}$. At the point $e^{it}$, the product of the first $J$ factors $g_j$ has the radial limit $2^{-J}$. As we pointed out in Section 2, the radius $[0, e^{it})$ supports a sequence $\{z_m\}$ that converges to $e^{it}$ and lies in $D \setminus T_{J+1}$. Because the sequence of sets $T_j$ decreases, the sequence $\{z_m\}$ lies in each of the sets $D \setminus T_j$ ($j = J+1, J+2, \ldots$). To study the behavior of the corresponding factors $g_j$ ($j = J+1, J+2, \ldots$) on the sequence of points $z_m$ ($m = 1, 2, \ldots$), we point out that if $|g_j(z_m)| = \rho$, then, by virtue of formula (5), the function $g_j$ carries the point $z_m$ to a point on the circle having a diameter with the endpoints

$$\frac{1-2\rho}{2-\rho} \quad \text{and} \quad \frac{1+2\rho}{2+\rho}$$
on the real line. Because the left-hand endpoint is nearer to the origin than the one on the right,

$$|g_j(z_m)| \geq \left| \frac{1-2\rho}{2-\rho} \right| .$$

The inequality $|f_j(z_m)| > 1/2$ resolves the ambiguity of sign, and we see that

$$1 - |g_j(z_m)| \leq 1 - \frac{2\rho - 1}{2 - \rho} = \frac{3(1 - \rho)}{2 - \rho} < 3(1 - |f_j(z_m)|).$$

Since the values of $g_j(z_m)$ are bounded away from zero, there exists a constant $c$ such that

$$-\log |g_j(z_m)| < c(1 - |g_j(z_m)|) < 3c(1 - |f_j(z_m)|).$$

It follows that

$$\log \prod_{j > J} \frac{1}{|g_j(z_m)|} < 3c \sum_{j > J} (1 - |f_j(z_m)|).$$

The inequality (4) implies that to each positive number $\epsilon$ there corresponds an integer $N$ ($N \geq J$) such that the inequality

$$\sum_{j > N} (1 - |f_j(z_m)|) < \epsilon$$

holds for each index $m$. The argument at the end of Section 2 shows that for all sufficiently large values of the index $m$, the number $\epsilon$ is also an upper bound for the sum of the analogous terms with the indices $j = J + 1, \ldots, N$. 
It follows that the radial cluster set at \( e^{it} \) of the product \( \prod_{j>1} g_j(z_m) \) contains at least one point of \( C \), and we see immediately that the radial cluster set at \( e^{it} \) of the function \( f \) contains at least one point of the circle \( |w| = 2^{-j} \). This concludes the proof that every subset of \( C \) that is a countable intersection of \( F_\alpha \)-sets of measure 0 is the set where an inner function has radial limit 0.

We point out that if some open subset \( A \) of \( C \) contains no points of the set \( E \), then each component of \( A \) lies in a domain (cuspidate at both ends of the component) that contains no zeros of any of the Blaschke products \( g_j \). Consequently, the Blaschke product \( f \) is holomorphic and has absolute value 1 on each component of \( A \). Moreover, corresponding to each positive \( \epsilon \) and each compact subset \( K \) of \( D \cup A \), we can choose our set \( \{ \delta_{jk} \} (j, k = 1, 2, \ldots) \) so that the inequality \( |f(z) - 1| < \epsilon \) holds everywhere in \( K \).

4. The cases where \( w \neq 0 \)

The following extension of our main theorem holds.

If \( w \in D \setminus \{0\} \), then a set \( E \) on \( C \) is the set \( E(f_w, w) \) for some Blaschke product \( f_w \) if and only if \( E \) is a countable intersection of \( F_\alpha \)-sets of measure 0.

The necessity of the condition is obvious. To prove sufficiency, suppose first that \( |w| \neq 2^{-1}, 2^{-2}, 2^{-3}, \ldots \). We construct the function \( f \) as in Sections 2 and 3 and define \( f_w \) by the formula

\[
f_w = \frac{w + f}{1 + \overline{w} f}.
\]

Then \( f_w \) has the radial limit \( w \) at all points \( e^{it} \) where \( f \) has the radial limit 0. Moreover, since \( -w \) is not a radial limit value for \( f \), the set \( E(f_w, 0) \) is empty, and therefore \( f_w \) is a Blaschke product.

If \( |w| \) has one of the values \( 2^{-m} (m = 1, 2, \ldots) \), we proceed in the same way, except that we now define the Blaschke products \( B_j \) by the formula

\[
B_j = \frac{1/3 + f_j}{1 + f_j/3}.
\]

In the case where \( f \) is an inner function and \( |w| = 1 \), the set \( E(f, w) \) is still of type \( F_\alpha \); also, the theorems of F. and M. Riesz and of Cargo mentioned in our introduction are still applicable. But it has been shown (see [BN]) that \( E(f, w) \) need not be contained in an \( F_\alpha \)-set of measure 0. In other words, the characterization of the sets \( E(f, w) \) for inner functions depends on whether \( w \in D \) or \( w \in C \).

An interesting question—perhaps easier than the problem of complete characterization of the sets \( E(f, 1) \)—is whether every meagre \( F_\alpha \)-set of measure 0 is contained in the set \( E(f, 1) \) for some nonconstant inner function \( f \).

Another natural extension of the radial-limit zero-set problem concerns the possibility of radial-limit interpolation. We refer to [BN] for results in this direction and a discussion of the literature.

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