A NOTE ON SINGULARITIES IN SEMILINEAR PROBLEMS

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Abstract. We consider the equation \( \Delta u - \frac{1}{2} x \cdot \nabla u - \frac{u}{q-1} + u^q = 0 \), for \( q > 1 \). We study the isolated singularities and present a nonlinear technique, and give a complete classification.

1. Introduction

In this note we study the isolated singularities of the positive solutions of the following nonlinear equation:

\[
\Delta u - \frac{1}{2} x \cdot \nabla u - \frac{u}{q-1} + u^q = 0.
\]

Let \( \Omega \) be an open subset of \( \mathbb{R}^N \), \( N \geq 2 \), containing 0, \( \Omega' = \Omega \setminus \{0\} \) and \( q > 1 \). We are concerned with the following question:

If \( u \in C^2(\Omega') \) is a positive solution of (1.1) in \( \Omega' \), what can be said about \( u(x) \) and about the equation as \( |x| \to 0 \)?

It is well known [15] that (1.1) has no nontrivial globally bounded solution for \( A = 1, 2 \) or \( A > 2 \), \( q < \frac{N}{N-2} \) except the constant solution \( u = \left( \frac{1}{q-1} \right)^{1/(q-1)} \).

If we look for a specific solutions of (1.1) under the form

\[ u(r) = \alpha r^\beta, \]

then we get

\[ \beta = -\frac{1}{q-1} \quad \text{and} \quad \alpha = \lambda_{N,q} = \left\{ \frac{2}{q-1} \left( N - \frac{2q}{q-1} \right) \right\}^{1/(q-1)} \]

where it is clear that \( \lambda_{N,q} \) only exists when \( N > 2 \) and \( q > \frac{N}{N-2} \). It follows by substitution that \( u(r) = \lambda_{N,q} r^\beta \) is also a solution of the Emden-Fowler equation

\[
\Delta w + w^q = 0.
\]

This equation has been intensively studied. When \( N > 2 \) two critical values \( \frac{N}{N-2} \) and \( \frac{N+2}{N-2} \) appear. The first studies in the radial case are due to Emden, then Fowler [3, 5, 6], Brezis and Lions [2] and Lions [22] (see also [18, 1]).

Let us briefly describe our results. We have two cases.
First case. $1 < q < \frac{N}{N-2}$. Then any positive solution of (1.1) in $\Omega'$ satisfies the equation $-\Delta u + \frac{1}{2} x \cdot \nabla u + \frac{u}{q-1} - u^q = \alpha \delta_0$ in $D'(\Omega)$ with $\alpha \geq 0$. Furthermore if $\alpha = 0$, then the “singularity is removable”.

Second case. $\frac{N}{N-2} < q < \frac{N+2}{N}$. Then for any positive radial solution of (1.1) in $\Omega'$ we have $\lim_{r \to 0} r^{2/(q-1)} u(r) = l \in \{0, \lambda_N, \} \quad \text{as} \quad r \to 0$, where $r = |x|$. If $l = 0$ and $q > 2$, the “singularity is removable”. Note that if $q \geq \frac{N}{N-2}$, any solution of (1.1) satisfies

$$-\Delta u + \frac{1}{2} x \cdot \nabla u + \frac{u}{q-1} - u^q = 0, \quad \text{in} \ D'(\Omega).$$

2. Main results

Let $N > 2$ and set $\Omega = B_R(0) = \{x \in \mathbb{R}^N, |x| < R\}, \quad R > 0$ and $\Omega' = \Omega \setminus \{0\}$. Our main result is the following:

Theorem 2.1. Assume that $u \in L^1_{\text{loc}}(\Omega')$,

\begin{align}
&(2.1) \quad \Delta u - \frac{1}{2} x \cdot \nabla u \in L^1_{\text{loc}}(\Omega'), \quad \text{in} \ D'(\Omega'), \\
&(2.2) \quad u \geq 0, \quad \Delta u - \frac{1}{2} x \cdot \nabla u \leq au + f \quad \text{a.e. in} \ \Omega,
\end{align}

where $a$ is a nonnegative constant and $f \in L^1_{\text{loc}}(\Omega)$. Then

\begin{equation}
(2.3) \quad u \in L^1_{\text{loc}}(\Omega'), \\
\text{and there exists} \ h \in L^1_{\text{loc}}(\Omega), \text{and} \ \alpha \geq 0 \ \text{such that}
\end{equation}

\begin{equation}
(2.4) \quad -\Delta u + \frac{1}{2} x \cdot \nabla u = h + \alpha \delta_0 \quad \text{in} \ D'(\Omega).
\end{equation}

For the proof, we use a nonlinear technique introduced by Serrin [23], [24] and a linear method by H. Brezis and P. L. Lions [2], [22].

The proof is divided into five steps.

Step 1. We claim that

$$x \cdot \nabla u = r \frac{\partial u}{\partial r},$$

where $x = (r, \sigma), \ r = |x|$.

Proof of the claim. Writing

\begin{align}
x \cdot \nabla u &= \sum_{i=1}^N x_i \left( \frac{\partial u}{\partial r} \frac{\partial r}{\partial x_i} + \sum_{j=1}^{N-1} \frac{\partial u}{\partial \sigma_j} \frac{\partial \sigma_j}{\partial x_i} \right), \\
x \cdot \nabla u &= r \frac{\partial (u(r, \sigma))}{\partial r} + \sum_{j=1}^{N-1} \frac{\partial u}{\partial \sigma_j} \sum_{i=1}^N x_i \frac{\partial \sigma_j}{\partial x_i},
\end{align}

as in [17], we have

$$\frac{\partial \sigma_j}{\partial x_i} = \begin{cases} \frac{x_i x_{j+1}}{r_j^i f_j^i} & \text{if} \ i \leq j, \\
-r_j^i & \text{if} \ i = j + 1, \\
0 & \text{if} \ i > j + 1,
\end{cases}$$
where \( r_j^2 = x_1^2 + x_2^2 + \cdots + x_j^2 \), \( r_N^2 = r^2 \). Thus
\[
\sum_{i=1}^{N} x_i \frac{\partial \sigma_i}{\partial x_i} = 0.
\]
And then
\[
(2.5) \quad x \cdot \nabla u = r \frac{\partial u}{\partial r}.
\]

**Step 2.** \( u \in L_{\text{loc}}^{1}(\Omega) \). We begin by considering the average
\[
\bar{u}(r) = \frac{1}{\omega_{N-1}} \int_{S^{N-1}} u(r, \sigma) \, d\sigma, \quad 0 < r < R,
\]
where \( \omega_{N-1} \) is the volume of the sphere \( S^{N-1} \). It follows from (2.1), (2.2) and (2.5) that
\[
(2.6) \quad \Delta \bar{u} - \frac{1}{2} r \bar{u}_r \in L_{\text{loc}}^{1}(0, R), \quad \bar{u} \geq 0,
\]
\[
(2.7) \quad \Delta \bar{u} - \frac{1}{2} r \bar{u}_r \leq a \bar{u} + \bar{f} \quad \text{on} \ (0, R),
\]
and
\[
(2.8) \quad \frac{1}{r^{N-1}} (r^{N-1} K(r) \bar{u}_r)_r \leq a \bar{u} + \bar{f}, \quad \text{for} \ r \in (0, R),
\]
where \( K(r) = e^{-r^2/4} \). In particular \( \bar{u} \in C^1(0, R) \).

Let \( R' < R \) be fixed. Integrating (2.8) over \( (r, R') \) we find as in [2] that
\[
(2.9) \quad u \in L_{\text{loc}}^{1}(\Omega)
\]
and
\[
(2.10) \quad \bar{u}(r) \leq \frac{C}{r^{N-2}} + C.
\]

**Step 3.** Let \( g(x) = -\text{div}(K(x)\nabla u) \) a.e. in \( \Omega \), where \( K(x) = e^{-|x|^2/4} \). Then \( g \in L_{\text{loc}}^{1}(\Omega) \) and for any \( \varphi \in D(\Omega) \), \( 0 \leq \varphi \leq 1 \), \( \varphi \equiv 1 \) near \( x = 0 \),
\[
(2.11) \quad \int_{\Omega} g \varphi^2 \, dx \leq -\int_{\Omega} u \cdot \text{div}(K(x)\nabla \varphi^2) \, dx.
\]

**Proof of the step 3.** From the definition of \( g \) we have
\[
(2.12) \quad \int_{\Omega} g \psi \, dx = \int_{\Omega} K(x)\nabla u \nabla \psi \, dx,
\]
for any \( \psi \in W^{1, \infty}(\Omega') \) with compact support in \( \Omega' \). Set
\[
P_k(t) = \begin{cases} 1 & \text{if} \ t > 1 + k, \\ t - k & \text{if} \ k \leq t \leq 1 + k, \\ 0 & \text{if} \ t < k \end{cases}
\]
for any \( t \geq 0 \) and \( k \geq 0 \).
Let $0 < \rho < R$ and $\varepsilon < \frac{\rho}{R}$. Let $\varphi \in D(\Omega)$ be such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on $B_{\rho} = \{x \in \mathbb{R}^N, |x| < \rho\}$ and $\psi_\varepsilon = \eta_\varepsilon \varphi$ with $\eta_\varepsilon \in C^\infty(\Omega)$, $0 \leq \eta_\varepsilon \leq 1$, $\eta_\varepsilon \equiv 0$ if $|x| < \varepsilon$, $\eta_\varepsilon(x) \equiv 1$ if $|x| > 2\varepsilon$ and $|\nabla \eta_\varepsilon| \leq \frac{C}{\varepsilon}$. We have, from (2.12),

$$
(2.13) \quad \int_\Omega g(1 - P_k(u))\psi_\varepsilon^2 \, dx + \int_{\{k < u < 1 + k\}} K(x)|\nabla u|^2\psi_\varepsilon^2 \, dx
$$

$$
= \int_\Omega (1 - P_k(u))K\nabla u\nabla \psi_\varepsilon^2 \, dx.
$$

Since

$$
\sum_{k=0}^n (1 - P_k(t))^+ = (n + 1 - t)^+,
$$

we have

$$
\int_{\{u < n+1\}} g(n + 1 - u)^+\psi_\varepsilon^2 \, dx + \int_{\{u < 1+n\}} K(x)|\nabla u|^2\psi_\varepsilon^2 \, dx
$$

$$
\leq \int_{\{u < n+1\}} (n + 1 - u)K\nabla u\nabla \varphi^2\eta_\varepsilon^2 \, dx
$$

$$
+ 2\int_{\{u < 1+n\} \cap B_{2\varepsilon}} (n + 1 - u)K(x)|\nabla u|\psi_\varepsilon|\nabla \eta_\varepsilon| \, dx.
$$

Now for any real $h > 0$, we have $n + 1 - u(x) > n + 1 - \frac{h}{n+1}$ a.e. in $\{u < \frac{n+1}{h+1}\}$. Thus, dividing by $n + 1$ and using the Hölder inequality, for any $\beta > 0$,

$$
\frac{h}{h + 1} \int_{\{u < \frac{n+1}{h+1}\}} g\psi_\varepsilon^2 + \frac{1}{n + 1} \int_{\{u < 1+n\}} K|\nabla u|^2\psi_\varepsilon^2
$$

$$
\leq \int_{\{u < 1+n\}} \left(1 - \frac{u}{n+1}\right)K\nabla u\nabla (\varphi^2)\eta_\varepsilon^2
$$

$$
+ \beta^2 \int_{\{u < 1+n\}} K(x)|\nabla u|^2\psi_\varepsilon^2 + \beta^{-2} \int_{B_{2\varepsilon}} K(x)|\nabla \eta_\varepsilon|^2.
$$

Let $\beta^2 = \frac{1}{2(n+1)}$; then

$$
\frac{h}{h + 1} \int_{\{u < \frac{n+1}{h+1}\}} g\psi_\varepsilon^2 + \frac{1}{2(n + 1)} \int_{\{u < 1+n\}} K|\nabla u|^2\psi_\varepsilon^2
$$

$$
\leq \int_{\{u < 1+n\}} \left(1 - \frac{u}{n+1}\right)K\nabla u\nabla (\varphi^2)\eta_\varepsilon^2 + 2(n + 1)C\varepsilon^{N-2}.
$$

From Fatou's Lemma—which is valid since $g \geq -au - f \in L^2_{loc}(\Omega)$—we deduce as $\varepsilon \to 0$, $n \to +\infty$, $h \to +\infty$ that $g\varphi^2 \in L^1(\Omega)$ and satisfies (2.11).

Step 4. Now since $u \in L^1(\Omega)$, we can define the distribution

$$
T = -\text{div}(K\nabla u) - g \quad \text{in } D'(\Omega).
$$

Then as in [2], we have

$$
T = \sum_{|p| \leq m} c_p D^p \delta_0.
$$

Let $\psi \in D(B_R)$ be any fixed function such that

$$
(-1)^{|p|} D^p \psi(0) = c_p \quad \text{for every } |p| \leq m
$$
and
\[ \psi_\varepsilon(x) = \psi \left( \frac{x}{\varepsilon} \right). \]

Then
\[ (2.14) \int_{B_R} u \text{div}(K \nabla \psi_\varepsilon) = \int_{B_R} g \psi_\varepsilon + \sum_{|p| \leq m} \frac{c_p^2}{|p|}. \]

On the other hand we have
\[ \int_{B_R} u \text{div}(K \nabla \psi_\varepsilon) \leq \frac{C}{\varepsilon^2} \int_{B_R} u \, dx + CR \int_{B_R} u \, dx \]
and therefore
\[ \int_{B_R} u \text{div}(K \nabla \psi_\varepsilon) \leq \frac{C}{\varepsilon^2} \int_{0}^{R} \overline{u} r^{n-1} \, dr + CR \int_{0}^{R} \overline{u} r^{n-1} \, dr. \]

We deduce from (2.10) that \( |\int_{B_R} u \text{div}(K \nabla \psi_\varepsilon)| \leq C \). Comparing this with (2.14) we conclude that \( c_p = 0 \) when \( |p| \geq 1 \).

Finally we choose \( \eta \in D(\Omega) \), \( 0 \leq \eta \leq 1 \), and \( \eta \equiv 1 \) near \( x = 0 \).

We have
\[ \langle T, \eta^2 \rangle = c_0 \int_{\Omega} u \text{div}(K \nabla \eta^2) - \int_{\Omega} g \eta^2, \]
and hence \( c_0 \geq 0 \) from (2.11).

Step 5. The end of the proof of Theorem 2.1.

Let \( h(x) = g(x)e^{\frac{|x|}{\varepsilon}} \) a.e. in \( \Omega \); then \( h \in L^1_{\text{loc}}(\Omega) \) and \(-\Delta u + \frac{1}{2} x. \nabla u = h + c_0 \delta_0 \) in \( D'(\Omega) \).

3. THE SUBCRITICAL CASE

Now we return to equation (1.1). Let \( \mu \) be the fundamental harmonic function in \( \mathbb{R}^N \setminus \{0\} \), \( N > 2 \), that is
\[ (3.1) \mu(x) = \frac{1}{N(N-2)\omega_N}|x|^{2-N}. \]

**Theorem 3.1.** Let \( N > 2 \), \( 1 < q < \frac{N}{N-2} \). Let \( u \in C^2(\Omega') \) be a nonnegative solution of equation
\[ (3.2) \Delta u - \frac{1}{2} x. \nabla u - \frac{u}{q-1} + u^q = 0 \quad \text{in } \Omega' = \Omega \setminus \{0\}. \]

Then
(i) either \( u \) can be extended to a smooth solution of (3.2) in \( \Omega \), or
(ii) there exists \( \alpha > 0 \) such that \( \lim_{x \to 0} \frac{u(x)}{\mu(x)} = \alpha \) and \( u \) satisfies the equation
\[ (3.3) -\Delta u + \frac{1}{2} x. \nabla u + \frac{u}{q-1} - u^q = \alpha \beta_0 \quad \text{in } D'(\Omega). \]

**Proof.** From (3.2) we have \( \Delta u - \frac{1}{2} x. \nabla u \leq \frac{u}{q-1} \); hence from Theorem 2.1 we have
\[ -\Delta u + \frac{1}{2} x. \nabla u + \frac{u}{q-1} - u^q = \beta \delta_0, \quad u \in L^1_{\text{loc}}(\Omega), u^q \in L^1_{\text{loc}}(\Omega). \]
Moreover equation (3.2) can be written in the form

\begin{equation}
\text{div}(K\nabla u) + d(x)u = 0 \quad \text{in } \Omega'
\end{equation}

where \( d(x) = k(x)(u^q - 1) \). Since \( q < \frac{N}{N-2} \) we can find an \( \varepsilon > 0 \) such that \( d \in L^{N/(2-\varepsilon)}(\Omega) \). We then deduce from [24] that if \( u \) is singular at \( 0 \) there exists a constant \( C > 0 \) such that

\begin{equation}
C\mu \leq u \leq \frac{1}{C\mu} \quad \text{near } 0.
\end{equation}

As in Guedda-Veron [18], we prove by scaling that there is an \( \alpha > 0 \) such that

\[ \lim_{x \to 0} \frac{u(x)}{\mu(x)} = \alpha; \]

we get that

\begin{equation}
\lim_{x \to 0} |x|^{N-1}\nabla u(x) = \frac{\alpha}{N\omega_N} \xi, \quad \text{where } \xi = \lim_{x \to 0} \frac{x}{|x|}
\end{equation}

and then (3.3).

### 4. The super critical case

We still assume that \( \Omega = B_R(0) = \{x \in \mathbb{R}^N, |x| < R\} \) and \( \Omega' = \Omega \setminus \{0\} \). In this section we present some results concerning the isolated singularities of the positive solutions of (1.1), that is, of

\begin{equation}
-\Delta u + \frac{1}{2}x \cdot \nabla u + \frac{u}{q-1} - u^q = 0 \quad \text{in } \Omega',
\end{equation}

where

\[ \frac{N}{N-2} < q < \frac{N+2}{N-2} \quad \text{and} \quad N > 2. \]

If we look for the solution of (3.2) under the form \( u(x) = u(|x|) = \alpha|x|^\beta \), then we get

\[ \beta = -\frac{1}{q-1} \quad \text{and} \quad \alpha = \lambda_{N,q} = \left( \frac{2}{q-1} \left( \frac{N-2q}{q} \right) \right)^{1/(q-1)}.
\]

**Theorem 4.1.** Let \( N > 2, q \geq \frac{N}{N-2} \). Let \( u \in C^2(\Omega') \) be a nonnegative solution of (4.1) in \( D'(\Omega') \). Then we have \( \alpha = 0 \) in (3.3), i.e.

\[ -\Delta u + \frac{1}{2}x \cdot \nabla u + \frac{u}{q-1} - u^q = 0 \quad \text{in } D'(\Omega).
\]

**Theorem 4.2.** Under the assumptions of Theorem 4.1, if

\[ u^{(q-1)N/2} \in L^1_{\text{loc}}(\Omega), \]

then \( u \) can be extended to \( \Omega \) as a solution of (4.1) in \( \Omega \).

The proofs of Theorems 4.1 and 4.2 are the same as in Guedda and Veron [18] and they are omitted.

The main result of this section is the following
Theorem 4.3. Let $N > 2$ and $\frac{N}{N-2} < q < \frac{N+2}{N-2}$ and $u$ be a positive radial solution of (4.1).

Then we have

(i) $r^{2/(q-1)}u(r)$ has a limit $l$ as $r \to 0$ and $l \in \{0, \lambda_N, q\}$.

(ii) If $l = 0$ and $q > 2$, then $u$ can be extended to $\Omega$ as a $C^2(\Omega)$ solution of (4.1) in $\Omega$.

We need the following

Lemma 4.1. Let $u$ be a positive radial solution of (4.1) in $\Omega'$. Then $r^{2/(q-1)}u(r)$ is bounded as $r \to 0$.

Proof. Let

\begin{equation}
(4.5) \quad t = \log(r^2), \quad v(t) = r^{2/(q-1)}u(r), \quad \gamma = \frac{4}{q-1} + 2 - N > 0;
\end{equation}

then (4.1) takes the equivalent form

\begin{equation}
(4.6) \quad \gamma^2 v'' + \frac{1}{2} \gamma e^{-2t/\gamma} v_t + \lambda v + v^q = 0
\end{equation}

where $\lambda = \frac{2}{q-1}(\frac{2}{q-1} + 2 - N) < 0$. Let

\begin{equation}
L(t) = \frac{1}{2} \gamma^2 v_t^2 + \lambda \frac{v^2}{2} + \frac{v^{q+1}}{q+1}, \quad \text{for } t \in [0, +\infty[.
\end{equation}

From (4.6) the function $L$ is nonincreasing. Hence for any $t_0 \leq t < +\infty$

\begin{equation}
(4.7) \quad \frac{1}{2} \gamma^2 v_t^2 + \lambda \frac{v^2}{2} + \frac{v^{q+1}}{q+1} \leq L(t_0).
\end{equation}

Hence $v$ is bounded, as $q > 1$.

Lemma 4.2. Suppose $\frac{N}{N-2} < q < \frac{N+2}{N-2}$. Let $v$ be any solution of (4.6). Then $\lim_{t \to +\infty} v_t = 0$ and $v$ has a limit $l$ at $+\infty$ and

\begin{equation}
(4.8) \quad l(\lambda + l^{q-1}) = 0.
\end{equation}

Proof. Using (4.6), (4.7) and Lemma 4.1 we have

\begin{equation}
(4.9) \quad v, v_t, v_{tt} \text{ and } v_{ttt} \text{ are bounded}.
\end{equation}

Hence $v_t \in L^2(t_0, +\infty)$. This implies $v_{tt} \in L^2(t_0, +\infty)$.

It follows from this and (4.9) that

\begin{equation}
\lim_{t \to +\infty} v_t = \lim_{t \to +\infty} v_{tt} = 0.
\end{equation}

For any $t \geq \tau \geq t_0$, we have

\begin{equation}
|v(t) - v(\tau)| \leq \|v_t\|_{L^2(t_0, +\infty)} \cdot \sqrt{(t - \tau)},
\end{equation}

and hence $v$ has a limit $l$ at $+\infty$. From (4.6) we get (4.8).

Proof of Theorem 4.3. From Lemmas 4.1 and 4.2 $r^{2/(q-1)}u(r)$ has a limit $l$ where $l = 0$ or $\lambda_N, q$.

Let $v$ be as in (4.5). Assume that $l = 0$. The proof of (ii) is divided into two steps.
Step 1. Assume \( v \) decreases to 0 as \( t \) tends to \( +\infty \); then
\[
v(t) \leq Ce^{-t},
\]
for \( t \geq 0 \) and some \( C > 0 \). In fact, from (4.6) we have
\[
\gamma^2(v_t + v) = \int_t^{+\infty} \frac{1}{2} ye^{-2t/\gamma} v_\gamma + \lambda v + v^q \, ds.
\]
Since \( v \) and \( v_t \) are bounded, \( \omega(t) = \frac{1}{2} ye^{-2t/\gamma} v_\gamma + \lambda v + v^q \) is integrable; then \( \lambda v + v^q \) is integrable and then \( v \) is integrable. Since \( v_t \leq 0 \) and \( \lambda < 0 \), we have
\[
v_t + v < Cv^q, \quad \text{for } t \text{ large},
\]
then
\[
(4.12) \quad v_t + v \leq Cv^{q-1}, \quad \text{for } t \text{ large},
\]
which implies, if \( q > 2 \), (4.10). Moreover, Theorem 4.2 implies that if \( v \) satisfies (4.10), then \( u \) can be extended to \( \Omega \) as a regular solution of (1.1) in \( \Omega \).

Step 2. Assume that \( v \) is not asymptotically monotone. Then there exists a sequence \( \{t_n\} \) such that \( v(t_n) = 0 \), \( v(t_{2n}) \) is local minimum and \( v(t_{2n+1}) \) is local maximum, and (from the equation)
\[
0 < v(t_{2n}) < \lambda v(t_{2n+1})
\]
which is not possible since \( \lim_{t \to +\infty} v(t) = 0 \), which ends the proof.

References


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