ELLIPITIC EQUATIONS OF ORDER $2m$ IN ANNULAR DOMAINS

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ABSTRACT. In this paper we study the existence of positive radial solutions for some semilinear elliptic problems of order $2m$ in an annulus with Dirichlet boundary conditions. We consider a nonlinearity which is either sublinear or the sum of a sublinear and a superlinear term.

1. Introduction

Let $\Omega(a, b)$ denote the annulus $\{x \in \mathbb{R}^n; a < |x| < b\}, 0 < a < b < \infty, n \geq 2$, and consider the semilinear elliptic problems

\begin{equation}
(-1)^m \Delta^m u = g(|x|)f(u) \quad \text{in} \quad \Omega(a, b)
\end{equation}

and

\begin{equation}
(-1)^m \Delta^m u = \lambda g(|x|)f(u) + k(|x|)h(u) \quad \text{in} \quad \Omega(a, b)
\end{equation}

with the boundary conditions

\begin{equation}
\frac{\partial u}{\partial \nu} = \cdots = \frac{\partial^{m-1} u}{\partial \nu^{m-1}} = 0 \quad \text{on} \quad \partial \Omega(a, b)
\end{equation}

where $\lambda > 0$ is a parameter, $\frac{\partial}{\partial \nu}$ is the outward normal derivative, $m$ is a positive integer and $f, g, h, k$ satisfy at least the following assumptions:

(H1) $f, h : [0, \infty) \to [0, \infty)$ are continuous functions;

(H2) $g, k : [a, b] \to [0, \infty)$ are continuous functions such that $g, k \neq 0$ in $[a, b]$.

When $m = 1$ the existence of a positive radial solution of problem (1.1), (1.3) has been intensively studied in the case where $f$ is superlinear at 0 and $\infty$ (see e.g. [2]-[4], [6], [11], [13]). The approach used in most papers was the shooting method. In contrast the result of [2] was obtained by a variational approach and the use of a priori estimates. The case $m \geq 1$ was treated in [8] and [9] using a priori estimates and well-known properties of compact mappings taking a cone in a Banach space into itself (see [10]).

When $m = 1$ and $f$ is sublinear at 0 and $\infty$ problem (1.1), (1.3) possesses at least one positive radial solution. This case was studied in [11] and [15] using the shooting method and the fixed point theorem in cones respectively.

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Finally, when \( m = 1 \), \( g = k = 1 \), \( \lambda > 0 \), \( f(u) = u^q \), \( 0 < q < 1 \), and \( h(u) = u^p \), \( p > 1 \), equation (1.2) in a smooth bounded domain with Dirichlet boundary condition was recently studied in [1]. Some results extend to the case where \( f \) is concave and behaves like \( u^q \), \( 0 < q < 1 \) near \( u = 0 \). Also if \( 1 < p < \frac{n+2}{n-2} \), \( u^p \) can be replaced by a function \( h \) with the same behavior near \( u = 0 \) and near \( u = \infty \).

In this paper we first prove an existence result for problem (1.1), (1.3) when \( f \) is sublinear at 0 and \( \infty \). We do not require any monotonicity assumptions on \( f \). Then we consider problem (1.2), (1.3) when \( f \) is sublinear at 0 and \( h \) is superlinear at 0 and possibly at \( \infty \). For this problem we also assume that \( f \) is nondecreasing.

Our main results are the following two theorems.

**Theorem 1.1.** Let \( f \) satisfy (H1) and let \( g \) satisfy (H2). Assume moreover that the following condition holds:

\[
(H_3) \quad \lim_{u \to 0} \frac{f(u)}{u} = \infty \quad \text{and} \quad \lim_{u \to \infty} \frac{f(u)}{u} = 0.
\]

Then problem (1.1), (1.3) has at least one positive radial solution in \( C^{2m}(\Omega(a, b)) \).

**Theorem 1.2.** Let \( f, h \) satisfy (H1) and let \( g, k \) satisfy (H2). Assume moreover that the following conditions hold:

\[
(H_4) \quad \lim_{u \to 0} \frac{h(u)}{u} = 0;
\]

\[
(H_5) \quad \lim_{u \to 0} \frac{f(u)}{u} = \infty;
\]

\[
(H_6) \quad f \text{ is nondecreasing.}
\]

Then there exists \( \lambda_0 > 0 \) such that for all \( \lambda \in (0, \lambda_0) \) problem (1.2), (1.3) has at least one positive radial solution in \( C^{2m}(\Omega(a, b)) \).

Since we are interested in positive radial solutions, problems (1.1), (1.3) and (1.2), (1.3) reduce to the one-dimensional boundary value problems

\[
(1.4) \quad (-1)^m \Delta^m u(t) = g(t)f(u(t)), \quad t \in (a, b),
\]

and

\[
(1.5) \quad (-1)^m \Delta^m u(t) = \lambda g(t)f(u(t)) + k(t)h(u(t)), \quad t \in (a, b),
\]

with the boundary conditions

\[
(1.6) \quad u^{(j)}(a) = u^{(j)}(b) = 0, \quad j = 0, \ldots, m-1,
\]

where \( \Delta \) denotes the polar form of the Laplacian, i.e.:

\[
\Delta = t^{1-n} \frac{d}{dt} \left( t^{n-1} \frac{d}{dt} \right).
\]

The proofs make use of some precise estimates for the Green's function of the corresponding linear two-point boundary value problem. The other tools are a fixed point theorem in cones and the Schauder fixed point theorem.

In Section 2 we give some simple inequalities of the Green's function. Various results concerning disconjugate operators are needed. We also give a priori bounds for positive solutions of problem (1.5), (1.6) when \( h \) is superlinear at
In Section 3 we prove Theorem 1.1. In Section 4 we prove Theorem 1.2 and we give a bound for $\lambda_0$ when in addition $h$ is superlinear at $\infty$.

2. Preliminaries

The homogeneous Dirichlet problem

$$\begin{cases}
\Delta^m v = 0 & \text{in } [a, b], \\
v^{(j)}(a) = v^{(j)}(b) = 0, & j = 0, \ldots, m - 1,
\end{cases}$$

has only the trivial solution. Then it is well-known (see e.g. [14], p. 29) that the operator $(-1)^m \Delta^m$ with Dirichlet boundary conditions has one and only one Green's function $G_m(t, s)$.

**Theorem 2.1.** $G_m(t, s) > 0$ for $a < t, s < b$.

**Proof.** Since $(-1)^m \Delta^m$ is a disconjugate operator on $[a, b]$, this is an immediate consequence of a theorem obtained in [7] (Theorem 11 on p. 108).

As we shall see in section 3 our next result provides very useful estimates for the norm of the integral operator associated with problem (1.4), (1.6).

**Theorem 2.2.** (i) There exists a positive constant $C_m$ such that

$$0 < G_m(t, s) < C_m(s - a)^m(b - s)^m, \quad a < t, s < b.$$

(ii) For any $\delta \in (0, (b - a)/2)$ there exists $\eta \in (0, 1)$ such that

$$G_m(t, s) > \eta C_m(s - a)^m(b - s)^m, \quad a < s < b \quad \text{and} \quad a + \delta < t < b - \delta.$$

In order to prove Theorem 2.2 we shall need some results obtained in [5].

Denote by $\Delta^*$ the adjoint of $\Delta$.

Let $v, v^*, w, w^* \in C^{2m}([a, b])$ be defined by the following relations:

**Equation (2.1)**

$$\begin{cases}
\Delta^m v = (\Delta^*)^m v^* = 0 & \text{in } [a, b], \\
v^{(j)}(a) = v^{(j)}(b) = 0, & j = 0, \ldots, m - 1, \\
v^{(j)}(b) = v^{(j)}(a) = 0, & j = 0, \ldots, m - 2 \text{ (if } m \geq 2), \\
v^{(m-1)}(b) = (-1)^{m-1}, & v^{(m-1)}(a) = 1,
\end{cases}$$

and

**Equation (2.2)**

$$\begin{cases}
\Delta^m w = (\Delta^*)^m w^* = 0, & \text{in } [a, b], \\
w^{(j)}(a) = w^{(j)}(b) = 0, & j = 0, \ldots, m - 2 \text{ (if } m \geq 2), \\
w^{(j)}(b) = w^{(j)}(a) = 0, & j = 0, \ldots, m - 1, \\
w^{(m-1)}(a) = 1, & w^{(m-1)}(b) = (-1)^{m-1}.
\end{cases}$$

The functions defined in (2.1), (2.2) are positive on $(a, b)$ because of the disconjugacy of the operators $\Delta^m$ and $(\Delta^*)^m$. Now define

$$K_m(t, s) = \begin{cases} 
\frac{G_m(t, s)}{v(t)v^*(s)} & \text{on } a < t \leq s < b \\
\frac{(-1)^m}{v^*(m)(b)} & \text{on } t = a \text{ or } s = b
\end{cases}$$
and

\[ L_m(t, s) = \begin{cases} 
\frac{G_m(t, s)}{w(t)w^*(s)} & \text{on } a < s \leq t < b \\
1 & \text{on } s = a \text{ or } t = b.
\end{cases} \]

Denote by \( T_u = \{ (t, s) \in [a, b] \times [a, b]; t \leq s \} \) the upper triangle and by \( T_l = \{ (t, s) \in [a, b] \times [a, b]; s \leq t \} \) the lower triangle. The proof of the following lemma can be found in [5], section 3.

**Lemma 2.1.** (i) \( K_1 \) is a positive constant on \( T_u \) and \( L_1 \) is a positive constant on \( T_l \).

(ii) If \( m \geq 2 \), \( K_m \) is bounded on \( T_u \) and \( K_m \) is continuous and positive on \( T_u \setminus \{(a, a), (b, b)\} \).

(iib) If \( m \geq 2 \), \( L_m \) is bounded on \( T_l \) and \( L_m \) is continuous and positive on \( T_l \setminus \{(a, a), (b, b)\} \).

**Proof of Theorem 2.2.** Since \( \Delta^m \) and \( (\Delta^*)^m \) are disconjugate operators on \([a, b]\) there exist \( \alpha, \alpha^*, \beta \) and \( \beta^* \) in \( C^1([a, b]) \) such that

\[ v(t) = (t-a)^m(b-t)^{m-1}\alpha(t), \quad a < t < b, \]
\[ v^*(s) = (s-a)^{m-1}(b-s)^m\alpha^*(s), \quad a \leq s \leq b, \]
\[ w(t) = (t-a)^{m-1}(b-t)^m\beta(t), \quad a < t < b, \]
\[ w^*(s) = (s-a)^m(b-s)^{m-1}\beta^*(s), \quad a < s < b, \]

and \( \alpha, \alpha^*, \beta \) and \( \beta^* > 0 \) on \([a, b]\).

(i) By virtue of Lemma 2.1 we can define

\[ M_m = \max_{(t,s) \in T_u} \max_{(t,s) \in T_l} \left( \max K_m(t, s), \max L_m(t, s) \right). \]

Then using Theorem 2.1 we get

\[ 0 \leq G_m(t, s) \leq M_m \| \alpha \|_\infty \| \alpha^* \|_\infty (t-a)^m(b-t)^{m-1}(s-a)^{m-1}(b-s)^m \]

for \((t, s) \in T_u\) and

\[ 0 \leq G_m(t, s) \leq M_m \| \beta \|_\infty \| \beta^* \|_\infty (s-a)^m(b-s)^{m-1}(t-a)^{m-1}(b-t)^m \]

for \((t, s) \in T_l\) and (i) follows with

\[ C_m = M_m \max(\| \alpha \|_\infty \| \alpha^* \|_\infty, \| \beta \|_\infty \| \beta^* \|_\infty)(b-a)^2(m-1). \]

(ii) Let \( \delta \in (0, (b-a)/2) \). By Lemma 2.1 we can define

\[ A_\delta = \min_{(t,s) \in T_u} \min_{a+\delta \leq t \leq b-\delta} \max_{(t,s) \in T_l} \min_{a+\delta \leq s \leq b-\delta} K_m(t, s) \]

and \( A_\delta > 0 \). Therefore if \( t \in [a+\delta, b-\delta] \) and \( s \in [a, b] \) we obtain

\[ G_m(t, s) \geq A_\delta \begin{cases} 
(t-a)^m(b-t)^{m-1}(s-a)^{m-1}(b-s)^m, & t \leq s, \\
(s-a)^m(b-s)^{m-1}(t-a)^{m-1}(b-t)^m, & s \leq t,
\end{cases} \]

for some positive constant \( C \) and (ii) is proved.
Now we give an example.

**Example.** When $m = 1$ the Green’s function $G_1(t, s)$ is easily obtained. We have

$$G_1(t, s) = \frac{st^{2-n}}{(n-2)(b^{n-2} - a^{n-2})} \left\{ \begin{array}{ll}
(t^{n-2} - a^{n-2})(b^{n-2} - s^{n-2}), & a \leq t \leq s \leq b,
(s^{n-2} - a^{n-2})(b^{n-2} - t^{n-2}), & a \leq s \leq t \leq b,
\end{array} \right.$$ 

if $n \geq 3$ and

$$G_1(t, s) = \frac{s}{\ln b - \ln a} \left\{ \begin{array}{ll}
(ln t - ln a)(ln b - ln s), & a \leq t \leq s \leq b,
(ln s - ln a)(ln b - ln t), & a \leq s \leq t \leq b,
\end{array} \right.$$ 

if $n = 2$.

We conclude this section with the following result.

**Theorem 2.3.** Assume $(H_1)$ and $(H_2)$. Suppose in addition that $h$ satisfies the following condition:

$$(H_7) \lim_{u \to \infty} \frac{h(u)}{u} = \infty.$$ 

Then there exist $M, M', M'' > 0$ such that

$$||u||_{\infty} \leq M \quad \text{and} \quad ||u'||_{\infty} \leq M' \lambda + M''$$

for all positive solutions $u \in C^{2m}([a, b])$ of (1.5), (1.6) where $M, M'$ and $M''$ are independent of $\lambda > 0$.

**Proof.** Define

$$\rho(t) = (t-a)^m(b-t)^m, \quad a \leq t \leq b.$$ 

Let $\varphi \in C^{2m}([a, b])$ be the solution of the boundary value problem

$$\left\{ \begin{array}{ll}
(-1)^m \Delta^m \varphi = k \rho & \text{in } [a, b],
\varphi^{(j)}(a) = \varphi^{(j)}(b) = 0, & j = 0, \ldots, m-1.
\end{array} \right.$$ 

By $(H_2)$ and Theorem 2.1 $\varphi > 0$ on $(a, b)$. Then using a proposition obtained in [7] (Proposition 13 on p. 109) we deduce that

$$\varphi^{(m)}(a) > 0 \quad \text{and} \quad (-1)^m \varphi^{(m)}(b) > 0.$$ 

Therefore there exist $c_1, c_2 > 0$ such that

$$c_1 \rho \leq \varphi \leq c_2 \rho \quad \text{on } [a, b].$$ 

By $(H_7)$ there exist $\mu > c_1^{-1}$ and a positive constant $c_3$ such that

$$h(u) \geq \mu u - c_3 \quad \text{for } u \geq 0.$$ 

Now let $u \in C^{2m}([a, b])$ be a positive solution of (1.5), (1.6) where $\lambda > 0$. If we multiply equation (1.5) by $t^{n-1} \varphi$ and integrate by parts $2m$ times we obtain

$$\int_a^b t^{n-1} \rho k u \, dt = \int_a^b t^{n-1} \varphi (\lambda g f(u) + kh(u)) \, dt.$$
From (2.4)-(2.6) we deduce that
\[ \int_a^b t^{n-1} \rho k u \, dt \geq \mu \int_a^b t^{n-1} \varphi k u \, dt - c_4 \geq \mu c_1 \int_a^b t^{n-1} \rho k u \, dt - c_4 \]
for some positive constant \( c_4 \), hence
\[ (2.7) \quad \int_a^b t^{n-1} \rho k u \, dt \leq \frac{c_4}{\mu c_1 - 1}. \]

Using Theorem 2.2(i), (2.4), (2.6) and (2.7) we get
\[ u(t) = \int_a^b G_m(t, s)(\lambda g(s)f(u(s)) + k(s)h(u(s))) \, ds \leq M, \quad a \leq t \leq b, \]
for some positive constant \( M \) independent of \( \lambda > 0 \). This gives the first estimate.

Now, if \( m = 1 \) we can write
\[ (2.8) \quad t^{n-1} u'(t) = - \int_c^t s^{n-1}(\lambda g(s)f(u(s)) + k(s)h(u(s))) \, ds \]
for \( t \in [a, b] \) with \( c \in (a, b) \) such that \( u'(c) = 0 \). When \( m > 2 \) we have
\[ (2.9) \quad u'(t) = \int_a^b \frac{\partial}{\partial t} G_m(t, s)(\lambda g(s)f(u(s)) + k(s)h(u(s))) \, ds. \]

Therefore the second estimate follows from (2.8), (2.9) and the \( a \) priori \( L^\infty \) bound already obtained for \( u \).

**Remark 1.** Note that in the proof of Theorem 2.3 the condition \( g \neq 0 \) is not needed and when \( g \equiv 0 \) in \([a, b]\) we can take \( M' = 0 \).

### 3. Proof of Theorem 1.1

As noted in the introduction it is enough to show that problem (1.4), (1.6) has at least one positive solution \( u \in C^{2m}([a, b]) \). The proof makes use of the following fixed point theorem due to Krasnosel'skii ([12]).

**Theorem 3.1.** Let \( X \) be a Banach space, \( K \) a cone in \( X \) and \( 0 < r < R \). Let \( T : \{ u \in K ; 0 < r \leq ||u|| \leq R \} \to K \) be a compact operator such that \( ||Tu|| \geq r \) for \( ||u|| = r \) and \( ||Tu|| \leq R \) for \( ||u|| = R \). Then \( T \) has a fixed point in \( \{ u \in K ; 0 < r \leq ||u|| \leq R \} \).

Now by (H2) there exists \( \delta \in (0, (b-a)/2) \) such that \( g \neq 0 \) in \([a+\delta, b-\delta]\). Let \( \eta \) be as in Theorem 2.2(ii). Let \( X \) be the Banach space \( C([a, b]) \) endowed with the sup norm and define the cone
\[ K = \{ u \in X ; u \geq 0, \, \min\{u(t) ; a + \delta \leq t \leq b - \delta\} \geq \eta ||u||_\infty \}. \]

For \( u \in K \) we define
\[ Tu(t) = \int_a^b G_m(t, s)g(s)f(u(s)) \, ds, \quad a \leq t \leq b. \]

We first show that \( TK \subset K \). By Theorem 2.2(i) we have
\[ (3.2) \quad ||Tu||_\infty \leq C_m \int_a^b \rho(s)g(s)f(u(s)) \, ds \]
where \( \rho \) is defined by (2.3). Using Theorem 2.2(ii) we obtain

\[
(3.3) \quad \min\{Tu(t); a + \delta \leq t \leq b - \delta\} \geq \eta C_m \int_a^b \rho(s)g(s)f(u(s))\,ds.
\]

From (3.2) and (3.3) we deduce that

\[
\min\{Tu(t); a + \delta \leq t \leq b - \delta\} \geq \eta \||Tu||_\infty\|
\]

Since by Theorem 2.1 \( Tu \geq 0 \) we conclude that \( TK \subset K \). It is well-known that \( T: K \to K \) is completely continuous.

By \((H_3)\) there exists \( r > 0 \) such that

\[
f(u) \geq Cu \quad \text{for} \quad 0 \leq u \leq r
\]

where \( C \) is a positive constant satisfying

\[
C \eta \int_{a+\delta}^{b-\delta} G_m(\frac{a+b}{2}, s)g(s)\,ds \geq 1.
\]

Now let \( u \in K \) be such that \( ||u||_\infty = r \). We have

\[
T(u)(\frac{a+b}{2}) = \int_a^b G_m(\frac{a+b}{2}, s)g(s)f(u(s))\,ds
\]

\[
\geq \int_{a+\delta}^{b-\delta} G_m(\frac{a+b}{2}, s)g(s)f(u(s))\,ds
\]

\[
\geq (C \eta \int_{a+\delta}^{b-\delta} G_m(\frac{a+b}{2}, s)g(s)\,ds)r
\]

\[
\geq r
\]

which implies that \( ||Tu||_\infty \geq r \).

By \((H_3)\) there exists \( r' > 0 \) such that

\[
f(u) \leq C'u \quad \text{for} \quad u \geq r'
\]

where \( C' \) is a positive constant satisfying

\[
C'C_m \int_a^b \rho(s)g(s)\,ds \leq 1.
\]

Suppose first that \( f \) is bounded. Then there exists \( B > 0 \) such that \( f(u) \leq B \)

for \( u \geq 0 \). Then choose \( R > r \) such that

\[
BC_m \int_a^b \rho(s)g(s)\,ds \leq R.
\]

Let \( u \in K \) be such that \( ||u||_\infty = R \). By (3.2) we have

\[
||Tu||_\infty \leq C_m \int_a^b \rho(s)g(s)f(u(s))\,ds
\]

\[
\leq BC_m \int_a^b \rho(s)g(s)\,ds
\]

\[
\leq R.
\]
Now if \( f \) is unbounded, we choose \( R \) such that \( R > \max\{r, r'\} \) and \( f(u) \leq f(R) \) for \( 0 \leq u \leq R \). Let \( u \in K \) be such that \( ||u||_\infty = R \). By (3.2) we have

\[
||Tu||_\infty \leq C_m \int_a^b \rho(s)g(s)f(u(s)) \, ds
\]

\[
\leq (C'C_m \int_a^b \rho(s)g(s) \, ds)R
\]

\[
\leq R.
\]

Therefore in both cases we get \( ||Tu||_\infty \leq R \) for \( u \in K \) such that \( ||u||_\infty = R \).

Thus we may apply Theorem 3.1 to conclude that \( T \) has a fixed point in \( \{u \in K; 0 < r \leq ||u||_\infty \leq R\} \). By Theorem 2.1, (H1), (H2) and the properties of the Green's function any nontrivial fixed point of \( T \) in \( K \) yields a positive solution of problem (1.4), (1.6) in \( C^2([a, b]) \). The proof of the theorem is complete.

**Remark 2.** Theorem 3.1 still holds if both inequalities are reversed. Then, using analogous arguments, we could treat the case where \( f \) is superlinear at 0 and \( \infty \). As noted in the introduction a different proof of the superlinear case was given in [9].

**Remark 3.** Clearly Theorem 1.1 remains true for a nonlinearity \( f(|x|, u) \) satisfying:

(i) \( f : [a, b] \times [0, \infty) \rightarrow [0, \infty) \) is a continuous function;

(ii) \( \lim_{u \to 0} \min_{t \in [a, b]} \frac{f(t, u)}{u} = \infty \) and \( \lim_{u \to \infty} \max_{t \in [a, b]} \frac{f(t, u)}{u} = 0 \).

4. PROOF OF THEOREM 1.2

Again it is enough to show that there exists \( \lambda_0 > 0 \) such that for all \( \lambda \in (0, \lambda_0) \) problem (1.5), (1.6) has at least one positive solution \( u \in C^2([a, b]) \). We begin with a lemma.

**Lemma 4.1.** Let \( N > 0 \). For all \( R > \frac{3||g||_\infty f(0)}{N} \) we can find \( \lambda_0 > 0 \) (depending on \( R \) and \( N \)) such that for all \( \lambda \in (0, \lambda_0) \) and \( u \in [0, R\lambda] \) we have

\[
NR\lambda \geq \lambda||g||_\infty f(u) + ||k||_\infty h(u).
\]

**Proof.** Since \( ||g||_\infty f(0) \leq \frac{NR}{3} \), there exists \( \lambda_1 > 0 \) such that

\[
||g||_\infty f(u) \leq \frac{NR}{2} \quad \text{for} \quad u \in [0, R\lambda_1].
\]

Let \( \varepsilon \in (0, \frac{N}{2||k||_\infty}] \). By (H4) there exists \( r > 0 \) such that \( h(u) \leq \varepsilon u \) for \( u \in [0, r] \). Define \( \lambda_0 = \min\left(\frac{r}{R}, \lambda_1\right) \) and let \( \lambda \in (0, \lambda_0) \) and \( u \in [0, R\lambda] \). Then we have

\[
\lambda||g||_\infty f(u) + ||k||_\infty h(u) \leq \frac{NR\lambda}{2} + ||k||_\infty \varepsilon u \leq NR\lambda.
\]

The proof of the lemma is complete.
Now let
\[ N = (C_m \int_a^b \rho(s)ds)^{-1} \]
where \( C_m \) is given by Theorem 2.2(i) and \( \rho \) is defined by (2.3). Fix \( R > \frac{3||g||\infty f(0)}{N} \) and let \( \lambda_0 \) be as in Lemma 4.1. Fix \( \lambda \in (0, \lambda_0] \). First consider the solution \( \psi \in C^{2m}([a, b]) \) of the boundary value problem
\[
\begin{cases}
(-1)^m \Delta^m \psi = g \rho & \text{in } [a, b], \\
\psi^{(j)}(a) = \psi^{(j)}(b) = 0, & j = 0, \ldots, m - 1.
\end{cases}
\]
As in the proof of theorem 2.3 \( \psi > 0 \) on \((a, b)\), \( \psi^{(m)}(a) > 0, (-1)^m \psi^{(m)}(b) > 0 \) and there exist \( d_1, d_2 > 0 \) such that
\[
d_1 \rho \leq \psi \leq d_2 \rho \quad \text{on } [a, b].
\]
By (H5) we can choose \( r = r(\lambda) \in (0, R \lambda] \) such that
\[
f(u) \geq \frac{1}{\lambda d_1} u \quad \text{for } 0 \leq u \leq r
\]
where \( d_1 \) is given by (4.2). Let \( c > 0 \) be such that
\[
c(b - a)^{2m} \leq r
\]
and consider the set of functions
\[
Z = \{ u \in C([a, b]); \ c \rho(t) \leq u(t) \leq R \lambda, \ a \leq t \leq b \}.
\]
Clearly, \( Z \) is a nonempty closed bounded convex subset of \( C([a, b]) \) equipped with the sup norm. For \( u \in Z \) we define
\[
F(u(t)) = \int_a^b G_m(t, s)(\lambda g(s)f(u(s)) + k(s)h(u(s))) \, ds
\]
for \( a \leq t \leq b \). We first prove that \( FZ \subset Z \). Indeed let \( u \in Z \). By Theorem 2.2(i) we have
\[
\|F(u)\|\infty \leq C_m \int_a^b \rho(s)(\lambda \|g\|\infty f(u(s)) + \|k\|\infty h(u(s))) \, ds.
\]
Using (4.5) and Lemma 4.1 we get
\[
\|F(u)\|\infty \leq C_m \int_a^b \rho(s)(\lambda ||g||\infty f(u(s)) + ||k||\infty h(u(s))) \, ds \\
\leq R \lambda.
\]
Now by virtue of (H6), (4.3) and (4.4) we have
\[
F(u(t)) \geq \lambda \int_a^b G_m(t, s)g(s)f(u(s)) \, ds \\
\geq \lambda \int_a^b G_m(t, s)g(s)f(cp(s)) \, ds \\
\geq cd_1^{-1} \int_a^b G_m(t, s)g(s)\rho(s) \, ds \\
\geq c \rho(t)
\]

for \( t \in [a, b] \) because the solution \( \psi \) of (4.1) is given by
\[
\psi(t) = \int_a^b G_m(t, s)g(s)p(s)\,ds.
\]
Therefore \( FZ \subset Z \). Since \( F \) is compact, the Schauder fixed point theorem implies that \( F \) has a fixed point \( u \in Z \). By the properties of the Green's function any fixed point of \( F \) in \( Z \) yields a positive solution of problem (1.5), (1.6) in \( C^{2m}([a, b]) \). The theorem is proved.

Now we shall show that if in addition \( h \) is superlinear at \( \infty \) we can give a bound for \( \lambda_0 \). Let us define
\[
A = \{ \mu > 0; \ (1.5), (1.6) \ has \ a \ positive \ solution \ for \ all \ \lambda \in (0, \mu) \}.
\]
By Theorem 1.2 \( A \neq \emptyset \). Thus, if we define
\[
\lambda^* = \sup A
\]
we have \( \lambda^* \in (0, \infty] \).

**Lemma 4.2.** Assume moreover that \( h \) satisfies (H7). Then \( \lambda^* < \infty \).

**Proof.** By Theorem 2.3 there exists \( M > 0 \) such that for all \( \lambda > 0 \) and all positive solutions \( u \in C^{2m}([a, b]) \) of problem (1.5), (1.6) we have
\[
(4.6) \quad ||u||_{\infty} \leq M.
\]
(H1), (H5) and (H6) imply that there exists \( \lambda > 0 \) such that
\[
u < \lambda f(u) \quad \forall \ u \in (0, M].
\]
Therefore we obtain
\[
(4.7) \quad g(s)u < \lambda g(s)f(u)
\]
for \( u \in (0, M] \) and \( s \in [a, b] \) such that \( g(s) \neq 0 \). Now let \( \psi \in C^{2m}([a, b]) \) be as in the proof of Theorem 1.2 and let \( \lambda > 0 \) be such that (1.5), (1.6) has a positive solution \( u \in C^{2m}([a, b]) \). Multiplying (1.5) by \( t^n-\nu \) and integrating by parts \( 2m \) times we obtain
\[
\int_a^b s^{n-1} \psi(s)(\lambda g(s)f(u(s)) + k(s)h(u(s)))\,ds = \int_a^b s^{n-1} g(s)p(s)u(s)\,ds.
\]
Since by (H2), (4.2), (4.6) and (4.7) we have
\[
\int_a^b s^{n-1} g(s)p(s)u(s)\,ds < \lambda d^{-1} \int_a^b s^{n-1} \psi(s)g(s)f(u(s))\,ds
\]
we deduce that \( \lambda < \lambda d^{-1} \), hence \( \lambda^* \leq \lambda d^{-1} \). The proof of the lemma is complete.

We conclude this section with a result concerning the limit case \( \lambda = \lambda^* < \infty \).

**Theorem 4.1.** Assume (H1) and (H2). Suppose in addition that \( h \) satisfies (H7) and that \( f(0) > 0 \). Let \( \lambda \in (0, \infty) \) be such that for all \( \lambda \in (0, \lambda) \) problem (1.5), (1.6) has a positive solution \( u \in C^{2m}([a, b]) \). Then for \( \lambda = \lambda \) problem (1.5), (1.6) has at least one positive solution in \( C^{2m}([a, b]) \).

**Proof.** Let \( (\lambda_n) \) be a sequence in \((0, \lambda)\) such that \( \lambda_n \to \lambda \). By our assumption for each \( n \in \mathbb{N} \) there exists a positive solution \( u_n \in C^{2m}([a, b]) \) of problem
(1.5), (1.6). By Theorem 2.3 \((u_n)\) is bounded in the sup norm. Since \((u_n')\) is also bounded in the sup norm we deduce that \((u_n)\) is equicontinuous. By virtue of the Ascoli theorem there is a subsequence \((u_{n_k})\) of \((u_n)\) which converges uniformly to a function \(u \in C([a, b])\) such that \(u \geq 0\). Clearly

\[
(4.8) \quad u(t) = \int_a^b G_m(t, s)(\hat{\lambda}g(s)f(u(s)) + k(s)h(u(s))) \, ds
\]

for \(t \in [a, b]\). Thus \(u \in C^{2m}([a, b])\) and \(u\) is a solution of problem (1.5), (1.6). Since \(f(0) > 0\), Theorem 2.1, \((H_2)\) and (4.8) imply that \(u(t) > 0\) for \(t \in (a, b)\). The proof of the theorem is complete.

REFERENCES