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L^p SPECTRA OF PSEUDODIFFERENTIAL OPERATORS GENERATING INTEGRATED SEMIGROUPS

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ABSTRACT. Consider the L^p-realization $\mathrm{Op}_p(a)$ of a pseudodifferential operator with symbol $a \in S^m_{\rho,0}$ having constant coefficients. We show that for a certain class of symbols the spectrum of $\mathrm{Op}_p(a)$ is independent of p. This implies that $\mathrm{Op}_p(a)$ generates an N-times integrated semigroup on $L^p(\mathbb{R}^n)$ for a certain N if and only if $\rho(\mathrm{Op}_p(a)) \neq \varnothing$ and the numerical range of a is contained in a left half-plane. Our method allows us also to construct examples of operators generating integrated semigroups on $L^p(\mathbb{R}^n)$ if and only if p is sufficiently close to 2.

1. INTRODUCTION

Let $\mathrm{Op}_p(a)$ be the L^p-realization of a pseudodifferential operator $\mathrm{Op}(a)$ with symbol $a \in S^m_{\rho,0}$ having constant coefficients. Consider the initial value problem

$$u'(t) = \mathrm{Op}_p(a)u(t), \quad u(0) = u_0, \tag{1.1}$$

in the space $L^p(\mathbb{R}^n)$, where $1 \leq p < \infty$. We are interested in the question how the location of the spectrum $\sigma(\mathrm{Op}_p(a))$ of $\mathrm{Op}_p(a)$ or the numerical range $a(\mathbb{R}^n)$ of the symbol a influences the existence and regularity theory of a solution of (1.1) for $u_0 \in D(\mathrm{Op}_p(a))^N$, $N \in \mathbb{N}$. If for all $t \geq 0$ the function $\xi \mapsto e^{ta(\xi)}$ is a Fourier multiplier for $L^p(\mathbb{R}^n)$ (with exponentially bounded norm), then a complete answer of the above question is obtained via the classic theory of strongly continuous semigroups (cf. [F], [G], [P]). Notice that, by results of Hörmander [Hö1], there exist however many examples of operators $\mathrm{Op}_p(a)$ which generate a C_0-semigroup on an L^p-space only for certain values of p. In fact, for simplicity let $\mathrm{Op}_p(a)$ be a differential operator of order $m > 1$ such that the real part of the principal part of its symbol a vanishes for all $\xi \in \mathbb{R}^n$. Then $\mathrm{Op}_p(a)$ generates a C_0-semigroup on $L^p(\mathbb{R}^n)$ only if $p = 2$. In particular, this holds for the operator $i\Delta$, where Δ denotes the Laplacian.

In this paper we examine the initial value problem (1.1) by means of integrated semigroups (cf. [A]). The relationship between (1.1) and integrated semigroups may be described as follows: a linear operator A generates an N-times integrated semigroup on E if and only if $\rho(A) \neq \varnothing$ and (1.1) admits a unique, exponentially bounded solution for all $u_0 \in D(A^{N+1})$. By estimating the order N of integration we thus obtain precise information on the regularity

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of the initial data needed in order to obtain a unique classical solution of (1.1). We carry out this approach for a certain class of symbols $a \in S^m_{\rho,0}$ having constant coefficients. The estimates obtained turn out to be optimal for a large class of symbols. Moreover, they illustrate the special role of the case $p = 1$ and show in particular the different regularity behavior of the solution for the cases $p = 1$ and $p \in (1, \infty)$. As an immediate consequence we obtain L^p-resolvent estimates for $\mathrm{Op}_p(a)$ in a right half-plane. Former results in this direction are contained in [AK], [BE], [Hi1] and [deL].

Similarly to the case of semigroups, there exist operators $\mathrm{Op}_p(a)$ generating integrated semigroups on $L^p(\mathbb{R}^n)$ for some but not for all values of p. The method to construct such examples is inextricably entangled with the spectral theory of the operators under consideration (cf. [Hi2], [IS], [KT]). It is shown that, for a certain class of symbols, the spectrum, being contained in a left half-plane for some value of p, may "explode" to be all of the complex plane for other values of p.

On the other hand it is natural to ask for conditions under which $\sigma(\mathrm{Op}_p(a))$ of $\mathrm{Op}_p(a)$ coincides with the numerical range $a(\mathbb{R}^n)$ of a. We show that

$$\sigma(\mathrm{Op}_p(a)) = \sigma(\mathrm{Op}_2(a)) = a(\mathbb{R}^n) \tag{1.2}$$

provided the symbol a and its derivatives satisfy certain growth conditions. In particular, in this case $\sigma(\mathrm{Op}_p(a))$ is independent of p. Since our assumptions are satisfied above all for elliptic polynomials, assertion (1.2) extends results of Balslev [Ba] and Iha and Schubert [IS] to our situation.

Finally, we illustrate our results by means of Dirac's equation on $L^p(\mathbb{R}^3)$.

2. Preliminaries and notations

Let $\mathscr{S} := \mathscr{S}(\mathbb{R}^n)$ be the space of all rapidly decreasing functions and $\mathscr{S}'$ its dual, the space of all tempered distributions. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$, let $\langle x, \xi \rangle = \sum_{i=1}^n x_i\xi_i$ and $\|x\| = \langle x, x\rangle^{1/2}$. The Fourier transform on $\mathscr{S}$ and its inverse transform are defined by

$$(\mathscr{F} f)(\xi) = \hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} f(x)\, dx, \qquad \xi \in \mathbb{R}^n,$$

and

$$(\mathscr{F}^{-1})f(x) := \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \hat{f}(\xi)\, d\xi.$$

Throughout this paper, α, β, γ will denote multi-indices and D^α is defined by $D^\alpha = (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n}$, where $|\alpha| := \sum_{i=1}^n \alpha_i$.

We denote by $\mathscr{M}_p$ $(1 \le p \le \infty)$ the set of all functions $u \in L^\infty(\mathbb{R}^n)$ such that $\mathscr{F}^{-1}(u\hat{\phi}) \in L^p(\mathbb{R}^n)$ for all $\phi \in \mathscr{S}$ and

$$\|u\|_{\mathscr{M}_p} := \sup\{\|\mathscr{F}^{-1}(u\hat{\phi})\|_{L^p}\,;\ \phi \in \mathscr{S},\ \|\phi\|_{L^p} \le 1\} < \infty.$$

We give this space the norm $\|\cdot\|_{\mathscr{M}_p}$ so that it becomes a Banach space. Then $\mathscr{M}_p = \mathscr{M}_q (\frac{1}{p} + \frac{1}{q} = 1\,;\ 1 \le p \le \infty)$ and we have

$$\sup_\xi \|u(\xi)\| = \|u\|_{\mathscr{M}_2} \le \|u\|_{\mathscr{M}_p} \le \|u\|_{\mathscr{M}_1}. \tag{2.1}$$

In order to determine whether or not a given function belongs to $\mathcal{M}_p$ the following fact is useful: there exists a function $\psi \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\psi \subseteq \{\frac{1}{2} < |\xi| < 2\}$ such that

$$\sum_{l=-\infty}^{\infty} \psi(2^{-l}\xi) = 1 \qquad (\xi \neq 0). \tag{2.2}$$

A very efficient sufficient criterion for a function u to belong to $\mathcal{M}_p$, $1 < p < \infty$, is given by the Mikhlin multiplier theorem (cf. [S, p. 96]). For the case $p = 1$ the following elementary bound for the $\mathcal{F}L^1(\mathbb{R}^n)$-norm is useful. Here we consider $\mathcal{F}L^1(\mathbb{R}^n)$ as a Banach space for the norm inherited by $L^1(\mathbb{R}^n)$.

Lemma 2.1. *Let $u \in H^j(\mathbb{R}^n)$ for some $j > \frac{n}{2}$. Then $u \in \mathcal{F}L^1(\mathbb{R}^n)$ and*

$$\|u\|_{\mathcal{F}L^1} \leq C_n \|u\|_2^{1-n/2j} \|u\|_{j,2}^{n/2j}$$

for some constant C_n depending only on n.

For a proof we refer to [Hi1, Lemma 2.1]. We call a function $a \in C(\mathbb{R}^n, \mathbb{C})$ a symbol if there exist constants $M > 0$, $m \in \mathbb{R}$ such that

$$|a(\xi)| \leq M(1 + |\xi|)^m \quad \text{for all } \xi \in \mathbb{R}^n.$$

Then we define the pseudodifferential operator $\operatorname{Op}(a) : \mathcal{S} \to \mathcal{S}'$ with symbol a by

$$\operatorname{Op}(a)u(x) := \int_{\mathbb{R}^n} e^{i\langle x, \xi\rangle} a(\xi)\hat{u}(\xi)\, d\xi.$$

Moreover, we denote by S the class of all symbols a such that $\operatorname{Op}(a)$ maps $\mathcal{S}$ into $\mathcal{S}$.

For $m \in \mathbb{R}$ and $\rho \in [0, 1]$, we define $S^m_{\rho,0}$ to be the set of all functions $a \in C^\infty(\mathbb{R}^n)$ such that for each multi-index α there exists a constant C_α such that

$$|D^\alpha a(\xi)| \leq C_\alpha (1 + |\xi|)^{m-\rho|\alpha|} \quad (\xi \in \mathbb{R}^n).$$

Obviously, $S^m_{\rho,0} \subset S$ and a polynomial of order m is of class $S^m_{1,0}$. Furthermore, for $a \in S^m_{\rho,0}$ we put $a(\mathbb{R}^n) := \{a(\xi)\,;\, \xi \in \mathbb{R}^n\}$.

For the time being, let $a \in S$. Then we associate with a a linear operator $\operatorname{Op}_p(a)$ on $L^p(\mathbb{R}^n)$ $(1 \leq p < \infty)$ as follows. Set

$$\begin{aligned} &D(\operatorname{Op}_p(a)) := \{f \in L^p(\mathbb{R}^n)\,;\, \mathcal{F}^{-1}(a\hat{f}) \in L^p(\mathbb{R}^n)\} \quad \text{and define} \\ &\operatorname{Op}_p(a)f := \mathcal{F}^{-1}(a\hat{f}) \quad \text{for all } f \in D(\operatorname{Op}_p(a)). \end{aligned} \tag{2.3}$$

Then it is not difficult to verify that $\operatorname{Op}_p(a)$ is closed.

We call a polynomial a of degree m elliptic if its principal part a_m given by $a_m(\xi) := \sum_{|\alpha|=m} a_\alpha (i\xi)^\alpha$ vanishes only at $\xi = 0$. Moreover, a is called hypoelliptic if

$$\frac{D^\alpha a(\xi)}{a(\xi)} \to 0 \quad \text{as } |\xi| \to \infty \text{ and } \alpha \neq 0.$$

Finally, if A is a linear operator acting on a Banach space E, we denote its resolvent set by $\rho(A)$ and its spectrum by $\sigma(A)$.

For the time being, let A be a linear operator on a Banach space E and $k \in \mathbb{N} \cup \{0\}$. Then A is called the generator of a k-times integrated semigroup if and only if $(\omega, \infty) \subset \rho(A)$ for some $\omega \in \mathbb{R}$ and there exists a strongly continuous mapping $S : [0, \infty) \to \mathscr{L}(E)$ satisfying $\|S(t)\| \le Me^{\omega t}$ $(t \ge 0)$ for some $M \ge 0$ such that

$$R(\lambda, A) = \lambda^k \int_0^\infty e^{-\lambda t} S(t)\, dt \qquad (\lambda > \omega).$$

In this case $(S(t))_{t\ge 0}$ is called the k-times integrated semigroup generated by A. In particular, a 0-times integrated semigroup is a C_0-semigroup. For more detailed information on semigroups and integrated semigroups we refer to [F], [G], [P] and [A], [deL], [Hi2] and [L]. The connection between integrated semigroups and the Cauchy problem

$$u'(t) = Au(t), \quad u(0) = 0 \tag{2.4}$$

is given by the following fact: Let A be a linear operator on a Banach space E and let $k \in \mathbb{N} \cup \{0\}$. Then A generates a k-times integrated semigroup on E if and only if $\rho(A) \neq \varnothing$ and there exists a unique, classical solution u of (2.4) for all $u_0 \in D(A^{k+1})$ satisfying $\|u(t)\| \le Me^{\omega t}$ for all $t \ge 0$ and some $M, \omega \ge 0$.

3. Fourier multipliers

We start this section with a sufficient criterion for a function a to belong to $\mathscr{M}_p$.

Theorem 3.1. *Let $a \in C^j(\mathbb{R}^n)$, $j > \frac{n}{2}$, and suppose that $a(\xi) = 0$ for all $\xi \in \mathbb{R}^n$ with $|\xi| \le 1$. Let $\varepsilon \ge 0$ and $\rho \in (-\infty, 1]$. Assume that there exist constants $M_0 > 0$, $M \ge 1$ such that*

$$\sup_{0<|\alpha|\le j} \left(\sup_{|\xi|\ge 1} |D^\alpha a(\xi)|\, |\xi|^{\varepsilon+\rho|\alpha|} \right)^{1/|\alpha|} \le M \quad \text{and} \quad \sup_{\xi\in\mathbb{R}^n} |a(\xi)|\, |\xi|^\varepsilon \le M_0.$$

(a) *Let $1 \le p \le \infty$. If $\varepsilon > n|\frac{1}{2} - \frac{1}{p}|(1-\rho)$, then $a \in \mathscr{M}_p$ and there exists a constant $C_{n,p,\rho}$ such that*

$$\|a\|_{\mathscr{M}_p} \le C_{n,p,\rho} \max(M_0, 1) M^{n|1/2-1/p|}.$$

(b) (*Miyachi*) *Let $1 < p < \infty$. Assume that $M_0 = 1$ and $\rho \neq 1$. If $\varepsilon \ge n|\frac{1}{2} - \frac{1}{p}|(1-\rho)$, then $a \in \mathscr{M}_p$ and there exists a constant C_p such that*

$$\|a\|_{\mathscr{M}_p} \le C_p M^{n|1/2-1/p|}.$$

Proof. (a) Without loss of generality we may assume that $1 \le p \le 2$. Let $\psi \in C_c^\infty(\mathbb{R}^n)$ be the function defined in (2.2). For $l \in \mathbb{N}$ put $a_l := a\psi_l$, where $\psi_l(x) := \psi(2^{-l}x)$ for all $x \in \mathbb{R}^n$. We claim that $\|a\|_{\mathscr{M}_p} \le \sum_{l=1}^\infty \|a_l\|_{\mathscr{M}_p} < \infty$. To this ena, observe that by Leibniz's rule

$$\begin{aligned} |D^\alpha a_l(\xi)| &= \left| \sum_{\beta\le\alpha} \binom{\alpha}{\beta} D^{\alpha-\beta} a(\xi) 2^{-l|\beta|} (D^\beta \psi)(2^{-l}\xi) \right| \\ &\le \begin{cases} C_0 M_0 2^{-l\varepsilon} & \text{if } \alpha = 0, \\ C_\alpha M^{|\alpha|} 2^{l(-\varepsilon-\rho|\alpha|)} & \text{if } \alpha \neq 0 \end{cases} \end{aligned}$$

for some constants C_0, $C_\alpha > 0$. Consequently, there exist constants $C_{\alpha,n}$ such that

$$\|D^\alpha a_l\|_2 \leq \begin{cases} C_{0,n} 2^{-l\varepsilon} 2^{ln/2} & (\alpha = 0), \\ C_{\alpha,n} 2^{l(-\varepsilon-\rho|\alpha|)} 2^{ln/2} & (\alpha \neq 0). \end{cases}$$

Now, choosing $j > \frac{n}{2}$, we conclude by Lemma 2.1 that

$$\begin{aligned} \|a_l\|_{\mathscr{F}L^1} &\leq C_n (M_0 2^{-l\varepsilon} 2^{ln/2})^{1-n/2j} (M^j 2^{l(-\varepsilon-\rho j)} 2^{ln/2})^{n/2j} \\ &\leq C_n M_0^{1-n/2j} M^{n/2} 2^{l(-\varepsilon+\frac{n}{2}(1-\rho))}. \end{aligned}$$

Setting $\theta := 2(1-\frac{1}{p})$ for some $p \in (1, 2)$ it follows from the Riesz-Thorin theorem that

$$\|a_l\|_{\mathscr{M}_p} \leq \|a_l\|_{\mathscr{M}_1}^{1-\theta} \|a_l\|_2^\theta \leq C_{n,p} \max\{M_0, 1\} M^{n|1/2-1/p|} 2^{l(-\varepsilon+(1-\rho)n|1/2-1/p|)}.$$

Finally, since $\|a\|_{\mathscr{M}_p} \leq \sum_{l=1}^\infty \|a_l\|_{\mathscr{M}_p} < \infty$, the proof of assertion (a) is complete. The assertion (b) follows immediately from [M, Theorem G]. □

Assume that the symbol a belongs to $S^m_{\rho,0}$. We consider the following hypothesis:

(H1) $\sup_{\xi \in \mathbb{R}^n} \operatorname{Re} a(\xi) \leq \omega$ for some $\omega \in \mathbb{R}$.

(H2) There exist constants $C, L, r > 0$ such that $|a(\xi)| \geq C|\xi|^r$ for all $\xi \in \mathbb{R}^n$ with $|\xi| \geq L$.

Remark 3.2. We note that by the Seidenberg-Tarski theorem Hypothesis (H2) above is in particular satisfied for all polynomials a satisfying $|a(\xi)| \to \infty$ as $|\xi| \to \infty$ (cf. [Hö2, Theorem 11.1.3]). Hence assumption (H2) holds especially for hypoelliptic polynomials.

Lemma 3.3. *Let* $N \in \mathbb{N}$, $m \in (0, \infty)$, $\rho \in [0, 1]$. *Suppose that* $a \in S^m_{\rho,0}$ *satisfies* (H2) *and that* $0 \notin a(\mathbb{R}^n)$.

(a) *If* $N > \frac{n}{2}(\frac{m-\rho-r-1}{r})$, *then* $a^{-N} \in \mathscr{M}_1$.

(b) *If* $1 < p < \infty$ *and* $N \geq n|\frac{1}{2} - \frac{1}{p}|(\frac{m-\rho-r-1}{r})$, *then* $a^{-N} \in \mathscr{M}_p$.

Proof. Let $\varphi \in C_c^\infty$ such that

$$\varphi(\xi) := \begin{cases} 1 & \text{for } |\xi| \leq L, \\ 0 & \text{for } |\xi| \geq L+1 \end{cases}.$$

Then, writing $a^{-N} = \varphi a^{-N} + (1-\varphi) a^{-N}$, we conclude by Lemma 2.1 that it suffices to prove that $(1-\varphi)a^{-N} \in \mathscr{M}_p$. Observe that by assumption

$$|D^\alpha (a^{-N})(\xi)| \leq C_\alpha |\xi|^{-rN+(m-r-\rho)|\alpha|} \quad (|\xi| \geq \max(L, 1)). \tag{3.1}$$

Hence the assertion follows from Theorem 3.1 provided $\rho \neq 1$. If $\rho = 1$, then the assertion follows from Mikhlin's theorem. □

Lemma 3.4. *Let* $N \in \mathbb{N}$, $m \in (0, \infty)$, $\rho \in [0, 1]$ *and let* $a \in S^m_{\rho,0}$. *Assume that* (H1) *and* (H2) *are satisfied and that* $0 \notin a(\mathbb{R}^n)$.

(a) *If* $N > \frac{n}{2}(\frac{1+m-\rho}{r})$, *then* $e^{ta}/a^N \in \mathscr{M}_1$ *and there exists a constant* $C_{N,n}$ *such that*

$$\left\| \frac{e^{ta}}{a^N} \right\|_{\mathscr{M}_1} \leq C_{N,n} (1+t)^{n/2} e^{\omega t} \qquad (t \geq 0).$$

(b) *If* $1 < p < \infty$ *and* $N \geq n|\frac{1}{2} - \frac{1}{p}|(\frac{1+m-\rho}{r})$, *then* $e^{ta}/a^N \in \mathcal{M}_p$ *and there exists a constant* $C_{N,n,p}$ *such that*

$$\left\| \frac{e^{ta}}{a^N} \right\|_{\mathcal{M}_p} \leq C_{N,n,p}(1+t)^{n|1/2-1/p|} e^{\omega t} \qquad (t \geq 0).$$

Proof. Note first that after rescaling we may assume that $\omega = 0$. Moreover, thanks to (2.1), we may restrict ourselves to the case $1 \leq p \leq 2$. Now, let $\varphi \in C_c^\infty$ such that $0 \leq \varphi(\xi) \leq 1$ $(\xi \in \mathbb{R}^n)$ and

$$\varphi(\xi) := \begin{cases} 1 & \text{for } |\xi| \leq L_1, \\ 0 & \text{for } |\xi| \geq L_1 + 1, \end{cases}$$

where $L_1 := \max(L, C^{-1/r})$. We put $v_t^N := e^{ta}/a^N$. Then Lemma 2.1 implies that $\varphi v_t^N \in \mathcal{M}_p$ and that

$$\left\| \varphi \frac{e^{ta}}{a^N} \right\|_{\mathcal{M}_1} \leq C_n(1+t)^{n/2}$$

for some constant C_n. Writing $v_t^N = \varphi v_t^N + (1-\varphi)v_t^N$, we conclude that it remains to prove the assertion for $(1-\varphi)v_t^N$. Now, by Leibniz's rule

$$D^\alpha(v_t^N) = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} D^\beta(e^{ta}) D^\gamma(a^{-N}).$$

Since $|(D^\gamma a^{-N})(\xi)| \leq C_\gamma |\xi|^{-rN+|\gamma|(m-r-\rho)}$ for all ξ with $|\xi| \geq L$ (see (3.1)) and since $|(D^\beta e^{ta})(\xi)| \leq C_\beta (1+t)^{|\beta|} |\xi|^{|\beta|(m-\rho)}$ it follows that there exists a constant $C > 0$ such that

$$\sup_{0<|\alpha|\leq j} \sup_{|\xi|\geq 1} (|(D^\alpha(1-\varphi)(\xi) v_t^N)(\xi)| \, |\xi|^{rN+|\alpha|(\rho-m)})^{1/|\alpha|} \leq C(1+t)$$

and

$$\sup_{|\xi|\geq 1} ((1-\varphi)v_t^N)(\xi)|\xi|^{rN} \leq 1$$

for all $t \geq 0$. Hence the assertion follows from Theorem 3.1. □

4. L^p SPECTRA OF PSEUDODIFFERENTIAL OPERATORS

We start this section with a result illustrating the close relationship between L^p multipliers and the L^p spectra of the pseudodifferential operators under consideration.

Lemma 4.1. *Let* $1 \leq p < \infty$ *and* $a \in S$. *Then* $\lambda \in \rho(\mathrm{Op}_p(a))$ *if and only if* $(\lambda - a)^{-1} \in \mathcal{M}_p$.

We note that if the symbol a is a polynomial, then Lemma 4.1 was first proved by Schechter [Sch, Theorem 4.4.1]. The generalization to symbols a belonging to S is a straightforward modification of Schechter's proof. We therefore omit the details.

In order to obtain a precise description of $\sigma(\mathrm{Op}_p(a))$ we need to decide whether or not the function $(\lambda - a)^{-1}$ is an L^p multiplier. In general this is not an easy matter; however if the symbol a satisfies the growth condition (H2), then the situation is fairly easy to describe. Indeed, in this case we obtain the following result.

Proposition 4.2. *Let* $1 \leq p < \infty$, $m \in (0, \infty)$ *and* $\rho \in [0, 1]$. *Suppose that* $a \in S^m_{\rho,0}$ *satisfies* (H2). *If* $\rho(\mathrm{Op}_p(a)) \neq \varnothing$, *then* $\sigma(\mathrm{Op}_p(a)) = \sigma(\mathrm{Op}_2(a)) = a(\mathbb{R}^n)$.

Related results on the p-independence of the spectrum of differential operators on $L^p(\mathbb{R}^n)$ are contained in [Ba] and [IS].

Remarks 4.3. (a) If a is a polynomial, then the Seidenberg-Tarski theorem implies that Hypothesis (H2) is fulfilled provided $|a(\xi)| \to \infty$ as $|\xi| \to \infty$.

(b) We emphasize that Hypothesis (H2) is essential for obtaining the above assertion. In fact, consider the example of the symbol a given by

$$a(\xi) := -i(\xi_1 + \xi_2^2 + \xi_3^2 - i).$$

Then $\sigma(\mathrm{Op}_2(a)) = \{z \in \mathbb{C};\, \mathrm{Re}\, z = -1\}$, but Kenig and Tomas [KT] showed that $a^{-1} \notin \mathscr{M}_p$ if $p \neq 2$. Hence, by Lemma 4.1, $0 \in \sigma(\mathrm{Op}_p(a))$ whenever $p \neq 2$.

Proof. We note first that Lemma 4.1 together with Hypothesis (H2) and the fact that $\mathscr{M}_2 = L^\infty$ implies that $\sigma(\mathrm{Op}_2(a))$ coincides with $a(\mathbb{R}^n)$. Therefore and in view of Lemma 4.1 we only have to prove that $\sigma(\mathrm{Op}_p(a)) \subset a(\mathbb{R}^n)$. By assumption we have $\mathbb{C}\backslash a(\mathbb{R}^n) \neq \varnothing$. Let $\lambda \in \mathbb{C}\backslash a(\mathbb{R}^n)$. By Lemma 3.3 there exists an integer N such that the function $r_\lambda^N := (\lambda - a)^{-N}$ belongs to $\mathscr{M}_p$ $(1 \leq p < \infty)$. Hence, by Lemma 4.1, $0 \in \rho(\mathrm{Op}_p(\lambda - a)^N)$. We claim that $0 \in \rho((\lambda - \mathrm{Op}_p(a))^N)$. Since $(\lambda - \mathrm{Op}_p(a))^N f = \mathrm{Op}_p((\lambda - a)^N) f$ for all $f \in \mathscr{S}$, we conclude by [HP, Theorem 2.16.4] that $(\lambda - \mathrm{Op}_p(a))^N$ is an extension of $\mathrm{Op}_p((\lambda - a)^N)$. Furthermore, $\ker(\lambda - \mathrm{Op}_p(a))^N = \{0\}$. In fact, assume that $(\lambda - \mathrm{Op}_p(a))^N u = 0$. Then $0 = ((\lambda - \mathrm{Op}_p(a))^N u, g) = (u, (\bar{\lambda} - \mathrm{Op}_p(\bar{a}))^N g)$ for all $g \in \mathscr{S}$. Hypothesis (H2) implies that, given $f \in \mathscr{S}$, we find $g \in \mathscr{S}$ such that $(\bar{\lambda} - \mathrm{Op}_p(\bar{a}))^N g = f$. Consequently $0 = (u, f)$ for all $f \in \mathscr{S}$ and hence $u = 0$. It follows that $\mathrm{Op}_p((\lambda - a)^N) = (\lambda - \mathrm{Op}_p(a))^N$ and hence $0 \in \rho((\lambda - \mathrm{Op}_p(a))^N)$. In a second step, we claim that $0 \in \rho(\lambda - \mathrm{Op}_p(a))$. Suppose the contrary. Then the spectral mapping theorem for closed operators (cf. [DS, p. 604]) implies that $\sigma((\lambda - \mathrm{Op}_p(a))^N) = (\sigma(\lambda - \mathrm{Op}_p(a)))^N$ which yields a contradiction. Hence $\sigma(\mathrm{Op}_p(a)) \subset a(\mathbb{R}^n)$. The proof is complete. □

We now give a quantitative version of Proposition 4.2.

Theorem 4.4. *Let* $1 \leq p < \infty$, $m \in (0, \infty)$ *and* $\rho \in [0, 1]$. *Suppose that* $a \in S^m_{\rho,0}$ *satisfies* (H2).

(a) *Then the following assertions hold.*
   (i) *If* $1 < p < \infty$ *and* $n|\frac{1}{2} - \frac{1}{p}|(\frac{m-\rho-r+1}{r}) \leq 1$, *then* $\sigma(\mathrm{Op}_p(a)) = \sigma(\mathrm{Op}_2(a))$.
   (ii) *If* $\frac{n}{2}(\frac{m-\rho-r+1}{r}) < 1$, *then* $\sigma(\mathrm{Op}_1(a)) = \sigma(\mathrm{Op}_2(a))$.

(b) *If* $\rho \neq 0$, *then the bounds in assertions* (i) *and* (ii) *are optimal*; *i.e. given* $p \in [1, \infty)$, *there exists* $a \in S^m_{\rho,0}$ $(m > 0,\ \rho \in (0, 1))$ *such that* $\sigma(\mathrm{Op}_p(a)) \neq \sigma(\mathrm{Op}_2(a))$ *whenever* $n|\frac{1}{2} - \frac{1}{p}|(\frac{m-\rho-r+1}{r}) > 1$ *or* $\frac{n}{2}(\frac{m-\rho-r+1}{r}) \geq 1$, *respectively.*

Proof. The assertion (a) follows by combining Lemma 3.3 and Proposition 4.2. In order to prove (b) let $\alpha \in (0, 1)$, $m \in (0, \frac{n\alpha}{2})$ and let $a : \mathbb{R}^n \to \mathbb{C}$ be a

C^∞-function such that

$$a(\xi) := \begin{cases} \dfrac{|\xi|^m}{e^{i|\xi|^\alpha}}, & |\xi| \geq 2, \\ 0, & |\xi| \leq 1. \end{cases}$$

Then $a \in S^m_{1-\alpha,0}$ and (H2) is satisfied with $r = m$. Hence in this case $\frac{m-\rho-r+1}{r} = \frac{\alpha}{m}$. It follows from the results in [FS, p. 160] that $a^{-1} \in \mathcal{M}_p$ ($\mathcal{M}_1$) if and only if $n|\frac{1}{2} - \frac{1}{p}| \leq \frac{m}{\alpha}$ ($\frac{n}{2} < \frac{m}{\alpha}$). Therefore, by Lemma 4.1, $0 \in \sigma(\mathrm{Op}_p(a))$ if and only if $n|\frac{1}{2} - \frac{1}{p}|\frac{\alpha}{m} > 1$ ($\frac{n}{2}\frac{\alpha}{m} \geq 1$). On the other hand, $0 \notin a(\mathbb{R}^n) = \sigma(\mathrm{Op}_2(a))$, which proves the assertion. □

Remark 4.5. The above assumptions are in particular satisfied for elliptic polynomials a, in which case we have $\rho = 1$ and $m = r$.

Suppose that $a \in S^m_{\rho,0}$ satisfies Hypothesis (H2) and let N_p be the smallest integer such that

$$N_p \begin{cases} \geq n\left|\dfrac{1}{2} - \dfrac{1}{p}\right|\left(\dfrac{1+m-\rho}{r}\right) & \text{if } 1 < p < \infty, \\ > \dfrac{n}{2}\left(\dfrac{1+m-\rho}{r}\right) & \text{if } p = 1. \end{cases} \tag{4.1}$$

Theorem 4.6. *Let* $1 \leq p < \infty$, $m \in (0, \infty)$, $N \in \mathbb{N} \cup \{0\}$ *and* $\rho \in [0, 1]$. *Suppose that* $a \in S^m_{\rho,0}$ *satisfies* (H2). *Then the following assertions are equivalent.*

(a) $\rho(\mathrm{Op}_p(a)) \neq \varnothing$ *and* $\sup_{\xi \in \mathbb{R}^n} \mathrm{Re}\, a(\xi) \leq \omega$ *for some* $\omega \in \mathbb{R}$.

(b) *The operator* $\mathrm{Op}_p(a)$ *generates an* N_p*-times integrated semigroup* $(S(t))_{t \geq 0}$ *on* $L^p(\mathbb{R}^n)$.

(c) $\sigma(\mathrm{Op}_p(a)) \subset \{z \in \mathbb{C}; \mathrm{Re}\, z \leq \omega\}$ *for some* $\omega \in \mathbb{R}$.

Proof. (a) $\Rightarrow$ (b). A rescaling argument shows that we may assume that $\omega = -1$. Hence, it follows from Proposition 4.2 that $0 \in \rho(\mathrm{Op}_p(a))$. For $t \geq 0$ and $k \in \mathbb{N}$ define the function $u_t^k : \mathbb{R}^n \to \mathbb{C}$ by

$$u_t^k(\xi) := \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} e^{sa(\xi)}\, ds.$$

Then, integrating by parts we obtain

$$u_t^k = \frac{e^{ta}}{a^k} - \sum_{j=1}^k \frac{1}{(k-j)!} \frac{t^{k-j}}{a^j}.$$

Since $\mathcal{M}_p$ is a Banach algebra, we conclude from Lemma 4.1 that there exists a constant $C_{k,p}$ such that

$$\left\| \sum_{j=1}^k \frac{1}{(k-j)!} \frac{t^{k-j}}{a^j} \right\|_{\mathcal{M}_p} \leq C_{k,p}(1+t)^{k-1} \qquad (t \geq 0). \tag{4.2}$$

By assumption, the symbol a satisfies Hypothesis (H2). Therefore, choosing N_p as in (4.1), it follows from Lemma 3.4 that

$$\left\| \frac{e^{ta}}{a^{N_p}} \right\|_{\mathcal{M}_p} \leq C_{N,p,n}(1+t)^{n|1/2-1/p|} e^{-t} \qquad (t \geq 0) \tag{4.3}$$

for some constant $C_{N,p,n}$. Combining (4.2) with (4.3) it follows that $u_t^{N_p} \in \mathscr{M}_p$ for all $t \geq 0$ and that

$$\|u_t^{N_p}\|_{\mathscr{M}_p} \leq C_{N,p,n}(1+t)^{\max\{n|1/2-1/p|, N_p-1\}} \qquad (t \geq 0)$$

for some constant $C_{N,p,n}$. Following the proof of [Hi2, Theorem 5.1] it is now not difficult to verify that the mapping $S : [0, \infty) \to \mathscr{L}(L^p(\mathbb{R}^n))$, $t \mapsto \mathscr{F}^{-1}(u_t^N)$ is strongly continuous and to prove that $\mathrm{Op}_p(a)$ is the generator of the integrated semigroup $(S(t))_{t\geq 0}$. Finally, a well-known perturbation argument completes the proof of this assertion.

The assertion (b) $\Rightarrow$ (c) follows from the definition of the integrated semigroup and assertion (c) $\Rightarrow$ (a) is a consequence of Proposition 4.2. □

Remarks 4.7. Suppose that the assumptions of Theorem 4.6 are fulfilled and that assertion (a) or (c) of Theorem 4.6 is satisfied for some $\omega \in \mathbb{R}$.

(a) It follows from the above proof that the N_p-times integrated semigroup $(S(t))_{t\geq 0}$ on $L^p(\mathbb{R}^n)$ satisfies an estimate of the form

$$\|S(t)\|_{\mathscr{L}(L^p)} \leq M(1+t)^{\max\{n|1/2-1/p|, N_p-1\}} e^{\max\{\omega', 0\}t} \qquad (t \geq 0)$$

for some constants $M > 0$ and $\omega' > \omega$.

(b) If in addition a is homogeneous, then

$$\|S(t)\|_{\mathscr{L}(L^p)} \leq Mt^k \qquad (t \geq 0)$$

for some constant $M > 0$ and some integer

$$k \begin{cases} \geq n\left|\dfrac{1}{2} - \dfrac{1}{p}\right| & \text{if } 1 < p < \infty, \\ > \dfrac{n}{2} & \text{if } p = 1. \end{cases}$$

In order to prove (b) note that $\mathscr{M}_p$ is isometrically invariant under affine transformations of $\mathbb{R}^n$. Thus

$$\|S(t)\|_{\mathscr{L}(L^p)} = \left\|\int_0^t \frac{(t-s)^{k-1}}{(k-1)!} e^{sa}\right\|_{\mathscr{M}_p} = t^k \left\|\int_0^1 \frac{(1-s)^{k-1}}{(k-1)!} e^{sa}\right\|_{\mathscr{M}_p} \quad \text{for all } t \geq 0.$$

Recalling the fact that $\mathscr{M}_1 \subset \mathscr{M}_p$, $1 \leq p \leq \infty$, we immediately obtain the following result.

Corollary 4.8. *Let $m \in (0, \infty)$ and $\rho \in [0, 1]$. Suppose that $a \in S^m_{\rho,0}$ satisfies* (H2). *Then the following assertions are equivalent.*

(a) $\mathrm{Op}_p(a)$ *generates a k-times integrated semigroup on $L^p(\mathbb{R}^n)$* $(1 < p < \infty)$ *for some integer k and $\rho(\mathrm{Op}_1(a)) \neq \varnothing$.*

(b) $\mathrm{Op}_1(a)$ *generates an l-times integrated semigroup on $L^1(\mathbb{R}^n)$ for some integer l.*

The numbers k and l in Corollary 4.8 are related to each other in the following manner.

Corollary 4.9. *Assume that the assumptions of Corollary* 4.8 *are satisfied. Then the following hold.*

(i) *If assertion* (a) *of Corollary* 4.8 *holds for some $k \in \mathbb{N} \cup \{0\}$, then assertion* (b) *is true for any integer $l > \frac{n}{2}(\frac{1+m-\rho}{r})$.*

(ii) *If assertion* (b) *of Corollary* 4.8 *holds for some* $l \in \mathbb{N} \cup \{0\}$, *then assertion* (a) *is true for any integer* $k \geq n|\frac{1}{2} - \frac{1}{p}|(\frac{1+m-\rho}{r})$.

Remark 4.10. It follows from Theorem 4.3 in [Hi1] that the orders of integration in Theorem 4.6 and Corollary 4.9, respectively, are optimal for a large class of operators including the operator $\mathrm{Op}_p(a) = i\Delta$. Indeed, in this case $\rho = 1$ and $m = r$. Thus $i\Delta$ generates an N-times integrated semigroup on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, if and only if $N \geq n|\frac{1}{2} - \frac{1}{p}|$ and on $L^1(\mathbb{R}^n)$ if and only if $N > \frac{n}{2}$. For more general results in this direction, see [Hi1].

Corollary 4.11. *Let* $1 \leq p < \infty$, $m \in (0, \infty)$, $N \in \mathbb{N}$ *and* $\rho \in [0, 1]$. *Assume that* $a \in S^m_{\rho,0}$ *satisfies* (H2). *If* $\sup_{\xi \in \mathbb{R}^n} \mathrm{Re}\, a(\xi) \leq \omega$ *for some* $\omega \in \mathbb{R}$, *then there exists a constant* $\delta_N > 0$ *such that* $\mathrm{Op}_p(a)$ *generates an* N*-times integrated semigroup on* $L^p(\mathbb{R}^n)$ *provided* $|\frac{1}{2} - \frac{1}{p}| < \delta_N$.

Example 4.12 (see [Hi2]). The example of the symbol a given by

$$a(\xi) := (-i)(\xi_1 - \xi_2^2 - \xi_3^2 - i)(\xi_1 + \xi_2^2 + \xi_3^2 + i)$$

shows that $\mathrm{Op}_p(a)$ generates an integrated semigroup on $L^p(\mathbb{R}^3)$ only for certain values of p. Indeed, we verify that $\sup_\xi \mathrm{Re}\, a(\xi) \leq 0$ and that $r = 1$. Hence, by Theorem 4.4 we see that $\rho(\mathrm{Op}_p(a)) \neq \varnothing$ provided $|\frac{1}{2} - \frac{1}{p}| \leq \frac{1}{9}$. Therefore $\mathrm{Op}_p(a)$ generates a once integrated semigroup on $L^p(\mathbb{R}^3)$ provided $|\frac{1}{2} - \frac{1}{p}| \leq \frac{1}{12}$. However, it follows from the results in [IS] that $\sigma(\mathrm{Op}_p(a)) \neq a(\mathbb{R}^n)$ if $|\frac{1}{2} - \frac{1}{p}| > \frac{3}{8}$. Hence by Proposition 4.2, $\rho(\mathrm{Op}_p(a)) = \varnothing$ if $|\frac{1}{2} - \frac{1}{p}| > \frac{3}{8}$. Consequently, in this case $\mathrm{Op}_p(a)$ does not generate an N-times integrated semigroup on $L^p(\mathbb{R}^3)$ for any N.

Recall that the Laplace transform of an exponentially bounded, strongly continuous function exists in a right half-plane of $\mathbb{C}$. Hence, as a consequence of Theorem 4.6 and Remark 4.7, we obtain the following L^p resolvent estimates for pseudodifferential operators with symbol $a \in S^m_{\rho,0}$ having constant coefficients.

Corollary 4.13. *Let* $1 \leq p < \infty$, $m \in (0, \infty)$ *and* $\rho \in [0, 1]$. *Assume that* $a \in S^m_{\rho,0}$ *satisfies* (H2) *and that* $\sup_\xi \mathrm{Re}\, a(\xi) \leq 0$.

(a) *If* $\rho(\mathrm{Op}_p(a)) \neq \varnothing$, *then* $(\lambda - \mathrm{Op}_p(a))$ *is invertible for all* $\lambda \in \mathbb{C} \backslash a(\mathbb{R}^n)$ *and for* $\varepsilon > 0$ *and* $N \geq N_p$ *there exists a constant* $C_{N,p,n} > 0$ *such that*

$$\|(\lambda - \mathrm{Op}_p(a))^{-1}\| \leq C_{N,p,n}|\lambda|^N \left(\frac{1}{\mathrm{Re}\,\lambda - \varepsilon} + \frac{1}{(\mathrm{Re}\,\lambda - \varepsilon)^{2N+1}}\right) \qquad (\mathrm{Re}\,\lambda > \varepsilon).$$

(b) *If in addition* a *is homogeneous, then* $(\lambda - \mathrm{Op}_p(a))$ *is invertible for all* $\lambda \in \mathbb{C} \backslash a(\mathbb{R}^n)$ *and for*

$$N \begin{cases} \geq n \left|\dfrac{1}{2} - \dfrac{1}{p}\right| & \text{if } 1 < p < \infty, \\ > \dfrac{n}{2} & \text{if } p = 1, \end{cases}$$

there exists a constant $C_{N,p,n} > 0$ *such that*

$$\|(\lambda - \mathrm{Op}_p(a))^{-1}\| \leq C_{N,p,n} \frac{|\lambda|^N}{(\mathrm{Re}\,\lambda)^{N+1}} \qquad (\mathrm{Re}\,\lambda > 0).$$

5. AN APPLICATION: DIRAC'S EQUATION ON $L^p(\mathbb{R}^3)$

The relativistic description of the motion of a particle of mass m with spin $1/2$ is provided by the Dirac equation (see [G] or [F])

$$\frac{\partial}{\partial t}u(x, t) = c\sum_{j=1}^{3} A_j D_j u(x, t) - A_4 \frac{mc^2}{ih} u(x, t) + Vu(x, t), \qquad x \in \mathbb{R}^3, t \geq 0.$$

Here u is a function defined on $\mathbb{R}^3 \times \mathbb{R}_+$ which takes values in $\mathbb{C}^4$, c is the speed of light, h is Planck's constant and A_j $(j = 1, 2, 3, 4)$ are 4×4 matrices given by

$$A_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \quad (j = 1, 2, 3) \quad \text{and} \quad A_4 = \begin{pmatrix} \sigma_4 & 0 \\ 0 & -\sigma_4 \end{pmatrix},$$

where σ_j are the Pauli spin matrices defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Suppose that $V = 0$ and all units are chosen so that all constants are equal to 1. Then Dirac's equation can be rewritten as a symmetric, hyperbolic system of the form

$$v'(t) = Dv(t), \quad v(0) = v_0$$

where

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad D := \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} + i \begin{pmatrix} \sigma_4 & 0 \\ 0 & -\sigma_4 \end{pmatrix}$$

and

$$A := \begin{pmatrix} D_3 & D_1 - iD_2 \\ D_1 + iD_2 & -D_3 \end{pmatrix}.$$

Here $D_j := \frac{\partial}{\partial x_j}$ $(j = 1, 2, 3)$. Let $E := L^p(\mathbb{R}^3, \mathbb{C})^4$ $(1 \leq p < \infty)$. We define the L_p-realization $\mathscr{D}_p$ of D by

$$D(\mathscr{D}_p) := D(\mathscr{A}_p) \times D(\mathscr{A}_p) \quad \text{and}$$
$$\mathscr{D}_p f := Df \quad \text{for all } f \in D(\mathscr{D}_p),$$

where $D(\mathscr{A}_p) := \{f \in L^p(\mathbb{R}^3)^2; Af \in L^p(\mathbb{R}^3)^2\}$. Then it is well known that the Dirac operator $\mathscr{D}_p$ generates a C_0-semigroup on $L^p(\mathbb{R}^3)^4$ if and only if $p = 2$ (cf. [Br]).

The symbol a if the Dirac operator $\mathscr{D}_p$ is similar (in the sense of matrices) to the function $b : \mathbb{R}^3 \to GL_4$ given by

$$b(\xi) := \operatorname{diag}(\lambda_1(\xi), \lambda_2(\xi), \lambda_3(\xi), \lambda_4(\xi)).$$

The values $\lambda_j(\xi)$ $(j = 1, 2, 3, 4)$ are the eigenvalues of $a(\xi)$ and are determined by

$$\lambda_1(\xi) = \lambda_2(\xi) = -\lambda_3(\xi) = -\lambda_4(\xi) = i(|\xi|^2 + 1)^{1/2}.$$

We immediately verify that for λ_j $(j = 1, 2, 3, 4)$ we have $\rho = 1$ and $m = r$. Thus, by Proposition 4.2

$$\sigma(\mathrm{Op}_p(\lambda_1)) = i[1, \infty) \quad \text{and} \quad \sigma(\mathrm{Op}_p(\lambda_3)) = i(-\infty, -1]$$

for all p satisfying $1 \leq p < \infty$. Moreover, it follows from Theorem 4.6 that $\mathrm{Op}_p(\lambda_j)$ $(j = 1, 2, 3, 4)$ generates a once integrated semigroup on $L^p(\mathbb{R}^3)$

provided $|\frac{1}{2}-\frac{1}{p}|\le\frac{1}{3}$ and a twice integrated semigroup for all other values of p. Finally, we may conclude from Remark 4.7 and Corollary 4.13 that

$$\|(\lambda-\mathrm{Op}_p(\lambda_j))^{-1}\|\le\begin{cases} C|\lambda|\left(\dfrac{1}{\mathrm{Re}\,\lambda}+\dfrac{1}{\mathrm{Re}\,\lambda^3}\right) & \text{if } \left|\dfrac{1}{2}-\dfrac{1}{p}\right|\le\dfrac{1}{3}, \\ C|\lambda|^2\left(\dfrac{1}{\mathrm{Re}\,\lambda}+\dfrac{1}{\mathrm{Re}\,\lambda^5}\right) & \text{if } \left|\dfrac{1}{2}-\dfrac{1}{p}\right|>\dfrac{1}{3}, \end{cases}$$

for all $j=1,2,3,4$ and all $\lambda\in\mathbb{C}$ with $\mathrm{Re}\,\lambda>0$.

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