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SECOND ROOT VECTORS FOR MULTIPARAMETER EIGENVALUE PROBLEMS OF FREDHOLM TYPE

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Abstract. A class of multiparameter eigenvalue problems involving (generally) non self-adjoint and unbounded operators is studied. A basis for the second root subspace, at eigenvalues of Fredholm type, is computed in terms of the underlying multiparameter system. A self-adjoint version of this result is given under a weak definiteness condition, and Sturm-Liouville and finite-dimensional examples are considered.

1. Introduction

We consider multiparameter operator pencils

$$W_i(\boldsymbol{\lambda}) = \sum_{j=0}^{n} \lambda_j V_{ij}, \tag{1.1}$$

for $i = 1, 2, \ldots, n$, where V_{ij}, $j = 1, 2, \ldots, n$, are bounded operators on a separable Hilbert space H_i and V_{i0} are closed, densely defined operators with domains $\mathcal{D}(V_{i0}) \subset H_i$. Eigenproblems of the form

$$W_i(\boldsymbol{\lambda}) x_i = 0 \neq x_i \tag{1.2}$$

arise in a variety of applications, for example to separation of variables for classical p.d.e. [3], to linearized bifurcation models [15] and to certain matrix inverse problems [14]. We refer to the books [5, 13, 23, 26] for background on multiparameter spectral theory. In order to introduce our topic, we suppose initially that $\dim H_i < \infty$, $i = 1, ..., n$.

When $n = 1$ and V_{11} is one-to-one we have a problem of the form

$$(\lambda_1 V_{11} + V_{10}) x_1 = 0 \neq x_1$$

which is equivalent to the ordinary eigenvalue problem for the matrix $\Gamma_1 = -V_{11}^{-1} V_{10}$. If the eigenvalues λ of Γ_1 are semisimple (e.g., if V_{ij} are Hermitean and $V_{11} > 0$) then the eigenvectors are complete in H_1, i.e., a basis of eigenvectors exists. If not, then root vectors are required; specifically, H_1 decomposes into a direct sum of "root subspaces" of the form $R(\lambda) = \mathcal{N}(\Gamma_1 - \lambda I)^{\nu}$. If so-called Jordan bases are used for the $R(\lambda)$, then Γ_1 is represented by a matrix in Jordan form.

When $n > 1$, it is natural to study completeness in the tensor product space $H = \bigotimes_{i=1}^{n} H_i$, by means of certain "determinantal" operators $\triangle_j$. Specifically, $\triangle_j$ is (up to a sign) the tensor determinant of the array $[V_{kl}]_{1 \leq k \leq n, 0 \leq l \leq n}$ with j-th

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column omitted. If $\triangle_0$ is one-to-one, then the operators $\Gamma_j = \triangle_0^{-1}\triangle_j$ commute and provide a joint spectral decomposition of H [5, Ch. 6]. If the V_{ij} are Hermitean and $\triangle_0 > 0$ then the eigenvalues are semisimple, and a basis of joint eigenvectors for the Γ_j exists for H. It is important to note that these eigenvectors can in fact be constructed out of (decomposable) tensors of the eigenvectors for the original operators W_i, so the Γ_j do not need constructing explicitly [5, Ch. 7].

In general, completeness requires "joint root subspaces" of the form

$$R(\boldsymbol{\lambda}) = \bigcap_{j=1}^{n} \mathcal{N}\left(\Gamma_j - \lambda_j I\right)^{\nu_j}$$

[5, Ch. 6] and it has been an open problem for many years, cf. [4], to describe bases for the $R(\boldsymbol{\lambda})$ in terms of the W_i. Here we shall carry this out for the "second" root subspace, where each $\nu_j \leq 2$. For precise definitions see §5. We remark that the desirability in finite dimensions of using W_i rather than Γ_j can be gauged from the relation $\dim H = \prod_{i=1}^{n} \dim H_i$. In typical infinite dimensional examples, the W_i are ordinary differential operators resulting from separation of variables in partial differential equations. The Γ_j are then also partial differential operators, so the main virtue of the technique disappears unless one has completeness statements in terms of the W_i.

In infinite dimensions, there are significant difficulties even in defining the Γ_j, at least when unbounded operators are involved, e.g., for differential equations. For self-adjoint problems this has been accomplished in the case when $\triangle_0$ is uniformly positive definite [25], and if the spectrum is discrete then again the eigenvectors are complete in the (Hilbert space) tensor product. Without uniformity, there are still open problems in this area, and we refer to [26] for some of the relevant problems and literature. An approach via rigged Hilbert spaces can be found in [6].

In this paper we shall set up an analogue of the Γ_j for a useful class of problems involving (generally) non self-adjoint and unbounded operators (see also [2, 16]). We remark that our construction of "second" root vectors is new even in finite dimensions, although certain self-adjoint and "simply separated" cases have been examined in [9, 11, 13, 22]. We shall compare our results with these in §9.

In §2 we set up our notation and assumptions and in §3 we show how to define our analogue of the Γ_j. In §4 we study the "first" root subspace, and we give a condition ensuring that it is spanned by eigenvectors. In §5 we define the "second" root subspace and in §6 we construct a basis for it in terms of the underlying system (1.1). This is our main result. We compute such a basis for a non-self-adjoint finite-dimensional example in §7. We conclude with a version of our main result under a very weak definiteness condition in §8 and we give an application to a semi-definite Sturm-Liouville example in §9. The Jordan chains in semi-definite examples are at most of length 2, so our results give complete bases for the corresponding root subspaces.

2. Regularity assumptions

The operators V_{ij}, $j = 1, 2, \ldots, n$, induce operators $V_{ij}^{\dagger}$ on the Hilbert space tensor product $H = H_1 \otimes H_2 \otimes \cdots \otimes H_n$ by means of

$$V_{ij}^{\dagger}\left(x_1 \otimes x_2 \otimes \cdots \otimes x_n\right) = x_1 \otimes \cdots \otimes x_{i-1} \otimes V_{ij} x_i \otimes x_{i+1} \otimes \cdots \otimes x_n$$

on decomposable tensors, extended by linearity and continuity to the whole of H. Similarly V_{i0} induces an operator $V_{i0}^{\dagger}$ with domain $\mathcal{D}\left(V_{i0}^{\dagger}\right) \subset H$ (see [23, §2.3]). We denote by $\mathcal{D}$ the intersection $\bigcap_{i=1}^{n} \mathcal{D}\left(V_{i0}^{\dagger}\right)$, which is a dense subspace of H. We then define the operator

$$\Delta_0 = \det \left[V_{ij}^{\dagger}\right]_{i,j=1}^{n} \tag{2.1}$$

on H and operators Δ_i $(1 \leq i \leq n)$ on $\mathcal{D}$ by replacing the i-th column in (2.1) by $\left[-V_{i0}^{\dagger}\right]_{i=1}^{n}$.

In what follows we make three regularity assumptions. We state two of them now and we formulate the third one in §5 when all the notions involved have been introduced.

Assumption I. There exists $\alpha \in \mathbb{C}$ such that the operator $\Delta_n' = \Delta_n + \alpha\Delta_0$ has a bounded inverse.

In the finite-dimensional case Assumption I is equivalent to the matrix pencil $\Delta_n + \lambda\Delta_0$ being regular, i.e., to its determinant being a nonzero polynomial in λ. This can be also formulated in terms of polynomials $\det W_i(\boldsymbol{\lambda})$ in $n+1$ variables $\lambda_0, \lambda_1, \ldots, \lambda_n$ (cf. [5, Ch. 8]). The two-parameter case was discussed in terms of Kronecker chains in [20].

In the Sturm-Liouville case of (1.2), Δ_n and Δ_0 are partial differential and multiplication operators respectively. We denote the null space of an operator T by $\mathcal{N}(T)$.

Proposition 2.1. *In the Sturm-Liouville case, if Δ_n is self-adjoint and uniformly elliptic, and if $\mathcal{N}(\Delta_0) \cap \mathcal{N}(\Delta_n) = \{0\}$, then Assumption* I *holds with $\alpha \in \mathbb{R}$.*

Proof. The hypotheses show that Δ_n is bounded below with compact resolvent, so by the minimax principle for the eigenvalues of Δ_n the cone

$$C = \{x \in \mathcal{D},\ (x, \Delta_n x) = 0\} \tag{2.2}$$

has a maximal subspace of finite dimension c, say. Suppose Assumption I fails. Then by Rellich's theorem (cf. [19, Theorem VII.3.9]) there are real analytic $\mu(\alpha) \equiv 0$ and $u(\alpha)$ satisfying $\|u(\alpha)\| = 1$ and $(\Delta_n - \alpha\Delta_0)\, u(\alpha) = \mu(\alpha)\, u(\alpha)$ for all real α.

Let $u_j = u(j)$ for integers j between 0 and c. Routine manipulations give $(u_j, \Delta_n u_k) = 0$ whenever $j \neq k$, and [10, Theorem 2.3] gives

$$0 = \mu'(j) = (u_j, \Delta_0 u_j)$$

whence

$$(u_j, \Delta_n u_j) = j\,(u_j, \Delta_0 u_j) = 0$$

for all j. Now the argument of [1, Theorem 2.0 (g)] shows that if Δ_0 corresponds to multiplication by δ_0 and $\sum c_j u_j = 0$, then $\delta_0 u_1 = 0$. Thus $\Delta_n u_1 = \Delta_0 u_1 = 0$, so by hypothesis $u_1 = 0$, contradicting $\|u_1\| = 1$. Thus the u_j are linearly independent, and they span a $(c+1)$-dimensional subspace of the cone C of (2.2), and this is a contradiction. □

We remark that the hypotheses in Proposition 2.1 hold, by virtue of unique continuation for Δ_n, in the uniformly elliptic case of (1.1) (see §8 and §9) provided

Δ_0 does not vanish (a.e.) on some open set. This is the setting used (for a one parameter problem) in [1, Theorem 2.0] cited above.

We assume, unless stated otherwise, that the operator Δ_n has a bounded inverse. This form of Assumption I can be obtained by a shift in parameters. Thus we can normalize our eigenvalue by assuming that $\lambda_n = 1$, so $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \dots, \lambda_{n-1}, 1)$ and

$$W_i(\boldsymbol{\lambda}) = \sum_{j=0}^{n-1} V_{ij}\lambda_j + V_{in}$$

for $i = 1, 2, \dots .n$. We denote the range of an operator T by $\mathcal{R}(T)$.

Assumption II. For a given eigenvalue $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \dots, \lambda_{n-1}, 1)$ of (1.2) the operators $W_i(\boldsymbol{\lambda})$, $i = 1, 2, \dots, n$, are Fredholm, [24]. In particular, $\dim \mathcal{N}(W_i(\boldsymbol{\lambda}))$ and $\operatorname{codim} \mathcal{R}(W_i(\boldsymbol{\lambda}))$ are both finite.

This assumption is satisfied, for example, in several cases arising from boundary value problems, e.g., of Sturm-Liouville type and also in the finite-dimensional case. We discuss such examples in §7 and §9.

Let $\mathcal{V}_{ij}^{\dagger}$ denote the restriction of $V_{ij}^{\dagger}$ to $\mathcal{D}$. The array $\mathcal{V} = \left[\mathcal{V}_{ij}^{\dagger}\right]_{i=1,\ j=0}^{n\quad\ n}$ then defines a linear map $\mathcal{V} : \mathcal{D}^{n+1} \longrightarrow H^n$. Here H^n is the direct sum of n copies of H. Omitting the j-th column we get a transformation $\mathcal{V}_j$ acting on the (algebraic) direct sum $\mathcal{D}^n$ for $j = 0, 1, \dots, n$. Note that $\Delta_j = (-1)^j \det \mathcal{V}_j$. Next we define the transformations $\mathcal{A}_j$ adjugate to $\mathcal{V}_j$, so $(\mathcal{A}_j)_{lk}$ is the (k, l)-th cofactor of $\mathcal{V}_j$.

The following result is a consequence of Assumption I.

Proposition 2.2. *The domain $\mathcal{D}(\mathcal{A}_n)$ has a decomposition into an (algebraic) direct sum*

$$\mathcal{D}(\mathcal{A}_n) = \mathcal{R}(\mathcal{V}_n) \oplus \mathcal{N}(\mathcal{A}_n). \tag{2.3}$$

Proof. From the construction of the transformations $\mathcal{A}_n$ and $\mathcal{V}_n$ it follows as in the finite-dimensional case (cf. [5, Thm. 6.4.1]) that

$$(-1)^n \mathcal{A}_n \mathcal{V}_n = \begin{bmatrix} \Delta_n & 0 & \cdots & 0 \\ 0 & \Delta_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Delta_n \end{bmatrix}. \tag{2.4}$$

(Note that each operator in the matrix on the right is defined on $\mathcal{D}$, since each summand in the matrix product is defined on $\mathcal{D}$.) Since $\mathcal{A}_n\mathcal{V}_n$ is one-to-one it follows that $\mathcal{R}(\mathcal{V}_n) \cap \mathcal{N}(\mathcal{A}_n) = \{0\}$. Because Δ_n has a bounded inverse, $\mathcal{A}_n\mathcal{V}_n$ also has a bounded inverse which is denoted by $\mathcal{B}$. Next we choose $\mathbf{x} \in \mathcal{D}(\mathcal{A}_n)$ and write $\mathbf{y} = \mathcal{V}_n\mathcal{B}A_n\mathbf{x}$ and $\mathbf{z} = \mathbf{x} - \mathbf{y}$. Then $\mathcal{A}_n\mathbf{z} = 0$, and therefore $\mathbf{x} = \mathbf{y} + \mathbf{z}$ is the required decomposition. □

3. Associated operators

The operators $\Gamma_j = \Delta_n^{-1}\Delta_j$, $j = 0, 1, \dots, n-1$, with domain $\mathcal{D}$ are called the *associated operators* of the multiparameter system (1.1). Note that $\mathcal{R}(\Gamma_j) \subset \mathcal{D}$ for all j. We also use the notation $\Gamma_n = I_{\mathcal{D}}$, and we write $\mathcal{C}_j$ for the j-th column of $\mathcal{V}$, $j = 0, 1, \dots, n$.

Proposition 3.1. *If* $\mathbf{x} = \begin{bmatrix} x_0 & x_1 & \cdots & x_n \end{bmatrix}^T \in \mathcal{D}^{n+1}$ *and* $\mathcal{V}\mathbf{x} = 0$ *then* $x_j = \Gamma_j x_n$ *for* $j = 0, 1, \ldots, n-1$.

Proof. By definition of the transformations $\mathcal{V}_n$ and $\mathcal{A}_n$ we have (2.4) and

$$(-1)^n \mathcal{A}_n \mathcal{C}_n = -\begin{bmatrix} \Delta_0 & \Delta_1 & \cdots & \Delta_{n-1} \end{bmatrix}^T. \tag{3.1}$$

Then $\mathcal{A}_n \mathcal{V}\mathbf{x} = 0$ implies that $\Delta_n x_j - \Delta_j x_n = 0$ and therefore $x_j = \Gamma_j x_n$. □

We note that the equation $\mathcal{V}\mathbf{x} = 0$ of Proposition 3.1 may be written in the form $\mathcal{V}_j \mathbf{y} = \mathcal{C}_j x$. (Here $\mathbf{y} = \begin{bmatrix} x_0 & \cdots & x_{j-1} & x_{j+1} & \cdots & x_n \end{bmatrix}^T$ and $x = x_j$.) The solvability of this equation for some $\mathbf{y} \in \mathcal{D}^n$ is one of the basic problems considered in multiparameter theory (cf. [2, 4], [5, eq. (11.8.8)], [12, 16, 18, 23]), $j = n$ being the most suitable for our hypotheses. We remark that the cited works assume $j = 0$, so the matrix $\mathcal{V}_j$ then contains only bounded operators, and the cited results cannot be taken over directly to our case. We write

$$\mathcal{K}_j = \left\{ x \in \mathcal{D}, \mathcal{V}_j \mathbf{x} = \mathcal{C}_j x \text{ for some } \mathbf{x} \in \mathcal{D}^n \right\}. \tag{3.2}$$

and we prove the following important property of $\mathcal{K}_n$:

Theorem 3.2. *The operators* Γ_j, $j = 0, 1, \ldots, n-1$, *commute on* $\mathcal{K}_n$, *i.e.,* $\Gamma_j \Gamma_k x = \Gamma_k \Gamma_j x$ *for all* $x \in \mathcal{K}_n$, *and*

$$\sum_{j=0}^{n} V_{ij}^{\dagger} \Gamma_j x = 0 \tag{3.3}$$

for $i = 1, 2, \ldots, n$ *and* $x \in \mathcal{K}_n$.

Proof. Suppose that $x \in \mathcal{K}_n$. By definition of $\mathcal{K}_n$ there exist vectors $x_i \in \mathcal{D}$, $i = 0, 1, \ldots, n-1$, such that

$$\mathcal{V} \begin{bmatrix} x_0 & x_1 & \cdots & x_{n-1} & x \end{bmatrix}^T = 0. \tag{3.4}$$

Then by Proposition 3.1 it follows that

$$x_j = \Gamma_j x \tag{3.5}$$

and hence (3.3) follows from (3.4). As for (2.4) and (3.1), we find that

$$(-1)^k \mathcal{A}_k \mathcal{V} = \begin{bmatrix} \Delta_k & \cdots & 0 & -\Delta_0 & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & \Delta_k & -\Delta_{k-1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -\Delta_{k+1} & \Delta_k & & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & -\Delta_n & 0 & \cdots & \Delta_k \end{bmatrix} \tag{3.6}$$

for $k = 0, 1, \ldots, n-1$. Therefore it follows from (3.3) and (3.5) that

$$0 = (-1)^k \mathcal{A}_k \mathcal{V} \begin{bmatrix} \Gamma_0 x \\ \Gamma_1 x \\ \vdots \\ \Gamma_{n-1} x \\ x \end{bmatrix} = \begin{bmatrix} \Delta_k \Gamma_0 x - \Delta_0 \Gamma_k x \\ \vdots \\ \Delta_k \Gamma_{k-1} x - \Delta_{k-1} \Gamma_k x \\ \Delta_k \Gamma_{k+1} x - \Delta_{k+1} \Gamma_k x \\ \vdots \\ \Delta_k \Gamma_{n-1} x - \Delta_{n-1} \Gamma_k x \\ \Delta_k x - \Delta_n \Gamma_k x \end{bmatrix}$$

and so we have $\Delta_k \Gamma_j x = \Delta_j \Gamma_k x$ for all j and k. Multiplying by Δ_n^{-1} on the left-hand side we see that $\Gamma_j,\ j = 0, 1, \dots, n-1$, commute on $\mathcal{K}_n$. □

4. Eigenvalues and eigenvectors

The subspace

$$\mathcal{N} = \bigcap_{j=0}^{n-1} \mathcal{N}\left(\Gamma_j - \lambda_j I\right) \quad (\subset \mathcal{D})$$

is called the (geometric) *eigenspace* corresponding to $\boldsymbol{\lambda}$. The direct sum decomposition (2.3) "induces" a direct sum decomposition of $\mathcal{N}$. This is described in the next two results and illustrated with an example. We write $\mathcal{N}' = \mathcal{N} \cap \mathcal{K}_n$.

Theorem 4.1. *Suppose that $\mathcal{N}_i = \mathcal{N}\left(W_i\left(\boldsymbol{\lambda}\right)\right)$, $i = 1, 2, \dots, n$. Then*

$$\mathcal{N}' = \mathcal{N}_1 \otimes \mathcal{N}_2 \otimes \cdots \otimes \mathcal{N}_n.$$

Proof. Choose $x_i \in \mathcal{N}_i$ and write $z = x_1 \otimes x_2 \otimes \cdots \otimes x_n$. Then $W_i\left(\boldsymbol{\lambda}\right)^\dagger z = 0$ for all i, so $\mathcal{V}_n \begin{bmatrix} \lambda_0 z & \lambda_1 z & \cdots & \lambda_{n-1} z \end{bmatrix}^T + \mathcal{C}_n z = 0$ and hence $z \in \mathcal{K}_n$. For $j = 0, 1, \dots, n-1$ we have

(4.1)

$$\begin{aligned} 0 &= \begin{vmatrix} V_{10}^\dagger & \cdots & W_1\left(\boldsymbol{\lambda}\right)^\dagger & \cdots & V_{1,n-1}^\dagger \\ V_{20}^\dagger & \cdots & W_2\left(\boldsymbol{\lambda}\right)^\dagger & \cdots & V_{2,n-1}^\dagger \\ \vdots & & \vdots & & \vdots \\ V_{n0}^\dagger & \cdots & W_n\left(\boldsymbol{\lambda}\right)^\dagger & \cdots & V_{n,n-1}^\dagger \end{vmatrix} z \\ &= \begin{vmatrix} V_{10}^\dagger & \cdots & \lambda_j V_{1j}^\dagger + V_{1n}^\dagger & \cdots & V_{1,n-1}^\dagger \\ V_{20}^\dagger & \cdots & \lambda_j V_{2j}^\dagger + V_{2n}^\dagger & \cdots & V_{2,n-1}^\dagger \\ \vdots & & \vdots & & \vdots \\ V_{n0}^\dagger & \cdots & \lambda_j V_{nj}^\dagger + V_{nn}^\dagger & \cdots & V_{n,n-1}^\dagger \end{vmatrix} z = (-1)^n \left(\lambda_j \Delta_n - \Delta_j\right) z, \end{aligned}$$

and therefore $\mathcal{N}_1 \otimes \mathcal{N}_2 \otimes \cdots \otimes \mathcal{N}_n \subset \mathcal{N}'$.

Assume now that $x \in \mathcal{N}'$. Then $x \in \mathcal{N}$ gives $\Gamma_j x = \lambda_j x$, so by (3.3) it follows that

$$0 = \sum_{j=0}^{n} V_{ij}^\dagger \Gamma_j x = W_i\left(\boldsymbol{\lambda}\right)^\dagger x. \tag{4.2}$$

Because $\mathcal{N}_i$ are finite-dimensional it follows that $H = \bigoplus_{\mathbf{j}} \mathcal{B}_{\mathbf{j}}$, where the sum is over all the n-tuples $\mathbf{j} = (j_1, j_2, \dots, j_n)$ of 0's and 1's and $\mathcal{B}_{\mathbf{j}} = \mathcal{B}_{j_1} \otimes \mathcal{B}_{j_2} \otimes \cdots \otimes \mathcal{B}_{j_n}$, where

$$\mathcal{B}_{j_i} = \begin{cases} \mathcal{N}_i, & \text{if } j_i = 0, \\ \left(\mathcal{N}_i\right)^\perp, & \text{if } j_i = 1. \end{cases}$$

Then we write $x = \sum_{\mathbf{j}} y_{\mathbf{j}}$, where $y_{\mathbf{j}} \in \mathcal{B}_{\mathbf{j}}$. Because $W_i\left(\boldsymbol{\lambda}\right)^\dagger y_{\mathbf{j}} = 0$ if and only if $j_i = 0$, it follows from (4.2) that $y_{\mathbf{j}} = 0$ if $\mathbf{j} \neq (0, 0, \dots, 0)$. Thus $x \in \mathcal{N}_1 \otimes \mathcal{N}_2 \otimes \cdots \otimes \mathcal{N}_n$. □

Corollary 4.2. *If $\mathcal{N} \subset \mathcal{K}_n$ then $\mathcal{N} = \mathcal{N}_1 \otimes \mathcal{N}_2 \otimes \cdots \otimes \mathcal{N}_n$.*

This conclusion is found in many places, e.g., [5, 7, 13, 23, 26], but under hypotheses guaranteeing $\mathcal{N} \subset \mathcal{K}_0$ instead.

We write $\mathcal{W}^\lambda = \left[W_i\left(\boldsymbol{\lambda}\right)^\dagger\right]_{i=1}^n : \mathcal{D} \longrightarrow H^n$.

Proposition 4.3. *Suppose that $\mathcal{J}$ is a subspace of $\mathcal{D}$ such that*

$$\mathcal{W}^\lambda \mid_{\mathcal{J}} : \mathcal{J} \longrightarrow \mathcal{N}\left(\mathcal{A}_n\right) \cap \mathcal{R}\left(\mathcal{W}^\lambda\right)$$

is bijective. Then the eigenspace has a direct sum decomposition

$$\mathcal{N} = \mathcal{N}' \oplus \mathcal{J}. \tag{4.3}$$

Proof. Suppose that $x \in \mathcal{N}$. By (3.6), $(-1)^n \mathcal{A}_n \mathcal{W}^\lambda x = \left[\left(\lambda_j \Delta_n - \Delta_j\right) x\right]_{j=0}^{n-1} = \mathbf{0}$, so $\mathcal{W}^\lambda x \in \mathcal{N}\left(\mathcal{A}_n\right) \cap \mathcal{R}\left(\mathcal{W}^\lambda\right)$ and therefore there exists $y \in \mathcal{J}$ such that $\mathcal{W}^\lambda\left(x - y\right) = 0$. Then it follows that $z = x - y$ is an element of $\mathcal{N}'$ as in the proof of Theorem 4.1. The decomposition $x = y + z$ is unique because $\mathcal{N}' \cap \mathcal{J} = \{0\}$. □

Note that the subspace $\mathcal{J}$ in (4.3) is not unique. The importance of the above result is that it tells us that in general it might happen that $\mathcal{N}' \underset{\neq}{\subset} \mathcal{N}$, as the following example confirms. Additional vectors to those from $\mathcal{N}'$ are needed to construct bases for $\mathcal{N}$ if and only if $\mathcal{N}\left(\mathcal{A}_n\right) \cap \mathcal{R}\left(\mathcal{W}^\lambda\right) \neq \{0\}$.

Example 4.4. Consider a separable Hilbert space H with an orthonormal basis $\{e_i\}_{i=0}^\infty$. For example, we could take $H = L^2\left[0, 2\pi\right]$ or $H = l^2$. We define a two-parameter system on $H_1 = H_2 = H$ by :

$$V_{10} e_m = e_{2m}, \qquad V_{11} e_m = e_{2m+1},$$

$$V_{12} e_m = \begin{cases} 0, & m = 0, \\ e_0, & m = 1, \\ e_5, & m = 2, \\ e_4, & m = 3, \\ e_{m+2}, & m \geq 4, \end{cases}$$

and

$$V_{20} e_k = \begin{cases} -e_{\frac{k}{2}}, & \text{if } k \text{ even}, \\ 0, & \text{if } k \text{ odd}, \end{cases} \qquad V_{21} e_k = \begin{cases} 0, & \text{if } k \text{ even}, \\ e_{\frac{k-1}{2}}, & \text{if } k \text{ odd}, \end{cases}$$

$$V_{22} e_k = \begin{cases} 0, & k = 0, \\ e_0, & k = 1, \\ e_k, & k \geq 2. \end{cases}$$

The operator Δ_2 is a special example of the operators considered in [2, Theorem 2] and [17, Lecture 1]. Because

$$\Delta_2\left(e_m \otimes e_{2k}\right) = e_{2m+1} \otimes e_k \text{ and } \Delta_2\left(e_m \otimes e_{2k+1}\right) = e_{2m} \otimes e_k \tag{4.4}$$

it follows that Δ_2 is bounded and has a bounded inverse. It is easy to see that $\boldsymbol{\lambda} = (0, 0, 1)$ is an eigenvalue and that V_{i2}, $i = 1, 2$, are Fredholm. Hence our Assumptions I and II are fulfilled.

Next we see that $\mathcal{N}' = \mathrm{Sp}\left\{e_0 \otimes e_0\right\}$, where $\mathrm{Sp}\left\{S\right\}$ is the linear span of the set S. Because we also have $\Delta_j\left(e_3 \otimes e_0 - e_2 \otimes e_1\right) = 0$ for $j = 0, 1$, it follows that $\mathcal{N}' \underset{\neq}{\subset} \mathcal{N}$. Observe that Theorem 4.1 and Proposition 4.3 then imply that $e_3 \otimes e_0 - e_2 \otimes e_1 \notin \mathcal{K}_2$. □

5. The second root subspace

The subspace

$$\mathcal{M} = \bigcap_{j=0}^{n-1} \bigcap_{k=0}^{n-1} \mathcal{N}\left[\left(\Gamma_j - \lambda_j I\right)\left(\Gamma_k - \lambda_k I\right)\right] \subset \mathcal{D}$$

is called *the second root subspace* of the multiparameter system (1.1) at the eigenvalue $\boldsymbol{\lambda}$. Note that by definition the Γ_j commute on $\mathcal{M}$.

Next we formulate our third assumption. The subspace $\mathcal{K}_n$ is defined by (3.2).

Assumption III. The second root subspace $\mathcal{M}$ is a subspace of $\mathcal{K}_n$.

By the definition of $\mathcal{K}_n$, Assumption III is equivalent to the solvability condition :

$$\mathcal{V}_n \mathbf{x} = \mathcal{C}_n x \text{ is solvable for all } x \in \mathcal{M}.$$

There are several possible levels of this condition, for instance :

(a) local as in Assumption III,
(b) column specific, i.e., $\mathcal{V}_j \mathbf{y} = \mathcal{C}_j x$ is solvable for all $x \in \mathcal{D}$,
(c) global, i.e., $\mathcal{V}_j \mathbf{y} = \mathbf{z}$ is solvable for all $\mathbf{z} \in H^n$.

For $j = 0$, (c) was considered in [12, 18, 23], (b) in [2, 5, 16, 17]. In the finite-dimensional case (c) (and so also (a) and (b)) with $j = n$ follows from the invertibility of Δ_n (cf. [5, Theorem 6.2.3]).

Assumption III also implies that $\mathcal{N} \subset \mathcal{K}_n$ (so Corollary 4.2 holds), but the reverse implication fails as the following example shows :

Example 5.1. Suppose that V_{ij} are as in Example 4.4 except

$$V_{12} e_m = \begin{cases} 0, & m = 0, \\ e_0, & m = 1, \\ e_5 + e_1, & m = 2, \\ e_4 + e_1, & m = 3, \\ e_{m+2,} & m \geq 4. \end{cases}$$

Note that $\mathcal{V}_2$ and Δ_2 remain as in Example 4.4 and that Assumptions I and II hold. We also have $\mathcal{N}' = \operatorname{Sp}\{e_0 \otimes e_0\}$. A simple calculation, as for (4.4), shows that $\mathcal{N}(\Delta_0) = H \otimes \operatorname{Sp}\{e_0\}$, and since $\Delta_1 e_m \otimes e_0 = -V_{12} e_m \otimes e_0$ vanishes only for $m = 0$, it follows that $\mathcal{N} = \operatorname{Sp}\{e_0 \otimes e_0\} = \mathcal{N}'$. Furthermore

$$(-1)^j \Delta_j \left(e_3 \otimes e_0 - e_2 \otimes e_1\right) = e_1 \otimes e_0 = \Delta_2 e_0 \otimes e_0, \qquad j = 0, 1,$$

implies that $e_3 \otimes e_0 - e_2 \otimes e_1 \in \mathcal{M}$. Because of the special form of the operators V_{10} and V_{11} the operator $\begin{bmatrix} V_{10}^\dagger & V_{11}^\dagger \end{bmatrix} : H^2 \to H$ is invertible. Then $V_{10}^\dagger x + V_{11}^\dagger y = V_{12}^\dagger \left(e_3 \otimes e_0 - e_2 \otimes e_1\right)$ has the solution $x = e_2 \otimes e_0$ and $y = e_0 \otimes e_0 - (e_0 + e_2) \otimes e_1$. However $V_{21}^\dagger x + V_{22}^\dagger y \neq V_{22}^\dagger \left(e_3 \otimes e_0 - e_2 \otimes e_1\right)$, so $e_3 \otimes e_0 - e_2 \otimes e_1 \notin \mathcal{K}_2$ and $\mathcal{M} \not\subset \mathcal{K}_2$. □

Lemma 5.2. *The subspace $\mathcal{M}$ is finite-dimensional.*

Proof. The subspace $\mathcal{M}$ is invariant for all $\Gamma_j - \lambda_j I$. The range $\mathcal{R}\left(\left(\Gamma_j - \lambda_j I\right)|_{\mathcal{M}}\right)$ is finite-dimensional because it is a subspace of $\mathcal{N}$ by Corollary 4.2. Then each kernel $\mathcal{N}\left(\left(\Gamma_j - \lambda_j I\right)|_{\mathcal{M}}\right)$ has finite codimension in $\mathcal{M}$, i.e., the orthogonal complement $\mathcal{Q}_j$ of $\mathcal{N}\left(\left(\Gamma_j - \lambda_j I\right)|_{\mathcal{M}}\right)$ in $\mathcal{M}$ is finite-dimensional. Hence the linear span $\mathcal{Q}$ of

the $\mathcal{Q}_j$ is finite dimensional. Because $\mathcal{N} = \bigcap_{j=1}^{n} \mathcal{N}\left((\Gamma_j - \lambda_j I)\mid_{\mathcal{M}}\right)$, $\mathcal{M} = \mathcal{N} \oplus \mathcal{Q}$ also has finite dimension. □

Our main objective is to construct a basis for the second root subspace under Assumptions I–III. This is done in the next section (see Theorem 6.3). First we introduce new notations and prove an auxiliary result.

We define a set of integer n-tuples

$$\mathbf{Q} = \{(k_1, k_2, \dots, k_n);\ 1 \le k_i \le n_i\},$$

where $n_i = \dim \mathcal{N}_i = \dim \mathcal{N}(W_i(\boldsymbol{\lambda}))$, and sets of integer $(n-1)$-tuples

$$\mathbf{Q}_i = \{(l_1, \dots, l_{i-1}, l_{i+1}, \dots, l_n),\ 1 \le l_j \le n_j\}.$$

We call elements $\mathbf{k} = (k_1, k_2, \dots, k_n) \in \mathbf{Q}$ and $\mathbf{l}' = (l_1, \dots, l_{i-1}, l_{i+1}, \dots, l_n) \in \mathbf{Q}_i$ multiindices. For $1 \le l \le n_i$ we use the notation $\mathbf{l}' \cup_i l$ for $(l_1, \dots, l_{i-1}, l, l_{i+1}, \dots, l_n) \in \mathbf{Q}$. We write $\mathcal{N}_i^* = \mathcal{N}\left(W_i(\boldsymbol{\lambda})^*\right)$.

In this setting we have :

Lemma 5.3. *Suppose that the vectors $x_{ij_i} \in H_i$, $j_i = 1, 2, \dots, n_i$, $i = 1, 2, \dots, k-1, k+1, \dots, n$ are linearly independent and suppose that for every y in the algebraic tensor product $H_1 \otimes_a \cdots \otimes_a H_{k-1} \otimes_a \mathcal{N}_k^* \otimes_a H_{k+1} \otimes_a \cdots \otimes_a H_n$*

(5.1)
$$\left(\sum_{\mathbf{j}' \in \mathbf{Q}_k} x_{1j_1} \otimes \cdots \otimes x_{k-1,j_{k-1}} \otimes x_k^{\mathbf{j}'} \otimes x_{k+1,j_{k+1}} \otimes \cdots \otimes x_{nj_n}, y\right) = 0,$$

where $x_k^{\mathbf{j}'} \in H_k$. Then $x_k^{\mathbf{j}'} \in (\mathcal{N}_k^*)^{\perp}$ for all $\mathbf{j}' \in \mathbf{Q}_k$.

Proof. Fix $\mathbf{j} \in \mathbf{Q}$. Assume that the vectors $v_{il_i} \in H_i$, $l_i = 1, 2, \dots, n_i$, are such that $(x_{ij_i}, v_{il_i}) = \delta_{j_i l_i}$, where δ_{jl} is the Kronecker symbol. Next we choose $y = v_{1j_1} \otimes \cdots \otimes v_{k-1,j_{k-1}} \otimes y_k \otimes v_{k+1,j_{k+1}} \otimes \cdots \otimes v_{nj_n}$ where $y_k \in \mathcal{N}_k^*$. Then (5.1) implies $\left(x_k^{\mathbf{j}'}, y_k\right) = 0$. Since $y_k \in \mathcal{N}_k^*$ was arbitrary, $x_k^{\mathbf{j}'} \in (\mathcal{N}_k^*)^{\perp}$. □

6. A basis for the second root subspace

We assume that the vectors $x_{i0}^k \in H_i$, $k = 1, 2, \dots, n_i$, form a basis for $\mathcal{N}_i$, $i = 1, 2, \dots, n$, and we introduce the vectors $z_0^{\mathbf{k}} = x_{10}^{k_1} \otimes x_{20}^{k_2} \otimes \cdots \otimes x_{n0}^{k_n}$. We can complete the basis $\mathcal{B}_0 = \left\{z_0^{\mathbf{k}},\ \mathbf{k} \in \mathbf{Q}\right\}$ for $\mathcal{N}$ to a basis $\mathcal{B} = \mathcal{B}_0 \cup \left\{z_1^l,\ l = 1, 2, \dots, d\right\}$ for the subspace $\mathcal{M}$, and then we have

$$(\Gamma_j - \lambda_j I)\, z_1^l = \sum_{\mathbf{k} \in \mathbf{Q}} a_j^{\mathbf{k}l} z_0^{\mathbf{k}} \tag{6.1}$$

for some $a_j^{\mathbf{k}l}$ for $j = 0, 1, \dots, n-1$ and $l = 1, 2, \dots, d$. We write $\mathbf{a}_j^l = \left(a_j^{\mathbf{k}l}\right)_{\mathbf{k} \in \mathbf{Q}}$. We regard $\mathbf{a}_j^l$ as an element of the tensor product space

$$H_\lambda = \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \cdots \otimes \mathbb{C}^{n_n} \tag{6.2}$$

and we regard

$$\mathbf{a}^l = \begin{bmatrix} \mathbf{a}_0^l & \mathbf{a}_1^l & \cdots & \mathbf{a}_{n-1}^l \end{bmatrix}^T \tag{6.3}$$

as an element of the n-tuple direct sum H_λ^n.

Then we have :

Proposition 6.1. *The elements $\mathbf{a}^l$, $l = 1, 2, \dots, d$, are linearly independent.*

Proof. Let us assume the contrary to obtain a contradiction. If $\mathbf{a}^l$ are linearly dependent, i.e. $\sum_{l=1}^{d} \alpha_l \mathbf{a}^l = 0$ and not all α_l equal 0, then there exists a vector $z \in \mathcal{M}\backslash\mathcal{N}$, i.e. $z = \sum_{l=1}^{d} \alpha_l z_1^l$, such that $(\Gamma_j - \lambda_j I)\, z = 0$ for all j. But this yields $z \in \mathcal{N}$, which contradicts $z \notin \mathcal{N}$. □

The following theorem describes the general form of a root vector in the second root subspace that is not an eigenvector, i.e., a vector $z \in \mathcal{M}\backslash N$. For convenience we write $\mathbf{a} = \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \end{bmatrix}^T \in \mathbb{C}^n$ and $U_i(\mathbf{a}) = \sum_{j=0}^{n-1} a_j V_{ij}$.

Theorem 6.2. *A vector z is in $\mathcal{M}\backslash N$ if and only if there exist n-tuples $\mathbf{a}^k \in \mathbb{C}^n$, not all 0, for $\mathbf{k} \in \mathbf{Q}$ and vectors $x_{i1}^{\mathbf{k}'} \in H_i,\ \mathbf{k}' \in \mathbf{Q}_i,\ i = 1, 2, \ldots, n$, such that*

$$\sum_{k_i=1}^{n_i} U_i\left(\mathbf{a}^{\mathbf{k}' \cup_i k_i}\right) x_{i0}^{k_i} + W_i(\boldsymbol{\lambda})\, x_{i1}^{\mathbf{k}'} = 0. \tag{6.4}$$

Then

$$z = \sum_{i=1}^{n} \sum_{\mathbf{k}' \in \mathbf{Q}_i} x_{10}^{k_1} \otimes \cdots \otimes x_{i-1,0}^{k_{i-1}} \otimes x_{i1}^{\mathbf{k}'} \otimes x_{i+1,0}^{k_{i+1}} \otimes \cdots \otimes x_{n0}^{k_n}. \tag{6.5}$$

It also follows that

$$(\Gamma_j - \lambda_j I)\, z = \sum_{\mathbf{k} \in \mathbf{Q}} a_j^{\mathbf{k}} z_0^{\mathbf{k}} \tag{6.6}$$

for $j = 0, 1, \ldots, n-1$.

Proof. Suppose that (6.4) and (6.5) hold. Then the following direct calculation shows that (6.6) holds, and since not all $\mathbf{a}^k$ are 0 it follows that $z \in \mathcal{M}\backslash N$. As for (4.1), we have

$$(-1)^n (\lambda_j \Delta_n - \Delta_j)\, z = \sum_{i=1}^{n} \sum_{\mathbf{k}' \in \mathbf{Q}_i} \begin{vmatrix} V_{10} x_{10}^{k_1} & \cdots & W_1(\boldsymbol{\lambda})\, x_{10}^{k_1} & \cdots & V_{1,n-1} x_{10}^{k_1} \\ \vdots & & \vdots & & \vdots \\ V_{i0} x_{i1}^{\mathbf{k}'} & \cdots & W_i(\boldsymbol{\lambda})\, x_{i1}^{\mathbf{k}'} & \cdots & V_{i.n-1} x_{i1}^{\mathbf{k}'} \\ \vdots & & \vdots & & \vdots \\ V_{n0} x_{n0}^{k_n} & \cdots & W_n(\boldsymbol{\lambda})\, x_{n0}^{k_n} & \cdots & V_{n,n-1} x_{n0}^{k_n} \end{vmatrix}$$

$$= \sum_{i=1}^{n} \sum_{\mathbf{k}' \in \mathbf{Q}_i} \begin{vmatrix} V_{10} x_{10}^{k_1} & \cdots & 0 & \cdots & V_{1,n-1} x_{10}^{k_1} \\ \vdots & & \vdots & & \vdots \\ V_{i0} x_{i1}^{\mathbf{k}'} & \cdots & -\sum_{k_i=1}^{n_i} U_i\left(\mathbf{a}^{\mathbf{k}' \cup_i k_i}\right) x_{i0}^{k_i} & \cdots & V_{i,n-1} x_{i1}^{\mathbf{k}'} \\ \vdots & & \vdots & & \vdots \\ V_{n0} x_{n0}^{k_n} & \cdots & 0 & \cdots & V_{n,n-1} x_{n0}^{k_n} \end{vmatrix}.$$

Because the above determinants are independent of $V_{il}x_{i1}^{\mathbf{k}'},\ l \neq j$, we obtain

$$\sum_{i=1}^{n}\sum_{k_i=1}^{n_i}\sum_{\mathbf{k}'\in\mathbf{Q}_j}\begin{vmatrix} V_{10}x_{10}^{k_1} & \cdots & 0 & \cdots & V_{1,n-1}x_{10}^{k_1} \\ \vdots & & \vdots & & \vdots \\ V_{i0}x_{i0}^{k_i} & \cdots & -U_i\left(\mathbf{a}^{\mathbf{k}'\cup_i k_i}\right)x_{i0}^{k_i} & \cdots & V_{i,n-1}x_{i0}^{k_i} \\ \vdots & & \vdots & & \vdots \\ V_{n0}x_{n0}^{k_n} & \cdots & 0 & \cdots & V_{n,n-1}x_{n0}^{k_n} \end{vmatrix}$$

$$\begin{aligned} &= -\sum_{\mathbf{k}\in\mathbf{Q}}\begin{vmatrix} V_{10}x_{10}^{k_1} & \cdots & U_1\left(\mathbf{a}^k\right)x_{10}^{k_1} & \cdots & V_{1,n-1}x_{10}^{k_1} \\ V_{20}x_{20}^{k_2} & \cdots & U_2\left(\mathbf{a}^k\right)x_{20}^{k_2} & \cdots & V_{2,n-1}x_{20}^{k_2} \\ \vdots & & \vdots & & \vdots \\ V_{n0}x_{n0}^{k_n} & \cdots & U_n\left(\mathbf{a}^k\right)x_{n0}^{k_n} & \cdots & V_{n,n-1}x_{n0}^{k_n} \end{vmatrix} \\ &= (-1)^{n+1}\sum_{\mathbf{k}\in\mathbf{Q}} a_j^{\mathbf{k}}\Delta_n z_0^{\mathbf{k}}. \end{aligned} \tag{6.7}$$

Now assume that $x \in \mathcal{M}\backslash N$. Then equations $(\Gamma_j - \lambda_j I)\,x = \sum_{\mathbf{k}\in\mathbf{Q}} a_j^{\mathbf{k}} z_0^{\mathbf{k}}$ hold for some $a_j^{\mathbf{k}} \in \mathbb{C},\ \mathbf{k} \in \mathbf{Q}$ and $j = 0, 1, \ldots, n-1$. We also write

$$\mathbf{a}^{\mathbf{k}} = \left[\begin{array}{cccc} a_0^{\mathbf{k}} & a_1^{\mathbf{k}} & \cdots & a_{n-1}^{\mathbf{k}} \end{array}\right]^T (\in \mathbb{C}^n).$$

By Assumption III we have $\mathcal{M} \subset \mathcal{K}_n$, and so relation (3.3) gives

$$-W_i\left(\boldsymbol{\lambda}\right)^{\dagger} x = \sum_{j=0}^{n} V_{ij}^{\dagger}\left(\Gamma_j - \lambda_j I\right)x = \sum_{j=0}^{n-1} V_{ij}^{\dagger}\sum_{\mathbf{k}\in\mathbf{Q}} a_j^{\mathbf{k}} z_0^{\mathbf{k}} = \sum_{\mathbf{k}\in\mathbf{Q}} U_i\left(\mathbf{a}^{\mathbf{k}}\right)^{\dagger} z_0^{\mathbf{k}}$$

for all i. For every $y \in H_1 \otimes_a \cdots \otimes_a H_{i-1} \otimes_a \mathcal{N}_i^* \otimes_a H_{i+1} \otimes_a \cdots \otimes_a H_n$ it follows that

$$\left(\sum_{\mathbf{k}\in\mathbf{Q}} x_{10}^{k_1} \otimes \cdots \otimes x_{i-1,0}^{k_{i-1}} \otimes U_i\left(\mathbf{a}^{\mathbf{k}}\right) x_{i0}^{k_i} \otimes x_{i+1,0}^{k_{i+1}} \otimes \cdots \otimes x_{n0}^{k_n}, y\right) = 0.$$

Since $x_{i0}^{k_i}$ are linearly independent it follows by Lemma 5.3 that

$$\sum_{k_i=1}^{n_i} U_i\left(\mathbf{a}^{\mathbf{k}'\cup_i k_i}\right) x_{i0}^{k_i} \in \left(\mathcal{N}_i^*\right)^{\perp}$$

for all $\mathbf{k}' \in \mathbf{Q}_i$ and every i. The ranges $\mathcal{R}_i = \mathcal{R}\left(W_i\left(\boldsymbol{\lambda}\right)\right)$ are closed (because the operators $W_i\left(\boldsymbol{\lambda}\right)$ are Fredholm, cf. [24, Theorem 5.10, p.217]), so $\mathcal{R}_i = \left(\mathcal{N}_i^*\right)^{\perp}$, and there exist vectors $x_{i1}^{\mathbf{k}'} \in \mathcal{D}\left(V_{i0}\right)$ such that relations (6.4) hold. Now we can construct a vector

$$z = \sum_{j=1}^{n}\sum_{\mathbf{k}'\in\mathbf{Q}_j} x_{10}^{k_1} \otimes \cdots \otimes x_{j-1,0}^{k_{j-1}} \otimes x_{j1}^{\mathbf{k}'} \otimes x_{j+1,0}^{k_{j+1}} \otimes \cdots \otimes x_{n0}^{k_n}. \tag{6.8}$$

The same calculation as for (6.7) shows that $\left(\Gamma_j - \lambda_j I\right)z = \sum_{\mathbf{k}\in\mathbf{Q}} a_j^{\mathbf{k}} z_0^{\mathbf{k}}$. Then we have $x - z \in \mathcal{N}$, and thus there exist complex numbers $\beta_{\mathbf{k}},\ \mathbf{k} \in \mathbf{Q}$, such that

$x = z + \sum_{\mathbf{k}\in\mathbf{Q}} \beta_{\mathbf{k}} z_0^{\mathbf{k}}$. If we substitute the vectors $x_{11}^{\mathbf{k}'} + \sum_{k_1=1}^{n_1} \beta_{\mathbf{k}'\cup k_1} x_{10}^{k_1}$ for the vectors $x_{11}^{\mathbf{k}'}$ in the expression (6.8), we obtain

$$x = \sum_{i=1}^{n} \sum_{\mathbf{k}'\in\mathbf{Q}_i} x_{10}^{k_1} \otimes \cdots \otimes x_{i-1,0}^{k_{i-1}} \otimes x_{i1}^{\mathbf{k}'} \otimes x_{i+1,0}^{k_{i+1}} \otimes \cdots \otimes x_{n0}^{k_n}$$

and, since (6.4) and (6.6) are unaffected by this substitution, the proof is complete. □

Because $W_i(\boldsymbol{\lambda})$ is Fredholm it follows that $n_i^* = \dim \mathcal{N}_i^*$ is finite. We choose vectors $y_{i0}^{k_i} \in H_i,\ k_i = 1, 2, \ldots, n_i^*$, so that they form a basis for $\mathcal{N}_i^*$. Then we write

$$V_{ij}^{\lambda} = \left[\left(V_{ij} x_{i0}^{k}, y_{i0}^{l}\right)\right]_{l=1,k=1}^{n_i^* \quad n_i} \in \mathbb{C}^{n_i^* \times n_i}$$

for $i = 1, 2, \ldots, n$ and $j = 0, 1, \ldots, n-1$. (Recall that the vectors x_{i0}^{k} form a basis for $\mathcal{N}_i$.) The matrix V_{ij}^{λ} induces a transformation $V_{ij}^{\lambda\dagger}$, which is defined on the tensor product space H_λ of (6.2). Finally, the array

$$\mathcal{D}_n^{\lambda} = \begin{bmatrix} V_{10}^{\lambda\dagger} & V_{11}^{\lambda\dagger} & \cdots & V_{1,n-1}^{\lambda\dagger} \\ V_{20}^{\lambda\dagger} & V_{21}^{\lambda\dagger} & \cdots & V_{2,n-1}^{\lambda\dagger} \\ \vdots & \vdots & & \vdots \\ V_{n0}^{\lambda\dagger} & V_{n1}^{\lambda\dagger} & \cdots & V_{n,n-1}^{\lambda\dagger} \end{bmatrix} \tag{6.9}$$

defines a transformation on the space H_λ^n. It plays an important role in the construction of a basis for the second root subspace completely in terms of the underlying multiparameter system (1.1) as described in the following theorem. This is our main result.

Theorem 6.3. *Suppose that $\mathbf{a}^l \in H_\lambda^n,\ l = 1, 2, \ldots, d$, are as in* (6.3) *and are determined by* (6.1)*. Then they form a basis for the kernel of $\mathcal{D}_n^{\lambda}$.*

Conversely, to any basis $\left\{\mathbf{a}^l,\ l = 1, 2, \ldots, d\right\} \subset H_\lambda^n$ for the kernel of $\mathcal{D}_n^{\lambda}$ we can associate a set of vectors $\mathcal{B}_1 = \left\{z_1^l,\ l = 1, 2, \ldots, d\right\} \subset H$ such that $\mathcal{B}_0 \cup \mathcal{B}_1$ is a basis for $\mathcal{M}$ and $(\Gamma_j - \lambda_j I)\, z_1^l = \sum_{\mathbf{k}\in\mathbf{Q}} a_j^{\mathbf{k}l} z_0^{\mathbf{k}}$ for all j and l.

Proof. Suppose that $\left\{z_1^l,\ l = 1, 2, \ldots, d\right\} \cup \left\{z_0^{\mathbf{k}},\ \mathbf{k} \in \mathbf{Q}\right\}$ is a basis for $\mathcal{M}$. Then we have $(\Gamma_j - \lambda_j I)\, z_1^l = \sum_{\mathbf{k}\in\mathbf{Q}} a_j^{\mathbf{k}l} z_0^{\mathbf{k}}$, and the relations (3.3) imply

$$\begin{aligned} -W_i(\boldsymbol{\lambda})^{\dagger} z_1^l &= \sum_{j=0}^{n-1} V_{ij}^{\dagger} (\Gamma_j - \lambda_j I)\, z_1^l = \sum_{j=0}^{n-1} V_{ij}^{\dagger} \sum_{\mathbf{k}\in\mathbf{Q}} a_j^{\mathbf{k}l} z_0^{\mathbf{k}} \\ &= \sum_{\mathbf{k}\in\mathbf{Q}} x_{10}^{k_1} \otimes \cdots \otimes x_{i-1,0}^{k_{i-1}} \otimes U_i\left(\mathbf{a}^{\mathbf{k}l}\right) x_{i0}^{k_i} \otimes x_{i+1,0}^{k_{i+1}} \otimes \cdots \otimes x_{n0}^{k_n}. \end{aligned}$$

Now it follows for every $y \in H_1 \otimes_a \cdots \otimes_a H_{k-1} \otimes_a \mathcal{N}_i^* \otimes_a H_{k+1} \otimes_a \cdots \otimes_a H_n$ that

$$\left(\sum_{\mathbf{k}\in\mathbf{Q}} x_{10}^{k_1} \otimes \cdots \otimes x_{i-1,0}^{k_{i-1}} \otimes U_i\left(\mathbf{a}^{\mathbf{k}l}\right) x_{i0}^{k_i} \otimes x_{i+1,0}^{k_{i+1}} \otimes \cdots \otimes x_{n0}^{k_n}, y\right) = 0.$$

Since $x_{i0}^{k_i}$ are linearly independent it follows by Lemma 5.3 that

$$\left(\sum_{k_i=1}^{n_i} \sum_{j=0}^{n-1} a_j^{\mathbf{k}'\cup_i k_i, l} V_{ij} x_{i0}^{k_i}, y_{i0}^{l_i}\right) = 0$$

for $\mathbf{k}' \in \mathbf{Q}_i$, $l = 1, 2, \ldots, d$ and $l_i = 1, 2, \ldots, n_i^*$. This is the entry-wise version of the equation

$$\sum_{j=0}^{n-1} V_{ij}^{\lambda\dagger} \mathbf{a}_j^l = 0.$$

Hence we have $\mathbf{a}^l \in \mathcal{N}\left(\mathcal{D}_n^\lambda\right)$ for all l. Proposition 6.1 implies that the $\mathbf{a}^l$ are linearly independent, and so it follows that

$$d \leq \dim \mathcal{N}\left(\mathcal{D}_n^\lambda\right). \tag{6.10}$$

Next we will show that to every $\mathbf{a} \in \mathcal{N}\left(\mathcal{D}_n^\lambda\right)$ we can associate a vector $z_1 \in \mathcal{M} \backslash \mathcal{N}$ such that

$$\left(\Gamma_j - \lambda_j I\right) z_1 = \sum_{\mathbf{k} \in \mathbf{Q}} a_j^{\mathbf{k}} z_0^{\mathbf{k}}. \tag{6.11}$$

Because $\mathbf{a}$ is in the kernel of $\mathcal{D}_n^\lambda$ it follows that $\sum_{j=1}^n V_{ij}^{\lambda\dagger} \mathbf{a}_j = 0$ for all i. This is equivalent to

$$0 = \left(\sum_{k_i=1}^{n_i} U_i \left(\mathbf{a}^{\mathbf{k}' \cup_i k_i} \right) x_{i0}^{k_i}, y_{i0}^{l_i} \right)$$

for $\mathbf{k}' = (k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_n) \in \mathbf{Q}_i$, $i = 1, 2, \ldots, n$ and $l_i = 1, 2, \ldots, n_i^*$. From the above equations it follows that $\sum_{k_i=1}^{n_i} U_i \left(\mathbf{a}^{\mathbf{k}' \cup_i k_i} \right) x_{i0}^{k_i} \in \left(\mathcal{N}_i^*\right)^\perp$. Because the ranges $\mathcal{R}_i$ are closed there exist vectors $x_{i1}^{\mathbf{k}'} \in H_i'$ such that

$$\sum_{k_i=1}^{n_i} U_i \left(\mathbf{a}^{\mathbf{k}' \cup_i k_i} \right) x_{i0}^{k_i} + W_i\left(\boldsymbol{\lambda}\right) x_{i1}^{\mathbf{k}'} = 0.$$

As in the proof of Theorem 6.2 it follows that the vector

$$z_1 = \sum_{j=1}^n \sum_{\mathbf{k}' \in \mathbf{Q}_j} x_{10}^{k_1} \otimes \cdots \otimes x_{j-1,0}^{k_{j-1}} \otimes x_{j1}^{\mathbf{k}'} \otimes x_{j+1,0}^{k_{j+1}} \otimes \cdots \otimes x_{n0}^{k_n} \tag{6.12}$$

is such that relations (6.11) hold. Then, if $\left\{\mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^{d'}\right\}$ is a basis for $\mathcal{N}\left(\mathcal{D}_n^\lambda\right)$, we can associate with every $\mathbf{a}^l$ a vector $z_1 = z_1^l$ as above. The vectors z_1^l, $l = 1, 2, \ldots, d'$, are linearly independent because

$$\left(\Gamma_j - \lambda_j I\right) z_1^l = \sum_{\mathbf{k} \in \mathbf{Q}} a_j^{\mathbf{k}l} z_0^{\mathbf{k}}$$

and $\mathbf{a}^l$ are linearly independent. Thus it follows $d \geq \dim \mathcal{N}\left(\mathcal{D}_n^\lambda\right)$ and together with (6.10) we obtain $d = \dim \mathcal{N}\left(\mathcal{D}_n^\lambda\right)$. The proof is complete. □

Corollary 6.4. *The dimension of the second root subspace $\mathcal{M}$ is*

$$\prod_{i=1}^n n_i + \dim \mathcal{N}\left(\mathcal{D}_n^\lambda\right).$$

7. A finite-dimensional example

First we illustrate Theorem 6.3 with a non-self-adjoint finite-dimensional example, for which $d = \dim \mathcal{N}\left(\mathcal{D}_n^\lambda\right) > \dim \mathcal{N}$. This is not possible in the cases treated in Sections 8 and 9.

Example 7.1. Consider the two-parameter system

$$W_1(\boldsymbol{\lambda}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \lambda_0 + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 1 \end{bmatrix} \lambda_1 + \begin{bmatrix} 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$W_2(\boldsymbol{\lambda}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \lambda_0 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \lambda_1 + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & -2 \end{bmatrix}.$$

The transformation Δ_2 is invertible and the matrices V_{12} and V_{22} are singular. Thus Assumptions I–III hold here and $\boldsymbol{\lambda}_0 = (0,0,1)$ is an eigenvalue. We have $\dim \mathcal{N}(V_{12}) = 1$ and $\dim \mathcal{N}(V_{22}) = 2$, so $\mathbf{Q} = \{(1,1),(1,2)\}$. We choose

$$x_{10} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T, \qquad y_{10} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T,$$

and

$$x_{20}^1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T, \qquad x_{20}^2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T,$$
$$y_{20}^1 = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix}^T, \qquad y_{20}^2 = \begin{bmatrix} 2 & 1 & 0 \end{bmatrix}^T.$$

Then it follows that the vectors

$$z_0^1 = x_{10} \otimes x_{20}^1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

and

$$z_0^2 = x_{10} \otimes x_{20}^2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

form a basis for $\mathcal{N}$, and we have

$$V_{10}^{\lambda_0} = V_{11}^{\lambda_0} = [0], \qquad V_{20}^{\lambda_0} = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}, \qquad V_{21}^{\lambda_0} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

The space $H_{\lambda_0} = \mathbb{C} \otimes \mathbb{C}^2 \equiv \mathbb{C}^2$, and so $H_{\lambda_0}^2 = \mathbb{C}^4$. Then

$$\mathcal{D}_2^{\lambda_0} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}.$$

Because the matrix $\mathcal{D}_2^{\lambda_0}$ has rank 1 it follows that $d = 3$, and we choose

$$\mathbf{a}^1 = \begin{bmatrix} a_0^{11} & a_0^{12} & a_1^{11} & a_1^{12} \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 & -2 \end{bmatrix}^T,$$
$$\mathbf{a}^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^T \quad \text{and} \quad \mathbf{a}^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T$$

to form a basis for $\mathcal{N}\left(\mathcal{D}_2^\lambda\right)$. Using (6.4) (indexed also by $l = 1$), we construct a vector z_1^1 corresponding to $\mathbf{a}^1$ by finding vectors $x_{11}^{11}\left(= x_{11}^{\mathbf{k}'l}\right), x_{11}^{21}$ and x_{21}^{11} such that

$$V_{10}x_{10} + V_{12}x_{11}^{11} = 0, \qquad -2V_{11}x_{10} + V_{12}x_{11}^{21} = 0$$

and

$$V_{20}x_{20}^1 - 2V_{21}x_{20}^2 + V_{22}x_{21}^{11} = 0.$$

A possible choice is

$$x_{11}^{11} = \begin{bmatrix} 0 & -1 & 1 \end{bmatrix}^T, \qquad x_{11}^{21} = \begin{bmatrix} 0 & 2 & -2 \end{bmatrix}^T, \qquad x_{21}^{11} = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}^T,$$

so (6.12) (indexed also by $l = 1$) gives

$$z_1^1 = x_{11}^{11} \otimes x_{20}^1 + x_{11}^{21} \otimes x_{20}^2 + x_{10} \otimes x_{21}^{11} = \begin{bmatrix} 0 & 0 & -1 & -1 & 2 & 0 & 1 & -2 & 0 \end{bmatrix}^T.$$

Similarly we find vectors

$$x_{11}^{12} = \begin{bmatrix} 0 & -1 & 1 \end{bmatrix}^T, \qquad x_{11}^{22} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T, \qquad x_{21}^{12} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T,$$

with $z_1^2 = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T$ that correspond to $\mathbf{a}^2$, and vectors

$$x_{11}^{13} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T, \qquad x_{11}^{23} = \begin{bmatrix} 0 & -1 & 1 \end{bmatrix}^T, \qquad x_{21}^{13} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$$

with $z_1^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \end{bmatrix}^T$ that correspond to $\mathbf{a}^3$. Then $\{z_0^1, z_0^2; z_1^1, z_1^2, z_1^3\}$ is a basis for $\mathcal{M}$. □

Note that Assumptions I–III hold in finite dimensions if there exists a linear combination of $\Delta_i,\ i = 0, 1, \ldots, n$, which is invertible. Then we can ensure with an appropriate shift of parameters λ_i that the transformation Δ_n is invertible. If $n = 2$ we can construct a basis $\mathcal{B}$ in a canonical way. This is discussed in [20]. (See also [21].)

8. Self-adjoint cases

In this section we assume that the operators V_{ij} are self-adjoint and that our Assumptions I–III hold with α in Assumption I a real number. In addition suppose that at least one of the cofactors $\Delta_{0in} > 0$; for convenience we assume that

$$\Delta_{0nn} > 0. \tag{8.1}$$

We remark that stronger positivity assumptions on the cofactors of Δ_0 are called ellipticity conditions (e.g. [11], [23, p.62]). We return to these later in this section.

Under the above assumptions the structure of root vectors corresponding to real eigenvalues becomes simpler than in Theorem 6.3. Because $W_i(\boldsymbol{\lambda})$ are self-adjoint we take $V_{ij}^{\lambda} = \left[\left(V_{ij} x_{i0}^{l_i}, x_{i0}^{k_i}\right)\right]_{l_i,k_i=1}^{n_i}$, where $\left\{x_{i0}^{k_i},\ k_i = 1, 2, \ldots, n_i\right\}$ is a basis for the kernel of $W_i(\boldsymbol{\lambda})$, $i = 1, 2, \ldots, n$. Then $\mathcal{D}_n^{\lambda}$ is defined as in (6.9) and we write $d = \dim\ker \mathcal{D}_n^{\lambda}$.

Theorem 8.1. *Suppose that V_{ij} are self-adjoint, $\Delta_{0nn} > 0$, and that $\boldsymbol{\lambda}$ is an eigenvalue in $\mathbb{R}^{n+1}$. Then there exist nonzero real n-tuples $\boldsymbol{\mu}^l = \left(\mu_0^l, \mu_1^l, \ldots, \mu_{n-1}^l\right)$, $l = 1, 2, \ldots, d$, with $d \le \dim \mathcal{N}$, and vectors $u_{i1}^l \in \mathcal{R}(W_i(\boldsymbol{\lambda}))$ and $u_{i0}^l \in \mathcal{N}(W_i(\boldsymbol{\lambda}))$ such that*

$$U_i\left(\boldsymbol{\mu}^l\right) u_{i0}^l + W_i(\boldsymbol{\lambda}) u_{i1}^l = 0. \tag{8.2}$$

Furthermore, the vectors

$$v_1^l = \sum_{i=1}^{n} u_{10}^l \otimes \cdots \otimes u_{i-1,0}^l \otimes u_{i1}^l \otimes u_{i+1,0}^l \otimes \cdots \otimes u_{n0}^l, \quad l = 1, 2, \ldots, d,$$

together with a basis for the eigenspace $\mathcal{N}$ at $\boldsymbol{\lambda}$, form a basis for the second root subspace $\mathcal{M}$ at $\boldsymbol{\lambda}$, and we have

$$(\Gamma_i - \lambda_i I) v_1^l = \mu_i^l v_0^l,$$

where $v_0^l = u_{10}^l \otimes u_{20}^l \otimes \cdots \otimes u_{n0}^l$.

Proof. We consider a finite-dimensional multiparameter system

$$
\begin{aligned}
W_i^\lambda(\boldsymbol{\mu}) &= \sum_{j=0}^{n-1} \mu_j V_{ij}^\lambda, \qquad i = 1, 2, \dots, n-1, \\
W_n^\lambda(\boldsymbol{\mu}) &= \sum_{j=0}^{n-1} \mu_j V_{nj}^\lambda + \mu_n I.
\end{aligned} \tag{8.3}
$$

We denote by $\mathcal{D}_j^\lambda$ the array

$$
\begin{bmatrix}
V_{10}^{\lambda\dagger} & V_{11}^{\lambda\dagger} & \cdots & V_{1,n-1}^{\lambda\dagger} & 0 \\
V_{20}^{\lambda\dagger} & V_{21}^{\lambda\dagger} & \cdots & V_{2,n-1}^{\lambda\dagger} & 0 \\
\vdots & \vdots & & \vdots & \vdots \\
V_{n0}^{\lambda\dagger} & V_{n1}^{\lambda\dagger} & \cdots & V_{n,n-1}^{\lambda\dagger} & I
\end{bmatrix}
$$

with j-th column omitted, and we write $\Delta_j^\lambda = (-1)^n \det \mathcal{D}_j^\lambda$. Because $\Delta_{0nn} > 0$ it follows that also $(-1)^n \Delta_0^\lambda > 0$. The transformations

$$
\Gamma_i^\lambda = \left(\Delta_0^\lambda\right)^{-1} \Delta_i^\lambda, \qquad i = 0, 1, \dots, n,
$$

act on the tensor product space H_λ (see (6.2)). Suppose that $\mathcal{A}_0^\lambda$ is the adjugate of $\mathcal{D}_0^\lambda$. Since Δ_0^λ is invertible, $\mathcal{A}_0^\lambda$ is also invertible. As for (3.6) with the last column omitted, we have that

$$
\mathcal{A}_0^\lambda \mathcal{D}_n^\lambda = \begin{bmatrix}
-\Delta_1^\lambda & \Delta_0^\lambda & 0 & \cdots & 0 \\
-\Delta_2^\lambda & 0 & \Delta_0^\lambda & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
-\Delta_{n-1}^\lambda & 0 & 0 & \ddots & \Delta_0^\lambda \\
-\Delta_n^\lambda & 0 & 0 & \cdots & 0
\end{bmatrix}.
$$

Then it follows that $(\mathbf{a}_i)_{i=1}^n \in \mathcal{N}\left(\mathcal{D}_n^\lambda\right)$ if and only if

$$
\Gamma_n^\lambda \mathbf{a}_1 = 0 \quad \text{and} \quad \mathbf{a}_{i+1} = \Gamma_i^\lambda \mathbf{a}_1, \tag{8.4}
$$

$i = 1, 2, \dots, n-1$. Note that then $d = \dim \ker \Gamma_n^\lambda \le \dim H_\lambda = \dim \mathcal{N}$ by the definition (6.2) of H_λ. Because $(-1)^n \Delta_0^\lambda > 0$ [5, Theorems 6.9.1 and 7.9.1] imply that all the eigenvalues of (8.3) are real and that there is a basis of decomposable eigenvectors in the joint eigenspaces of the Γ_i^λ. Since the spectrum of $\left\{\Gamma_i^\lambda\right\}_{i=0}^n$ coincides with the spectrum of (8.3) (see [5, Theorem 6.9.1]) it follows by (8.4) that there are precisely d nonzero eigenvalues $\widetilde{\boldsymbol{\mu}} = \widetilde{\boldsymbol{\mu}}^l$ $\left(\in \mathbb{R}^{n+1}\right)$ of (8.3) with $\mu_0 = 1$ and $\mu_n = 0$, repeated according to multiplicity. These are the eigenvalues corresponding to the elements in the kernel of Γ_n^λ. Note also that $\Gamma_0^\lambda = I$. Let $\mathbf{b}^l = \mathbf{b}_1^l \otimes \mathbf{b}_2^l \otimes \cdots \otimes \mathbf{b}_n^l$ be a decomposable eigenvector corresponding to $\widetilde{\boldsymbol{\mu}}^l$. Since $\Delta_n^\lambda \mathbf{b}^l = 0$, it follows from (8.4) that $\left(\Gamma_{i-1}^\lambda \mathbf{b}^l\right)_{i=1}^n$, $l = 1, 2, \dots, d$, form a basis for $\mathcal{N}\left(\mathcal{D}_n^\lambda\right)$. Next we write $\mathbf{b}_i^l = \begin{bmatrix} b_{i1}^l & b_{i2}^l & \cdots & b_{in_i}^l \end{bmatrix}^T$,

$$
u_{i0}^l = \sum_{k=1}^{n_i} b_{ik}^l x_{i0}^k,
$$

and $\boldsymbol{\mu}^l$ for $\widetilde{\boldsymbol{\mu}}^l$ with μ_n omitted. Since $W_i^\lambda\left(\widetilde{\boldsymbol{\mu}}^l\right)\mathbf{b}_i^l = 0$, it follows that the vector $U_i\left(\boldsymbol{\mu}\right)u_{i0}^l$ is orthogonal to the kernel of $W_i\left(\boldsymbol{\lambda}\right)$, and thus there exists a vector u_{i1}^l such that (8.2) holds. A calculation similar to that for (6.7) shows that $(-1)^n\left(\lambda_i\Delta_n - \Delta_i\right)v_1^l = (-1)^{n+1}\Delta_n v_0^l$, and we prove that v_1^l, $l = 1, 2, \ldots, d$, together with a basis for the eigenspace at $\boldsymbol{\lambda}$, form a basis for the second root subspace as in Theorem 6.3. □

Note that the structure of second root vectors described in the above theorem is the same as the structure of these vectors in [9]. One consequence of the above theorem is for the semi-definite, uniformly elliptic case. Let us first state the definitions and recall some of the results from [11].

A multiparameter system (1.1) is called *uniformly elliptic* if V_{ij}, $i = 1, 2, \ldots, n$, $j = 0, 1, \ldots, n$, are self-adjoint and $\Delta_{0ij} \gg 0$ (i.e. $\Delta_{0ij} > 0$ with Δ_{0ij}^{-1} bounded) for $i, j = 1, 2, \ldots, n$. A uniformly elliptic multiparameter system is called *semi-definite* if $V_{i0} \geq 0$ for $i = 1, 2, \ldots, n$. Because this assumption is made about the original operators V_{i0}, in the rest of this section we do *not* assume that parameters are shifted : we assume that $\Delta_n + \alpha\Delta_0$ has bounded inverse, and we define $\Gamma_i = \left(\Delta_n + \alpha\Delta_0\right)^{-1}\Delta_i$ for $i = 1, 2, \ldots, n$.

Now we recall some of the properties of the uniformly elliptic, semi-definite systems (1.1) from [11, §5] :

(i) the eigenvalues of Γ_i are all real and they are semi-simple except possibly 0 for Γ_i, $i = 1, 2, \ldots n$,

(ii) $\mathcal{N}\left(\Delta_i\right) = \mathcal{N}\left(V_{10}\right) \otimes \mathcal{N}\left(V_{20}\right) \otimes \cdots \otimes \mathcal{N}\left(V_{n0}\right)$ for $i = 1, 2, \ldots, n$, and

(iii) $\bigcap_{j=1}^{n} \mathcal{N}\left(\Gamma_j^l\right) = \mathcal{N}\left(\Gamma_i^2\right)$ for all $l \geq 2$ and any $i = 1, 2, \ldots, n$. Also the second root subspace at eigenvalue $\mathbf{0}$ is equal to $\mathcal{N}\left(\Gamma_i^2\right)$.

Assertion (i) follows by [11, Lemma 5.2] and assertion (ii) was proved in [11, Lemma 5.3]. By [11, Lemma 5.2(iii)] we have that $\mathcal{N}\left(\Gamma_i^l\right) = \mathcal{N}\left(\Gamma_i^2\right)$ for $l \geq 2$ and $i = 1, 2, \ldots, n$. Then $\Gamma_i\Gamma_j x = 0$ implies that $\Gamma_j x \in \mathcal{N}\left(\Gamma_i\right)$, which is equal to $\mathcal{N}\left(\Gamma_k\right)$ by (ii) for every i, j, k, and similarly $\Gamma_j\Gamma_k x = 0$ implies that $\Gamma_k x \in \mathcal{N}\left(\Gamma_j\right) = \mathcal{N}\left(\Gamma_k\right)$. Thus we have that $0 = \Gamma_i\Gamma_j x = \Gamma_k\Gamma_j x = \Gamma_j\Gamma_k x = \Gamma_k^2 x$, and assertion (iii) follows. □

From the above properties it follows that our Theorem 8.1 also describes a basis for the entire root subspace $\mathcal{S}$ at $\mathbf{0}$ (which is the only non semi-simple eigenvalue) for the uniformly elliptic, semi-definite case completely in terms of the underlying system (1.1). We remark that a calculation of $\dim\mathcal{S}$ can be found in [11].

After a possible rotation of the $\boldsymbol{\lambda}$-axes, condition (8.1) includes most of the problems studied in the literature, e.g., those of left and right definite [26] and simply separated [22] types. Also separation of variables in the Helmholtz equation in $\mathbb{R}^n$ $(n \leq 3)$ automatically leads to systems satisfying (8.1), cf. [3].

9. A Sturm-Liouville example

We illustrate our results on a simple example, chosen so that several other methods also apply to it. The example is detailed in 9.1, and the root subspace is obtained in 9.2 and 9.3 by various methods which are compared in 9.4. As in the last part of §8, we do not assume that parameters are shifted.

9.1. **The problem.** We consider a two-parameter system

$$W_i(\boldsymbol{\lambda}) = \lambda_0 T - (-1)^i \lambda_1 I - \lambda_2 R_i, \tag{9.1}$$

$i = 1, 2$, where T is the Sturm-Liouville operator in $H_i = L_2(\mathcal{I})$, $\mathcal{I} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, given by $Ty = -y'' - y$ on

$$\mathcal{D}(T) = \left\{ y \in AC(\mathcal{I}) : \ y' \in AC(\mathcal{I}), \ y'' \in L_2(\mathcal{I}), \ y\left(\pm\frac{\pi}{2}\right) = 0 \right\},$$

and R_i is the multiplication on H_i given by $R_i y(t_i) = r_i(t_i) y_i(t_i)$, where $r_i \in L_\infty(\mathcal{I})$. Then Δ_0 is the operator of multiplication by $\delta_0(t_1, t_2) = -r_1(t_1) - r_2(t_2)$, and we assume (cf. the remark after Proposition 2.1) that there is an open set in $\mathcal{I}^2$ on which this does not vanish (a.e.). If the r_i are regular enough (e.g., piecewise continuous) this is the same as requiring $\Delta_0 \neq 0$, i.e., that r_1 and r_2 are not constant functions adding to zero.

We take $\boldsymbol{\lambda} = (1, 0, 0)$, so $W_i(\boldsymbol{\lambda}) = T \geq 0$ and $y_i(t_i) = \cos t_i$ spans $\mathcal{N}(W_i(\boldsymbol{\lambda}))$. Explicit calculations will be carried out on the case

$$r_1(t_1) = t_1 \text{ and } r_2(t_2) = 0.$$

It will be convenient to have a basis for the root subspace $\mathcal{R}$ of the one-parameter problem (9.1) with $i = 1$, $\lambda_0 = 1$, $\lambda_2 = 0$ and λ_1 suppressed. By [10, Corollary 6.2] and the fact that $\int_{\mathcal{I}} t_1 \cos^2 t_1 = 0$, $\dim \mathcal{R} = 2$. To find a (second) root vector z, we solve

$$Tz = R_1 y_1, \tag{9.2}$$

i.e., the boundary value problem

$$(-z'' - z)(t_1) = t_1 y_1(t_1), \qquad z\left(\pm\frac{\pi}{2}\right) = 0.$$

It is easily seen that $z = \frac{1}{16}\left[\pi^2 \sin t_1 - 4t_1 (t_1 \sin t_1 + \cos t_1)\right]$ is a solution, so $\{y_1, z\}$ is a basis for $\mathcal{R}$.

9.2. **Other methods.** Although their completeness result is different from ours, we shall describe briefly the method of [22, §1.2], since it uses some of the same constructions. Meixner, Schäfke and Wolf set up continuous operators $A = T - \lambda_2 R_1$, $B = T - \lambda_2 R_2$ and $C = \Delta_2 - \lambda_2 \Delta_0$, explaining how to generate a basis for the root subspaces at 0 of $C = A \otimes I + I \otimes B$ from those of A and B. This root subspace differs from $\mathcal{N}(C)$ only for certain "exceptional values" [22] of λ_2, which are nonreal since C is self-adjoint. We remark that A, B and C are treated as operators, so a second root vector z for A, say, satisfies $Az = y \in \mathcal{N}(A)$, as opposed to (9.2).

Faierman [13] also works with A, B and C, but they are now unbounded on appropriate L_2 spaces. He constructs self-adjoint boundary conditions for the uniformly elliptic partial differential expression C (which again turns out to be $A \otimes I + I \otimes B$, but see 9.3). He regards A, B and C as pencils, so a second root vector for A, say, satisfies (9.2). (Actually he uses the boundary condition $z\left(\frac{\pi}{2}\right) = z'\left(-\frac{\pi}{2}\right) = 0$ in [13, equ. (3.3)] but deduces (9.2).) As in [22] he aims for an expansion theorem [13, Theorem 5.7], but one can recover the root subspace $\mathcal{S}$ at $\boldsymbol{\lambda}$ from [13, Lemma 3.1], and it turns out that

$$\mathcal{S} = \mathcal{M} = \mathrm{Sp}\{y_1 \otimes y_2, \ z \otimes y_2\} \tag{9.3}$$

where z comes from (9.2).

Binding [9] uses an abstract formulation and assumes uniform ellipticity, which can be achieved by a rotation of $\boldsymbol{\lambda}$ axes. For the semi-definite case, [9, eq. (4.1)] coincides with (9.2) and hence gives the same basis (9.3).

9.3. **Our method.** We need to verify Assumptions I–III. The operator $W_i(\boldsymbol{\lambda})$ is self-adjoint with compact resolvent on H_i, and hence Assumption II is satisfied (with Fredholm index 0).

As noted above, Δ_2 is self-adjoint. One can demonstrate this by characterizing $\mathcal{D}$ as the W_2^2 functions with Dirichlet conditions on $\mathcal{I}^2$, as in [13], or via the identity $\mathcal{D} = \mathcal{D}(T \otimes I + I \otimes T)$. (This can be proved by using orthonormal bases of H_i consisting of eigenvectors for T, cf. [7] for the left definite case, and cf. the example in [25]. Incidentally [25] also shows that some caution is needed if ellipticity is not assumed.) Then Assumption I follows from Proposition 2.1 and the ensuing remark. Thus there is a real number α such that $\Delta_2 + \alpha\Delta_0$ has a bounded inverse.

To prove Assumption III, it suffices to establish solubility condition (b) (see §5), i.e., to show that

$$\begin{bmatrix} -\frac{\partial^2}{\partial t_1^2} - 1 - \alpha r_1 & -1 \\ -\frac{\partial^2}{\partial t_2^2} - 1 - \alpha r_2 & 1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} r_1 f \\ r_2 f \end{bmatrix} \tag{9.4}$$

(with appropriate boundary conditions) is solvable for y and z in $\mathcal{D}$. Eliminating z from (9.4), we have a Dirichlet problem $-\Delta y - (2 + \alpha(r_1 + r_2))\, y = (r_1 + r_2)\, f$. By [13, Theorem 2.2] there exists a solution $y \in \mathcal{D}$, and so also $z = r_2 f + \frac{\partial^2 y}{\partial t_2^2} + (1 + \alpha r_2)\, y$ exists. Thus (9.4) is satisfied; the boundary conditions for z then follow, and hence so does Assumption III.

Because $r_i \in L_\infty(\mathcal{I})$ we can assume by a rotation of $\boldsymbol{\lambda}$ axes that (9.1) is uniformly elliptic. Since $T \geq 0$ we may apply Theorem 8.1 with $\boldsymbol{\lambda} = (0, 0, 1)$ and $y_i = \cos t_i$. Thus

$$\mathcal{D}_2^\lambda = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix},$$

so there is a root vector generated by

$$\mathbf{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then (6.4) coincides with (9.2), and this leads to the same basis (9.3).

9.4. **Comparison.** First we compare the assumptions of the four approaches above. Continuity of A, B and C forces the use of a smooth function space (analytic in [22]) and hence an implicit assumption that the r_i are smooth. Faierman [13] explicitly assumes the r_i to be Lipschitz. While Binding [9] allows $r_i \in L_\infty$, as here, he assumes Δ_0 to be one-to-one, which means $r_1 + r_2 \neq 0$ a.e. Self-adjointness is crucial in [9] and is used heavily in [13], whereas in [22] and here, it makes verification of the assumptions easier, but is not essential to the methods.

We now compare the methods used. [22] uses a linear topological space setting, while [13] employs indefinite inner product space theory. [13] and [22] both approach completeness via the tensor product (partial differential) expression C. In particular, the geometric eigenspace is calculated via C in [13, 22]. Faierman

[13, Theorem 2.6] relates this to the V_{ij}, but certain tensor product constructions remain in his analysis, e.g., in the assumptions of [13, Lemma 3.1 and Theorem 5.7]. [9] and our §8 both use the full set of commuting Γ_i to reduce the size of the root subspace, and all concepts needed are derived from the V_{ij}. In [9] analytic perturbation theory is used, and the dimensions and bases are determined by certain derivatives and certain algebraic relations are deduced . In the approach here, algebraic relations are central, and since we do not need differentiability, we can treat more general problems in a more direct fashion.

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