

A PROBLEM OF FÖLDES AND PURI ON THE WIENER PROCESS

Z. SHI

ABSTRACT. Let W be a real-valued Wiener process starting from 0, and $\tau(t)$ be the right-continuous inverse process of its local time at 0. Földes and Puri [3] raise the problem of studying the almost sure asymptotic behavior of $X(t) = \int_0^{\tau(t)} \mathbf{1}_{\{|W(u)| \leq \alpha t\}} du$ as t tends to infinity, i.e. they ask: how long does W stay in a tube before “crossing very much” a given level? In this note, both limsup and liminf laws of the iterated logarithm are provided for X .

1. INTRODUCTION

Let W be a real-valued Wiener process with $W(0) = 0$. Define its local time process

$$\ell(t, x) = \frac{d}{dx} \int_0^t \mathbf{1}_{\{W(u) < x\}} du,$$

which is almost surely continuous (up to a modification, which is taken for granted) in (t, x) according to Trotter’s theorem ([7]). For a modern account of Wiener local times, see for example Revuz and Yor [6, Chap. VI] from stochastic analysis viewpoint, and Révész [5, Part I] through sample paths. We denote $\ell(t, 0)$ by $\ell(t)$ for notational convenience, and write

$$\tau(s) = \inf\{t > 0 : \ell(t) > s\}, \quad s > 0,$$

the right-continuous inverse process of $\ell(t)$. In their recent paper on Wiener processes, Földes and Puri [3] raise the following problem: let

$$(1.1) \quad X(t) = \int_0^{\tau(t)} \mathbf{1}_{\{|W(u)| \leq \alpha t\}} du, \quad t > 0,$$

with a fixed constant $\alpha > 0$, what can be said about the almost sure asymptotics of X ?

Our answer to the “limsup” part of the question is a law of the iterated logarithm (LIL).

Theorem 1. *For any $\alpha > 0$,*

$$(1.2) \quad \limsup_{t \rightarrow \infty} \frac{X(t)}{t^2 \log \log t} = \frac{8\alpha^2}{\pi^2} \quad a.s.$$

For the “liminf” behavior of X , we have an integral test characterizing its lower functions.

Received by the editors December 7, 1994.

1991 *Mathematics Subject Classification.* Primary 60J65; Secondary 60G17.

Key words and phrases. Wiener process (Brownian motion), law of the iterated logarithm.

©1996 American Mathematical Society

Theorem 2. *Let $\alpha > 0$, and let $f > 0$ be a non-decreasing function. Then*

$$(1.3) \quad \mathbb{P}\left[X(t) < \frac{t^2}{f(t)}, \text{ i.o.}\right] = \begin{cases} 0 \\ 1 \end{cases} \iff \int^\infty \frac{dt}{t} f^{1/2}(t) e^{-f(t)/2} \begin{cases} < \infty \\ = \infty \end{cases}.$$

In particular, the following Chung-type LIL holds:

$$(1.4) \quad \liminf_{t \rightarrow \infty} \frac{\log \log t}{t^2} X(t) = \frac{1}{2} \quad \text{a.s.}$$

Remarks. (i) In (1.3), the symbol “i.o.” means “infinitely often” as t tends to infinity.

(ii) It is somewhat surprising that the lower functions of X do not depend on the value of α . Formally (i.e. informally!) taking $\alpha = \infty$ (thus $X(t) = \tau(t)$) in (1.3) and several lines of elementary calculation yield

$$\mathbb{P}\left[\ell(t) > (t f(t))^{1/2}, \text{ i.o.}\right] = \begin{cases} 0 \\ 1 \end{cases} \iff \int^\infty \frac{dt}{t} f^{1/2}(t) e^{-f(t)/2} \begin{cases} < \infty \\ = \infty \end{cases},$$

for any non-decreasing function $f > 0$. We therefore recover the Erdős-Feller-Kolmogorov-Petrov test for the upper functions of W , since the classical Lévy’s identity asserts that $\{\ell(t); t \geq 0\}$ and $\{\sup_{0 \leq s \leq t} W(s); t \geq 0\}$ have the same law.

Theorem 1 is proved in Section 2, and Theorem 2 in Section 3.

2. PROOF OF THEOREM 1

Let $X(t)$ be defined as in (1.1). It follows from Brownian scaling that

$$(2.1) \quad X(t) \stackrel{(d)}{=} t^2 X(1),$$

for any $t > 0$ (the symbol “ $\stackrel{(d)}{=}$ ” denoting identity in distribution). We are first interested in the Laplace transform of $X(1)$.

Lemma 1. *For any $\theta > 0$,*

$$(2.2) \quad \mathbb{E} \exp\left(-\frac{\theta^2}{2} X(1)\right) = \exp\left(-\theta \tanh(\alpha\theta)\right).$$

Remark. This has previously been obtained by Földes and Puri [3, p. 536] by solving the corresponding Sturm-Liouville equation. Our proof, based on the Ray-Knight theorem, allows us to make use of a Lévy-type formula, thus avoiding further computation.

Proof of Lemma 1. By occupation times formula for local times (see for example Revuz and Yor [6, Corollary VI.1.6]), $X(1) = \int_{-\alpha}^\alpha \ell(\tau(1), x) dx$. The Brownian excursion theory confirms the independence between the processes

$$\{\ell(\tau(1), x), x \geq 0\} \text{ and } \{\ell(\tau(1), -x), x \geq 0\},$$

which, according to the second Ray-Knight theorem (Revuz and Yor [6, Theorem XI.2.3]), behave both like a square Bessel process of dimension 0 (i.e. a linear diffusion process with generator $2x \frac{d^2}{dx^2}$, absorbed at 0), starting from 1. Thus by scaling,

$$\begin{aligned} \mathbb{E} \exp\left(-\frac{\theta^2}{2} X(1)\right) &= \left[\mathbb{E} \exp\left(-\frac{\theta^2}{2} \int_0^\alpha R^2(x) dx\right) \right]^2 \\ &= \left[\mathbb{E} \exp\left(-\frac{(\theta\alpha)^2}{2} \int_0^1 \hat{R}^2(s) ds\right) \right]^2, \end{aligned}$$

where R and \hat{R} are Bessel processes of dimension 0, with $R(0) = 1$ and $\hat{R}(0) = 1/\sqrt{\alpha}$. Now Lemma 1 is a particular case (taken for dimension 0) of a formula for square Bessel processes presented in Revuz and Yor [6, Corollary XI.1.8]. \square

Remark. For connections between quadratic functionals of Bessel processes and the Sturm-Liouville equation, see for example Pitman and Yor [4].

Although it does not seem trivial to deduce the exact upper tail of $X(1)$ from its Laplace transform (2.2), one can manage to obtain the following useful estimate.

Lemma 2. *For any $\alpha > 0$, we have*

$$(2.3) \quad \lim_{\lambda \rightarrow \infty} \frac{1}{\lambda} \log \mathbb{P}\left(X(1) > \lambda\right) = -\frac{\pi^2}{8\alpha^2}.$$

Proof of Lemma 2. We begin with the upper bound. Fix $0 < \varepsilon < 1$. From (2.2) it follows using analytic continuation that

$$\mathbb{E} \exp\left(\frac{\theta^2}{2} X(1)\right) = \exp\left(\theta \operatorname{tg}(\alpha\theta)\right),$$

for $0 < \theta < \pi/2\alpha$. Consequently, for $\lambda > 0$, we have

$$\begin{aligned} \mathbb{P}\left(X(1) > \lambda\right) &\leq \exp\left(-\frac{\pi^2(1-\varepsilon)^2}{8\alpha^2} \lambda\right) \mathbb{E} \exp\left(\frac{\pi^2(1-\varepsilon)^2}{8\alpha^2} X(1)\right) \\ &= \exp\left(-\frac{\pi^2(1-\varepsilon)^2}{8\alpha^2} \lambda + \frac{\pi(1-\varepsilon)}{2\alpha} \operatorname{tg}\left(\frac{\pi(1-\varepsilon)}{2}\right)\right), \end{aligned}$$

which yields

$$\limsup_{\lambda \rightarrow \infty} \frac{1}{\lambda} \log \mathbb{P}\left(X(1) > \lambda\right) \leq -\frac{\pi^2(1-\varepsilon)^2}{8\alpha^2}.$$

The upper bound is proved by sending ε to 0^+ . For the lower bound, let

$$\begin{aligned} H_\alpha &= \inf\{t > 0 : |W(t)| = \alpha\}; \\ \sigma_0 &= \inf\{t > H_\alpha : W(t) = 0\}. \end{aligned}$$

Observe that when $t \in (H_\alpha, \sigma_0)$, the Wiener process W stays away from 0, and thus $\ell(t)$ does not increase. Consequently, $\sigma_0 \leq \tau(1)$ in case $H_\alpha < \tau(1)$. Define

$\hat{W}(t) = W(t+H_\alpha) - W(H_\alpha)$ which is again a standard Wiener process, independent of $\{W(u); 0 \leq u \leq H_\alpha\}$. For any x , write $\hat{T}_x = \inf\{t > 0 : \hat{W}(t) = x\}$. Obviously,

$$\begin{aligned} X(1) &\geq \mathbf{1}_{\{H_\alpha < \tau(1)\}} \int_{H_\alpha}^{\sigma_0} \mathbf{1}_{\{|W(u)| \leq \alpha\}} du \\ &= \mathbf{1}_{\{H_\alpha < \tau(1), W(H_\alpha) = -\alpha\}} \int_0^{\hat{T}_\alpha} \mathbf{1}_{\{\hat{W}(u) \geq 0\}} du \\ &\quad + \mathbf{1}_{\{H_\alpha < \tau(1), W(H_\alpha) = \alpha\}} \int_0^{\hat{T}_{-\alpha}} \mathbf{1}_{\{\hat{W}(u) \leq 0\}} du. \end{aligned}$$

By the strong Markov property and symmetry, we obtain:

$$\begin{aligned} \mathbb{P}\left(X(1) > \lambda\right) &\geq \mathbb{P}\left(H_\alpha < \tau(1)\right) \mathbb{P}\left(\int_0^{T_\alpha} \mathbf{1}_{\{W(u) \geq 0\}} du > \lambda\right) \\ &= \mathbb{P}\left(H_\alpha < \tau(1)\right) \mathbb{P}\left(\int_0^{T_1} \mathbf{1}_{\{W(u) \geq 0\}} du > \frac{\lambda}{\alpha^2}\right), \end{aligned}$$

with $T_x = \inf\{t > 0 : W(t) = x\}$. According to the first Ray-Knight theorem (Revuz and Yor [6, Theorem XI.2.2]), $\{\ell(T_1, 1-x); 0 \leq x \leq 1\}$ is a two-dimensional square Bessel process (i.e. the square of Euclidean modulus of a two-dimensional Wiener process), starting from 0. Since

$$\int_0^{T_1} \mathbf{1}_{\{W(u) \geq 0\}} du = \int_0^1 \ell(T_1, 1-x) dx,$$

and since a two-dimensional square Bessel process is (stochastically) greater than the square of a real-valued Wiener process, we have

$$\mathbb{P}\left(X(1) > \lambda\right) \geq \mathbb{P}\left(H_\alpha < \tau(1)\right) \mathbb{P}\left(\int_0^1 W^2(u) du > \frac{\lambda}{\alpha^2}\right).$$

Recall the Cameron-Martin formula ([1]):

$$\mathbb{P}\left(\int_0^1 W^2(u) du > x\right) \sim \frac{\sqrt{32}}{\pi^2} x^{-1/2} \exp\left(-\frac{\pi^2}{8}x\right), \quad x \rightarrow \infty,$$

(the symbol “ $a(x) \sim b(x)$ ” ($x \rightarrow x_0$) denoting $\lim_{x \rightarrow x_0} a(x)/b(x) = 1$). Consequently, for sufficiently large λ ,

$$\mathbb{P}\left(X(1) > \lambda\right) \geq \frac{C(\alpha)}{\lambda^{1/2}} \exp\left(-\frac{\pi^2 \lambda}{8\alpha^2}\right),$$

with $C(\alpha) > 0$ a finite constant depending only on α . This implies the lower bound in Lemma 2. □

Proof of Theorem 1. Let us begin with the upper bound. Fix a small $\varepsilon > 0$. Define $t_n = (1 + \varepsilon)^n$. Then $(1 + 5\varepsilon)t_n^2/t_{n+1}^2 \geq 1 + 2\varepsilon$. By scaling property (2.1), we have

$$\begin{aligned} \mathbb{P}\left(X(t_{n+1}) > (1 + 5\varepsilon) \frac{8\alpha^2}{\pi^2} t_n^2 \log \log t_n\right) \\ &= \mathbb{P}\left(X(1) > (1 + 5\varepsilon) \frac{8\alpha^2}{\pi^2} \frac{t_n^2}{t_{n+1}^2} \log \log t_n\right) \\ &\leq \mathbb{P}\left(X(1) > (1 + 2\varepsilon) \frac{8\alpha^2}{\pi^2} \log \log t_n\right). \end{aligned}$$

Using (2.3), the above expression is, when n is sufficiently large, bounded above by

$$\exp\left(- (1 + \varepsilon) \log \log t_n\right) = \frac{1}{n^{1+\varepsilon} (\log(1 + \varepsilon))^{1+\varepsilon}},$$

which sums for n . It follows from the Borel-Cantelli lemma that (almost surely) for large n , $X(t_{n+1}) \leq (1 + 5\varepsilon)(8\alpha^2/\pi^2)t_n^2 \log \log t_n$. Let $t \in [t_n, t_{n+1}]$. Then

$$\frac{X(t)}{t^2 \log \log t} \leq \frac{X(t_{n+1})}{t_n^2 \log \log t_n} \leq (1 + 5\varepsilon) \frac{8\alpha^2}{\pi^2}.$$

Accordingly,

$$\limsup_{t \rightarrow \infty} \frac{X(t)}{t^2 \log \log t} \leq (1 + 5\varepsilon) \frac{8\alpha^2}{\pi^2},$$

for any $\varepsilon > 0$. Letting ε tend to 0^+ gives the upper bound in Theorem 1. It remains to verify the lower bound part. Fix again an $\varepsilon > 0$. Let $t_n = 2^n$ and let

$$A_n = \left\{ \int_{\tau(t_{n-1})}^{\tau(t_n)} \mathbf{1}_{\{|W(u)| \leq \alpha t_n\}} du > (1 - \varepsilon) \frac{8\alpha^2}{\pi^2} t_n^2 \log \log t_n \right\}.$$

The measurable events (A_n) are obviously independent. Moreover,

$$\mathbb{P}(A_n) = \mathbb{P}\left(\int_0^{\tau(t_n - t_{n-1})} \mathbf{1}_{\{|W(u)| \leq \alpha t_n\}} du > (1 - \varepsilon) \frac{8\alpha^2}{\pi^2} t_n^2 \log \log t_n \right),$$

using the strong Markov property. By scaling, we have

$$\begin{aligned} \mathbb{P}(A_n) &= \mathbb{P}\left(\int_0^{\tau(1)} \mathbf{1}_{\{|W(u)| \leq \alpha t_n / (t_n - t_{n-1})\}} du > (1 - \varepsilon) \frac{8\alpha^2}{\pi^2} \frac{t_n^2}{(t_n - t_{n-1})^2} \log \log t_n \right) \\ &= \mathbb{P}\left(\int_0^{\tau(1)} \mathbf{1}_{\{|W(u)| \leq 2\alpha\}} du > (1 - \varepsilon) \frac{32\alpha^2}{\pi^2} \log \log t_n \right), \end{aligned}$$

using the fact that $t_n / (t_n - t_{n-1}) = 2$. Applying (2.3) to 2α (instead of α) implies that for large n ,

$$\mathbb{P}(A_n) \geq \exp(- \log \log t_n) = \frac{1}{n \log 2},$$

which forms a divergent series. Since the A_n 's are independent, the Borel-Cantelli lemma tells $\mathbb{P}(A_n, \text{i.o.}) = 1$. Therefore

$$\liminf_{t \rightarrow \infty} \frac{X(t)}{t^2 \log \log t} \geq (1 - \varepsilon) \frac{8\alpha^2}{\pi^2} \quad \text{a.s.}$$

Since the constant $\varepsilon > 0$ can be arbitrarily small, this yields the lower bound in Theorem 1. □

3. PROOF OF THEOREM 2

Let us keep the notation introduced previously. The first step is to establish the lower tail of $X(1)$.

Lemma 3. *We have, for any $\alpha > 0$,*

$$(3.1) \quad \mathbb{P}\left(X(1) < x\right) \sim \left(\frac{2}{\pi}\right)^{1/2} x^{1/2} \exp\left(-\frac{1}{2x}\right), \quad x \rightarrow 0.$$

Consequently, there exists a finite constant $C > 0$ such that

$$(3.2) \quad \mathbb{P}\left(X(1) < x\right) \leq C \exp\left(-\frac{1}{2x}\right), \quad \forall x > 0.$$

The proof of Lemma 3 is based on a general complex analysis argument I have learnt from Csáki [2, p. 210], which is stated as follows.

Lemma 4. *Let $g(z)$ be holomorphic on $\{(\operatorname{Im}(z))^2 > 4\varepsilon(\varepsilon - \operatorname{Re}(z))\}$ for any sufficiently small $\varepsilon > 0$, and let $G(z) = g(z^2)$. Assume that for $t \rightarrow \infty$ there exist finite constants $\beta \geq 0$, $\gamma > 1$ and $\eta > 1/2$ such that*

$$(3.3) \quad G(t + iu) = o\left(G(t) \exp(\beta|u|t^{-1/2})\right) \quad \text{uniformly for } |u| \geq t^\eta,$$

$$(3.4) \quad G''(t + iu) = o\left(t^{-\gamma}G(t)\right) \quad \text{uniformly for } |u| \leq t^\eta.$$

Then for any $c > 0$,

$$(3.5) \quad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} p^{-1/2} g(p) \exp(ps - 2p^{1/2}) dp = \frac{1 + o(s^{(\gamma-1)/2})}{(\pi s)^{1/2}} g(s^{-2}) \exp\left(-\frac{1}{s}\right),$$

as $s \rightarrow 0^+$.

Proof of Lemma 3. From (2.2), we have, using integration by parts,

$$\int_0^\infty dx e^{-\theta x} \mathbb{P}\left(X(1) < x\right) = \frac{1}{\theta} \exp\left(- (2\theta)^{1/2} \tanh(\alpha(2\theta)^{1/2})\right),$$

for $\theta > 0$. Inverting the Laplace transform yields

$$\begin{aligned} \mathbb{P}\left(X(1) < x\right) &= \frac{1}{2\pi i} \int_{2c-i\infty}^{2c+i\infty} \frac{d\theta}{\theta} e^{x\theta} \exp\left(- (2\theta)^{1/2} \tanh(\alpha(2\theta)^{1/2})\right) \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{dp}{p} e^{2px} \exp\left(- 2p^{1/2} \tanh(2\alpha p^{1/2})\right) \\ (3.6) \quad &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} p^{-1/2} g(p) \exp(2px - 2p^{1/2}) dp, \end{aligned}$$

where

$$g(p) = p^{-1/2} \exp\left(2p^{1/2} - 2p^{1/2} \tanh(2\alpha p^{1/2})\right).$$

Obviously, $g(z)$ is holomorphic on $\{(\operatorname{Im}(z))^2 > 4\varepsilon(\varepsilon - \operatorname{Re}(z))\}$ for any $\varepsilon > 0$. Let

$$G(z) = g(z^2) = \frac{1}{z} \exp\left(2z - 2z \tanh(2\alpha z)\right) = \frac{1}{z} \exp\left(\frac{4z}{e^{4\alpha z} + 1}\right).$$

We now verify conditions (3.3) and (3.4) for G . Write $G(z) = z^{-1} \exp(H(z))$, with $H(z) \equiv 4z/(e^{4\alpha z} + 1)$. Then

$$\begin{aligned}
 \operatorname{Re}(H(t + iu)) - H(t) &\leq |H(t + iu) - H(t)| \\
 &= \frac{|4te^{4\alpha t}(1 - e^{4i\alpha u}) + 4iu(e^{4\alpha t} + 1)|}{|e^{4\alpha(t+iu)} + 1| (e^{4\alpha t} + 1)} \\
 (3.7) \qquad \qquad \qquad &\leq \frac{8t + 4|u|}{e^{4\alpha t} - 1}.
 \end{aligned}$$

Since $(8t + 4|u|)/(e^{4\alpha t} - 1) \leq |u|t^{-1/2}$ for any $|u| \geq t \geq t_0$ (t_0 depending on the value of α), the above inequality trivially implies (3.3) with $\beta = 1$ and $\eta = 1$. For (3.4), observe that $G''(z) = z^{-2}(2 - 2zH'(z) + z^2H''(z) + z^2(H'(z))^2)G(z)$. Assume $|u| \leq t$ and $t \geq t_0$. By (3.7), we have $\operatorname{Re}(H(t + iu)) - H(t) \leq 1$. Thus $|G(t + iu)| \leq 3G(t)$. Several lines of elementary calculation show that

$$\left| 2 - 2(t + iu)H'(t + iu) + (t + iu)^2H''(t + iu) + (t + iu)^2(H'(t + iu))^2 \right| \leq 3.$$

Consequently, (3.4) holds with $\gamma = 3/2$ (actually any constant strictly smaller than 2 will do). Applying (3.5) to $s = 2x$ and using (3.6) completes the proof of Lemma 3. □

Proof of Theorem 2. The convergent half is an easy consequence of Lemma 3. Indeed, assume f to be non-decreasing such that

$$(3.8) \qquad \int_0^\infty \frac{dt}{t} f^{1/2}(t) e^{-f(t)/2} < \infty.$$

Thus $f(t) \uparrow \infty$ as $t \uparrow \infty$. Pick a large initial value t_0 and define the sequence (t_n) by recurrence: $t_{n+1} = t_n(1 + 1/f(t_n))$ for $n = 0, 1, 2, \dots$. Obviously (t_n) increases to infinity. A standard argument shows that (3.8) implies

$$(3.9) \qquad \sum_n f^{-1/2}(t_n) e^{-f(t_n)/2} < \infty.$$

By scaling property (2.1) and small deviation probability estimate (3.1), we have

$$\begin{aligned}
 \mathbb{P}\left(X(t_n) < \frac{t_n^2}{f(t_n) - 3}\right) &= \mathbb{P}\left(X(1) < \frac{1}{f(t_n) - 3}\right) \\
 &\leq (f(t_n) - 3)^{-1/2} \exp\left(-\frac{f(t_n) - 3}{2}\right) \\
 &\leq 5f^{-1/2}(t_n) e^{-f(t_n)/2},
 \end{aligned}$$

which, according to (3.9), is summable. Applying the Borel-Cantelli lemma gives that (almost surely) for sufficiently large n , $X(t_n) \geq t_n^2/(f(t_n) - 3)$. Let $t \in [t_n, t_{n+1}]$. Then

$$X(t) \geq X(t_n) \geq \frac{t_n^2}{f(t_n) - 3} = \frac{t_{n+1}^2}{(1 + 1/f(t_n))^2 (f(t_n) - 3)} \geq \frac{t_{n+1}^2}{f(t_n)} \geq \frac{t^2}{f(t)},$$

implying the convergent half of Theorem 1. To check the divergent half, let $f > 0$ be non-decreasing with

$$(3.10) \quad \int^{\infty} \frac{dt}{t} f^{1/2}(t) e^{-f(t)/2} = \infty.$$

In light of (1.4), let us assume without loss of generality that

$$(3.11) \quad \log \log t \leq f(t) \leq 3 \log \log t.$$

For an elegant argument justifying (3.11), we refer to Csáki [2]. Let $\rho > 0$ and $\varepsilon > 0$. Fix a large initial value $i_0 \equiv i_0(\rho, \varepsilon)$. Define $t_i = \exp(\rho i / \log i)$ (for $i \geq i_0$). By means of (3.11), we have

$$(3.12) \quad 1 + \frac{\rho}{2 \log i} \leq \frac{t_i}{t_{i-1}} \leq 1 + \frac{2\rho}{\log i},$$

$$(3.13) \quad \frac{1}{2} \log i \leq f(t_i) \leq 3 \log i, \quad i > i_0.$$

Moreover, it is easily seen from (3.10) that $\sum_i f^{-1/2}(t_i) \exp(-f(t_i)/2) = \infty$. Now consider

$$A_i = \left\{ \frac{t_{i-1}^2}{f(t_i)} \leq X(t_i) < \frac{t_i^2}{f(t_i)} \right\}.$$

Using (2.1), (3.1), (3.12) and (3.13) gives

$$\begin{aligned} \mathbb{P}(A_i) &= \mathbb{P} \left[\frac{t_{i-1}^2}{t_i^2 f(t_i)} \leq X(1) < \frac{1}{f(t_i)} \right] \\ &\geq (1 - \varepsilon) \left(\frac{2}{\pi} \right)^{1/2} \left(f^{-1/2}(t_i) e^{-f(t_i)/2} - \left(\frac{t_{i-1}^2}{t_i^2 f(t_i)} \right)^{1/2} \exp \left(-\frac{t_i^2 f(t_i)}{2 t_{i-1}^2} \right) \right) \\ (3.14) \quad &\geq (1 - \varepsilon) (1 - e^{-\rho/4}) (2/\pi)^{1/2} f^{-1/2}(t_i) e^{-f(t_i)/2}, \end{aligned}$$

for any $n > i_0$. Consequently,

$$(3.15) \quad \sum_i \mathbb{P}(A_i) = \infty.$$

Pick $i_0 < i < j$. In view of the strong Markov property, we have

$$\begin{aligned} \mathbb{P}(A_i A_j) &\leq \mathbb{P} \left(A_i, \int_{\tau(t_i)}^{\tau(t_j)} \mathbf{1}_{\{|W(u)| \leq \alpha t_j\}} du < \frac{t_j^2}{f(t_j)} - \frac{t_{i-1}^2}{f(t_i)} \right) \\ &\leq \mathbb{P}(A_i) \mathbb{P} \left(\int_0^{\tau(t_j - t_i)} \mathbf{1}_{\{|W(u)| \leq \alpha(t_j - t_i)\}} du < \frac{t_j^2 - t_{i-1}^2}{f(t_j)} \right). \end{aligned}$$

The above estimate together with scaling property (2.1) readily yield

$$(3.16) \quad \mathbb{P}(A_i A_j) \leq \mathbb{P}(A_i) \mathbb{P} \left(X(1) < \frac{t_j^2 - t_{i-1}^2}{(t_j - t_i)^2 f(t_j)} \right),$$

for any $i_0 < i < j$. Define

$$\begin{aligned} \mathcal{E}(n) &= \left\{ i_0 < i < j \leq n : j - i < (\log i)^3 \right\}, \\ \mathcal{F}(n) &= \left\{ i_0 < i < j \leq n : j - i \geq (\log i)^3 \right\}. \end{aligned}$$

Remark that when $i < j < i + (\log i)^3$,

$$\frac{j}{\log j} - \frac{i}{\log i} = \frac{(j - i) \log i - i \log(1 + (j - i)/i)}{\log i \log j} \sim \frac{j - i}{\log j} \sim \frac{j - i}{\log i},$$

as $i \rightarrow \infty$. Let $(i, j) \in \mathcal{E}(n)$. Then by the above observation, we have

$$\exp\left(-\frac{2\rho(j - i)}{\log i}\right) \leq \frac{t_{i-1}}{t_j} \leq \frac{t_i}{t_j} \leq \exp\left(-\frac{\rho(j - i)}{2 \log i}\right).$$

Therefore by (3.13),

$$\begin{aligned} \frac{t_j^2 - t_{i-1}^2}{(t_j - t_i)^2 f(t_j)} &\leq \frac{t_j^2 - t_{i-1}^2}{(t_j - t_i)^2 f(t_i)} \leq \frac{1 - \exp(-4\rho(j - i)/\log i)}{(1 - \exp(-\rho(j - i)/2 \log i))^2} \frac{2}{\log i} \\ &\leq \frac{C_1}{\min(j - i, \log i)}, \end{aligned}$$

for some finite constant $C_1 > 0$ depending only on ρ and ε (i_0 depending on ε). Using (3.16) and (3.2), we obtain (writing $C_2 \equiv 1/(2C_1)$ in the sequel):

$$\begin{aligned} \mathbb{P}(A_i A_j) &\leq C \mathbb{P}(A_i) \exp(-C_2 \min(j - i, \log i)) \\ &\leq C \mathbb{P}(A_i) \exp(-C_2(j - i)) + C \mathbb{P}(A_i) \exp(-C_2 \log i), \end{aligned}$$

(C being the constant introduced in (3.2)) which in turn implies

$$\begin{aligned} \sum_{(i,j) \in \mathcal{E}(n)} \mathbb{P}(A_i A_j) &\leq C \sum_{i=i_0+1}^n \mathbb{P}(A_i) \sum_{j>i} e^{-C_2(j-i)} \\ &\quad + C \sum_{i=i_0+1}^n \mathbb{P}(A_i) \sum_{i < j < i + (\log i)^3} i^{-C_2} \\ (3.17) \qquad \qquad \qquad &\leq C_3 \sum_{i=i_0+1}^n \mathbb{P}(A_i), \end{aligned}$$

for some finite constant $C_3 > 0$. Now let $(i, j) \in \mathcal{F}(n)$. In this case, $j - (\log j)^2 \geq i + (\log i)^3 - (\log(i + (\log i)^3))^2 \geq i$. Thus $j - i \geq (\log j)^2$. It is noticed that

$$\frac{j}{\log j} - \frac{i}{\log i} = \frac{(j - i) \log i - i \log(1 + (j - i)/i)}{\log i \log j} \geq \frac{j - i}{2 \log j}.$$

Therefore, we have

$$\begin{aligned} \frac{t_j^2 - t_{i-1}^2}{(t_j - t_i)^2 f(t_j)} &\leq \frac{1}{(1 - t_i/t_j)^2 f(t_j)} \leq \frac{1}{(1 - \exp(-\rho(j-i)/2 \log j))^2 f(t_j)} \\ &\leq \frac{1}{(1 - (\log j)^{-2}) f(t_j)}. \end{aligned}$$

According to (3.16) and (3.1), we get

$$\mathbb{P}(A_i A_j) \leq (1 + \varepsilon) \mathbb{P}(A_i) \left(\frac{2}{\pi}\right)^{1/2} f^{-1/2}(t_j) \exp\left(-\frac{(1 - (\log j)^{-2}) f(t_j)}{2}\right).$$

Since $(\log j)^{-2} f(t_j)/2 \leq 3/2 \log j \leq \varepsilon$ for $j \geq i_0$, the above estimate together with (3.14) confirm

$$\mathbb{P}(A_i A_j) \leq \frac{e^\varepsilon (1 + \varepsilon)}{(1 - \varepsilon)(1 - e^{-\rho/4})} \mathbb{P}(A_i) \mathbb{P}(A_j),$$

which, with the aid of (3.17) and (3.15), yields

$$\liminf_{n \rightarrow \infty} \sum_{i=i_0+1}^n \sum_{j=i_0+1}^n \mathbb{P}(A_i A_j) \Big/ \left(\sum_{i=i_0+1}^n \mathbb{P}(A_i) \right)^2 \leq \frac{1}{1 - e^{-\rho/4}}.$$

Since $\sum_i \mathbb{P}(A_i)$ diverges, using a well-known version of the Borel-Cantelli lemma (see for example Révész [5, p. 28]) gives $\mathbb{P}(A_i, \text{i.o.}) \geq 1 - e^{-\rho/4}$. A fortiori, we have

$$\mathbb{P}\left[X(t) < \frac{t^2}{f(t)}, \text{i.o.}\right] \geq 1 - e^{-\rho/4},$$

for any $\rho > 0$. The proof of the divergent half is completed by sending ρ to infinity. \square

ACKNOWLEDGEMENTS

I am grateful to Marc Yor for helpful comments on the first draft of this work. Many thanks are due to an anonymous referee for a careful reading of the manuscript and for valuable suggestions.

REFERENCES

1. R.H. Cameron and W.T. Martin, *The Wiener measure of Hilbert neighborhoods in the space of real continuous functions*, J. Math. Phys. **23** (1944), 195–209. MR **6**:132a
2. E. Csáki, *An integral test for the supremum of Wiener local time*, Probab. Th. Rel. Fields **83** (1989), 207–217. MR **91a**:60206
3. A. Földes and M.L. Puri, *The time spent by the Wiener process in a narrow tube before leaving a wide tube*, Proc. Amer. Math. Soc. **117** (1993), 529–536. MR **93d**:60131
4. J.W. Pitman and M. Yor, *A decomposition of Bessel bridges*, Z. Wahrscheinlichkeitstheorie verw. Gebiete **59** (1982), 425–457. MR **84a**:60091
5. P. Révész, *Random Walk in Random and Non-Random Environments*, World Scientific, Singapore, 1990. MR **92c**:60096
6. D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, 2nd ed., Springer, Berlin, 1994. CMP 95:04
7. H.F. Trotter, *A property of Brownian motion paths*, Illinois J. Math. **2** (1958), 425–433. MR **20**:2795

UNIVERSITÉ PARIS VI, L.S.T.A. - CNRS URA 1321, UNIVERSITÉ PARIS VI, TOUR 45-55, 4 PLACE JUSSIEU, F-75252 PARIS CEDEX 05, FRANCE
E-mail address: shi@ccr.jussieu.fr