SOME NEW HOMOGENEOUS EINSTEIN METRICS ON SYMMETRIC SPACES

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Abstract. We classify homogeneous Einstein metrics on compact irreducible symmetric spaces. In particular, we consider symmetric spaces with rank$(M) > 1$, not isometric to a compact Lie group. Whenever there exists a closed proper subgroup $G$ of Isom$(M)$ acting transitively on $M$ we find all $G$-homogeneous (non-symmetric) Einstein metrics on $M$.

1. Introduction

In this paper we look at compact symmetric spaces presented homogeneously, i.e. as $M = G/H$, where $G = \text{Isom}_0(M)$ is simple, and we consider the cases where there exists a closed subgroup $G' \subset G$ which acts transitively on $M$. Denote by $H'$ the isotropy subgroup in $G'$, then $M = G'/H'$. Since $G'$ is smaller than $G$, we expect more $G'$-invariant metrics on $M$ than $G$-invariant metrics, and thus we can hope for non-symmetric $G'$-invariant Einstein metrics on our symmetric space $M$. We find the following.

Lemma 1.1. Let $M$ be a compact irreducible symmetric space of rank $> 1$, not isometric to a compact Lie group with a biinvariant metric. Let $G = \text{Isom}_0(M)$, and $M = G/H$. Then there exists a subgroup $G' \subset G$ acting transitively on $M$ if

1. $G = \text{SO}(2n)$, $H = \text{U}(n)$, $G' = \text{SO}(2n - 1)$, $H' = \text{U}(n - 1)$ ($n \geq 4$);
2. $G = \text{SU}(2n)$, $H = \text{Sp}(n)$, $G' = \text{SU}(2n - 1)$, $H' = \text{Sp}(n - 1)$ ($n \geq 3$);
3. $G = \text{SO}(7)$, $H = \text{SO}(2) \text{SO}(5)$, $G' = G_2$, $H' = \text{U}(2) (\text{U}(2) \subset \text{SU}(3))$;
4. $G = \text{SO}(8)$, $H = \text{SO}(3) \text{SO}(5)$, $G' = \text{Spin}(7)$, $H' = \text{SO}(4) (\text{SO}(4) \subset G_2)$.

Theorem 1.2. Among the compact irreducible symmetric spaces of rank $> 1$, not isometric to a Lie group with a biinvariant metric, only $\text{G}_2^+(\mathbb{R}^7)$, $\text{G}_3^+(\mathbb{R}^8)$, and SO$(2n)/\text{U}(n)$, for $n \geq 4$, carry non-symmetric homogeneous Einstein metrics. The Grassmannians $G_2^+(\mathbb{R}^7)$ and $G_3^+(\mathbb{R}^8)$ each carry two and SO$(2n)/\text{U}(n)$ carries one; the only homogeneous Einstein metric on SU$(2n)/\text{Sp}(n)$ is the symmetric metric.

Let $\mathcal{M}_{G'}$ denote the space of $G'$-invariant metrics of volume one. Our results are summarized in the following table.

<table>
<thead>
<tr>
<th>$G/H$</th>
<th>$G'/H'$</th>
<th>dim$\mathcal{M}_{G'}$</th>
<th>no. of Einstein</th>
</tr>
</thead>
<tbody>
<tr>
<td>SO$(2n)/\text{U}(n)$</td>
<td>SO$(2n - 1)/\text{U}(n - 1)$</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>SU$(2n)/\text{Sp}(n)$</td>
<td>SU$(2n - 1)/\text{Sp}(n - 1)$</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>SO$(7)/\text{SO}(2) \text{SO}(5)$</td>
<td>G$_2$/U$(2)$</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>SO$(8)/\text{SO}(3) \text{SO}(5)$</td>
<td>Spin$(7)/\text{SO}(4)$</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>
The analogous results for symmetric spaces of rank 1 were studied in [Z] and for compact Lie groups with biinvariant metrics in [DA-Z].

Remark 1.3. The following result should be true for all compact irreducible symmetric spaces $M$, up to diffeomorphism: If $G$ is a compact connected Lie group acting transitively and effectively on $M$, then $G$ is conjugate to a subgroup of $\text{Isom}_0(M)$. This would imply that Theorem 1.2 classifies all homogeneous Einstein metrics on compact irreducible symmetric spaces of rank $> 1$, not isometric to a Lie group. Such a result is well known for rank 1 symmetric spaces, but does not seem to be known for all symmetric spaces of rank $> 1$. Partial results can be found in [O2], [O3], [S], [T].

For example, in [O3, Thm. 1] Oniščik showed that if $M$ is diffeomorphic to $G_{2k}(\mathbb{R}^n)$ for $n$ even, $n > 5$, $1 < k < \frac{n-2}{2}$, or for $n$ odd, $2 < k < \frac{n-3}{2}$, and if a compact connected Lie group $G$ acts transitively and effectively on $M$, then $G$ is conjugate to $\text{SO}(n)$ with the standard action. Tsukada proved in [T] that if $M$ is diffeomorphic to $G_{2k+1}(\mathbb{R}^{2n})$, and $G$ is compact, connected, and simple, then if $G$ acts transitively and effectively, the action of $G$ is conjugate to the standard action.

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2. Preliminaries

A Riemannian manifold $(M,g)$ is Einstein if $\text{Ric}(X,Y) = \lambda g(X,Y)$ for some constant $\lambda$, for all vector fields $X,Y$.

We say a Riemannian manifold $(M,g)$ is a symmetric space if for all $p \in M$ there exists an isometry $\sigma_p : M \to M$ such that $\sigma_p(p) = p$ and $(d\sigma_p)_p = -\text{Id}$. Symmetric spaces make up a class of manifolds which includes spheres, projective spaces, and Grassmannians; their geometry is well understood. In fact, every symmetric space is homogeneous.

A manifold $M$ is defined to be $G$-homogeneous if we have a Riemannian metric $g$ and a closed subgroup $G \subset \text{Isom}(M,g)$ such that for any $p$ and $q \in M$, there exists a $g \in G$ with $g(p) = q$. We write $H_p = \{g \in G \mid g(p) = p\}$, called the isotropy subgroup corresponding to $p$. Notice $H_p$ is compact, since $H_p \subset O(T_pM)$. Via the map $g \mapsto g(p)$ we identify the two manifolds $G/H_p$ and $M$. Any two isotropy subgroups $H_p$ and $H_q$ are conjugate: if $q = g(p)$, then $g^{-1}H_qg = H_p$, hence we will usually suppress the point $p$.

Given a homogeneous manifold $G/H$, where $G$ is compact and $H$ is closed, what metrics can we put on $G/H$ so that $G$ acts by isometries?

Just as a left-invariant metric on a Lie group is determined by any inner product on its Lie algebra, a $G$-invariant metric on $G/H$ is determined by an inner product on $g/\mathfrak{h} \cong T_H(G/H)$, with the additional requirement that the inner product be $\text{Ad}(H)$-invariant. We identify the quotient $g/\mathfrak{h}$ with an $\text{Ad}(H)$-invariant complement $\mathfrak{p}$ to $\mathfrak{h}$ in $g$; compactness of $H$ guarantees such a $\mathfrak{p}$ exists. If $g$ is semisimple then the Killing form is $\text{Ad}(H)$-invariant and we can use it to define $\mathfrak{p} = \mathfrak{h}^\perp$. We want to consider all $\text{Ad}(H)$-invariant inner products on $\mathfrak{p}$.

A homogeneous space $M = G/H$ is said to be isotropy irreducible if the isotropy action, denoted $\chi : H \to \text{GL}(T_pM)$, or equivalently $\text{Ad} : H \to \text{GL}(\mathfrak{p})$, is an irreducible representation of $H$. When this is the case, the $G$-invariant metric on $G/H$ is unique, up to scaling, and it is Einstein. When $G/H$ is a symmetric space
with $G$ simple and $G = \text{Isom}_0(G/H)$, then $G/H$ is an irreducible symmetric space. (In fact, the only irreducible symmetric space with $G$ not simple is $(K \times K)/\Delta K$, for $K$ a compact simple Lie group, and $(K \times K)/\Delta K$ is isometric to $K$ with a biinvariant metric.)

In 1962 A.L. Oniščik classified all simple compact Lie algebras $\mathfrak{g}$ with subalgebras $\mathfrak{g}'$ and $\mathfrak{g}''$, such that $\mathfrak{g} = \mathfrak{g}' + \mathfrak{g}''$. In terms of transitive group actions, let $G$ be the simply connected compact Lie group corresponding to $\mathfrak{g}$ and let $G'$, $G''$ be subgroups corresponding to $\mathfrak{g}'$, $\mathfrak{g}''$, respectively, then $G/G' = G''/(G' \cap G'')$ and $G/G'' = G'/G''$. When $G/G'$ or $G/G''$ is a symmetric space, Oniščik's list tells us when a subgroup of $G$ still acts transitively. Here are the symmetric spaces on his list $[O1]$: (See the appendix of this paper for the non-symmetric homogeneous spaces on his list.)

\[
\begin{align*}
\text{SO}(2n)/\text{SO}(2n-1) &= \text{U}(n)/\text{U}(n-1) = S^{2n-1} \\
\text{SO}(2n)/\text{SO}(2n-1) &= \text{SU}(n)/\text{SU}(n-1) = S^{2n-1} \\
\text{SO}(4n)/\text{SO}(4n-1) &= \text{Sp}(n)/\text{Sp}(n-1) = S^{4n-1} \\
\text{SO}(4n)/\text{SO}(4n-1) &= \text{Sp}(n) \text{U}(1)/\text{Sp}(n-1) \text{U}(1) = S^{4n-1} \\
\text{SO}(4n)/\text{SO}(4n-1) &= \text{Sp}(n) \text{Sp}(1)/\text{Sp}(n-1) \text{Sp}(1) = S^{4n-1} \\
\text{SO}(7)/\text{SO}(6) &= \text{G}_2/\text{SU}(3) = S^6 \\
\text{SO}(8)/\text{SO}(7) &= \text{Spin}(7)/\text{G}_2 = S^7 \\
\text{SO}(16)/\text{SO}(15) &= \text{Spin}(9)/\text{Spin}(7) = S^{15} \\
\text{SU}(2n)/\text{U}(2n-1) &= \text{Sp}(n)/\text{Sp}(n-1) \text{U}(1) = \mathbb{C}P^{2n} \\
\text{SO}(2n)/\text{U}(n) &= \text{SO}(2n-1)/\text{U}(n-1) = \text{spec. orth. ex. str. on } \mathbb{R}^{2n} \\
\text{SU}(2n)/\text{Sp}(n) &= \text{SO}(2n-1)/\text{Sp}(n-1) = \text{spec. orth. quat. str. on } \mathbb{C}^{2n} \\
\text{SO}(7)/\text{SO}(2) \text{SO}(5) &= \text{G}_2/\text{U}(2) = G_2^+(\mathbb{R}^7) \\
\text{SO}(8)/\text{SO}(3) \text{SO}(5) &= \text{Spin}(7)/\text{SO}(4) = G_3^+(\mathbb{R}^8).
\end{align*}
\]

Each of these symmetric spaces in the left-hand presentation is an irreducible symmetric space. Up to scaling, each has exactly one Einstein metric, the symmetric metric, homogeneous with respect to the left-hand presentation. However, with respect to the right-hand presentation, only the sixth and seventh symmetric spaces are isotropy irreducible.

The first nine examples are discussed in [Z]. In this paper we consider the last four examples in the table. Of these, the first is originally described in [W-Z]. On $\text{SU}(2n)/\text{Sp}(n)$, the only homogeneous Einstein metric is the original one. However, the last two spaces each carry two new Einstein metrics, homogeneous with respect to the right-hand presentation.

For any homogeneous space $M = G/H$, with $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ on the Lie algebra level, we parametrize the space of $G$-invariant metrics on $M$ by decomposing $\mathfrak{p}$ into its $\text{Ad}(H)$-irreducible subspaces, $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \cdots \oplus \mathfrak{p}_k$. If the $\mathfrak{p}_i$'s are pairwise inequivalent representations, a $G$-homogeneous metric is determined by an inner product on $\mathfrak{p}$ of the form $\langle \cdot, \cdot \rangle = x_1Q|_{\mathfrak{p}_1} + x_2Q|_{\mathfrak{p}_2} + \cdots + x_kQ|_{\mathfrak{p}_k}$, for $Q$ an $\text{Ad}(H)$-invariant inner product and $x_i > 0$ for all $i$. If $\mathfrak{p}_i$ and $\mathfrak{p}_j$ are equivalent for some $i$
and \( j \), then \( \langle \mathbf{p}_i, \mathbf{p}_j \rangle \) does not necessarily vanish; however, in each of the examples in this paper we have \( \mathbf{p}_i \neq \mathbf{p}_j \) for all \( i \neq j \).

Assume \( G/H \) is compact, and let \( S(g) \) denote the scalar curvature of \( g \). Einstein metrics are the critical points of the total scalar curvature functional

\[
T(g) = \int_M S(g) \, d\text{vol}_g
\]
on the space \( M \) of Riemannian metrics of volume one [Ber], [H]. Let \( \mathcal{M}_G \) denote the set of all \( G \)-invariant metrics of volume one on \( M \). Notice that on \( \mathcal{M}_G \), \( T(g) \simeq S(g) \). Furthermore, critical points of \( T|_{\mathcal{M}_G} \) are precisely \( G \)-invariant Einstein metrics of volume one [Bes, p.121].

If for our homogeneous space \( G/H \), every homogeneous metric is diagonal, i.e., \( \langle \cdot, \cdot \rangle = x_1 Q|_{\mathbf{p}_1} \perp x_2 Q|_{\mathbf{p}_2} \perp \cdots \perp x_k Q|_{\mathbf{p}_k} \), with \( x_i > 0 \) for all \( i \), then we use equation (1.3) for the scalar curvature given in [W-Z].

\[
S = \frac{1}{2} \sum_i \frac{d_i b_i}{x_i} - \frac{1}{4} \sum_{i,j,k} \left( k \right)_{ij} x_k x_j.
\]

In their formula, for each \( i \), \(-B|_{\mathbf{p}_i} = b_i Q|_{\mathbf{p}_i} \), where \( B \) denotes the Killing form, and \( d_i = \dim(\mathbf{p}_i) \); the triple \( \left( k \right)_{ij} = \sum Q([X_\alpha, X_\beta], X_\gamma)^2 \), summed over \( \{X_\alpha\} \), \{X_\beta\}, and \{X_\gamma\}; \( Q \)-orthonormal bases for \( \mathbf{p}_i \), \( \mathbf{p}_j \), and \( \mathbf{p}_k \), respectively. Notice \( \left( k \right)_{ij} \) is symmetric in all three entries. We are now ready to prove the theorem.

3. \( \text{SO}(2n)/U(n) \)

We begin with the symmetric space \( \text{SO}(2n)/U(n) \). Consider the space of orthogonal complex structures on \( \mathbb{R}^{2n} \), and let \( M_0 \) be the connected component containing \( J_0 \), the complex structure represented by \( \begin{pmatrix} 0 & -\text{Id}_n \\ \text{Id}_n & 0 \end{pmatrix} \) with respect to the standard basis on \( \mathbb{R}^{2n} \). We will show that \( M_0 \cong \text{SO}(2n)/U(n) \).

Let \( J \in M_0 \). Since \( J \) is an orthogonal complex structure \( \| Jv \| = \| v \| \) and \( J^2 = -\text{Id} \). We construct an orthonormal basis \( \{v_n\} \) for \( \mathbb{R}^{2n} \) such that for \( 1 \leq i \leq n \), \( Jv_i = v_{n+i} \) and \( Jv_{n+i} = -v_i \). Let \( v_1 = e_1 \), and let \( v_{n+1} = Jv_1 \). We have \( \langle v_1, Jv_1 \rangle = \langle v_1, J^2v_1 \rangle = -\langle v_1, v_1 \rangle \), hence \( \{v_1, v_{n+1}\} \) is an orthonormal basis for a \( J_0 \)-invariant subspace. Let \( v_2 \) be any unit vector in span \( \{v_1, v_{n+1}\} \), and let \( v_{n+2} = Jv_2 \).

Continue up to \( v_n, v_{2n} \). With respect to the basis \( \{v_n\} \), \( J = \begin{pmatrix} 0 & -\text{Id}_n \\ \text{Id}_n & 0 \end{pmatrix} \). Let \( P \) be the change of basis transformation from the standard basis to \( \{v_n\} \), then \( J = PJ_0P^{-1} \). The hypothesis that \( J \) be in the connected component containing \( J_0 \) corresponds exactly to the fact that \( P \) must be in \( \text{SO}(2n) \). Via conjugation, \( \text{SO}(2n) \) acts transitively on \( M_0 \).

The isotropy subgroup of \( J_0 \) is the set of all \( P \in \text{SO}(2n) \) such that \( PJ_0 = J_0P \). If we identify \( \mathbb{R}^n \oplus \mathbb{R}^n \cong \mathbb{C}^n \) via \( (u, v) \mapsto u + iv \) then \( J_0 \) is multiplication by \( i \) and hence \( PJ_0 = J_0P \) implies \( P \in \text{SO}(2n) \cap \text{GL}(n, \mathbb{C}) = U(n) \). Thus \( M_0 \cong \text{SO}(2n)/U(n) \), where we have \( U(n) \) embedded in \( \text{SO}(2n) \) in the following way: \( A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \). It is well known that \( \text{SO}(2n)/U(n) \) is an irreducible symmetric space [W, p.287]. Let \( \mu_n \) denote the standard complex \( n \)-dimensional representation of \( U(n) \); the isotropy representation of \( U(n) \) is \( [\wedge^2 \mu_n]_R \). Since \( \wedge^2 \mu_n \) is unitary,
\[ [\Lambda^2 \mu_n]|_{\mathbb{R}} \] is the irreducible real representation whose complexification is isomorphic to the direct sum of \( \Lambda^2 \mu_n \) and its dual.

If we look at the low dimensional examples, for \( n \leq 4 \), we find \( \text{SO}(4)/\text{U}(2) = S^2 \), \( \text{SO}(6)/\text{U}(3) = \mathbb{C}P^3 \), and \( \text{SO}(8)/\text{U}(4) = G_2^+(\mathbb{R}) \). For \( n \geq 4 \) the rank of the symmetric space is greater than one.

Notice that in our description above we let \( v_1 = e_1 \), so the subgroup \( \text{SO}(2n-1) \subset \text{SO}(2n) \) fixing \( \text{span}(e_1) \) also acts transitively on \( M_0 \). The isotropy subgroup of \( \text{SO}(2n-1) \) corresponding to \( J_0 \) is \( \text{U}(n-1) \subset \text{SO}(2n-2) \), where \( \text{SO}(2n-2) \) is the subgroup fixing \( e_1 \) and \( e_{n+1} \).

On the Lie algebra level we have, for \( X, Y \in \mathfrak{gl}(n-1, \mathbb{R}) \),

\[
\mathfrak{u}(n-1) \cong \left\{ \begin{pmatrix} X & 0 & -Y \\ 0 & 0 & 0 \\ Y & 0 & X \end{pmatrix} \right| X = -X^t, \ Y = Y^t \right\} \subset \mathfrak{so}(2n-1);
\]

therefore, \( \mathfrak{p} \cong \left\{ \begin{pmatrix} X & v & Y \\ -v^t & 0 & -w^t \\ Y & w & -X \end{pmatrix} \right| X = -X^t, \ Y = -Y^t, \ v, w \in \mathbb{R}^{n-1} \right\} \).

We find that \( \mathfrak{p} \) decomposes into the sum of two irreducible representations. In fact \( \text{U}(n-1) \subset \text{SO}(2n-2) \subset \text{SO}(2n-1) \) gives rise to the following fibration:

\[
\text{SO}(2n-2)/\text{U}(n-1) \to \text{SO}(2n-1)/\text{U}(n-1) \to \text{SO}(2n-1)/\text{SO}(2n-2) \cong S^{2n-2}.
\]

Both base and fibre are irreducible symmetric spaces. Let \( \mathfrak{p}_1 \) denote the \( \text{Ad} \text{SO}(2n-2) \)-invariant complement to \( \mathfrak{so}(2n-2) \) in \( \mathfrak{so}(2n-1) \); in our fibration \( \mathfrak{p}_1 \) corresponds to the tangent space of the base. The representation of \( \text{U}(n-1) \) on \( \mathfrak{p}_1 \) is the restriction of the standard representation of \( \text{SO}(2n-2) \) on \( \mathfrak{p}_1 \cong \mathbb{R}^{2n-2} \) to \( \text{U}(n-1) \), which is \( [\mu_{n-1}]_{\mathbb{R}} \), again irreducible. Let \( \mathfrak{p}_2 \) denote the \( \text{Ad} \text{U}(n-1) \)-invariant complement to \( \mathfrak{u}(n-1) \) in \( \mathfrak{so}(2n-2) \); \( \mathfrak{p}_2 \) corresponds to the tangent space of the fibre. This representation of \( \text{U}(n-1) \) on \( \mathfrak{p}_2 \) is the irreducible isotropy representation of the fibre symmetric space, \( [\Lambda^2 \mu_{n-1}]_{\mathbb{R}} \) [W, p.287]. We have \( \mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \).

The dimensions of \( \mathfrak{p}_1 \) and \( \mathfrak{p}_2 \) are \( 2(n-1) \) and \( (n-1)(n-2) \), respectively. We see that \( \mathfrak{p}_1 \) and \( \mathfrak{p}_2 \) are clearly inequivalent representations of \( \text{U}(n-1) \) for \( n \neq 4 \), and for \( n = 4 \), while \( \mu_3 \) and \( \Lambda^2 \mu_3 \) are equivalent representations on \( \text{SU}(3) \), they are inequivalent on the center of \( \text{U}(3) \). We apply Schur’s lemma to know that \( \langle \mathfrak{p}_1, \mathfrak{p}_2 \rangle \) and \( \text{Ric}(\mathfrak{p}_1, \mathfrak{p}_2) \) must vanish. Thus any \( \text{SO}(2n-1) \)-homogeneous metric on \( M_0 \) must be of the form \( \langle , \rangle = x_1 Q|_{\mathfrak{p}_1} \perp x_2 Q|_{\mathfrak{p}_2} \), where \( Q(X,Y) = -\frac{1}{2} \text{tr}(XY) \) and \( x_1, x_2 > 0 \). We express the scalar curvature in terms of \( x_1 \) and \( x_2 \) using [W-Z, (1.3)]. The equation is

\[
S = \frac{1}{2} \sum_i \frac{d_i b_i}{x_i} - \frac{1}{4} \sum_{i,j,k} \left( \begin{array}{c} k \\ i \\ j \end{array} \right) \frac{x_k}{x_i x_j}.
\]

Since for \( \mathfrak{so}(k) \), \( -B(X,Y) = (k-2) \text{tr}(XY) \), we have \( b_1 = b_2 = 2(2n-3) \). From the fibration it follows that \( [\mathfrak{p}_1, \mathfrak{p}_1] \subset \mathfrak{u}(n-1) \oplus \mathfrak{p}_2 \), \( [\mathfrak{p}_2, \mathfrak{p}_2] \subset \mathfrak{u}(n-1) \), hence the only nonzero triple (up to rearrangements) is \( \left( \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right) \).

Let \( E_{ij} \) denote the skew-symmetric matrix in \( \mathfrak{so}(2n-1) \) with 1 in the \( ij \)th entry and \( -1 \) in the \( ji \)th entry, and zeros everywhere else.

\[
\mathfrak{p}_1 = \text{span}\{-E_{in}, E_{n,n+i} \mid 1 \leq i \leq n-1\},
\]

\[
\mathfrak{p}_2 = \text{span}\{E_{ij} - E_{n+i,n+j}, E_{i,n+j} - E_{j,n+i} \mid 1 \leq i < j \leq n-1\}.
\]
We find \( \frac{3}{11} = 2(n-1)(n-2) \), and then we substitute into the scalar curvature equation to find \( S \). Let \( \tilde{S} \) be the equation for \( S \) with the boundary constraint that volume = 1:

\[
S = \frac{2(n-1)(2n-3)}{x_1} + \frac{2(n-1)(n-2)^2}{x_2} - \frac{(n-1)(n-2)x_2}{2x_1^2},
\]

\[
\tilde{S} = S - \lambda(x_1^{2(n-1)}x_2^{(n-1)(n-2)-1}).
\]

We find the partial derivatives of \( \tilde{S} \):

\[
\frac{\partial \tilde{S}}{\partial x_1} = -\frac{2(n-1)(2n-3)}{x_1^2} + \frac{(n-1)(n-2)x_2}{x_1^2} - 2(n-1)\lambda x_1^{2n-3}x_2^{(n-1)(n-2)},
\]

\[
\frac{\partial \tilde{S}}{\partial x_2} = -\frac{2(n-1)(n-2)^2}{x_2^2} - \frac{(n-1)(n-2)}{2x_1^2}
\]

\[= -\frac{(n-1)(n-2)}{2} \lambda x_1^{2(n-1)}x_2^{(n-1)(n-2)-1}.\]

Setting both equations equal to zero is equivalent to the following equation:

\[
\frac{n-1}{2} x_2 + 2(n-2) \frac{x_1^2}{x_2} = (2n-3)x_1.
\]

We find that the solutions are \( x_1 = \frac{1}{2} x_2 \), and \( x_1 = \frac{(n-1)}{2(n-2)} x_2 \). The second solution is a non-symmetric \( SO(2n-1) \)-invariant Einstein metric, discovered earlier in [W-Z, §3,Ex.6]. The first solution is the \( SO(2n) \)-invariant symmetric metric, but this is not obvious until we see how to compare them.

Let \( \tilde{p} \) denote the \( Q \)-orthogonal complement to \( u(n) \) in \( so(2n) \):

\[
\tilde{p} = \text{span}\{ E_{ij} - E_{n+i,n+j}, E_{i,n+j} - E_{j,n+i} \mid 1 \leq i < j \leq n \}.
\]

We must project \( p \) to \( \tilde{p} \). We take a basis element of \( p_1 \): \( -E_{in} \). Under the embedding of \( so(2n-1) \) in \( so(2n) \), \( -E_{in} \mapsto -E_{i+1,n+1} \). Next we write \( -E_{i+1,n+1} \) as the sum of an element in \( u(n) \) and an element in \( \tilde{p} \):

\[
-E_{i+1,n+1} = -\frac{1}{2} (E_{i+1,n+1} + E_{1,n+i+1}) + \frac{1}{2} (E_{i+1,n+1} - E_{1,n+i+1}).
\]

This shows that an element of norm \( \sqrt{x_1} \) is sent to an element of norm \( \frac{1}{\sqrt{2}} \). A basis element of \( p_2 \) is \( \frac{1}{\sqrt{2}} (E_{ij} - E_{n+i,n+j}) \), which the embedding sends to \( \frac{1}{\sqrt{2}} (E_{i+1,j+1} - E_{n+i+1,n+j+1}) \), already in \( \tilde{p} \); hence an element of norm \( \sqrt{x_2} \) is sent to an element of norm \( 1 \). The symmetric metric on \( SO(2n)/U(n) \) is given by the restriction of \( Q \) to \( \tilde{p} \). Hence it corresponds to \( \frac{2}{n} \), i.e., \( x_1 = \frac{1}{2} x_2 \).

To see that these metrics are distinct, we can compare the scale-invariant product \( (S)_{\frac{n}{2}} (V)_{\frac{1}{2}} \), where \( S \) is the scalar curvature, \( V \) is the volume, and \( d \) is the dimension of \( M \). The first metric has \( S = \frac{2n(n-1)^2}{x_2} \) and \( V = x_1^{2(n-1)}x_2^{(n-1)(n-2)} \), so that

\[
(S)_{\frac{n}{2}} (V)_{\frac{1}{2}} = 2^{\frac{(n-1)(n-2)}{2}} (n-1)^2 \frac{n(n-1)}{2}.
\]

The second metric has \( S = \frac{2n(n-2)(n^2-n-1)}{(n-1)x_2} \), and \( V = (\frac{n-1}{2(n-2)})^{2(n-1)}x_2^{n(n-1)} \), hence
\[
(S)_\frac{n(n-1)}{2} (V)^\frac{n}{2} = \left(\frac{2n(n-2)(n^2 - n - 1)}{n-1}\right)^\frac{n(n-1)}{2} \left(\frac{n-1}{2(n-2)}\right)^{n-1}.
\]

4. SU(2n)/Sp(n)

Our next example is the symmetric space \( M = SU(2n)/Sp(n) \), an analogue of the previous example. This is the set of special orthogonal quaternionic structures on \( \mathbb{C}^{2n} \). We identify \( \mathbb{R}^{4n} \cong \mathbb{C}^{2n} \) via a fixed orthogonal complex structure \( I \) on \( \mathbb{R}^{4n} \). An orthogonal quaternionic structure on \( \mathbb{C}^{2n} \) is given by \( J \in SO(4n) \) such that \( J^2 = -\text{Id} \) and \( IJ = -JI \). As a map from \( \mathbb{C}^{2n} \) to itself, \( J \) is complex anti-linear, i.e., \( J(\lambda v) = \bar{\lambda} J(v) \). We show that the set of all orthogonal quaternionic structures can be written homogeneously as \( U(2n)/Sp(n) \), and we call the submanifold \( M = SU(2n)/Sp(n) \) the set of special orthogonal quaternionic structures.

We first observe that if \( I = \begin{pmatrix} 0 & -\text{Id}_{2n} \\ \text{Id}_{2n} & 0 \end{pmatrix} \) and if we identify \( \mathbb{C}^n \oplus \mathbb{C}^n = \mathbb{H} \) via \( (u,v) \mapsto u + jv \), then multiplication by \( j \) on \( \mathbb{C}^{2n} = \mathbb{R}^{4n} \) becomes

\[
J_0 = \begin{pmatrix} 0 & -\text{Id}_n & 0 & 0 \\ \text{Id}_n & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{Id}_n \\ 0 & 0 & -\text{Id}_n & 0 \end{pmatrix},
\]

which is the standard orthogonal quaternionic structure. We want to show that \( U(2n) \) acts transitively on \( \{ J \in SO(4n) \mid J^2 = -\text{Id} \text{ and } IJ = -JI \} \), and the isotropy subgroup is \( Sp(n) \). Since \( U(2n) = \text{GL}(2n, \mathbb{C}) \cap SO(4n) \), \( A \) is unitary when \( A \in \text{SO}(4n) \) and \( AI = IA \), and for \( A \) unitary, \( AJA^{-1} \) is a quaternionic structure if \( J \) is: \( (AJA^{-1})(AJA^{-1}) = -\text{Id} \) and \( (AJA^{-1})I = AJIA^{-1} = -AIJA^{-1} = -I(AJA^{-1}) \). Furthermore, \( IJ \) is also a quaternionic structure:

\[
(IJ)(IJ) = -I(IJ)J = -(\text{Id})^2, \text{ and } (IJ)I = -I(IJ).
\]

Given an orthogonal quaternionic structure \( J \), we construct a unitary basis of \( \mathbb{C}^{2n} \) in which \( J \) is represented by the matrix \( J_0 \). Let \( v_1 = e_1 \), the first element of the standard basis. Let \( v_{n+1} = Jv_1, v_{2n+1} = Iv_1, \) and \( v_{3n+1} = Jv_1 \). Clearly \( \{v_1, Jv_1, Iv_1, Jv_1\} \) is an orthonormal basis for a 4-plane invariant under \( I, J, \) and \( II \). Choose \( v_2 \) to be any unit vector in the orthogonal complement, and repeat the process above, continuing up to \( v_n, v_{2n}, v_{3n}, v_{4n} \). Notice this is a unitary basis for \( \mathbb{R}^{4n} \), since \( v_{2n+i} = Iv_i \) for all \( 1 \leq i \leq 2n \).

The isotropy subgroup of \( U(2n) \) corresponding to \( J_0 \) is all \( A \in U(2n) \) such that \( AJ_0 = J_0 A \). i.e., \( A \) commutes with \( I, J, \) and \( II \): \( A \) is quaternionic linear. We embed \( Sp(n) \subset U(2n) \) via \( A + jB \mapsto \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix} \). The image of this embedding is contained in \( SU(2n) \), and we now restrict ourselves to the orbit of \( SU(2n) \), which is the symmetric space of special orthogonal quaternionic structures on \( \mathbb{R}^{4n} \), or \( SU(2n)/Sp(n) \). This symmetric space is irreducible; up to scaling the symmetric metric is the unique \( SU(2n) \)-invariant metric, and it is Einstein. Notice that when \( n = 2 \), \( SU(4)/Sp(2) = \mathbb{S}^5 \); for \( n \geq 3 \) the rank of the symmetric space is greater than one.
Let \( \tilde{p} \) be the orthogonal complement to \( \mathfrak{sp}(n) \) in \( \mathfrak{su}(2n) \) with respect to the inner product \( Q(X, Y) = -\frac{1}{2} \text{tr}(XY) \).

We have \( \mathfrak{su}(2n) = \left\{ \begin{pmatrix} X & Y \\ -Y^t & Z \end{pmatrix} \mid X, Z \in \mathfrak{u}(n), \ Y \in \mathfrak{gl}(n, \mathbb{C}), \ \text{tr} \ Z = -\text{tr} \ X \right\} \)

and \( \mathfrak{sp}(n) \cong \left\{ \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \mid X \in \mathfrak{u}(n), \ Y = Y^t \in \mathfrak{gl}(n, \mathbb{C}) \right\} \),

hence \( \tilde{p} = \left\{ \begin{pmatrix} X & -Y \\ Y & -X \end{pmatrix} \mid X \in \mathfrak{su}(n), \ Y = Y^t \in \mathfrak{gl}(n, \mathbb{C}) \ (\text{and } X = 0) \right\} \).

The isotropy representation of \( \text{Sp}(n) \) on \( \tilde{p} \) is \([\wedge^2 \nu_n - \text{Id}]_\mathbb{R}\), where \( \nu_n \) is the standard representation of \( \text{Sp}(n) \) on \( \mathbb{H}^n \cong \mathbb{C}^{2n} \). (The representation \( \wedge^2 \nu_n \) is the sum of a complex \( (2n + 1)(n - 1) \)-dimensional irreducible representation and a one-dimensional trivial representation. We denote by \( \wedge^2 \nu_n - \text{Id} \) the non-trivial summand.) We write \([\wedge^2 \nu_n - \text{Id}]_\mathbb{R}\) for the real representation whose complexification is \( \wedge^2 \nu_n - \text{Id} \).

The subgroup \( \text{SU}(2n - 1) \subset \text{SU}(2n) \) fixing \( e_1 \) acts transitively on \( M \), just as in the previous example. The isotropy subgroup of \( \text{SU}(2n - 1) \) corresponding to \( J_0 \) is

\[
H = \left\{ \begin{pmatrix} A & 0 & -B \\ 0 & 1 & 0 \\ B & 0 & A \end{pmatrix} \in \text{SU}(2n - 1) \mid A + jB \in \text{Sp}(n - 1, \mathbb{C}) \right\} = \text{Sp}(n - 1) \subset \text{SU}(2n - 2)
\]

fixing \( e_{2n+1} \). In \( \mathfrak{su}(2n - 1) \), for \( X, Y \in \mathfrak{gl}(n - 1, \mathbb{C}) \),

\[
\mathfrak{h} = \mathfrak{sp}(n - 1) = \left\{ \begin{pmatrix} X & 0 & -Y \\ 0 & 0 & 0 \\ Y & 0 & X \end{pmatrix} \mid X \in \mathfrak{u}(n - 1), \ Y = Y^t \right\}.
\]

We denote by \( \mathfrak{p} \) the \( Q \)-orthogonal complement to \( \mathfrak{sp}(n - 1) \) in \( \mathfrak{su}(2n - 1) \):

\[
\mathfrak{p} = \left\{ \begin{pmatrix} X & -\bar{u}^t & \bar{v} \\ u & z & v^t \\ \bar{Y} & -\bar{v} & -\bar{X} \end{pmatrix} \mid X \in \mathfrak{u}(n - 1), \ Y = -Y^t, \ u, v \in \mathbb{C}^{n-1}, \ z = -2 \text{tr} \ X \right\}.
\]

We have the following fibration of our symmetric space, which tells us how to decompose \( \mathfrak{p} \) into irreducible \( \text{Ad} \, \text{Sp}(n - 1) \)-invariant subrepresentations:

\[
\text{SU}(2n - 2)/\text{Sp}(n - 1) \rightarrow \text{SU}(2n - 1)/\text{Sp}(n - 1) \rightarrow \text{SU}(2n - 1)/\text{SU}(2n - 2) = S^{4n-3}.
\]

From the fibration we see that \( \mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}' \), where \( \mathfrak{p}_1 \) is tangent to the fibre and the \( \text{Ad} \, \text{Sp}(n - 1) \) action on \( \mathfrak{p}_1 \) is \([\wedge^2 \nu_{n-1} - \text{Id}]_\mathbb{R}\), and \( \text{dim}(\mathfrak{p}_1) = (2n - 1)(n - 2) \). The subspace \( \mathfrak{p}' \) is tangent to the base, and \( \text{Ad} \, \text{SU}(2n - 2) \) acts on \( \mathfrak{p}' \) by \([\mu_{2n-2}]_\mathbb{R} ^\oplus \text{Id}\), which when restricted to \( \text{Sp}(n - 1) \) is \([\nu_{n-1}]_\mathbb{R} ^\oplus \text{Id}\). That is, \( \mathfrak{p}' = \mathfrak{p}_2 \oplus \mathfrak{p}_3 \); \( \text{dim}(\mathfrak{p}_2) = 4(n - 1) \) and \( \text{dim}(\mathfrak{p}_3) = 1 \) (the \( \text{Ad} \, \text{Sp}(n - 1) \) action on \( \mathfrak{p}_3 \) is trivial). The set of elements listed below gives a \( Q \)-orthogonal basis for \( \mathfrak{p} \). We write \( E_{ij} \) for the skew-symmetric \((2n - 1) \times (2n - 1)\) matrix with 1 in the \( ij^{th} \) entry and \(-1\) in the \( ji^{th} \) entry, and zeros elsewhere. We denote by \( F_{ij} \) the symmetric \((2n - 1) \times (2n - 1)\) matrix with 1 in both the \( ij^{th} \) and \( ji^{th} \) entries.
\(p_1 = \text{span}\{ (E_{kl} - E_{n+k,n+l}), i(F_{kl} + F_{n+k,n+l}), \}
\)
\(\quad (E_{k,n+l} - E_{i,n+k}), i(F_{k,n+l} - F_{i,n+k}) \mid 1 \leq k < l \leq n - 1 \}\n\oplus \text{span}\{ i(F_{kk} - F_{n-1,n-1} + F_{n+k,n+k} - F_{2n-1,2n-1}) \mid 1 \leq k < n - 1 \}\n\oplus \text{span}\{ E_{nk}, iE_{nk}, E_{n+k}, iF_{n+n+k} \mid 1 \leq k \leq n - 1 \}\n\oplus \text{span}\{ \text{diag}(\eta, \ldots, \eta, -2(n-1)\eta, \eta, \ldots, \eta) \}, \text{where } \eta = \frac{i}{\sqrt{(2n-1)(n-1)}}.

Since \(p_1, p_2, \text{ and } p_3\) are inequivalent irreducible representations of \(\text{Sp}(n-1)\), any \(\text{SU}(2n-1)\)-invariant metric on \(M\) must take the form

\[ (\cdot, \cdot) = x_1Q_{p_1} \downarrow x_2Q_{p_2} \downarrow x_3Q_{p_3}, \text{ with } x_i > 0 \text{ for } i = 1, 2, 3. \]

To find all \(\text{SU}(2n-1)\)-invariant Einstein metrics on \(M\), we solve for the critical points of the scalar curvature equation in terms of \(x_1, x_2, \text{ and } x_3\) (restricting to unit volume). As in the previous case we use the formula given in [W-Z, (1.3)]:

\[ S = \frac{1}{2} \sum_i \frac{d_i b_i}{x_i} - \frac{1}{4} \sum_{i,j,k} \left( \begin{array}{c} k \\ i \\ j \end{array} \right) \frac{x_k}{x_i x_j}. \]

In our example we have \(b_i = 4(2n-1)\) for all \(i\), since for \(\text{su}(k)\), \(-B(X,Y) = 2k \text{ tr}(XY)\). And \(d_1 = (2n-1)(n-1), d_2 = 4(n-1), d_3 = 1. \) We find that

\[ [p_1, p_1] \subset \text{sp}(n-1), \quad [p_1, p_2] \subset p_2, \]
\[ [p_2, p_2] \subset \text{sp}(n-1) + p_1 + p_3, \quad [p_1, p_3] = 0, \quad [p_2, p_3] \subset p_2. \]

Therefore \((\frac{1}{2}, 2, 2)\) (and rearrangements) are the only nonzero triples. We compute \((\frac{3}{2}, 2) = 4(2n-1)(n-2)\) and \((\frac{3}{2}, 3) = 4(2n-1). \) We now have the equation for the scalar curvature of \(M\) in terms of \(x_1, x_2, \text{ and } x_3, \)

\[ S = (2n-1) \left( \frac{4(n-1)(n-2)}{x_1} + \frac{8(n-1)}{x_2} - \frac{x_3}{x_2^2} \right) \left( \frac{(n-2)x_1}{x_2} \right). \]

We normalize for volume 1 metrics: \(\tilde{S} = S - \lambda(x_1^d_1 x_2^d_2 x_3 - 1) \).

\[ \frac{\partial \tilde{S}}{\partial x_1} = -\frac{4(2n-1)(n-1)(n-2)}{x_1^2} - \frac{2(n-1)(n-2)}{x_2^2} - (2n-1)(n-2)\lambda x_1^{d_1} x_2^{d_2} x_3, \]
\[ \frac{\partial \tilde{S}}{\partial x_2} = \frac{8(2n-1)(n-1)}{x_2^4} + \frac{2(2n-1)((n-2)x_1 + x_3)}{x_2^2} - 4(n-1)\lambda x_1^{d_1} x_2^{d_2-1} x_3, \]
\[ \frac{\partial \tilde{S}}{\partial x_3} = -\frac{(2n-1)}{x_2^2} - \lambda x_1^{d_1} x_2^{d_2}. \]

Setting \(\frac{\partial \tilde{S}}{\partial x_1} = \frac{\partial \tilde{S}}{\partial x_2} = \frac{\partial \tilde{S}}{\partial x_3} = 0\) simultaneously is equivalent to

\[ 4(n-1)x_2^2 + x_3^2 = (2n-1)x_1 x_3 = 2(2n-1)x_1 x_2 - \frac{2n-1}{2n-2}((n-2)x_1^2 + x_1 x_3). \]

There is only one solution; it is \(x_2 = \frac{1}{2} x_1\) and \(x_3 = \frac{n^2}{2n-2} x_1\), unique up to scaling. This is not a new metric, rather it is the symmetric metric, which we knew must solve our equations. (It is \(\text{SU}(2n)\)-invariant, hence \(\text{SU}(2n-1)\)-invariant.) Thus the
only homogeneous Einstein metric on \( M = \text{SU}(2n-1)/\text{Sp}(n-1) \cong \text{SU}(2n)/\text{Sp}(n) \) is the symmetric metric.

5. \( G_2^2(\mathbb{R}^7) \)

The Grassmann manifold of oriented two-planes through the origin in \( \mathbb{R}^7 \) is generally written homogeneously \( G_2^2(\mathbb{R}^7) \cong \text{SO}(7)/\text{SO}(2)\text{SO}(5) \). It is irreducible: the symmetric metric is not only Einstein, it is the only \( \text{SO}(7) \)-invariant metric. We will show that this Grassmannian manifold can also be written homogeneously as \( \text{G}_2 / U(2) \) and we find it carries two non-symmetric \( \text{G}_2 \)-invariant Einstein metrics.

First we must see how \( \text{G}_2 \subset \text{SO}(7) \), following [M, p. 190]. We identify \( \mathbb{R}^8 \) with the Cayley numbers, or octonians, the normed division algebra \( \mathbb{O} = \mathbb{H} \oplus \mathbb{H}_c \). Then \( \text{G}_2 \) is the set of automorphisms of \( \mathbb{O} \). Any automorphism of the Cayley numbers must take 1 to itself and must preserve the inner product, so elements of \( \text{G}_2 \) also preserve \( \text{Im}(\mathbb{O}) \), the space of imaginary Cayley numbers, the orthogonal complement to 1. In this way we see \( \text{G}_2 \subset \text{SO}(7) \). To see that \( \text{G}_2 \) acts transitively on \( G_2^2(\mathbb{R}^7) \), we use the following observation [M, p. 186].

**Lemma 5.1.** Given three imaginary orthogonal unit octonions: \( v_1, v_2, \) and \( v_3 \in \{v_1, v_2, v_1v_2\}^\perp \), there exists a unique automorphism \( A \) of \( \mathbb{O} \) with \( A(i) = v_1, A(j) = v_2, \) and \( A(e) = v_3 \).

Using the lemma we take any \( P = \text{span}\{v, w\} \) to the oriented two-plane \( P_0 = \text{span}\{i, j\} \), where we take \( v \) and \( w \) an orthonormal basis for \( P \). We must find the isotropy subgroup fixing \( P_0 \). It will be useful to recall the following well known fact.

**Lemma 5.2.** The quotient \( \text{G}_2 / \text{SU}(3) \cong S^6 \).

**Proof.** By the previous lemma, \( \text{G}_2 \) acts transitively on \( S^6(1) \subset \text{Im}(\mathbb{O}) \). We need to show that the isotropy subgroup \( H_v \) of \( \text{G}_2 \) corresponding to any \( v \in S^8 \) is \( \text{SU}(3) \). We observe first that \( A(v) = v \) implies the map \( L_v \) is a complex structure on \( \mathbb{O} \) and on \( V = \text{span}\{1, v\}^\perp \). This shows that \( H_v \subset \text{U}(V) \cong \text{U}(3) \). Furthermore, \( \dim(H_v) = 8 \). Since \( \text{G}_2 \) and \( S^6 \) are connected and simply connected, \( H_v \) is connected.

Consider the homomorphism \( \det : H_v \rightarrow S^1 \); it must be either trivial or onto. If it is trivial, \( H_v \cong \text{SU}(3) \). If it is onto, then let \( H' \) denote the kernel. Then \( H' \) is a normal subgroup of \( H_v \) of dimension seven, and if \( H'_0 \) is the connected component of the identity, \( \text{rank}(H'_0) \leq \text{rank}(\text{SU}(3)) = 2 \). But the only compact connected seven-dimensional Lie groups, up to finite cover, are \( T^7, S^3 \times T^4, \) and \( S^3 \times S^3 \times S^1 \), and each of these has rank \( > 2 \):

\[
\text{rank}(T^7) = 7, \quad \text{rank}(S^3 \times T^4) = 5, \quad \text{rank}(S^3 \times S^3 \times S^1) = 3.
\]

This shows \( H' = H_v \), and thus \( H_v \cong \text{SU}(3) \). \( \square \)

We are now ready to describe the isotropy subgroup \( H \) corresponding to the oriented two-plane \( P_0 \). Since \( \text{G}_2 \) and \( G_2^2(\mathbb{R}^7) \) are connected and simply connected, we know \( H \) is connected. If we have \( A \in G_2 \) such that \( A(P_0) = P_0 \) (with orientation), then \( A(i) = i \cos \theta - j \sin \theta, \) \( A(j) = i \sin \theta + j \cos \theta \), thus \( A(k) = A(i)A(j) = k \). The isotropy subgroup \( H \subset \{A \in G_2 | A(k) = k \} \cong \text{SU}(3) \), and since the \( i,j \)-plane is a complex line with respect to our complex structure \( L_k \), \( H \) must preserve this complex line and the complex two-plane perpendicular to it. Hence \( H \subset S(U(1)U(2)) \subset \text{SU}(3) \). A dimension count shows \( H = S(U(1)U(2)) \cong U(2) \).
If we look on the Lie algebra level, and we take \( \{i, j, k, \varepsilon, i\varepsilon, j\varepsilon, k\varepsilon\} \) as our basis for \( \text{Im}(Q) \), then using the automorphism property we can write \( g_2 \subset \mathfrak{so}(7) \) in the following way:

\[
\mathfrak{g}_2 = \text{span}\{E_{12} + E_{56}, E_{47} + E_{56}, E_{45} + E_{67}, E_{46} - E_{57}, E_{46} + E_{57} + 2E_{13}, E_{67} - E_{45} + 2E_{23}, E_{14} - E_{27} - 2E_{36}, E_{15} + E_{26} - 2E_{37}, E_{16} - E_{25} + 2E_{34}, E_{17} + 2E_{35}, E_{14} + E_{27} - E_{15} + E_{26}, E_{16} + E_{25} - E_{17} + E_{24}\}.
\]

We use the inner product on \( g_2 \) given by \( Q(X, Y) = -\frac{1}{2} \text{tr}(XY) \), in which the basis for \( g_2 \) above is orthogonal. The subalgebra \( \mathfrak{h} \) corresponds to \( H \cong U(2) \):

\[
u(2) \cong \text{span}\{2E_{12} + E_{56} - E_{47}, E_{47} + E_{56}, E_{45} + E_{67}, E_{46} - E_{57}\}.
\]

The isotropy representation of U(2) is the action of \( \text{Ad} \ U(2) \) on \( \mathfrak{p} \), the \( Q \)-orthogonal complement of \( \mathfrak{u}(2) \) in \( g_2 \). We can use the following fibrations to decompose \( \mathfrak{p} \) into its irreducible \( \text{Ad} \ U(2) \) representations: First,

\[
\mathbb{C}P^2 \cong \text{SU}(3)/U(2) \to G_2 / U(2) \to G_2 / \text{SU}(3) \cong S^6.
\]

The tangent space to the base is isomorphic to \([\mu_3]_{\mathbb{R}}\); when restricted to \( U(2) \) it gives \([\mu_3 \otimes \text{Id}] \oplus (\text{Id} \otimes \mu_2)]_{\mathbb{R}} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \). Thus, \( \mathfrak{p}_1 = [\mu_3 \otimes \text{Id}]_{\mathbb{R}} \), \( \mathfrak{p}_2 = [\text{Id} \otimes \mu_2]_{\mathbb{R}} \). The tangent space to the fibre is \( \mathfrak{p}_3 = [\mu_1 \otimes \mu_2]_{\mathbb{R}} \). We have \( \mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3 \).

\[
\begin{align*}
\mathfrak{p}_1 &= \text{span}\{E_{46} + E_{57} + 2E_{13}, E_{67} - E_{45} + 2E_{23}\}, \\
\mathfrak{p}_2 &= \text{span}\{E_{14} - E_{27} - 2E_{36}, E_{15} + E_{26} - 2E_{37}, E_{16} - E_{25} + 2E_{34}, E_{17} + 2E_{35}\}, \\
\mathfrak{p}_3 &= \text{span}\{E_{14} + E_{27}, E_{26} - E_{15}, E_{16} + E_{25}, E_{24} - E_{17}\}.
\end{align*}
\]

We have a second fibration of our manifold: we claim \( U(2) \subset \text{SO}(4) \subset G_2 \). Before showing this, we briefly discuss the embedding \( \text{SO}(4) \subset G_2 \) and the irreducible symmetric space \( G_2 / \text{SO}(4) \).

**Lemma 5.3.** The quotient \( G_2 / \text{SO}(4) \) is the space of quaternionic subalgebras of the Cayley numbers, \( Q \).

**Proof.** We have \( \text{SO}(4) \cong (\text{Sp}(1) \times \text{Sp}(1))/\{(q_1, q_2) \simeq (-q_1, -q_2)\} \), and it acts on \( Q \cong \mathbb{H} \oplus \mathbb{H}\varepsilon \) by

\[
(q_1, q_2) : a + b\varepsilon \mapsto q_1 a\bar{q}_1 + (q_2 b\bar{q}_1)\varepsilon.
\]

A calculation shows that \( \text{SO}(4) \subset G_2 \), and this embedding of \( \text{SO}(4) \) in \( G_2 \) can also be described as the subgroup of \( G_2 \) which leaves the subalgebra \( \mathbb{H} \cong \text{span}\{1, i, j, k\} \) invariant. \( \square \)

Since \( U(2) \) is the subgroup of \( G_2 \) preserving the plane spanned by \( i \) and \( j \), elements of \( U(2) \) take 1 to itself and \( k \) to itself, hence they preserve \( \text{span}\{1, i, j, k\} \). This also shows \( U(2) \subset \text{SO}(4) \cap \text{SU}(3) \). We also have \( \text{SO}(4) \cap \text{SU}(3) \subset U(2) \): under \( G_2 \), \( 1 \mapsto 1 \); under \( \text{SO}(4) \), \( \text{span}\{1, i, j, k\} \mapsto \text{span}\{1, i, j, k\} \); under \( \text{SU}(3) \), \( k \mapsto k \). Thus \( \text{SO}(4) \cap \text{SU}(3) = U(2) \). Our second fibration is

\[
\text{SO}(4)/U(2) \to G_2 / U(2) \to G_2 / \text{SO}(4).
\]

Here \( \mathfrak{p}_1 \) is tangent to the fibre, \( \mathfrak{p}_2 \) and \( \mathfrak{p}_3 \) are tangent to the base: From the two fibrations we obtain the following Lie bracket relations among the \( \mathfrak{p}_i \)’s: \([\mathfrak{p}_1, \mathfrak{p}_1] \subset \mathfrak{u}(2) \), \([\mathfrak{p}_1, \mathfrak{p}_2] \subset \mathfrak{p}_2 \oplus \mathfrak{p}_3 \), \([\mathfrak{p}_2, \mathfrak{p}_2] \subset \mathfrak{u}(2) \oplus \mathfrak{p}_1 \), and \([\mathfrak{p}_3, \mathfrak{p}_3] \subset \mathfrak{u}(2) \).

Since \( \mathfrak{p}_1 \), \( \mathfrak{p}_2 \), and \( \mathfrak{p}_3 \) are mutually inequivalent, any \( G_2 \)-invariant metric on our space is of the form \( \langle , \rangle = x_1 Q|_{\mathfrak{p}_1} \perp x_2 Q|_{\mathfrak{p}_2} \perp x_3 Q|_{\mathfrak{p}_3} \), with \( x_i > 0 \), for \( i = 1, 2, 3 \).
As in the previous cases we can write the scalar curvature on $G_2^+(\mathbb{R}^7)$ as function of $x_1$, $x_2$, and $x_3$ via the formula given in [W-Z]:

$$S = \frac{1}{2} \sum_i d_i b_i - \frac{1}{4} \sum_{i,j,k} \left( \frac{k}{x_i x_j} \right) x_k.$$  

When we compute the non-zero Lie bracket relations between the $p_i$'s we find that $(\frac{3}{12}) = 4$ and $(\frac{1}{22}) = \frac{16}{3}$. Also we compute (using the basis elements for $g_2$) $b_i = 8$ for each $i$, and of course $d_1 = 2$, $d_2 = 4$, and $d_3 = 4$. This gives us

$$S = 8 \left( \frac{1}{x_1} + \frac{2}{x_2} + \frac{2}{x_3} \right) - 4 \left( \frac{x_1}{x_2 x_3} + \frac{1}{x_1} \right) - 2 \left( \frac{x_1}{x_2 x_3} + \frac{x_2}{x_1 x_3} + \frac{x_3}{x_1 x_2} \right).$$

Then $\tilde{S} = S - \lambda(x_1^2 x_2^2 x_3^2 - 1)$ includes our boundary condition, volume = 1. Einstein metrics on $G_2 / U(2)$ will be critical points of $\tilde{S}$.

$$\frac{\partial \tilde{S}}{\partial x_1} = \frac{16}{3x_1^2} - \frac{4}{3x_2^2} + 2 \left( \frac{x_2}{x_2 x_3} + \frac{x_3}{x_1 x_2} - \frac{1}{x_2 x_3} \right) - 2\lambda x_1 x_2 x_3,$$

$$\frac{\partial \tilde{S}}{\partial x_2} = -\frac{16}{x_2^2} + 8\lambda x_1 + 2 \left( \frac{x_1}{x_2 x_3} + \frac{1}{x_1 x_2} \right) - 4\lambda x_2 x_3 x_4,$$

$$\frac{\partial \tilde{S}}{\partial x_3} = -\frac{16}{x_3^2} + 2 \left( \frac{x_1}{x_2 x_3} + \frac{x_2}{x_1 x_3} - \frac{1}{x_1 x_2} \right) - 4\lambda x_1 x_2 x_3.$$

Now we look for all solutions to $\frac{\partial \tilde{S}}{\partial x_1} = \frac{\partial \tilde{S}}{\partial x_2} = \frac{\partial \tilde{S}}{\partial x_3} = 0$. We solve these using Maple, and we find the following:

- either $x_2 = \frac{1}{2} x_1$ and $x_3 = \frac{3}{2} x_1$
- or $x_2 = \zeta x_1$ and $x_3 = \left( \frac{7}{120} \zeta^4 - \frac{7}{60} \zeta^3 - \frac{151}{96} \zeta^2 + \frac{39}{10} \zeta - \frac{21}{40} \right) x_1$,

where $\zeta$ is a root of $56\zeta^5 - 532\zeta^4 + 1570\zeta^3 - 1891\zeta^2 + 776\zeta - 60 = 0$.

We need both a positive solution to this polynomial in order for $x_2 > 0$, and also

$$\left( \frac{7}{120} \zeta^4 - \frac{7}{60} \zeta^3 - \frac{151}{96} \zeta^2 + \frac{39}{10} \zeta - \frac{21}{40} \right) > 0,$$

so that $x_3 > 0$.

There are exactly three real solutions to the quintic polynomial. We give the approximate values for $x_2$ and $x_3$, setting $x_1 = 1$:

$$x_2 = 0.09953, \quad x_3 = -0.15252,$$
$$x_2 = 0.59713, \quad x_3 = 1.22554,$$
$$x_2 = 5.35063, \quad x_3 = 5.25153.$$

We must eliminate the first of the solutions from the quintic, since it gives a negative value for $x_3$. The solution $x_2 = \frac{1}{2} x_1$ and $x_3 = \frac{3}{2} x_1$ is the symmetric metric; that is, we will see that it is SO(7)-invariant, after projection. If we denote by $\mathfrak{p}$ the Q-orthogonal complement to $\mathfrak{so}(2) \oplus \mathfrak{so}(5)$ in $\mathfrak{so}(7)$, then $\mathfrak{p} = \{ E_{ij} \mid 1 \leq i \leq 2, \ 3 \leq j \leq 7 \}$. When we project $\mathfrak{p}$ to $\mathfrak{p}$, we see that in $\mathfrak{p}_1$, the basis element $\frac{1}{\sqrt{6}} (2E_{13} + E_{46} + E_{57})$ projects to $\frac{2}{\sqrt{6}} E_{13}$, i.e., an element of norm $= \sqrt{2} x_1$ projects to an element of norm $= \sqrt{2} \frac{\sqrt{6}}{3}$. In $\mathfrak{p}_2$, the basis element $\frac{1}{\sqrt{6}} (2E_{36} - E_{14} + E_{27})$ projects to $\frac{1}{\sqrt{6}} (-E_{13} + E_{27})$.  


so norm $= \sqrt{x_2}$ is projected to norm $= \sqrt{\frac{3}{2}}$. In $\mathfrak{p}_3$, basis element $\frac{1}{\sqrt{2}}(E_{14} + E_{27})$ is already an element of $\mathfrak{p}$, so norm $= \sqrt{x_3}$ is projected to norm $= 1$. This shows that the symmetric metric in our basis satisfies $\frac{x_2}{x_1} = \frac{1}{2}$ and $\frac{x_A}{x_1} = \frac{3}{2}$. Thus we end up with two new $G_2$-homogeneous Einstein metrics on the Grassmannian $G_2^+ (\mathbb{R}^7)$.

Remark 5.4. Notice none of these is a fibration metric, since a metric of the first fibration would require $x_1 = x_2$ and a metric of the second fibration would require $x_2 = x_3$. Many examples of Einstein metrics have been obtained by using fibrations over Einstein spaces and with Einstein fibres [Bes, ch.9]. In our example we have two fibrations; in each the fibre and base space are isotropy irreducible, so they must be Einstein. However, in each fibration the isotropy representation of the base, when restricted to $U(2)$, decomposes into the sum of two irreducible subrepresentations. Recall $Q(X,Y) = -\frac{1}{2} \text{tr} XY$ is our comparison metric. The Casimir constant corresponding to $Q$ and to the restriction to $U(2)$ of the isotropy representations for these homogeneous spaces differs on these two subrepresentations. This implies that the O’Neill tensor restricted to horizontal vectors is not a scalar multiple of $Q$. Using Besse’s Proposition 9.70 [Bes, p.253] we see that therefore our fibrations cannot give rise to Einstein metrics.

To see that we have three non-isometric solutions we compare the scale-invariant product: $(S)^{\frac{d}{2}}(V)^{\frac{d}{2}}$, where $S$ is the scalar curvature and $V$ is the volume, and $d$ is the dimension of our homogeneous space, for our three metrics. If we let $x_2 = \frac{1}{2} x_1$ and $x_3 = \frac{3}{2} x_1$, we obtain $S = \frac{100}{167}$ and $V = \frac{3^4}{2^6} x_1^{10}$, so $(S)^{\frac{5}{2}}(V)^{\frac{1}{2}} = \frac{2^7 3^{10}}{167}$. This is approximately $2.3148 \times 10^7$. The second solution gives $(S)^{\frac{5}{2}}(V)^{\frac{1}{2}} \approx 2.3044 \times 10^7$, and the third solution gives $(S)^{\frac{5}{2}}(V)^{\frac{1}{2}} \approx 1.5836 \times 10^7$.

We note that these have been found previously in [A] and [K]. They observe that one of the three metrics is Kähler and the other two are not Kähler for any complex structure on $M$. Neither author observes that the Kähler Einstein metric is the symmetric metric.

6. $G_4^+ (\mathbb{R}^8)$

We can write the Grassmann manifold of oriented three-planes in $\mathbb{R}^8$ homogeneously $G_4^+ (\mathbb{R}^8) \cong SO(8)/SO(3)SO(5)$, but we can also write it as $G_4^+ (\mathbb{R}^8) \cong Spin(7)/SO(4)$. With respect to $SO(8)$, $G_4^+ (\mathbb{R}^8)$ is irreducible, and therefore the symmetric metric is Einstein and it is the unique $SO(8)$-invariant metric, up to scaling. However, we find there are two more $Spin(7)$-invariant Einstein metrics which are not symmetric.

We first describe how $Spin(7)$ sits inside $SO(8)$. We again identify $\mathbb{R}^8$ with the Cayley numbers $\Omega$; from Murakami [M] we know

$$Spin(7) = \{ A \in SO(8) | \exists B \in SO(8) \text{ such that } B(x)A(y) = A(xy) \forall x, y \in \Omega\}.$$  

In this definition notice $B(1) = 1$, so $B \in SO(7)$ and if $A$ corresponds to $B$, $-A$ corresponds to $B$ as well, which shows $Spin(7)$ is indeed a double cover of $SO(7)$. We also remark that that $\{L_a | a \in \text{Im}(\Omega), |a| = 1\} \subset Spin(7)$. (The corresponding $B$ is conjugation by $a$.) We need to show that $C_a(x)L_a(y) = L_a(xy)$ for any $x, y \in \Omega$. Because $a$ is a unit imaginary octonian, $a^{-1} = -a$, thus $C_a(x)L_a(y) = (axa^{-1})(ay) = -(axa)(ay)$. Then the first Moufang identity tells us that $-(axa)(ay) = -a(x(aa)y)$, and using that $aa = -1$, we have $-a(x(aa)y) = a(xy) = L_a(xy)$.
It is also convenient to identify two subgroups of \( \text{Spin}(7) \), they are \( G_2 \), the automorphisms of the Cayley numbers, and \( \text{SU}(4) \), complex linear maps with respect to \( L_i \). We see \( G_2 \subset \text{Spin}(7) \) by letting \( B = A \) in the definition of \( \text{Spin}(7) \). Murakami shows how to see that \( \text{SU}(4) \subset \text{Spin}(7) \) with the following lemma [M].

**Lemma 6.1.** In \( \text{SO}(8) \), \( U(4) = \{ A \in \text{SO}(8) \mid iA(x) = A(ix) \ \forall x,y \in \mathbb{O} \} \) and \( U(4) \cap \text{Spin}(7) = \text{SU}(4) \).

**Proof.** Let \( p^+ : \text{Spin}(7) \to \text{SO}(7) \) be the homomorphism sending \( A \mapsto B \), for \( A \) and \( B \) in the definition of \( \text{Spin}(7) \). For every \( A \in U(4) \cap \text{Spin}(7) \) we have \( B(i) = i \), so \( p^+(U(4) \cap \text{Spin}(7)) \subset \text{SO}(6) \). And for every \( B \in \text{SO}(6) \) \( (B(i) = i) \), the corresponding \( A \) must be in \( U(4) \cap \text{Spin}(7) \), hence
\[
\text{SO}(6) \subset p^+(U(4) \cap \text{Spin}(7)).
\]
Furthermore, \( p^+ \) is a local isomorphism, thus \( U(4) \cap \text{Spin}(7) \) is a 15-dimensional connected Lie group with a simple Lie algebra. Observe that \( \text{SU}(4) \) is the commutator subgroup of \( U(4) \). Since its Lie algebra is simple, \( U(4) \cap \text{Spin}(7) \) is its own commutator subgroup, thus \( U(4) \cap \text{Spin}(7) \subset \text{SU}(4) \), and a dimension count tells us that these subgroups are equal. \( \Box \)

We embed \( \text{SU}(4) \subset \text{SO}(8) \) via \( A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \). To embed \( \text{SU}(4) \) into \( \text{SO}(8) \) in this way, we want \( \mathbb{R}^4 \oplus \mathbb{R}^4 \cong \mathbb{C}^4 \) with \( (u,v) \mapsto u + iv \); this restricts our choice of ordered bases for \( \mathbb{O} \): we choose \( \{1, j, \varepsilon, j\varepsilon, i, k, i\varepsilon, -k\varepsilon \} \). The intersection of our two subgroups is \( G_2 \cap \text{SU}(4) = \text{SU}(3) \), and this time \( \text{SU}(3) \) in \( G_2 \) fixes \( i \) instead of \( k \).

We check that \( \text{Spin}(7) \) acts transitively on \( G_3^+(\mathbb{R}^8) \): Let \( P_0 = \text{span}\{i, j, k\} \), an oriented three-plane through the origin. Let \( P \) be given by \( \text{span}\{v_1, v_2, v_3\} \), where \( v_1, v_2 \) and \( v_3 \) are ordered orthonormal vectors. Without loss of generality we may assume \( v_1, v_2 \in \text{Im}(\mathbb{O}) \), since \( P \) is a three-plane, so \( \dim(P \cap \text{Im}(\mathbb{O})) \geq 2 \). We know from the previous example that we can find an element \( A \in G_2 \) such that \( A(i) = v_1, A(j) = v_2 \). Let \( x = A^{-1}(v_3) \). Observe that
\[
\langle x, i \rangle = \langle A^{-1}(v_3), i \rangle = \langle v_3, v_1 \rangle = 0,
\]
\[
\langle x, j \rangle = \langle A^{-1}(v_3), j \rangle = \langle v_3, v_2 \rangle = 0.
\]
We claim there exists \( A' \in \text{Spin}(7) \) such that \( A'(i) = i, A'(j) = j \), and \( A'(k) = x \). This is because the subgroup of \( \text{Spin}(7) \) fixing one unit octonian is conjugate to \( G_2 \), and the subgroup of \( \text{Spin}(7) \) fixing two orthonormal octonians is conjugate to \( \text{SU}(3) \), which acts transitively on \( S^6(1) \subset \text{span}\{i, j\}^\perp \). The composition \( A \circ A' \in \text{Spin}(7) \) is our map taking \( i \mapsto v_1, j \mapsto v_2 \), and \( k \mapsto v_3 \), so that \( P_0 \) goes to \( P \), and this shows \( \text{Spin}(7) \) acts transitively on \( G_3^+(\mathbb{R}^8) \).

Next we must determine the isotropy subgroup \( H \) of \( P_0 \). We claim that \( H \subset G_2 \). To see this, we note that if \( A(P_0) = P_0 \), then if \( B \) is the element of \( \text{SO}(7) \) in the definition of \( \text{Spin}(7) \) corresponding to \( A \), since \( B(i)A(j) = A(k) \) we know that \( B(i) \in \text{span}\{i, j, k\} \). Thus \( A(1) = -B(i)A(i) \in \text{span}\{1, i, j, k\} \), and furthermore \( A(1) \perp P_0 \), so \( A(1) = \pm 1 \). Since \( \text{Spin}(7) \) and \( G_3^+(\mathbb{R}^8) \) are connected and simply connected, we know \( H \) is connected, hence \( A(1) = 1 \). From the definition of \( \text{Spin}(7) \) it follows that \( A = B \) and this implies \( H \subset G_2 \). Furthermore, any element of \( H \) takes 1 to itself and preserves the standard quaternionic subalgebra \( \text{span}\{1, i, j, k\} \). Thus \( H \subset \text{SO}(4) \subset G_2 \), and by a dimension count \( H = \text{SO}(4) \).
Now we are ready to find the isotropy representation. On the Lie algebra level we have

\[
\text{spin}(7) = \text{span}\{E_{ij} + E_{i+j,i+j}, E_{i,i+j} + E_{j,i+j} \mid 1 \leq i < j \leq 4\}
\]

\[\oplus \text{span}\{E_{i,i+j} - E_{i+j,i} \mid 1 \leq i \leq 3\}\]

\[\oplus \text{span}\{E_{27} - E_{45}, E_{23} + E_{58}, E_{24} - E_{57}, E_{28} + E_{35},
E_{56} - E_{78}, 2E_{25} - E_{38} + E_{47}\}.
\]

The subalgebra corresponding to the isotropy subgroup is \(\mathfrak{h} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)\):

\[
\mathfrak{h} = \text{span}\{E_{37} - E_{48}, E_{34} + E_{78}, E_{38} + E_{47}\}
\]

\[\oplus \text{span}\{2E_{56} + E_{34} - E_{78}, 2E_{26} - E_{48} - E_{37}, 2E_{25} - E_{38} + E_{47}\}.
\]

Notice each copy of \(\mathfrak{su}(2)\) is an ideal in \(\mathfrak{h}\) and its basis vectors above are orthogonal with respect to the inner product on \(\text{spin}(7)\) given by \(Q(X,Y) = -\frac{1}{2} \text{tr}(XY)\). As usual we denote by \(\mathfrak{p}\) the \(Q\)-orthogonal complement of \(\mathfrak{h}\) in \(\text{spin}(7)\).

There are two fibrations of our symmetric space \(\text{Spin}(7)/\text{SO}(4)\). The first is

\[G_2/\text{SO}(4) \to \text{Spin}(7)/\text{SO}(4) \to \text{Spin}(7)/G_2 \cong S^7.
\]

Let \(p'\) be the subspace tangent to the fibre; let \(p''\) be the subspace tangent to the base. Let \(\theta_k\) denote the unique irreducible complex representation of \(\mathfrak{su}(2)\) in \(k\) dimensions; the first fibration tells us that \(p' = [\theta_2 \oplus \theta_4]_{\text{R}}\), since this is the representation of the symmetric space \(G_2/\text{SO}(4)\) [W, p.287]. The isotropy representation of \(\text{Spin}(7)/G_2\) is the seven-dimensional representation of \(G_2 \subset \text{SO}(7)\). We restrict this representation to \(\text{SO}(4)\), to see that \(p'' = [\rho_3 \oplus \text{Id}] \oplus [\theta_2 \oplus \theta_2]_{\text{R}}\), where \(\rho_3\) denotes the standard representation of \(\mathfrak{so}(3)\) on \(\mathbb{R}^3\). We let \(p_1 = [\rho_3 \oplus \text{Id}]\), and \(p_2 = [\theta_2 \oplus \theta_2]_{\text{R}}\), and \(p_3 = [\theta_2 \oplus \theta_4]_{\text{R}}\). We have \(\dim(p_1) = 3\), \(\dim(p_2) = 4\), and \(\dim(p_3) = 8\).

For the second fibration, we first need some explanation.

**Lemma 6.2.** The compact group \((\text{Spin}(4) \times \text{Spin}(3))/\{\pm \text{Id}, \text{Id}\}\) satisfies \(\text{SO}(4) \subset (\text{Spin}(4) \times \text{Spin}(3))/\{\pm \text{Id}, \text{Id}\} \subset \text{Spin}(7)\).

**Proof.** Recall, for any \(n\), \(\text{Spin}(n)\) is the simply connected double cover of \(\text{SO}(n)\); let \(\pi : \text{Spin}(n) \to \text{SO}(n)\) denote the two-fold homomorphism. Because \(\pi_1(\text{SO}(k)) \to \pi_1(\text{SO}(n))\) is a surjection, \(\pi(\text{Spin}(k)) = \text{SO}(k)\) for all \(k \leq n\). Observe that \(\ker(\pi) = \{\pm \text{Id}, \text{Id}\} \cap \text{Spin}(n - k)\), thus it is \((\text{Spin}(k) \times \text{Spin}(n - k))/\{\pm \text{Id}, \text{Id}\} \subset \text{Spin}(n)\). In \(\text{Spin}(7)\) we can consider the subgroup \((\text{Spin}(3) \times \text{Spin}(3))/\{\pm \text{Id}, \text{Id}\}\). We know \(\text{Spin}(4) \cong \text{Spin}(3) \times \text{Spin}(3)\) (and \(\text{Spin}(3) \cong \text{SU}(2)\)). Thus \(\text{Spin}(7)\) has a subgroup \((\text{Spin}(3) \times \text{Spin}(3) \times \text{Spin}(3))/\{\pm \text{Id}, \text{Id}\}\). Our isotropy subgroup \(\text{SO}(4) \subset \text{Spin}(7)\) is exactly \((\Delta \text{Spin}(3) \times \text{Spin}(3))/\{\pm \text{Id}, \text{Id}\}\), where \(\Delta \text{Spin}(3)\) is the diagonal subgroup of \(\text{Spin}(3) \times \text{Spin}(3)\) isomorphic to \(\text{Spin}(3)\). This can be seen via the restriction to \(\text{SO}(4)\) of the homomorphism from \(\text{Spin}(7)\) to \(\text{SO}(7)\) taking \(A\) to \(B\) (\(A\) and \(B\) in the definition of \(\text{Spin}(7)\)).

We obtain the following fibration:

\[
S^3 \cong \frac{\text{Spin}(4) \times \text{Spin}(3)}{\{\pm \text{Id}, \text{Id}\}} \to \frac{\text{Spin}(3) \times \Delta \text{Spin}(3)}{\{\pm \text{Id}, \text{Id}\}} \to \frac{\text{Spin}(7)/\text{SO}(4)}{G_3^+ (\mathbb{R}^7)} \cong G_3^+ (\mathbb{R}^8).
\]

\[
\frac{\text{Spin}(7)/\text{Spin}(4) \times \text{Spin}(3)}{\{\pm \text{Id}, \text{Id}\}} \cong G_3^+ (\mathbb{R}^7).
\]
In the second fibration, it is $p_1$ which is the subspace tangent to the fibre, while $p_2 \oplus p_3$ is the subspace tangent to the base.

$$p_1 = \text{span}\{3E_{12} - E_{34} + E_{56} + E_{78}, 3E_{15} - E_{26} - E_{37} - E_{48}, 3E_{16} + E_{25} + E_{38} - E_{47}\},$$

$$p_2 = \text{span}\{3E_{13} + E_{24} + E_{57} - E_{68}, 3E_{14} - E_{23} + E_{58} + E_{67}, 3E_{17} - E_{28} + E_{35} + E_{46}, 3E_{18} + E_{27} - E_{36} + E_{45}\},$$

$$p_3 = \text{span}\{E_{23} + E_{57}, E_{24} + E_{68}, E_{27} + E_{36}, E_{28} + E_{46}, 2E_{57} - E_{24} + E_{68}, 2E_{68} - E_{23} - E_{67}, 2E_{35} + E_{28} - E_{46}, 2E_{45} - E_{27} + E_{36}\}.$$ 

Since each of the $p_i$‘s has a different dimension, they are inequivalent representations. This means any Spin(7)-invariant metric on $G_3^+(\mathbb{R}^8)$ is determined by an inner product on $p$ satisfying

$$\langle , \rangle = x_1 Q|_{p_1} \perp x_2 Q|_{p_2} \perp x_3 Q|_{p_3}, \text{ for } x_1, x_2, x_3 > 0.$$ 

From the fibrations we obtain the following Lie bracket relations:

$$[p_1, p_1] \subset \mathfrak{h} \oplus p_1, \quad [p_1, p_2] \subset p_2 \oplus p_3, \quad [p_2, p_2] \subset \mathfrak{h} \oplus p_1, \quad [p_3, p_3] \subset \mathfrak{h}.$$ 

Recall the scalar curvature formula from [W-Z]:

$$S = \frac{1}{2} \sum_i d_i b_i - \frac{1}{4} \sum_{i,j,k} (k) x_i x_j x_k.$$ 

On Spin(7) we find that $b_i = 10$ for $i = 1, 2, 3$, and we know $d_1 = 3$, $d_2 = 4$, and $d_3 = 8$. From the Lie bracket relations we know $(1, 1), (1, 2), (1, 3) \neq 0$; all other triples (except rearrangements) are zero. We find that $(1, 1) = 2$, $(1, 2) = 4$, and $(1, 3) = 8$.

We now have the scalar curvature function in $x_1, x_2, x_3$:

$$S = \frac{25}{2x_1^2} + \frac{20}{x_2} + \frac{40}{x_3} - \frac{x_1}{x_2^2} - 4 \left( \frac{x_1}{x_2 x_3} + \frac{x_2}{x_1 x_3} + \frac{x_3}{x_1 x_2} \right).$$ 

We want the critical points of the scalar curvature function with the constraint equation of volume 1: $\bar{S} = S - \lambda (x_1^3 x_2 x_3^5 - 1)$.

$$\frac{\partial \bar{S}}{\partial x_1} = -\frac{25}{2x_1^2} - \frac{1}{x_2^2} - \frac{4}{x_2 x_3} + \frac{4x_2}{x_1^2 x_3} + \frac{4x_3}{x_1^2 x_2} - 3\lambda x_1^3 x_2^2 x_3^4,$$

$$\frac{\partial \bar{S}}{\partial x_2} = -\frac{20}{x_2} + \frac{2x_1}{x_2^2} + \frac{4x_1}{x_1^2 x_3} - \frac{4}{x_1 x_3} + \frac{x_3}{x_1 x_2} - 4\lambda x_1^3 x_2^3 x_3^8,$$

$$\frac{\partial \bar{S}}{\partial x_3} = -\frac{40}{x_3} + \frac{4x_1}{x_2 x_3} + \frac{4x_2}{x_1 x_3} - \frac{4}{x_1 x_2} - 8\lambda x_1^3 x_2^4 x_3^7.$$ 

A solution to $\frac{\partial \bar{S}}{\partial x_1} = \frac{\partial \bar{S}}{\partial x_2} = \frac{\partial \bar{S}}{\partial x_3} = 0$ is equivalent to the simultaneous solution of the following two polynomials:

$$10x_1 x_2^2 - 10x_1 x_2 x_3 + x_1^2 x_2 x_3 + x_1^2 x_3 - 3x_2^3 = 0,$$

$$-11x_1^2 x_2 + 11x_2 x_3^2 + 5x_3^3 - 25x_2^3 x_3 + 30x_1 x_2^2 - 2x_1^2 x_2 = 0.$$ 

We obtain three solutions, using Maple. The first is the symmetric solution: $x_1 = \frac{3}{4} x_3$, $x_2 = \frac{1}{4} x_3$. Let $\mathfrak{p}$ denote the $Q$-orthogonal complement to $\mathfrak{so}(3) \oplus \mathfrak{so}(5)$ in
The polynomial has three real roots, of which two are positive and yield two positive \( \eta \) for \( x \). We give the approximate values, setting \( x_3 = 1 \):

\[
\begin{align*}
&x_1 = -0.241854, \quad x_2 = -4.177304, \\
&x_1 = 0.25179, \quad x_2 = 0.902192, \\
&x_1 = 1.100300, \quad x_2 = 0.369813.
\end{align*}
\]

These are two new Einstein metrics on \( G_3^+ (\mathbb{R}^8) \).

Remark 6.3. None of these is a fibration metric, since the first fibration required \( x_1 = x_2 \), and the second required \( x_2 = x_3 \). Just as in the previous example in both fibrations the fibre and base space are isotropy irreducible, therefore Einstein. However, the isotropy representation of each base, when restricted to \( SO(4) \), again decomposes into the sum of two irreducible subrepresentations, where the Casimir constant corresponding to \( Q \) and to the restriction to \( SO(4) \) of the isotropy representations for these homogeneous spaces differs. Again this implies the O’Neill tensor restricted to horizontal vectors is not a scalar multiple of \( Q \). Using Besse’s Proposition 9.70 [Bes, p.253] we know our fibrations cannot give rise to Einstein metrics.

We verify that they are all distinct by estimating the (scale-invariant) product:

\[
(S)_{\frac{\pi}{2}}^\frac{1}{2} (V)^\frac{1}{2},
\]

where \( S \) is the scalar curvature of the metric and \( V \) is the volume of the metric. For the symmetric metric, \( S = \frac{80}{x_3} \eta \) and \( V = \frac{3\pi}{2x_3^3} x_3^{15} \), and so \( (S)_{\frac{\pi}{2}}^\frac{1}{2} (V)^\frac{1}{2} \approx 1.84200 \times 10^{13} \). For the two new metrics, we find that \( (S)_{\frac{\pi}{2}}^\frac{1}{2} (V)^\frac{1}{2} \approx 1.80936 \times 10^{13} \), and \( (S)_{\frac{\pi}{2}}^\frac{1}{2} (V)^\frac{1}{2} \approx 1.61159 \times 10^{13} \), respectively. This shows that they are non-isometric.

7. Appendix

Oniščik in fact lists more triples of Lie algebras \( (\mathfrak{g}, \mathfrak{g}', \mathfrak{g}'') \), but the extra triples can be obtained by combining the information given on page 155. For example, in addition to \( SO(2n)/SO(2n-1) = SU(n)/SU(n-1) \) and \( SO(4n)/SO(4n-1) = Sp(n)/Sp(n-1) \), he lists \( SU(2n)/SU(2n-1) = Sp(n)/Sp(n-1) \), which follows...
from the inclusions $\text{Sp}(n) \subset \text{SU}(2n) \subset \text{SO}(4n)$. Many of the triples on his list come from the subgroups of $\text{SO}(8)$: In addition to $\text{SO}(7) \subset \text{SO}(8)$, we have

$$\text{Sp}(2) \subset \text{Sp}(2) \text{U}(1) \subset \text{Sp}(2) \text{Sp}(1) \subset \text{SO}(8),$$

$$\text{U}(2) \subset \text{SU}(3) \subset \text{SU}(4) \subset \text{U}(4) \subset \text{SO}(8),$$

and $\text{SO}(4) \subset \text{G}_2 \subset \text{Spin}(7) \subset \text{SO}(8)$.

We note that $\text{SO}(8)$ contains two copies of $\text{Spin}(7)$ and that there is an outer automorphism of $\text{SO}(8)$ of order three, called the triality automorphism, which interchanges the $\text{Spin}(7)'s$, and on the Lie algebra level, it interchanges the $\text{spin}(7)'s$ and the standard embedding of $\text{so}(7)$. This yields equalities like the following:

$$\text{SO}(8)/\text{Spin}(7) = \text{SO}(6)/\text{SU}(3)$$

(with double cover $\text{SO}(8)/\text{SO}(7) = \text{SU}(4)/\text{SU}(3)$)

$$\text{SO}(8)/\text{Spin}(7) = \text{SO}(5)/\text{SU}(2)$$

(with double cover $\text{SO}(8)/\text{SO}(7) = \text{Sp}(2)/\text{Sp}(1)$).

We include some intersections of subgroups of $\text{SO}(8)$ and related equalities:

$$\text{G}_2 = \text{SO}(7) \cap \text{Spin}(7) \text{ implies } \text{SO}(8)/\text{SO}(7) = \text{Spin}(7)/\text{G}_2;$$

$$\text{Sp}(1) \text{Sp}(1) = \text{SO}(7) \cap \text{Sp}(2) \text{Sp}(1) \text{ implies } \text{SO}(8)/\text{SO}(7) = \text{Sp}(2) \text{Sp}(1)/\text{Sp}(1) \text{Sp}(1);$$

$$\text{SU}(3) = \text{SO}(7) \cap \text{SU}(4) \text{ implies } \text{SO}(8)/\text{SO}(7) = \text{SU}(4)/\text{SU}(3).$$

Here are the non-symmetric homogeneous spaces on Oniščik’s list [O1]:

$$\text{SO}(7)/\text{SO}(5) = \text{G}_2/\text{SU}(2) = V_2(\mathbb{R}^7)$$

$$\text{SO}(8)/\text{SO}(6) = \text{Spin}(7)/\text{SU}(3) = V_2(\mathbb{R}^8)$$

$$\text{SO}(8)/\text{SO}(5) = \text{Spin}(7)/\text{SU}(2) = V_3(\mathbb{R}^8)$$

$$\text{SO}(8)/\text{SO}(2) \text{SO}(5) = \text{Spin}(7)/\text{SO}(2) \text{SU}(2)$$

$$\text{SO}(16)/\text{Spin}(9) = \text{SO}(15)/\text{Spin}(7)$$

$$\text{SO}(2n)/\text{SU}(n) = \text{SO}(2n-1)/\text{SU}(n-1)$$

$$\text{SO}(4n)/\text{Sp}(n) = \text{SO}(4n-1)/\text{Sp}(n-1)$$

$$\text{SO}(4n)/\text{Sp}(n) \text{U}(1) = \text{SO}(4n-1)/\text{Sp}(n-1) \text{U}(1)$$

$$\text{SO}(4n)/\text{Sp}(n) \text{Sp}(1) = \text{SO}(4n-1)/\text{Sp}(n-1) \text{Sp}(1).$$

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