A TRANVERSALITY THEOREM FOR HOLOMORPHIC MAPPINGS AND STABILITY OF EISENMAN-KOBEYASHI MEASURES

SH. KALIMAN AND M. ZAIDENBERG

Abstract. We show that Thom’s Transversality Theorem is valid for holomorphic mappings from Stein manifolds. More precisely, given such a mapping \( f : S \rightarrow M \) from a Stein manifold \( S \) to a complex manifold \( M \) and given an analytic subset \( A \) of the jet space \( J^k(S, M) \), \( f \) can be approximated in neighborhoods of compacts by holomorphic mappings whose \( k \)-jet extensions are transversal to \( A \). As an application the stability of Eisenman-Kobayshi intrinsic \( k \)-measures with respect to deleting analytic subsets of codimension \( > k \) is proven. This is a generalization of the Campbell-Howard-Ochiai-Ogawa theorem on stability of Kobayashi pseudodistances.

1. Introduction

1.1. Let \( X \) and \( M \) be connected complex manifolds. Denote by \( \text{Hol}(X, M) \) the space of holomorphic mappings \( X \rightarrow M \) and by \( J^k(X, M) \) the space of \( k \)-jets of holomorphic mappings \( X \rightarrow M \). We say that \( A \subset J^k(X, M) \) is a stratified analytic subset if \( A \) is a closed analytic subset in \( J^k(X, M) \) (regarded as a complex manifold) with a stratification which satisfies Whitney’s condition (a). Recall that for every closed analytic subset such stratification always exists [W]. The symbol \( j_k(f) \mathrel{\mathbin{\mathop:}^A} A \) where \( f \in \text{Hol}(X, M) \) means that the \( k \)-jet extension \( j_k(f) : X \rightarrow J^k(X, M) \) of \( f \) is transversal to each stratum of the given stratification of \( A \).

The main result of this paper is the following analytic version of Thom’s Transversality Theorem [T]. (To our surprise we did not find this fact in the literature, and it must be formulated.)

1.2. Transversality Theorem. Let \( X \) and \( M \) be as above. Suppose that \( X \) is Stein and \( M \) is endowed with a Hermitian metric \( h \). Let \( A_i \subset J^k(X, M) \) be a sequence of stratified analytic subsets where \( k_i \) is a nonnegative integer and \( i = 1, 2, \ldots \). Let \( f \in \text{Hol}(X, M) \) and let \( Y \) be a closed analytic subset of \( X \) such that \( j^{k_i}(f) \mathrel{\mathbin{\mathop:}^Y} A_i \) for every \( i \) (where the restriction is regarded as a mapping from \( Y \) to \( J^{k_i}(X, M) \)). Then for each compact subset \( K \subset X \), every natural \( k \), and every \( \varepsilon > 0 \) there exist an open neighborhood \( \Omega \) of \( K \) in \( X \) and a holomorphic mapping \( \tilde{f} = f_{\varepsilon, K, k} : \Omega \rightarrow M \) such that

\[
(1) \quad j^{k_i}(\tilde{f}) \mathrel{\mathbin{\mathop:}^{\Omega}} A_i, \quad i = 1, 2, \ldots;
\]

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(2) \( j^k(f)_{|Y \cap \Omega} = j^k(f)_{|Y \cap \Omega} \);
(3) \( \text{dist}_h(f_{|\Omega}, f) < \varepsilon \) uniformly in \( \Omega \).

In the algebraic setting this theorem can be strengthened due to the Demailly-Lempert-Shiffman Approximation Theorem [DLS].

1.3. Corollary. Let the assumption of Theorem 1.2 hold. Suppose also that \( X \) is an affine algebraic variety, \( M \) is a smooth quasiprojective variety, \( Y \) is a closed algebraic subvariety of \( X \), and for some \( k > 0 \) the mapping \( j^k(f)_{|Y} : Y \to J^k(X, M) \) is a regular mapping. Then for each compact \( K \subset X \), every \( \varepsilon > 0 \), and every natural \( n \) there exist a neighborhood \( \Omega \) of \( K \) in \( X \) and a Nash algebraic mapping \( \bar{f} : \Omega \to M \) so that conditions (1) for \( i = 1, \ldots, n \), (2), (3) from Theorem 1.2 hold.

Recall that \( \bar{f} \) being Nash algebraic means that the graph of \( \bar{f} \) is contained in an algebraic subvariety \( N \subset X \times M \) where \( \dim N = \dim X \); the same holds for the image of \( \bar{f} \).

1.4. Let \( M \) be a connected complex manifold of complex dimension \( m > 0 \). Then for any \( k = 1, \ldots, m \) the Eisenman \( k \)-measure \( E^k_M \) is intrinsically defined on the set of decomposable \( k \)-wedge vectors in \( \Lambda^k TM \) and it has the following basic properties:
- every mapping of the \( k \)-wedge vector bundles of complex manifolds generated by a holomorphic mapping does not increase Eisenman measures;
- \( E^k_M \) coincides with the Bergman volume in the case when \( M \) is the unit ball \( B^k \) in \( \mathbb{C}^k \);

Moreover, the Eisenman \( k \)-measure is the largest measure on decomposable \( k \)-wedge vectors with the above two properties. Further information on Eisenman measures and their applications may be found in [E], [P], [Ko 1], [Ko 2], [LZ], [PS], [GW], [Ka 2]. Mention also that \( E^k_M \) is nothing but the Kobayashi-Royden infinitesimal pseudometric \( K_M \) on \( TM \), and \( E^m_M \) is the Eisenman-Kobayashi pseudovolume form on \( \Lambda^m TM \).

By Campbell-Howard-Ochiai-Ogawa’s theorem [CHO], [CO] the Kobayashi pseudodistance \( k_M \) of \( M \) does not change under deleting analytic subsets of \( M \) of codimension at least 2. From the Transversality Theorem 1.2 we obtain a generalization of this result.

1.5. Corollary. Let \( A \) be a closed analytic subset of \( M \) of codimension at least \( k+1 \). Then \( E^k_M \mid A = E^k_M \mid M \setminus A \). In particular, for \( k = 1 \) we have \( K_{M \setminus A} = K_M \mid M \setminus A \) if \( \text{codim}_M A \geq 2 \).

By Royden’s Theorem [Ro] the Kobayashi pseudodistance \( k_M \) is the integrated form of \( K_M \). Therefore, \( k_{M \setminus A} = k_M \mid M \setminus A \) if \( \text{codim}_M A \geq 2 \) which is the Campbell-Howard-Ochiai-Ogawa Stability Theorem.

1.6. It is easily seen that the Eisenman-Kobayashi volume of the unit ball does change after deleting the origin (see [Ra] for qualified estimates). Thus in general condition \( \text{codim}_M A \geq k+1 \) of Corollary 1.5 cannot be made weaker. However, by a result of [Ka 1] if \( A \) is a smooth closed submanifold of \( M \) of codimension \( \ell \geq 2 \) then in some cases deleting \( A \) affects the Eisenman \( k \)-measure \( E^k_M \) exactly in the same way as blowing up \( \sigma_A : M_A \to M \) of \( M \) with centrum at \( A \). More precisely, if either \( \ell \geq \left\lceil \frac{k+1}{2} \right\rceil + 2 \) or the \( (\ell-1) \)-st Chern class of the normal bundle of \( A \) is trivial then \( \sigma_A^* E^k_M \mid M_A \setminus E = E^k_M \mid M_A \setminus E \) where \( E \) is the exceptional divisor of \( \sigma_A \) in \( M_A \). In other words, the Eisenman \( k \)-measure does not change under deleting of a
smooth divisor which can be contracted onto a smooth submanifold of codimension \(\geq \left\lceil \frac{c}{d} \right\rceil + 2\). In particular, deleting of the exceptional divisor of a blow-up with centrum at a point does not affect any of Eisenman \(k\)-measures. This stability result was a motivation for the present paper.

1.7. Returning to the Transversality Theorem note that in general an approximation of a given holomorphic mapping from a Stein manifold \(X\) by transversal ones (which does exists in neighborhoods of compacts, by Theorem 1.2) cannot be found on the whole \(X\) due to certain rigidity phenomena. This can be shown by simple examples.

1.8. Examples. a) Consider a smooth complex surface \(S\) with a \((-1)\)-curve \(E \cong \mathbb{P}^1\). Let \(f: \mathbb{C} \to \mathbb{P}^1 \cong E \hookrightarrow S\) be an embedding. Put \(A = f(0) \in S\). Then \(f\) cannot be uniformly approximated by holomorphic mappings \(\tilde{f}: \mathbb{C} \to S\) transversal to \(A\), since after blowing \(E\) down the image of \(\tilde{f}(\mathbb{C})\) would be contained in a ball with center at the image of \(E\), but not coincide with this point. Thus, \(\tilde{f}\) itself must be constant which is absurd. Moreover, if \(S\) is a blow-up of a hyperbolic surface (for instance, \(S\) is the blow-up of the unit ball \(B^2\) in \(\mathbb{C}^2\) at the origin) then the above \(f\) cannot be approximated (not necessarily uniformly) on compacts by \(f \in \text{Hol}(\mathbb{C}, S)\) transversal to \(A\), by the same reason.

b) Let \(\pi: S \to C\) be a morphism of a smooth projective surface \(S'\) onto a curve \(C\) whose generic fibers are pairwise non-isomorphic curves of genus \(g \geq 2\). Fix a generic fiber \(F = \pi^{-1}(c_0)\) over a point \(c_0 \in C\) and two distinct points \(A, B \in F\). Put \(R = F - \{B\}\). Then \(R\) is an open Riemann surface, i.e. it is Stein. The identical embedding \(f: R \to F \hookrightarrow S\) cannot be approximated uniformly by holomorphic mappings \(\tilde{f}: R \to S\) transversal to \(A\). Indeed, \(\pi \circ f \equiv \text{const} = c \neq c_0\) and, therefore, \(f(R) \subset F_c \neq \pi^{-1}(c)\). Since \(c\) can be assumed to be sufficiently close to \(c_0\) the fiber \(F_c\) is a hyperbolic curve which is not isomorphic to \(F\). But, by the big Picard theorem, the mapping \(f: R \to F_c\) can be extended to an isomorphism \(F \to F_c\) which is impossible.

c) Consider the Stein domain \(D = \{z = (x, y) \in \mathbb{C}^2 | |x| < 1, |xy| < 1\}\). Clearly, the image of every non-constant holomorphic mapping \(\mathbb{C} \to D\) is contained in the \(y\)-axis. Hence the identical embedding of this axis into \(D\) cannot be approximated on compacts by holomorphic mappings \(\mathbb{C} \to D\) transversal to the origin.

d) In [BG] a family of surfaces in \(\mathbb{P}^3\) was constructed which gives a smooth proper morphism \(\pi: M \to \Delta\) of a threefold \(M\) onto the unit disc so that all fibers of \(\pi\) except for the central one \(F_0 = \pi^{-1}(0)\) are hyperbolic, and \(F_0\) is a Fermat surface of degree 50. It is known [G] that any entire curve \(f: \mathbb{C} \to F_0\), \(f \neq \text{const}\), is contained in a finite union of projective lines in \(F_0\). Thus the same is true for every entire curve in \(M\) due to the Liouville theorem and the hyperbolicity of nonzero fibers. Therefore, none of these curves can be pushed away from a point on it by means of small deformations.

2. Preliminaries

Let, as above, \(X\) and \(M\) be connected complex manifolds. By \(J^k_b(X, M)\) we denote the stalk at \(x\) of \(k\)-jets of holomorphic mappings \(X \to M\). If \(V\) is an open subset in \(X\) then the jet space \(J^k_b(V, M)\) can be viewed as a domain in \(J^k_b(X, M)\). For a holomorphic vector bundle \(\zeta = (T, \pi, X)\) we treat \(k\)-jets of its holomorphic sections as \(k\)-jets of holomorphic mappings \(X \to T\).
Let $A$ be a closed analytic subset of a complex manifold $N$ and $\Sigma$ be an analytic stratification of $A$ which satisfies Whitney’s condition (a). By $A_{\Sigma,l}$ we denote the union of all strata of $\Sigma$ of dimension $\leq l$. It is known that $A_{\Sigma,l}$ is a closed analytic subset of $N$. By Whitney’s condition (a), if $f \in \text{Hol}(X,N)$ is transversal to $A_{\Sigma,l}$ then $f$ is transversal to $A$ in a neighborhood of $A_{\Sigma,l}$ as well as any $\tilde{f} \in \text{Hol}(X,N)$ which is sufficiently uniformly close to $f$ in $C^1(X,N)$-topology.

### 3. SOME LEMMAS ON HOLONOMIC VECTOR FIELDS AND COLLECTIVE TRANSVERSALITY

As in the smooth case the proof of the Transversality Theorem is based on the Collective Transversality Lemma (see [T], [AVG]) which is the main result of this section. To prove it we need several simple lemmas on holomorphic vector fields. In the first of them we show that fixing the $k$-jet of a vector field at a point yields fixing the $k$-jet of the associated phase flow at this point.

#### 3.1. Lemma. Let $\nu$ be a germ of a holonomic vector field at the origin $\mathfrak{o} \in \mathbb{C}^r$ with the phase flow $\varphi_{\nu,t}$, where $t$ is the complex time in a neighborhood of the origin in $\mathbb{C}$. If $j^k(\nu)(\mathfrak{o}) = \mathfrak{o}$ for some $k \geq 0$ then for any $t$ with $|t| \ll 1$ one has $j^k(\varphi_{\nu,t})(\mathfrak{o}) = j^k(\text{id}_{\mathbb{C}^r})(\mathfrak{o})$.

**Proof.** Since the phase flow $\varphi_{\nu,t} : (\mathbb{C}^r \times \mathbb{C}, \mathfrak{o} \times 0) \to (\mathbb{C}^r, \mathfrak{o})$ is holomorphic, it has a convergent power series expansion

$$
\varphi_{\nu,t}(x) = x + t\nu(x) + \sum_{i \geq 2} t^i \nu_i(x)
$$

in some neighborhood of the origin $\mathfrak{o} \times 0 \subset C^r \times \mathbb{C}$, where $x \in \mathbb{C}^r$ and $\nu_i(x)$, $i = 2, ..., r$, are germs of holonomic vector fields at $\mathfrak{o} \in \mathbb{C}^r$. Therefore,

$$
j^k(\varphi_{\nu,t})(x) = j^k(\text{id}_{\mathbb{C}^r})(x) + t j^k(\nu)(x) + \sum_{i \geq 2} t^i j^k(\nu_i)(x).
$$

Thus we have to show that the condition $j^k(\nu)(\mathfrak{o}) = \mathfrak{o}$ implies that $j^k(\nu_i)(\mathfrak{o}) = \mathfrak{o}$ for all $i \geq 2$. Note that $\varphi_{\nu,t+s} = \varphi_{\nu,s} \circ \varphi_{\nu,t}$ when $|s|$ and $|t|$ are small enough. Put $\nu_t(x) = \nu(x)$ and compare the two convergent expansions:

$$
\varphi_{\nu,t+s}(x) = x + \sum_{i \geq 1} (t + s)^i \nu_i(x)
$$

and

$$
\varphi_{\nu,s} \circ \varphi_{\nu,t}(x)
$$

$$
= x + \sum_{i \geq 1} t^i \nu_i(x) + s \nu_1 \left( x + \sum_{i \geq 1} t^i \nu_i(x) \right) + \sum_{i \geq 2} s^i \nu_i \left( x + \sum_{n \geq 1} t^n \nu_n(x) \right) + \text{(higher order terms in } t + s). \tag{3}
$$

Suppose by induction that $j^k(\nu_i)(\mathfrak{o}) = \mathfrak{o}$ for $i \leq l$ where $l \geq 1$. Assume that $j^k(\nu_{l+1})(\mathfrak{o}) \neq \mathfrak{o}$. By (3), we obtain

$$
j^k(\varphi_{\nu,t+s})(\mathfrak{o}) = j^k(\text{id}_{\mathbb{C}^r})(\mathfrak{o}) + (t + s)^{l+1} j^k(\nu_{l+1})(\mathfrak{o})
$$

+ (higher order terms in $(t + s)$).
At the same time (4) implies
\[ j^k(\varphi_{v,t} \circ \varphi_{v,t})(\bar{\nu}) = j^k(id_{C^r})(\bar{\nu}) + (t^{s+1} + s^{l+1})j^k(\nu_{t+1})(\bar{\nu}) + \text{(higher order terms in } t \text{ and } s). \]

Since \( l + 1 \geq 2 \) this leads to contradiction. \( \square \)

3.2. Definition. Let \( T \rightarrow N \) be a holomorphic vector bundle on a complex manifold \( N \), \( O(T) \) be the sheaf of germs of holomorphic sections of \( T \), and \( p \) be a point of \( N \). A linear subspace \( V \subset O_p(T) \) will be called \( k \)-\textit{sufficient} if the set of \( k \)-jets of germs from \( V \) coincides with the whole stalk \( J^k_p(T) \), i.e. for any germ \( \nu \in O_p(T) \) there exists a germ \( \mu \in V \) such that \( j^k(\mu)(p) = j^k(\nu)(p) \). A linear subspace \( W \subset H^0(N, O(T)) \) will be called \( k \)-\textit{sufficient} at \( p \) if the subspace \( V \) of germs at \( p \) of sections from \( W \) is \( k \)-sufficient.

3.3. Denote by \( T_p \) the \( \mathcal{O} \)-module of germs of holomorphic vector fields at the origin \( \bar{\nu}_p \in C^s \). If \( V \subset T_p \) is a finite dimensional subspace then all germs from \( V \) can be represented by vector fields holomorphic in a common neighborhood \( U \ni \bar{\nu}_p \in C^s \). Moreover, one can choose \( U \) so that all phase flows \( \varphi_{\nu,t} \) are defined correctly on \( U \) for every \( \nu \in V \) and sufficiently small \( |t| \). Furthermore, we may suppose that \( \varphi_{\nu,t} \) is defined correctly on \( U \) for every \( t \in \mathbb{C} \setminus \{ t \in \mathbb{C} \mid |t| < 1 \} \) as soon as \( \nu \in \omega \), where \( \omega \) is a neighborhood of the zero germ \( \bar{\nu}_p \) in \( V \).

Consider the holomorphic mapping \( \rho_V : \omega \ni \nu \mapsto \varphi_{\nu,1} \in \text{Hol}(U, C^s) \) and its \( k \)-jet extension at the origin \( \rho_{V,k} : \omega \ni \nu \mapsto j^k(\varphi_{\nu,1})(\bar{\nu}) \in J^k_0((C^s, \bar{\nu}), C^s) \).

3.4. Lemma. Let a finite dimensional subspace \( V \subset T_p \) be \( k \)-sufficient. Then the differential \( d\rho_{V,k} \) at the origin \( \bar{\nu}_V \in \omega \subset V \) is surjective.

Proof. Since \( \varphi_{t_1,t_2} = \varphi_{t_1,t_2,t_3} \) we have \( \varphi_{t_1,1} = \varphi_{t_2,1} \). Applying expansions (1) and (2) one obtains that for \( \nu \in V \)

\[ \rho_{V,k}(\tau) = j^k(id_{C^s})(\bar{\nu}) + tj^k(\nu)(\bar{\nu}) + \sum_{i \geq 2} t^i j^k(\nu_i)(\bar{\nu}). \]

Therefore, the derivative \( \partial \rho_{V,k}/\partial \nu \) at the direction \( \nu \) at the origin \( \bar{\nu}_V \in V \) is equal to \( j^k(\nu)(\bar{\nu}) \). Since \( V \) is \( k \)-sufficient the image of the differential \( d\rho_{V,k}(\bar{\nu}_V) \) coincides with the whole stalk \( J^k_0(TC^s) = J^k_0((C^s, \bar{\nu}), C^s) \). \( \square \)

3.5. For \( r \leq s \) we regard \( C^r \) as a coordinate subspace in \( C^s : C^r = \{ \tau' \in C^s \mid \tau' = (x_1,...,x_r,0,...,0) \} \). The restriction to this subspace defines for any nonnegative integer \( k \) the projection \( \pi_{s,r}^k : J^k((C^s, \bar{\nu}), C^s) \to J^k((C^r, \bar{\nu}), C^s) \).

Let \( V \) and \( \omega \) be as in 3.3. Consider the holomorphic mapping \( \Phi = \Phi_{V,r,k} : \omega \times (C^r, \bar{\nu}) \to J^k((C^r, \bar{\nu}), C^s) \) given by the formula

\[ \Phi(\nu, \tau') = \pi_{s,r}^k \circ j^k(\varphi_{\nu,1})(\tau'). \]

We have the following commutative diagram:

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Choose a basis Proof. there exists \( \nu \)

\( M = \operatorname{span}(V) \)

\( \sigma \)

section \( \sigma \) cohomology sequence implies that the section \( T \) at \( x_0 \) may be viewed as a global holomorphic section \( T \equiv 0 \). Suppose that \( T \) is a holomorphic vector bundle on \( X \). Then for every nonnegative integer \( k \) there exists a holomorphic section \( \sigma \) of \( T \) with a given \( k \)-jet \( \nu \) at \( x_0 \) and such that \( j^k(\sigma)|_V \equiv 0 \).

Proof. Let \( I_Y \) (resp. \( I_{x_0} \)) be the ideal sheaf of \( Y \) (resp. \( x_0 \)) in \( \mathcal{O}_X \). Put

\[
T = \mathcal{O}_X(T), \quad T' = T \otimes_{\mathcal{O}_X} (I_Y)^k, \quad T'' = T' \otimes_{\mathcal{O}_X} (I_{x_0})^{k+1}.
\]

Note that \( T' \) (resp. \( T'' \)) can be regarded as a subset of \( T \) (resp. \( T'' \)). Put \( J = T'/T'' \). Then \( J|_{X \setminus \{x_0\}} \) is trivial. It is also clear that \( J|_{\{x_0\}} \) coincides with the stalk \( J^k_{x_0}(T) \) of \( k \)-jets of germs at \( x_0 \) of holomorphic sections of \( T \). Thus every \( k \)-jet \( \nu \) at \( x_0 \) may be viewed as a global holomorphic section \( \sigma_0 \) of \( J \). By the Cartan Theorem B applied to the coherent sheaf \( T'' \), we have \( H^1(X,T'') = 0 \). Hence the exact cohomology sequence implies that the section \( \sigma_0 \) is the image of a holomorphic section \( \sigma \in H^0(X,T') \). Clearly, \( \sigma \) is a section of \( T \) with the desired properties. ■

3.8. Corollary. Under the assumption of Lemma 3.7 there exists a finite dimensional subspace \( V \subset H^0(X,\mathcal{O}(T)) \) such that \( j^k(\sigma)|_V \equiv 0 \) for all \( \sigma \in V \) and \( V \) is \( k \)-sufficient at \( x_0 \), i.e. \( J^k_{x_0}(T) = j^k_{x_0}(V) \).

Proof. Choose a basis \( v_1, \ldots, v_N \) of \( J^k_{x_0}(T) \). By Lemma 3.7, for each \( i = 1, \ldots, N \) there exists \( v_i \in H^0(X,\mathcal{O}(T)) \) such that \( j^k(v_i)|_V \equiv 0 \) and \( j^k(v_i)(x_0) = v_i \). Put \( V = \operatorname{span}(v_1, \ldots, v_N) \).

Denote by \( T_M = H^0(M,\mathcal{O}(TM)) \) the space of global holomorphic vector fields on a complex manifold \( M \).

3.9. Lemma. Let \( X \) be a closed submanifold of a Stein manifold \( M \), \( Y \) be a closed analytic subset of \( X \), and \( K \subset X \), \( K' \subset K \setminus \{x_0\} \) be compact subsets. Then there exist a neighborhood \( U \) of \( K \) in \( X \) (resp. \( U' \) of \( K' \) in \( X \setminus \{x_0\} \)), a finite dimensional subspace \( V \subset T_M \), and a neighborhood \( \omega \) of the origin \( \mathcal{O}_V \in V \) such that the following conditions hold:

(i) \( j^k(\nu)|_V \equiv 0 \) for each \( \nu \in V \);
(ii) for every \( \nu \in \omega \) the associated phase flow \( \varphi_{\nu,t} \) is defined on \( U \times \mathbf{X} \);

(iii) if \( \Phi \) is the holomorphic mapping \( \Phi : \omega \times U \ni (\nu,x) \mapsto j^k(\varphi_{\nu,1})(x) \in J^k(U,M) \) then the differential \( d\Phi \) is surjective at each point \( (\nu,x) \in \omega \times U \).

**Proof.** By Corollary 3.8, for each point \( x' \in K' \) there exists a finite dimensional subspace \( V_{x'} \subset J_M \) which is \( k \)-sufficient at \( x' \) and satisfies the above condition (i).

Of course, \( V_{x'} \) is \( k \)-sufficient at each point of some small neighborhood \( U'_{x'} \) of \( x' \) in \( X \). Let \( \{U'_{x'_i}\}_{i=1}^{N'} \) be a finite covering of \( K' \) by such neighborhoods. Put \( U' = \bigcup_{i=1}^{N'} U'_{x'_i} \) and \( V = \bigoplus_{i=1}^{N'} V_{x'_i} \). Then \( V \) is a finite dimensional subspace of \( T_M \) which satisfies (i) and is \( k \)-sufficient at each point \( x' \in U' \).

Furthermore, for each point \( x \in K \) there exist a neighborhood \( U_x \) in \( X \) and a neighborhood \( \omega_x \supseteq \Theta_x \) in \( V \) such that for every \( \nu \in \omega \) the associated phase flow \( \varphi_{\nu,t} \) is holomorphic in \( U_x \times \mathbf{X} \). Choose a finite covering \( \{U_x\}_{j=1}^{N} \) of the compact \( K \) in \( X \) and put \( U = \bigcup_{j=1}^{N} U_x \), \( \omega = \bigcap_{j=1}^{N} \omega_x \). Then \( U \) and \( \omega \) satisfy condition (ii). By Corollary 3.6, condition (iii) also holds.

Next we give an analytic version of the Collective Transversality Lemma [AVG].

**3.10. Lemma.** Under the assumption of Lemma 3.9 for every stratified analytic subset \( A \subset J^k(X,M) \) there exists a dense subset \( \omega_0 \) of \( \omega \) such that for each \( \nu \in \omega_0 \) the mapping

\[
j^k(\varphi_{\nu,1})|_{U'} : U' \to J^k(U',M) \subset J^k(X,M)
\]

is transversal to \( A \).

**Proof.** Denote by \( (A_\sigma)_{\sigma \in \Sigma} \) the collection of strata of the given stratification \( \Sigma \) of \( A \) which satisfies Whitney’s condition (a). Put \( \hat{A} = \Phi^{-1}(A) \subset \omega \times U' \). Since \( d\Phi \) is surjective at each point \( (\nu,x') \in \omega \times U' \) it is clear that \( \{\hat{A}_\sigma = \Phi^{-1}(A_\sigma)\}_{\sigma \in \Sigma} \) is an analytic stratification of \( \hat{A} \) which also satisfies Whitney’s condition (a). Furthermore, the mapping \( j^k(\varphi_{\nu,1})|_{U'} : U' \to J^k(X,M) \) is transversal to \( A \) iff the identical embedding \( U'_\nu := (\nu,x') \in \omega \times U' \leadsto \omega \times U' \) is transversal to \( \hat{A} \). The latter means that for each \( \sigma \in \Sigma \) the differential of the canonical projection \( pr_1 : \omega \times U' \to \omega \) restricted to \( \hat{A} \) is surjective at each of the sets \( U'_\nu \cap \hat{A}_\sigma \). Put \( \pi_\sigma = pr_1|_{A_\sigma} : A_\sigma \to \omega \) and \( S_\sigma = \{x \in A_\sigma \text{rank } d\pi_\sigma(x) < \dim \omega \} \). By Proposition 1.3.8 in [Ch], the image \( \pi_\sigma(S_\sigma) \subset \omega \) is contained in at most countable union of proper analytic subsets of \( \omega \) (not necessarily closed). The same is valid for the “discriminant” set \( D = \bigcup_{\sigma \in \Sigma} \pi_\sigma(S_\sigma) \subset \omega \). Therefore, the desired transversality condition holds for any \( \nu \) in the dense subset \( \omega_0 = \omega \setminus D \) of \( \omega \).

4. **Proof of the Transversality Theorem**

**4.1. Lemma.** Let \( X \) be a closed submanifold of a Stein manifold \( M \), \( Y \) be a closed analytic subset of \( X \), and \( K \) be a compact in \( X \). Suppose that \( A \) is a stratified analytic subset of \( J^k(X,M) \). Let \( f \) be the identical embedding and let \( j^k(f)|_Y : Y \to A \).

Then in a neighborhood \( \Omega \) of \( K \) in \( X \) the mapping \( f \) can be uniformly approximated...
(with respect to a given Hermitian metric $h$ on $M$) by holomorphic mappings $\tilde{f} : \Omega \to M$ such that $j^k(\tilde{f})|_{\Omega} : A \to \Omega$ and $j^k(\tilde{f})|_{\Omega} = j^k(f)|_{\Omega}$.

**Proof.** Since the given stratification $\Sigma$ of $A$ satisfies Whitney’s condition (a) and $j^k(\tilde{f})|_{\Omega} : A \to \Omega$ there exists a neighborhood $W$ of $Y$ in $X$ for which $j^k(\tilde{f})|_{\Omega} : A \to \Omega$. Put $K' = K \setminus W'$, where $W' \subset W$ is a smaller neighborhood of $Y$ such that $W' \subset W$. Let $U \supset K$, $U' \supset K'$, $V \subset T_M$, and $\omega \subset V$ be the same as in Lemma 3.9. Let $\Omega$ be a neighborhood of $K$ so that $\Omega \subset \subset (U' \cup W') \cap U$. Put $f_0 = \varphi_{\nu,1}|_{\Omega}$ for $\nu \in \omega$. Since $(A, \Sigma)$ satisfies Whitney’s condition (a) and $j^k(f_0) \to j^k(f)|_{\Omega}$ as $\nu \to \varphi_{\nu,1} \in V$ there exists a smaller neighborhood $\omega' \subset \omega$ of the origin $\varphi_{\nu,1} \in V$ for which $j^k(f_0)|_{W' \cap \nu} : \Omega \to \Omega$ for all $\nu \in \omega'$. By Lemma 3.10, we also have $j^k(f_0)|_{U' \cap \nu} : \Omega \to \Omega$ for every $\nu$ in a dense subset $\omega'_0$ of $\omega'$. So, $j^k(f_0)|_{\Omega} : \Omega \to \Omega$ for every $\nu \in \omega'_0$. Condition $j^k(\nu)|_{\nu} \equiv \varphi$ and Lemma 3.1 imply that $j^k(f_0)|_{\Omega} = j^k(\varphi_{\nu,1})|_{\Omega} = j^k(id_X)|_{\Omega} = j^k(f)|_{\Omega}$. We conclude the proof by noting that $\tilde{f} = f_0$ uniformly converges to $f|_{\Omega} = id$ on $\Omega$ when $\nu \in \omega'_0$ approaches $\varphi_{\nu,1}$.

The rest of this section is a reduction of Theorem 1.2 to Lemma 4.1. The first step is to replace $f = id_X$ in Lemma 4.1 by an arbitrary holomorphic mapping $f : X \to M$.

**4.2. Lemma.** Let $X$ be a Stein manifold, $M$ be a complex manifold endowed by a Hermitian metric $h$, $f : X \to M$ be a holomorphic mapping. Let $Y, K, A$ be the same as in Lemma 4.1, and as before $j^k(f)|_{\Omega} : A \to \Omega$. Then the conclusion of Lemma 4.1 holds for such $f$.

**Proof.** Let $F = (id_X, f)$ be the embedding of $X \leftrightarrow X \times M$ onto the graph of $f$. In sequel we identify $X$ with its image under $F$ and consider $F$ as the identical embedding. Denote by $pr_* : j^k(X, X \times M) = J^k(X, X) \times J^k(X, M) \to J^k(X, M)$ the forgetting projection. It is easily seen that the analytic subset $\tilde{A} = (pr_*)^{-1}(A) \subset J^k(X, X \times M)$ with the induced stratification is a stratified analytic subset, i.e. the Whitney condition (a) holds. Furthermore, the assumption $j_k(f)|_{\Omega} : A \to \Omega$ yields $j^k(F)|_{\Omega} : \tilde{A} \to \tilde{A}$. Let $g$ be any Hermitian metric on $X$ and let $\tilde{h}$ be the Hermitian metric on $X \times M$ that is the Euclidian sum of $g$ and $h$. By Siu’s theorem [S], the Stein submanifold $X = F(X) \hookrightarrow X \times M$ admits a Stein neighborhood $M' \subset X \times M$. By Lemma 4.1 applied to the identical embedding $F$ of $X = F(X)$ into the Stein manifold $M'$, in a neighborhood $\Omega$ of a compact $K$ the mapping $F$ can be uniformly approximated (with respect to the metric $\tilde{h}$) by holomorphic mappings $\tilde{F} : \Omega \to M' \hookrightarrow X \times M$ such that $j^k(\tilde{F})|_{\Omega} = j^k(F)|_{\Omega}$ and $j^k(\tilde{F})|_{\tilde{A}}$. Put $\tilde{f} = pr_M \circ \tilde{F}$ where $pr_M : X \times M \to M$ is the canonical projection. Clearly, $\tilde{f}$ uniformly approximates $f|_{\Omega}$ with respect to the metric $h$ on $M$, $j^k(\tilde{f})|_{\Omega} = j^k(f)|_{\Omega}$, and $j^k(\tilde{f})|_{\Omega} : A \to \Omega$. Therefore, all conclusions of Lemma 4.1 hold.

**4.3. Proof of Theorem 1.2.** For any holomorphic submersion $\varphi : E \to B$ of complex manifolds which is a smooth fiber bundle and for every Hermitian metric $h$ on $B$, using smooth partition of unity, one can construct a Hermitian metric $\tilde{h}$ on $E$ so that $\varphi^* \tilde{h} \leq h$. By this simple remark, we may fix a sequence of Hermitian metrics
$h_1$ on $J^i(X, M)$ such that all natural projections $\pi_{m,n} : J^m(X, M) \to J^n(X, M)$ (where $n < m$) and $\pi_i : J^i(X, M) \to M$ ($M$ being endowed with a Hermitian metric $h$) are contractions with respect to these metrics. Let $U \subset X$ be an open subset and $g : U \to M$ be a holomorphic mapping. If $A \subset J^m(X, M)$ is a stratified analytic subset and for $m > n$ the preimage $\pi_{m,n}(A)$ is endowed with the induced stratification, then it is easily seen that conditions $j^m(g), A$ and $j^m(g) \cap \pi_{m,n}(A)$ are equivalent. Hence, passing to appropriate preimages of the given stratified analytic subsets $A_i \subset J^k(X, M)$, $i = 1, \ldots$, we may suppose that $k \leq k_i < k_j$ for all $i < j, i, j = 1, \ldots$. Next we fix a relatively compact neighborhood $\Omega$ of $K$ in $X$ which is a Runge domain in $X$. To see that $\Omega$ exists it is enough to take the intersection of the Stein manifold $X$ embedded in $\mathbb{C}^N$ with a large ball $B_N^0$ containing the image of $K$. Fix also a bigger Runge neighborhood $\Omega_0$ (for instance, put $\Omega_0 = X \cap B_N^0$). Replacing $K$ by the compact $L = \overline{\Omega_0}$, we will construct by induction a decreasing sequence $\Omega_1 \subset \Omega_2 \subset \cdots \Omega_1$ of neighborhoods of $L$ in $X$ and a sequence of holomorphic mappings $f_l : \Omega_l \to M$ which satisfy

$$(1) \quad j^{k_i}(f_l)^1 \mathcal{A}, A_1,$$

$$(2) \quad j^{k_i}(f_l)|_{V \cap \Omega_l} = j^{k_i}(f_l)|_{V \cap \Omega_l},$$

$$(3) \quad \text{dist}_{h_k}(j^{k_i}(f_l), j^{k_i}(f_{l-1})) < \delta_l \leq 2^{-l/\varepsilon} \text{ uniformly on } \Omega_l,$$

where $\delta_l > 0$ will be defined later on, and where for $l = 1$ in $(3)$ $f_0 := f$. By our choice of Hermitian metrics $h_k$, given such a sequence $\{f_l\}$, we have

$$(3') \quad \text{dist}_{h_k}(f_l, f_{l-1}) < 2^{-l/\varepsilon}$$

for all $l$ uniformly in $\Omega_0$. Therefore, $f_l \xrightarrow{l \to \infty} \tilde{f}$ uniformly in $\Omega_0$ where $\tilde{f} \in \text{Hol}(\Omega_0, M)$ satisfies condition (2) and (3) of Theorem 1.2. To ensure condition (1) we need to choose $\{\delta_l\}$ in an appropriate way.

By the Cauchy integral formula, given an open subset $U \subset X$ and a sequence $g_l \in \text{Hol}(U, M)$ convergent to $g \in \text{Hol}(U, M)$ uniformly on a compact subset $R \subset U$, we have a uniform convergence on $R$ of the $l$-jet extensions $j^l(g_l)$ to $j^l(g)$ for every natural $l$. Thus, by Lemma 4.2, there exists a neighborhood $\Omega_1$ of $L$ in $X$ and a holomorphic mapping $f_1 : \Omega_1 \to M$ such that $j^{k_i}(f_1)^1 \mathcal{A}, A_1$, $j^{k_i}(f_1)|_{V \cap \Omega_1} = j^{k_i}(f)|_{V \cap \Omega_1}$, and $\text{dist}_{h_k}(j^{k_i}(f_1), j^{k_i}(f)) < \varepsilon/2$. Hence $f_1$ satisfies $(1_1) - (3_1)$ with $\delta_1 = \varepsilon/2$. Passing to a smaller neighborhood of $L$ we may suppose that $\Omega_1$ is Stein. Let $\delta_i$, $i = 1, \ldots, l$, be already constructed so that $f_i$ satisfies $(1_i) - (3_i)$ for $i \leq l$. By Lemma 4.2 and the note above applied to the holomorphic mapping $f_l : \Omega_l \to M$, for every $\delta_{l+1}$ such that $0 < \delta_{l+1} < 2^{-(l+1)\varepsilon}$ there exist a smaller Stein neighborhood $\Omega_{l+1} \subset \subset \Omega_l$ of the compact $L$ in $X$ and a holomorphic mapping $f_{l+1} : \Omega_{l+1} \to M$ which satisfy $(1_{l+1}) - (3_{l+1})$. Now for each $i = 1, 2, \ldots, l$ we have

$$\text{dist}_{h_{k_{i+1}}}(j^{k_{i+1}}(f_i)|_{\Omega_{l+1}}, j^{k_{i+1}}(f_{l+1})) < \sum_{p=i+1}^{\infty} \delta_p.$$  

Since $k_i < k_{i+1}$ and, by condition

$$(1) \quad j^{k_i}(f_i)^1 \mathcal{A}, A_1,$$

there exists $\mu_i > 0$ such that for any $g \in \text{Hol}(\Omega_0, M)$ the condition

$$(5) \quad \text{dist}_{h_{k_{i+1}}}(j^{k_{i+1}}(g), j^{k_{i+1}}(f_i)) < \mu_i$$

in $\Omega_0$ yields $j^{k_i}(g)|_{\Omega_1} \mathcal{A}, A_1$. So, we will choose $\{\delta_i\}$ in such a way that

$$(6) \quad \sum_{p=i+1}^{\infty} \delta_p < \mu_i$$
for every $i = 1, \ldots$. By induction, we may suppose that the condition

$$
\rho_i := \sum_{p=i+1}^{l} \delta_p < \mu_i/2, \quad i = 1, \ldots, l-1,
$$
is already true. Choose $\delta_{l+1}, 0 < \delta_{l+1} < 2^{-(l+1)} \varepsilon$ so that $\delta_{l+1} < \min_{1 \leq i \leq l-1} (\mu_i/2 - \rho_i)$ and $\delta_{l+1} < \mu_l/2$. This implies (6) for every $i = 1, \ldots$. Therefore, the limit mapping $g := \tilde{f} = \lim_{l \to \infty} f_l \in \text{Hol}(\Omega_0, M)$ satisfies (5). Hence $j^k(\tilde{f})|_{\Omega} \in A$, which completes the proof of Theorem 1.2.

5. Stability of Eisenman intrinsic $k$-measures

5.1. Proof of Corollary 1.5. Denote by $\overline{v}_k$ the $k$-vector

$$
\frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_k} \in \Lambda^k T_{\overline{v}_k} B^k
$$
at the origin $\overline{v}_k \in B^k \subset \mathbb{C}^k$. Recall [E], [P], [Ko 1], [GW] that for a complex manifold $M$ and a decomposable vector $\pi = v_1 \wedge \cdots \wedge v_k \in \Lambda^k T_p M$, where $p \in M$ and $1 \leq k \leq m = \dim M$, the Eisenman $k$-measure of $\pi$ is defined by the formula

$$
E^k_M(p, \pi) = \inf \{ \lambda^2 \mid \lambda > 0 \text{ is such that there exists } f \in \text{Hol}(B^k, M) \text{ with } f(\overline{v}_k) = p \text{ and } df(\lambda \pi_k) = \pi \}.
$$

If $A$ is a closed analytic subset of $M$ of codimension at least $k + 1$ and $p \in M \setminus A$ then, by Transversality Theorem 1.2, every holomorphic mapping $f : B^k \to M$ with $f(\overline{v}_k) = p$ and $df(\lambda \pi_k) = \pi$ can be approximated by holomorphic mappings of smaller balls $f_\varepsilon : B^k_{\varepsilon} \to M \setminus A$ with $f_\varepsilon(\overline{v}_k) = p$ and $df_\varepsilon(\lambda \pi_k) = \pi$. This observation implies Corollary 1.5 in the Introduction.

Indeed, consider the mapping $\tilde{f}_\varepsilon(z) = f_\varepsilon((1 - \varepsilon)z)$, $\tilde{f}_\varepsilon \in \text{Hol}(B^k, M \setminus A)$, $\tilde{f}_\varepsilon(\overline{v}_k) = p$, $d\tilde{f}_\varepsilon(\overline{v}_k) = (1 - \varepsilon)\pi_\varepsilon$. By the above definition, we obtain

$$
E^k_{M \setminus A}(p, \pi) \leq E^k_M(p, \pi).
$$
The opposite inequality holds since embeddings do not increase Eisenman measures.

In [PS] it was mentioned that the Campbell-Howard-Ochiai-Ogawa Stability Theorem is still true for any subset of $M$ of Hausdorff $(2m - 2k)$-measure zero. Here we present the similar generalization of Corollary 1.5.

5.2. Proposition. Let $M$ be a connected complex manifold of complex dimension $m$ and let $A$ be its subset of Hausdorff $(2m - 2k)$-measure zero for some $k \in \{1, 2, \ldots, m\}$. Then $E^k_{M \setminus A} = E^k_M|_{M \setminus A}$.

The proof is similar to the proof of Corollary 1.5, but instead of Theorem 4.3 we have to use the following modification of it.

5.3. Theorem. Let $M$ be a connected complex manifold of dimension $m = m$ with a Hermitian metric $h$, and let $A$ be its closed subset of Hausdorff $(2m - 2k)$-measure zero for some $k \in \{1, 2, \ldots, m\}$. Let $X$ be a Stein manifold of dimension $X = k$, $Y$ be a closed analytic subset of $X$, and $f : X \to M$ be a holomorphic mapping such that $f(Y) \subset M \setminus A$. Then for any compact $K \subset X$ the mapping $f$ can be uniformly (with respect to $h$) approximated in a neighborhood $\Omega$ of $K$ by holomorphic mappings $\tilde{f} : \Omega \to M \setminus A$ such that $j^k(\tilde{f})|_{\Omega \cap \Omega} = j^k(f)|_{\Omega \cap \Omega}$. License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Proof. We follow the lines of the proof of Theorem 1.2. As in Lemma 4.2, replacing $f$ by the embedding $F = (\text{id}_X, f)$ onto the graph of $f$ and replacing $M$ by a Stein neighborhood of the graph in $X \times M$, we may suppose that $M$ is Stein, $X$ is a closed submanifold of $M$, and $f$ is the identical embedding $X \to M$. Note that under this replacement the codimension of $X$ in $M$ is still equal to the dimension of vanishing Hausdorff measure of $A$ ($A$ being replaced by $pr^{-1}_2(A) \subset X \times M$).

Since $f(Y) \cap A = \emptyset$ there exists a neighborhood $W$ of $Y \cap K$ in $X$ such that $f(W) \cap A = W \cap A = \emptyset$. Fix a smaller relatively compact neighborhood $W' \subset W$ and $\epsilon > 0$ such that for each $f \in \text{Hol}(W', M)$ with $\text{dist}_K(f, f|_{W'}) < \varepsilon$ uniformly in $W'$ one has $f(W') \cap A = \emptyset$.

By Lemma 3.9, there exist neighborhoods $U$ of $K$ in $X$, $U'$ of $K'$ in $X$, a finite dimensional subspace $V \subset T_M$, and a neighborhood $\omega'$ of the origin $\tilde{0}_V \in V$ such that the following conditions hold:

(i) $f^1(\nu)|_V \equiv 0$ for every $\nu \in V$;

(ii) $\Phi \in \text{Hol}(\omega' \times U, M)$ where $\Phi(\nu, z) = \varphi_{\nu, 1}(z)$;

(iii) $\text{dist}_K(\varphi_{\nu, 1}|_U, f|_U) < \varepsilon$ uniformly in $U$;

(iv) the rank of $\Phi$ is $m = \dim_{\mathbb{C}} M$ at each point $z \in U'$.

By Lemma 3.1, (i) and (ii) imply that $\varphi_{\nu, 1}|_{U' \cap Y} = \int_{U' \cap Y} = \text{id}|_{U' \cap Y}$, and, by (iii), $\varphi_{\nu, 1}(U' \cap W') \subset M \setminus A$ for each $\nu \in \omega'$. By virtue of Theorem 2 from [Ch, Appendix, 2] condition (iv) implies that the mapping $\Phi|_{\omega' \times U'}$ may be locally treated as a projection.

From Proposition 7 in [Ch, Appendix, 6] it easily follows that the preimage $A' = (\Phi|_{\omega' \times U'})^{-1}(A) \subset \omega' \times U'$ has Hausdorff 2-measure zero where $l = \dim_{\mathbb{C}} V$. Therefore, by Property 4 of Hausdorff measures as it was listed in [Ch, Appendix, 6], we have that the image $A''$ of $A'$ under the projection $\omega' \times U' \to \omega'$ also has Hausdorff 2-measure zero. It follows that $\omega' \setminus A''$ is a dense subset of $\omega'$.

For each $\nu \in \omega' \setminus A''$ the image $\varphi_{\nu, 1}(U') = \Phi(\nu) \times U'$ is closed, and thus as replaced by $pr^{-1}_2(A) \subset X \times M$.

Thus for this $\nu$ we have $\varphi_{\nu, 1}(U' \cup W') \cap A = \emptyset$. Take a neighborhood $\Omega$ of $K$ in $X$ such that $\Omega \subset (U' \cup W') \cap U$. Then $\tilde{f} := \varphi_{\nu, 1}|_{\Omega} \in \text{Hol}(\Omega, M)$, where $\nu \in \omega' \setminus A''$ and $\nu \to \tilde{0}_V$, gives the desired approximation of $f$.

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