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A CASCADE DECOMPOSITION THEORY WITH APPLICATIONS TO MARKOV AND EXCHANGEABLE CASCADES

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ABSTRACT. A multiplicative random cascade refers to a positive T-martingale in the sense of Kahane on the ultrametric space $T = \{0, 1, \dots, b-1\}^{\mathbf{N}}$. A new approach to the study of multiplicative cascades is introduced. The methods apply broadly to the problems of: (i) non-degeneracy criterion, (ii) dimension spectra of carrying sets, and (iii) divergence of moments criterion. Specific applications are given to cascades generated by Markov and exchangeable processes, as well as to homogeneous independent cascades.

1. POSITIVE T-MARTINGALES

Positive T-martingales were introduced by Jean-Pierre Kahane as the general framework for independent multiplicative cascades and random coverings. Although originating in statistical theories of turbulence, the general framework also includes certain spin-glass and random polymer models as well as various other spatial distributions of interest in both probability theory and the physical sciences.

For basic definitions, let T be a compact metric space with Borel sigmafield $\mathcal{B}$, and let $(\Omega, \mathcal{F}, P)$ be a probability space together with an increasing sequence $\mathcal{F}_n, n = 1, 2, \dots,$ of sub-sigmafields of $\mathcal{F}$. A *positive T-martingale* is a sequence $\{Q_n\}$ of $\mathcal{B} \times \mathcal{F}-$measurable non-negative functions on $T \times \Omega$ such that (i) For each $t \in T, \{Q_n(t, \cdot) : n = 0, 1, \dots\}$ is a martingale adapted to $\mathcal{F}_n, n = 0, 1, \dots;$ (ii) For P-a.s. $\omega \in \Omega, \{Q_n(\cdot, \omega) : n = 0, 1, \dots\}$ is a sequence of Borel measurable non-negative real-valued functions on T.

Let $M^+(T)$ denote the space of positive Borel measures on T and suppose that $\{Q_n(t)\}$ is a positive T-martingale. For $\sigma \in M^+(T)$ such that $q(t) := EQ_n(t) \in L^1(\sigma)$, let $\sigma_n \equiv Q_n\sigma$ denote the random measure defined by $Q_n\sigma << \sigma$ and $\frac{dQ_n\sigma}{d\sigma}(t) := Q_n(t), t \in T$. Then, essentially by the martingale convergence theorem, one obtains a random Borel measure $\sigma_\infty \equiv Q_\infty\sigma$ such that for $f \in C(T)$,

$$\lim_{n\to\infty} \int_T f(t)Q_n(t, \omega)\sigma(dt) = \int_T f(t)Q_\infty\sigma(dt, \omega)\, a.s. \tag{1.1}$$

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In general the random measure $Q_\infty\sigma$ will not be absolutely continuous with respect to σ; i.e. there is no corresponding "$Q_\infty(t)$". However, some further insight into the structure of $Q_\infty\sigma$ is furnished by the basic decomposition Theorem 1.1 of Kahane [K4], quoted below. By this decomposition Kahane also extended (1.1) to all bounded Borel functions f. This extension of (1.1) as well as the decomposition theorem itself lie at the heart of several results of the theory to be developed in this paper, so it is convenient to record a statement of it here.

Theorem 1.1 (T-Martingale Decomposition). *Let $\{Q_n\}$ be a positive T-martingale. Then corresponding to $\sigma \in M^+(T)$ such that $q \in L^1(\sigma)$, there is a unique decomposition of $\{Q_n\}$ as a sum of two positive T-martingales $\{Q'_n\}$, $\{Q''_n\}$, such that for each $t \in T$*

$$Q_n(t) = Q'_n(t) + Q''_n(t)$$

with

$$\text{(i)} \qquad EQ'_\infty\sigma(B) = q'\sigma(B), \qquad B \in \mathcal{B},$$

where $q' := EQ'_n$, and

$$\text{(ii)} \qquad Q''_\infty\sigma = 0 \ a.s.$$

Whenever $EQ_\infty\sigma(B) = q\sigma(B), B \in \mathcal{B}$, where $q := EQ_n$, then one says that $\{Q_n\}$ (or Q_∞) *lives on* σ. On the other hand, if $Q_\infty\sigma = 0$ a.s. then $\{Q_n\}$ (or Q_∞) is said to *die on* σ. Thus Theorem 1.1 is a unique decomposition into two positive T-martingales, one of which lives and the other dies on σ. We will refer to $\{Q'_n\}$ and $\{Q''_n\}$ as the *strong* and *weak* terms of the decomposition, respectively.

As noted in [WW2, Proposition 1.1], by introducing a notion of *martingale ratio sequences* the positive T-martingales may be represented multiplicatively, i.e. $Q_n(t) = P_0(t) \times P_1(t) \times \cdots \times P_n(t)$ and $E[P_{k+1}(t)|\mathcal{F}_k] = 1$. Positive T-martingales on the metric space $T = \{0, 1, \ldots, b-1\}^{\mathbf{N}}$, equipped with the ultrametric $\rho(s,t) = b^{-\inf\{n:s_n \neq t_n\}}, s = (s_1, s_2, \ldots), t = (t_1, t_2, \ldots) \in T$, are referred to as *multiplicative cascades.* In this case, the martingale ratio factors $P_n(t), t = (t_1, \ldots), n \geq 1$, are non-negative mean one random constants $W_{\epsilon_1 \ldots \epsilon_n}$, say, on rectangles $\Delta(\epsilon_0, \ldots, \epsilon_n) := \{t \in T : t_i = \epsilon_i, i \leq n\}$. The denumerable family $\{W_{\epsilon_1 \ldots \epsilon_n} : \epsilon_i \in \{0, \ldots, b-1\}\}$ comprises the *generator* of the cascade. The fundamental theorem of Kahane and Peyrière [KP] concerns the structure of *homogeneous independent cascades,* defined by i.i.d. generators, for $\sigma = \lambda$ the Haar measure on T. In particular, in this case [KP] provides necessary and sufficient conditions on the distribution of the generators for: (i) *Degeneracy/non-degeneracy,* (ii) *Hausdorff carrying dimension of the support,* and (iii) *Divergence of Moments of* $\lambda_\infty(T)$. The nondegeneracy problem (i) refers to the conditions for the existence of a nonzero limit measure, i.e. survival. It should be noted that conditions for degeneracy are of interest as well, e.g. in certain covering problems (see [SH], [K4]), and in certain low temperature spin-glass problems ([DER],[DERSP], [CK], [WW3]). The divergence of moments problem (iii) refers to a determination of finiteness of moments of the total mass distribution of a nondegenerate cascade measure. Finally, for fairly obvious reasons the cascade measures are generally carried by thin sets. It is of interest to determine the size of supporting sets, i.e. problem (ii). There are a variety of fine scale structure computations studied in the literature. For example,

a commonly studied problem is the computation of the so-called *singularity spectra;* see [FALC], [HW], etc. However a nontrivial singularity spectrum does not preclude measures having a unidimensional carrying set, e.g. nondegenerate homogeneous independent cascades as in Theorem 4.1(ii) below. In the framework of the present paper one obtains a more general "multifractal" structure which is best described in terms of the *dimension spectral* theory given in [CU], [KK]. In particular, one may first compute the dimension spectrum and then compute the singularity spectra of the spectral modes.

The purpose of the present paper is to develop methods for problems (i)–(iii) in a generality which to the extent possible does not depend on specific dependence structure, i.e. not beyond general martingale (ratio) structure. While the [KP] approach certainly exploits martingale structure of the total mass $\lambda_\infty(T)$ in solving these problems, it also makes clever use of the independence and recursive equations in key places. It is interesting to note that these random recursive equations have also been studied in the context of certain interacting particle systems in [HL], [DL],[GUI]. The applications to the corresponding recursive equations implicit in our analysis of dependence may also be of interest in these and other contexts. The essential ingredients of our general approach are provided in the next section 2, with proofs deferred to section 3. Applications to independent cascades, Markov cascades, and exchangeable cascades are then given in sections 4,5, and 6 respectively. Here are two simple concrete examples of dependence whose solutions were not possible by previous methods.

Example 1.1 (Markov Cascades). Suppose that the generator sequences $W_\emptyset, W_{t_1}$, $W_{t_1t_2}, \dots, t = (t_1, t_2, \dots) \in T$, are identically distributed Markov chains on state space $[0, \infty)$ having mean one transitions and such that for $s, t \in T$ the pair of Markov chains $W_\emptyset, W_{s_1}, W_{s_1s_2}, \dots$ and $W_\emptyset, W_{t_1}, W_{t_1t_2}, \dots$ are conditionally independent given the sigmafield generated by $\{W_{\epsilon_1\dots\epsilon_n} : \epsilon_i \in \{0, \dots, b-1\}, i \le n\}$, where $n = \inf\{k : s_k \ne t_k\}$. One might conjecture that if the Markov chain is irreducible and positive recurrent with invariant probability π, then the solution to problems (i)–(iii) should coincide with those for the homogeneous independent cascade with generators distributed as π. We will see that this fails to be true in general, though it is true under a time-reversibility condition on the Markov chain; in the finite state space case this may also be obtained from the recursive equations for $\lambda_\infty(T)$ noted above; see [WW1]. In addition, while homogeneous independent cascades always die at criticality we will see that this need not be the case for Markov cascades. To anticipate the spectral disintegration for Markov cascades think about a reducible Markov chain whose recurrent classes as well as transient classes are homogeneous independent cascades which may or may not survive. For example, a $\beta-$model is a transient Markov chain which may or may not survive. Once survival classes (criteria) are known then one is naturally led to consider the hitting probabilities (harmonic measure) for these classes.

Example 1.2 (Polya Type Exchangeable Cascades). Take $b = 2$ and fix $a \in [0, 1)$. The conceptualization of the cascade to be defined here is as follows. Imagine an urn initially containing a "1-ball" and a "flip-ball". Select a ball at random. If the selected ball is a 1-ball, then $W_\emptyset = 1$. If the selected ball is a flip-ball, then flip a fair coin, labeled on one side by a and the other by $2 - a$, and record $W_\emptyset$ as a or $2 - a$ according to the outcome of the toss. Define a new urn state by returning

the ball selected together with another of the same type (1-ball or flip-ball). This experiment is now repeated for *this urn state* to get two independent values for W_0, W_1, and so on down the tree. For this example we will see that the cascade survives on λ and the dimension spectrum is absolutely continuous with respect to λ. The explicit form of the spectral modes are also furnished by the general theory.

The general approach to these problems and more detailed organization of the sections of the paper are as follows. Although the homogeneous independent case is a special case of the general multiplicative theory, it is illuminating to see how the methods of section 2 apply. So a new proof of the Kahane-Peyrière theorem announced in [WW4] is provided in section 4 which makes the essential aspects of the general theory transparent. Specifically, one sees that this approach does not require independence beyond an ergodic theorem (law of large numbers) for the non-degeneracy problem away from the critical parameter. At the critical parameter independence surfaces in the form of a Chung-Fuchs criterion for null-recurrence. The carrying dimension problem is then solved by a percolation method based on a composition theorem from [WW2] and using the non-degeneracy criterion. The moment problem is solved by bootstrapping off a natural supermartingale bound obtained under the size-bias transform described in section 2. Apart from this, the main new results in section 4 concern other applications of the composition theorem. In particular, a notion of a measure-valued continuous parameter Markov process is introduced based on infinitely divisible generators and the composition theorem which, in the spirit of superprocess theory, we refer to as a *supercascade* (e.g. cf. [DP],[DYN] and references therein). These are used to construct examples of random measures having prescribed dimension spectral disintegrations in Theorem 4.4. The percolation theorem may be used quite generally to obtain dimension spectra. As a result of Proposition 4.2 one sees that this method provides carrying dimension for a wider class than obtained under the $E\lambda_\infty(T)\log_b \lambda_\infty(T) < \infty$ hypothesis of Theorem 4.1. It should be remarked that, while omitting the details, it was already realized by Kahane in [K1] that the relaxation of this condition would follow from such an approach.

Markov cascades are of interest for a variety of reasons. The tree approximations to spin-glass and random polymer models are based on a judicious relaxation of correlations; see [DERSP],[DER],[KOU], [WW3]. Also estimates on data sets associated with the spatial distribution of rainfall are improved by correlations of the form considered here, e.g. [OG], [GW]. So it is of general interest to study the effect of correlations of various types. As noted in Example 1.1 above, interesting new phenomena occur under simple Markov dependence. In section 5, the problems (i) and (ii) are solved for the $\psi-$recurrent generators on general non-negative state spaces in Theorem 5.2 and Corollary 5.1. Countable state, ergodic, mean-reversible generators are shown to die at criticality. However, under null-recurrence, examples are provided to show that at criticality the cascade may live or die; i.e. one may have measures carried by zero-dimensional sets. We use the percolation theory to give the spectral dimension decomposition of (reducible) finite state Markov generators in Theorem 5.7. The general divergence of moments problems is more difficult. The role of a good large deviation rate is explored for the moment problem in Theorem 5.5.

For exchangeable cascades one does not generally have available ergodicity, large deviation rates, etc. In this sense exchangeable cascades provide yet a different

dependence structure on which to test the theory developed in section 2. The degeneracy problem and spectral dimension decomposition problems are solved in Theorems 6.5, Corollary 6.1, and Theorems 6.6, 6.7. Application of the general theory is given to the Polya type urn scheme described in Example 1.2. Finally the divergence of moments problem is treated in Theorem 6.7 and Corollary 6.2.

2. Definitions and Statements of Main Results

Let $b \geq 2$ be a natural number, referred to as *branching number,* and let $T = \{0, 1, \dots, b-1\}^{\mathbf{N}}$ be the metric space for the ultrametric $\rho(s,t) = b^{-a(s,t)}$ where $a(s,t) = \inf\{n : s_n \neq t_n\}, s = (s_1, s_2, \dots), t = (t_1, t_2, \dots) \in T$. Let us first record some frequently used notation. For $t \in T, n \geq 1, j \in \{0, 1, \dots, b-1\}$ write $t|0 = \emptyset, t|n := (t_1, t_2, \dots, t_n)$, $n \geq 1$. For $\gamma = (t_1, \dots, t_n), |\gamma| := n$ and $\gamma * j := (t_1, \dots, t_n, j)$. Also denote the nth generation partition by $\Delta_n(t) \equiv \Delta_{t|n} := [\sum_{i=0}^n t_i b^{-i}, \sum_{i=0}^n t_i b^{-i} + b^{-n}) \subset [0,1)$. Throughout we reserve λ to denote the normalized Haar measure on T.

To define the general *multiplicative cascade* let $q_n(B|x_0, \dots, x_{n-1})$, $B \in \mathcal{B}[0,\infty)$, $x_i \geq 0, n \geq 1$, be a family of functions such that for each $B \in \mathcal{B}[0,\infty)$,

$$(x_0, \dots, x_{n-1}) \to q_n(B|x_0, \dots, x_{n-1}) \quad \text{is } \mathcal{B}^n[0,\infty) - \text{measurable.} \tag{2.1a}$$

For $(x_0, \dots, x_{n-1}) \in [0,\infty)^n$,

$$B \to q_n(B|x_0, \dots, x_{n-1}) \quad \text{is a probability measure on } \mathcal{B}[0,\infty). \tag{2.1b}$$

For $(x_0, \dots, x_{n-1}) \in (0,\infty)^n$,

$$\int_{[0,\infty)} x q_n(dx|x_0, \dots, x_{n-1}) = 1. \tag{2.1c}$$

Definition 2.1. A *tree distribution* corresponding to a family of probability kernels q_n satisfying (2.1,a–c) is a family of probability measures $p_n, n \geq 0$, on the product sigmafields $\mathcal{B}^{T^*(n)}[0,\infty), n \geq 0$, respectively, where

$$T^*(0) = \{\emptyset\}, \quad T^*(n) = \bigcup_{k=0}^{n} \{0, 1, \dots, b-1\}^k, \quad T^* = T^*(\infty),$$

defined inductively by

$$\text{(i)} \qquad p_0 \text{ is a probability measure on } \mathcal{B}[0,\infty), \quad \int_{[0,\infty)} x p_0(dx) = 1,$$

$$\text{(ii)} \qquad \begin{aligned} & p_n\Big(\prod_{\gamma \in T^*(n)} B_\gamma\Big) \\ & := \int_{\prod_{\eta \in T^*(n-1)} B_\eta} \prod_{|\gamma|=n} q_n(B_\gamma | x_\emptyset, x_{\gamma|1} \dots, x_{\gamma|n-1}) p_{n-1}\Big(\prod_{\eta \in T^*(n-1)} dx_\eta\Big), \end{aligned}$$

for $B_\gamma \in \mathcal{B}[0,\infty), \gamma \in T^*(n)$.

The following theorem is a simple consequence of the above definitions and the Kolmogorov consistency theorem.

Theorem 2.1. *Let* $\Omega := [0,\infty)^{T^*}, \mathcal{F} := \mathcal{B}^{T^*}[0,\infty)$. *Then there is a unique probability measure* P *on* $(\Omega, \mathcal{F})$ *such that*

$$P(\prod_{\gamma \in T^*(n)} B_\gamma \times [0,\infty)^{T^* - T^*(n)}) = p_n(\prod_{\gamma \in T^*(n)} B_\gamma).$$

The canonical cascade generators for this model, i.e. the coordinate projection maps $W_\gamma(\omega) := \omega_\gamma, \omega \in \Omega, \gamma \in T^*$, are then (jointly) distributed according to the probability measure P on $[0,\infty)^{T^*}$.

The model constructed in Theorem 2.1 will be referred to as the *cascade generator* corresponding to a given tree distribution. One may check that for any $t \in T = \{0, 1, \ldots, b-1\}^{\mathbf{N}}$, the sequence of random variables $X_0 := W_\emptyset, X_1 := W_{t|1}, X_2 := W_{t|2}, \ldots, X_n := W_{t|n}, \ldots$ has a distribution which does not depend on t. This sequence of random variables (or its distribution) will be referred to as the *generator process.*

Given a tree distribution we define the general *multiplicative cascade* by the positive T-martingale given by

$$P_n(t) := W_{t|n}, \tag{2.2}$$

$$Q_n(t) := P_0(t)P_1(t) \times \cdots \times P_n(t), \qquad t \in T, \tag{2.3}$$

$$\mathcal{F}_n := \mathcal{B}^{T^*(n)}[0,\infty). \tag{2.4}$$

Our approach to the study of cascades in this generality is based on three elements; namely (i) *a weight system perturbation,* (ii) *a size-biased transform probability,* and (iii) *a general percolation method.*

Definition 2.2. A *weight system* is a family $\mathbf{F}$ of real-valued functions $F_\gamma : \Omega \to [0,\infty)$, where for each $\gamma \in T^*$, F_γ is $\sigma\{W_{\gamma|j} : j \le |\gamma|\}$ measurable, such that $Q_{\mathbf{F},n}(t) := Q_n(t)F_{t|n}$ is a positive T-martingale, referred to as a *weighted cascade.* A weight system $\mathbf{F}$ is called a *weight decomposition* in the case $\mathbf{F}^c := \{1 - F_\gamma : \gamma \in T^*\}$ is also a weight system.

Note that a weight system is a weight decomposition if and only if the weights are bounded between 0 and 1.

In a certain general sense it is clear that nondegenerate cascades owe their structure to large deviations from average behavior; e.g. with P-probability one a product of i.i.d. mean one variables along a given path tends to zero. Fortunately, the tree T contains uncountably many paths. We take advantage of this by transforming the probability measure in such a way that the essential computations become law of large number computations. The trick is to find the "right" probability measure! In the present framework a natural choice is to choose a path at random according to the cascade measure at level n and then average and pass to the limit as $n \to \infty$. More precisely, for a given weight system $\mathbf{F}$, define a sequence $\mathcal{Q}_{\mathbf{F},n}$ of measures on $\Omega_n \times T, \Omega_n := [0,\infty)^{T^*(n)}$, by

$$\int_{\Omega_n \times T} f(\omega,t)\mathcal{Q}_{\mathbf{F},n}(d\omega \times dt) := E_P \int_T f(\omega,t)Q_{\mathbf{F},n}(t)\lambda(dt), \tag{2.5}$$

for bounded measurable functions f. Then one normalizes the masses of the $\mathcal{Q}_{\mathbf{F},n}$ by a factor $Z_\emptyset := EW_\emptyset F_\emptyset$ and extends to a probability $\mathcal{Q}_{\mathbf{F},\infty}$ using the Kolmogorov extension theorem. Of course, the nontrivial initialization $Z_\emptyset > 0$ will be required wherever $\mathcal{Q}_{\mathbf{F},\infty}$ is applied. The so-called *Peyrière probability* by Kahane was introduced in [P] under the a priori condition $P(\lambda_\infty(T) > 0)$ of nondegeneracy of the cascade. Apart from the weight system perturbations, the main difference here with [P] is that an a priori nondegeneracy is not required. As a result $\mathcal{Q}_{\mathbf{F},\infty}$ may be used to study the degeneracy vs. non-degeneracy problem. The third ingredient to this theory is a corollary to a general *composition theorem* proved in [WW2] which we refer to as the *percolation theorem* in the spirit of terminology introduced in [LY]. The difference with [LY] is that here we consider a "measure valued"percolation in place of "set valued"percolation. The idea is still that multiplication by an independent cascade with i.i.d. zero/non-zero valued generators, i.e. the so-called $\beta-$model at (2.6) below, corresponds to an independent pruning of the tree under which one studies the critical parameters governing the survival of mass. This is an extremely powerful tool for analyzing dimension spectra. However, as will be amply shown in sections 4,5, the more general composition theorem is also useful in a variety of diverse applications,e.g. supercascades, independent composition cascades, etc.

Theorem 2.2 (A Lebesgue Decomposition). *Let* $\mathbf{F}$ *be an arbitrary weight system with* $Z_\emptyset > 0$. *Let* π_Ω *denote the coordinate projection map on* $\Omega \times T$. *Then,*

$$\mathcal{Q}_{\mathbf{F},\infty} \circ \pi_\Omega^{-1}(d\omega) = Z_\emptyset^{-1}\lambda_{\mathbf{F},\infty}(T)(\omega)P(d\omega) + \mathbf{1}(\lambda_{\mathbf{F},\infty}(T) = \infty)\mathcal{Q}_{\mathbf{F}} \circ \pi_\Omega^{-1}(d\omega),$$

where $\lambda_{\mathbf{F},\infty} = Q_{\mathbf{F},\infty}\lambda$.

Corollary 2.1. *If* $Z_\emptyset > 0$, *then*

$$\begin{aligned} \lambda_{\mathbf{F},n}(T) \to \lambda_{\mathbf{F},\infty}(T) \text{ in } L^1 &\iff \lambda_{\mathbf{F},\infty}(T) < \infty \ \mathcal{Q}_{\mathbf{F},\infty}\text{-a.s.} \\ &\iff E_P\lambda_{\mathbf{F},\infty}(T) = Z_\emptyset. \end{aligned}$$

Remark 2.1. The Lebesgue decomposition and Corollary 2.1 are quite general and may also be derived for general positive T-martingales with no difference in the proof. The following theorem and its corollary exploit the special structure of T.

Theorem 2.3 (A Size-Bias Transform Disintegration). *Given a weight system* $\mathbf{F}$, *with* $Z_\emptyset > 0$, *for each* $t \in T, n \geq 1$ *define a probability* $P_{\mathbf{F},t} << P$ *on* $\mathcal{F}_n$ *by*

$$\frac{dP_{\mathbf{F},t}}{dP}|_{\mathcal{F}_n} = Z_\emptyset^{-1}\prod_{i=0}^{n} W_{t|i}F_{t|n}.$$

Also let $P_{\mathbf{F},t}$ *denote the Kolmogorov extension to* $\mathcal{F}$. *Then,*

$$\mathcal{Q}_{\mathbf{F},\infty}(d\omega \times dt) = P_{\mathbf{F},t}(d\omega)\lambda(dt).$$

Moreover, defining $\mathcal{F}_{t,n} := \sigma\{W_{t|n}, n \geq 0, W_\gamma, |\gamma| \leq n\}, t \in T, n \geq 0,$ *one has*

$$\lambda_{\mathbf{F},n}(T) = \sum_{j=0}^{n} b^{-j}\prod_{i=0}^{j} W_{t|i}M_{n,j}(t),$$

where each $M_{n,j}(t) := \sum_{\substack{|\gamma|=n-j \\ \gamma|1\neq t|j+1}} b^{-(n-j)} \prod_{i=1}^{n-j} W_{t|j*\gamma|i} F_{t|j*\gamma}, 1 \leq j \leq n$, *is the total mass in the (slightly irregular) subtree started at* $t|j$, *and given* $\mathcal{F}_{t,0}$, *the* $M_{n,j}(t), j = 0, 1, \ldots, n$, *are independent. Moreover,*

$$E_P[\lambda_{\mathbf{F},n}(T)|\mathcal{F}_{t,0}] = \sum_{j=0}^{n} b^{-j} \prod_{i=0}^{j} W_{t|i} F_{t|j} (\frac{b-1}{b} + \frac{1}{b}\delta_{j,n}),$$

where $\delta_{j,n}$ *is the Kronecker delta.*

Note. In the decomposition represented by the above Theorem 2.3, at each node, say $t|j$, of the t-path one has a subtree generated at that site. However, there is a slight irregularity in the subtree since, at the first generation, $\emptyset \equiv t|j$ subsequently generates only $b-1$ offspring, while each of these will generate b offspring. This is of no major consequence, however it tends to have subtle (arithmetic) effect on some calculations which will be noted as they occur. Also, $\lambda_{\mathbf{F},n}(T)$ need not be in $L^1(P_{\mathbf{F},t})$, in which case the "conditional expectation" formula is an abuse of notation which is to be interpreted as a disintegration formula.

Remark 2.2. Observe that as a result of the factor $F_{t|n}$ in the size-bias transform for $t \in T$, one has $F_{t|n} > 0$ $P_{\mathbf{F},t}$−a.s. In particular one is free to consider the reciprocal. This simple property applies to the generator factors $W_{t|i}$ as well. This property will be seen to contribute to the utility of the size-bias transform in various considerations of reciprocals throughout.

Corollary 2.2. *If* $Z_\emptyset > 0$, *then*

$$\begin{aligned} &\mathcal{Q}_{\mathbf{F},\infty}(d\omega \times dt) \\ &= Z_\emptyset^{-1} \lambda_{\mathbf{F},\infty}(dt) P(d\omega) + \mathbf{1}(\lambda_{\mathbf{F},\infty}(T) = \infty) \delta_{\tau(\omega)}(dt) \mathcal{Q}_{\mathbf{F},\infty} \circ \pi_\Omega^{-1}(d\omega), \end{aligned}$$

where τ *is a* T*-valued random path.*

Remark 2.3. The proof of Corollary 2.2 makes the following heuristic precise. By definition the construction of $\mathcal{Q}_{\mathbf{F},\infty}$ involves renormalization of the mass by the total mass at each stage of the cascade and then passing to the limit. If the total is finite in the limit, then one simply gets the total mass as the renormalization of P. On the other hand, if the limit is infinite, then the renormalization results in a delta mass along a single surviving path. This is similar to the description of the backbone of the "incipient infinite cluster" in [KES].

Theorem 2.4 (A 1st Departure Submartingale Bound). *Consider a weight system* $\mathbf{F}$, *such that* $Z_\emptyset > 0$. *Let* $\mathcal{F}_{t,n} := \sigma\{W_{t|i}, W_\gamma : i \geq 0, |\gamma| \leq n\}$. *Fix* $c_k \geq 0$, *such that* c_k *is* $\mathcal{F}_{t,0}$*−measurable,* $E_{\mathcal{Q}_{\mathbf{F},\infty}} \sum_k c_k < \infty$. *Then, for arbitrary* $t \in T$ *(with* $\frac{0}{0} := 1$*),*

$$b^{-n} \prod_{i=0}^{n} W_{t|i} F_{t|n} \leq \lambda_{\mathbf{F},n}(T) \leq b^{-n} \prod_{i=0}^{n} W_{t|i} F_{t|n} + \sup_j (b^{-j} \frac{\prod_{i=1}^{j} (W_{t|i})}{c_j} F_{t|j}) M_n,$$

where

$$M_n = \sum_{j=0}^{n-1} c_j b^{-(n-j)} F_{t|j}^{-1} \sum_{\substack{|\gamma|=n-j \\ \gamma|1\neq t|j+1}} \prod_{i=1}^{n-j} W_{t|j*[\gamma|i} F_{t|j*\gamma}$$

is a non-negative submartingale with respect to $\mathcal{F}_{t,n}$, *whose Doob decomposition has the predictable part* $A_n = \frac{b-1}{b} \sum_{j \leq n} c_j$, *and* $\lim M_n$ *exists* $\mathcal{Q}_{\mathbf{F},\infty}$*−a.s.*

Let us refer to c_j's satisfying the conditions of the previous theorem as *admissible.*

Corollary 2.3. *Fix $t \in T$. If*

$$\lambda\{t : P_{\mathbf{F},t}(\sum_j b^{-j} \prod_{k=1}^{j} W_{t|k} F_{t|j} < \infty) > 0\} > 0,$$

then

$$P(\lambda_{\mathbf{F},\infty}(T) > 0) > 0.$$

If for λ-a.e. t,

$$P_{\mathbf{F},t}(\sum_j b^{-j} \prod_{k=1}^{j} W_{t|k} F_{t|j} < \infty) = 1,$$

then

$$E_P \lambda_{\mathbf{F},\infty}(T) = Z_\emptyset$$

and

$$\lambda_{\mathbf{F},n}(T) \to \lambda_{\mathbf{F},\infty}(T) \text{ in } L^1(P).$$

If for λ-a.e. t,

$$P_{\mathbf{F},t}(\limsup_{n\to\infty} b^{-n} \prod_{k=1}^{n} W_{t|k} F_{t|n} = \infty) = 1,$$

then P-a.s.

$$\lambda_{\mathbf{F},n}(T) \to 0.$$

Remark 2.4. To obtain the convergence in Corollary 2.3 it is often useful to do a "root test" as the computation of $\sqrt[j]{b^{-j} \prod_{k\le j} W_{t|k} F_{t|j}}$ is often calculable using an ergodic theorem. Alternatively, it is sometimes convenient to do a "comparison test" based on the simple fact that for admissible c_j's,

$$\sup_j \frac{b^{-j} \prod_{k\le j} W_{t|k} F_{t|j}}{c_j} < \infty \Rightarrow \sum_{j=1}^{\infty} b^{-j} \prod_{k\le j} W_{t|k} F_{t|j} < \infty,$$

and conversely.

Although the proof is given in [WW2] we quote the composition theorem and its corollary (the percolation theorem) here for completeness.

Theorem 2.5 (Composition Theorem). *Let T be an arbitrary compact metric space and let $\{R_n\}$ and $\{S_n\}$ be independent arbitrary positive T-martingales defined on a probability space $(\Omega, \mathcal{F}, P)$, with respect to independent increasing sigmafield sequences $\{\mathcal{R}_n\}$ and $\{\mathcal{S}_n\}$, respectively. Let $\mathcal{F}_{n,m} := \sigma\{\mathcal{R}_n \cup \mathcal{S}_m\}$. Define $Q_n(t) := R_n(t)S_n(t), t \in T, n \ge 1$. Then for $Q_{-1}, S_{-1} \in L^1(\sigma)$, P-a.s. $R_{-1} \in L^1(S_\infty(\sigma))$, $\{Q_n\}$ is a positive T-martingale with respect to $\mathcal{F}_{n,n}$ and one has $R_\infty(S_\infty\sigma) = Q_\infty\sigma$ a.s.*

The *percolation theorem* is the composition theorem with $\{R_n\}$ given by the i.i.d. generator process with

$$W = \begin{cases} b^\beta & \text{with probability } b^{-\beta}, \\ 0 & \text{with probability } 1 - b^{-\beta}. \end{cases} \tag{2.6}$$

The usual terminology from the physics literature for this cascade is the $\beta-$*model*, e.g. see [SLSL], and will be denoted $\{Q_n^{(\beta)}\}$.

The following theorem provides the percolated cascade distribution.

Theorem 2.6. *Let $\{S_n\}$ be a general multiplicative cascade having generator process distributed as $q_n(B) := P((W_{t|0}, \dots, W_{t|n}) \in B), B \in B^{n+1}[0,\infty)$, $n \geq 0$. Let $\{R_n \equiv Q_n^{(\beta)}\}$ be a $\beta-$model independent of $\{S_n\}$. Then $\{Q_n\}$ is a multiplicative cascade with generator process distributed as*

$$\begin{aligned}
&\tilde{q}_n(B_0 \times B_1 \times \cdots \times B_n) \\
&\quad = \sum_{(\epsilon_0,\dots,\epsilon_n)\in\{0,1\}^{n+1}} b^{-\beta\sum_{k=0}^n \epsilon_k}(1-b^{-\beta})^{n+1-\sum_{k=0}^n \epsilon_k} \\
&\quad \times \mathbf{B}\prod_{k=0}^n \delta_0^{1-\epsilon_k}(B_k) q_n(b^{-\beta\epsilon_0}B_{0,\epsilon_0} \times b^{-\beta\epsilon_1}B_{1,\epsilon_1} \times \cdots \times b^{-\beta\epsilon_n}B_{n,\epsilon_n}),
\end{aligned}$$

where $B_{k,\epsilon_k} = B_k$ if $\epsilon_k = 1$, else $B_{k,\epsilon_k} = [0,\infty)$.

The above theorems are the essential new tools for analyzing multiplicative cascades in the generality of this paper. However, before specializing to specific types of dependence (e.g. Markov, exchangeable), we record some results which also apply in some generality to the specific problems we wish to address for these dependence classes; namely the *degeneracy/nondegeneracy problem,* the *supporting dimension spectra,* and the *divergence of moments problem.*

The first result is essentially a computation of Kahane's T-martingale decomposition for multiplicative cascades and is the basic tool for obtaining the nondegeneracy criterion.

Theorem 2.7 (A T-martingale decomposition). *Let λ be the Haar measure on $T = \{0,1,\dots,b-1\}^{\mathbf{N}}$ and let $Q_n(t) = Q_n'(t) + Q_n''(t)$, the T-martingale decomposition with respect to λ,* [K2, K4]. *Then, $\{F_{t|n} = \frac{Q_n'(t)}{Q_n(t)} : t \in T\}$ is a well-defined weight decomposition ($\frac{0}{0} := 1$).*

The next five results are useful in analyzing the dimension spectra of multiplicative cascades. The first of these provides a computation of the spectral disintegration formula in this setting.

Theorem 2.8 (Spectral Decomposition). *The spectral decomposition $\lambda_\infty = \int \mu_\alpha \nu(d\alpha)$, is equivalent to a family of weight decompositions, $\mathbf{F}(s) := \{F_\gamma(s) : \gamma \in T^*\}$, parameterized by $s \in [0,1]$, defined by*

$$F_\gamma(s) = \frac{E[\nu_\gamma([0,s])|\mathcal{F}_n]}{\lambda_n(\Delta_n(\gamma))},$$

where

$$\nu_\gamma([0,s]) := \int_{[0,s]} \mu_\alpha(\Delta_n(\gamma))\nu(d\alpha), \quad n = |\gamma|.$$

Moreover,

(i) $$\lambda_{\mathbf{F}(s),n} \Rightarrow \int_{[0,s]} \mu_\beta \nu(d\beta) \ a.s.,$$

and

(ii) $$\lambda_{\mathbf{F}(s),n}(T) \to \nu([0,s]) \ in \ L^1.$$

Remark 2.5. We refer to the weights obtained in Theorem 2.8 as the *dimension spectral weights.* The dimension spectral weight $F_\gamma(s)$ is simply the expected proportion of mass of smaller carrying dimension than s in $\Delta(\gamma)$. One may note from (ii) that the Lebesgue-Stieltjes measures $s \to \lambda_{\mathbf{F}(s),n}$ converge vaguely to the dimension spectral measure ν a.s.

Theorem 2.9. *Suppose that* $\mathbf{F}(s), s \in [0,1]$, *is an arbitrary family of weight decompositions. Then* $\mathbf{F}(s), s \in [0,1]$, *is a spectral decomposition in the sense of* (i) *and* (ii) *of Theorem* 2.9 *if and only if the following properties hold:*

(0) $\mathbf{F}(1)$ *is the strong weight of the T-martingale decomposition of* λ_∞, $s \to F_\gamma(s)$ *is right continuous,* $F_\gamma(s) \le F_\gamma(1)$ *for each* $\gamma \in T^*$, *and letting* $Q_\infty^{(\alpha)}$ *be independent of the* $\lambda_{\mathbf{F},n}, n \ge 0$,

(1) $Q_\infty^\alpha \lambda_{\mathbf{F}(s),\infty} = 0$ *a.s. if* $s < \alpha$, *and*

(2) $Q_\infty^\alpha(\lambda_{\mathbf{F}(1),\infty} - \lambda_{\mathbf{F}(s),\infty})$ *lives if* $s \ge \alpha$.

Corollary 2.4. *Let* $\mathbf{F}$ *be a weight system with* $F_{-1} > 0$ *and let* $a > 0$.

(1) *If for* $\lambda-$*a.e.* t, $P_{\mathbf{F},t}(\sum_{n=0}^\infty b^{-n} b^{an} \prod_{i=0}^n W_{t|i} F_{t|n} < \infty) = 1$, *then P-a.s., for every Borel set B of Hausdorff dimension less than a one has* $\lambda_{\mathbf{F},\infty}(B) = 0$.

(2) *If for* $\lambda-$*a.e.* t, $P_{\mathbf{F},t}(\limsup_{n\to\infty} b^{-n} b^{an} \prod_{i=0}^n W_{t|i} F_{t|n} = \infty) = 1$ *then P-a.s.* $\lambda_{\mathbf{F},\infty}$ *is supported on a set of dimension at most a.*

The computation of the *spectral modes* μ_β in the spectral disintegration of λ_∞ as weighted cascades can be made under some additional structure as summarized by the following two theorems. That such structure is nonvaccuous is illustrated by a version of an exchangeable Polya cascade in section 6, and by an application to finite state Markov cascades in section 5.

Definition 2.3. Let X be a Polish space with Borel sigmafield $\mathcal{B}$ and let $\sigma \in \mathbf{P}rob(X)$. Suppose for each $A \in \mathcal{B}$ that $\mathbf{F}(A) = \{F_\gamma(A) : \gamma \in T^*\}$ is a weight system. If $A \to F_\gamma(A)$ is countably additive and absolutely continuous with respect to σ then $\mathbf{F}(A), A \in \mathcal{B}$ posseses a *local structure* given by the *derived weights* defined for $\gamma \in T^*$ at each $x \in X$ by the functions $\partial F_\gamma(x)$ on $[0,\infty)^{|\gamma|+1}$ where

$$\partial F_\gamma(x) := \frac{dF_\gamma}{d\sigma}(x).$$

We say that $\mathbf{F}(A)$ is *differentiable* (*with respect to* σ) in this case.

Note: By definition each $F_\gamma(A)$ is a measurable function of $(w_0, \ldots, w_{|\gamma|})$; however as is customary this dependence is often suppressed. The same is true for the local structure.

Theorem 2.10. *Given a system of derived weights one has for* $\sigma-$*a.e.* x, $\partial F(x) := \{\partial F_\gamma(x) : \gamma \in T^*\}$ *is a weight system.*

Theorem 2.11. *Suppose that* $\partial F(x), x \in X$, *is a family of derived weight systems such that for* $\sigma-$*a.e.* $x \in X$,

$$\lambda_{\partial F(x),n}(T) \to \lambda_{\partial F(x),\infty}(T) \quad \text{in } L^1.$$

Then P-a.s., for all $A \in \mathcal{B}(X)$

$$\lambda_{\mathbf{F}(A),\infty} = \int_A \lambda_{\partial F(x),\infty} \sigma(dx)$$

One may observe that the disintegration formula in Theorem 2.11 is more general than the dimension spectral disintegration formula of [KK]. The latter formula follows according to the special case contained in the following corollary.

Remark 2.6. It is interesting to notice that $Q_n(x) := \lambda_{\partial F(x),n}(T)$ "almost" defines a X-martingale, in the sense of being a positive martingale for σ-a.e. $x \in X$. That is, there is an $A \subset X$ such that $\sigma(A) = 1$ and $\{Q_n(x) : x \in A\}_n$ provides a positive T-martingale in the sense of Kahane.

Corollary 2.5. *Let $X = [0,1]$ and $\mathbf{F}([0,s]) = \mathbf{F}(s)$ where $\mathbf{F}(s)$ is the dimension spectral weight decomposition defined in Theorem 2.8. If $\mathbf{F}([0,s])$ is differentiable with local structure $\partial\mathbf{F}(s)$ such that*

$$\lambda_{\partial F(s),n}(T) \to \lambda_{\partial F(s),\infty}(T) \quad in \quad L^1,$$

then P-a.s. $\nu << \sigma$ with

$$\frac{d\nu}{d\sigma}(s) = \lambda_{\partial F(s),\infty}(T).$$

Moreover, for $\nu - a.e.\ \beta$

$$\mu_\beta = \frac{\lambda_{\partial F(\beta),\infty}}{\lambda_{\partial F(\beta),\infty}(T)}.$$

Finally the next two results are useful for determining moments of the total mass distribution.

Theorem 2.12. *For an arbitrary weight system one has for $0 < \epsilon < 1$,*

$$\sup_n E_P[\lambda^{1+\epsilon}_{\mathbf{F},n}(T)] \le (\frac{b-1}{b})^\epsilon \sum_{j=0}^{\infty} \frac{1}{b^{j\epsilon}} E_{P_{\mathbf{F},t}} \prod_{i=0}^{j} W^\epsilon_{t|i} F^\epsilon_{t|j}.$$

In particular, if for $1 < h \le 2$,

$$\sum_{j=0}^{\infty} \frac{1}{b^{j(h-1)}} E_{P_{\mathbf{F},t}} \prod_{i=0}^{j} W^{h-1}_{t|i} F^{h-1}_{t|j} < \infty,$$

then

$$E\lambda^h_{\mathbf{F},\infty}(T) < \infty.$$

In instances where one has available a good large deviation rate in the sense of general large deviation theory, e.g. see [DEUST], it is possible to express the criterion of Theorem 2.12 in a form more familiar to cascade theory. Specifically, for fixed but arbitrary $t \in T$, the P_t-distribution of the random sum

$$S_{t,n} := \frac{1}{n} \sum_{i=0}^{n-1} \log W_{t|i} \tag{2.7}$$

does not depend on t in view of Theorem 2.3. Therefore, if a large deviation principle holds for the generator process under the size-bias transform, then the rate function $\tilde{I} : S^+ \to [0,\infty]$ will not depend on t. We denote the Legendre transform of I by I^* where

$$I^*(t) = \sup_x \{tx - I(x)\} \tag{2.8}$$

Corollary 2.6. *Consider a general multiplicative cascade for which one has a large deviation principle with rate function I together with Varadahn's formula*

$$\frac{1}{n}\log E_{P_t} e^{n\phi_h(S_{t,n})} \to \tilde{I}^*(h-1),$$

for $1 < h \leq 2$, and $\phi_h(x) = (h-1)x$.

(i) *If $\frac{\tilde{I}^*(h-1)}{\log b} - (h-1) < 0$, then $E_P\lambda_\infty^h(T) < \infty$.*

(ii) *If $\frac{\tilde{I}^*(h-1)}{\log b} - (h-1) > 0$, then $\lim_n E_P\lambda_n^h(T) = \infty$. In particular, if $\{\lambda_n\}$ lives (i.e. fully acting) then $E_P\lambda_\infty^h(T) = \infty$.*

A natural setting for the large deviation formulation is in the case of Markov cascades to be given in section 5; this includes the homogeneous independent cascades.

3. Proofs of Main Results

The proof of Theorem 2.1 is a standard application of Kolmogorov's consistency theorem and is omitted. The proof of Theorem 2.2 will require the following lemma.

Lemma 3.1. *Let $\mathbf{F}$ be an arbitrary weight system with $Z_\emptyset > 0$. Then $\{\frac{1}{\lambda_{\mathbf{F},n}(T)} : n \geq 1\}$ is a $\mathcal{Q}_{\mathbf{F},\infty}$ positive supermartingale on $(\Omega \times T, \mathcal{F} \times \mathcal{B}, \mathcal{Q}_{\mathbf{F},\infty})$ with respect to the σ-field $\tilde{\mathcal{F}}_n = \sigma\{\tilde{W}_k : k \leq n\}$, where $\tilde{W}_k(\omega, t) = W_{t|k}(\omega), (\omega, t) \in \Omega \times T$. If $P(\lambda_{\mathbf{F},n}(T) > 0) = 1$ for all n, then $\{\frac{1}{\lambda_{\mathbf{F},n}(T)} : n \geq 1\}$ is a $\mathcal{Q}_{\mathbf{F},\infty}$-martingale.*

Proof. Note first that $\lambda_{\mathbf{F},n}(T)$ is a P-martingale. Also $\frac{1}{\lambda_{\mathbf{F},n(T)}}$ is independent of $t \in T$. Let G be a non-negative bounded $\tilde{\mathcal{F}}_{n-1}$-measurable function. Then for arbitrary $M > O$, by definition of $\mathcal{Q}_{\mathbf{F},\infty}$ and using consistency, one has

$$\begin{aligned} E_{\mathcal{Q}_{\mathbf{F},\infty}}(\{\frac{1}{\lambda_{\mathbf{F},n}(T)} \wedge M\}G) &= E_{\mathcal{Q}_{\mathbf{F},n}}(\frac{G}{\lambda_{\mathbf{F},n}(T) \vee \frac{1}{M}}) \\ &= Z_\phi^{-1} E_P(\frac{G\lambda_{\mathbf{F},n}(T)}{\lambda_{\mathbf{F},n}(T) \vee \frac{1}{M}}) \leq E_P G. \end{aligned} \tag{3.1}$$

Let $M \to \infty$ to get, by the monotone convergence theorem in the first expression and the dominated convergence theorem in the second expression, that

$$\begin{aligned} &E_{\mathcal{Q}_{\mathbf{F},\infty}}(\frac{1}{\lambda_{\mathbf{F},n}(T)}G) \\ &= Z_\phi^{-1} E_P(\mathbf{1}[\lambda_{\mathbf{F},n}(T) > 0] \cdot G) \leq Z_\phi^{-1} E_P([\mathbf{1}[\lambda_{\mathbf{F},n-1}(T) > 0]G]) \\ &= E_{\mathcal{Q}_{\mathbf{F},\infty}}(\frac{1}{\lambda_{\mathbf{F},n-1}(T)}G), \end{aligned} \tag{3.2}$$

since $[\lambda_{\mathbf{F},n}(T) > 0] \subset [\lambda_{\mathbf{F},n-1}(T) > 0]$. Thus, $E[\frac{1}{\lambda_{\mathbf{F},n}(T)}|\tilde{\mathcal{F}}_{n-1}] \leq \frac{1}{\lambda_{\mathbf{F}n-1}(T)}$. □

Proof of Theorem 2.2. In view of Lemma 3.1 and the martingale convergence theorem, $\lim_{n\to\infty} \frac{1}{\lambda_{\mathbf{F},n(T)}} = A < \infty$ (exists) $\mathcal{Q}_{\mathbf{F},\infty}$-a.s. Now, by (3.1) one has for any bounded $\tilde{\mathcal{F}}_n$-measurable function G and $M > 0$,

$$E_{\mathcal{Q}_{\mathbf{F},\infty}}(\{\frac{1}{\lambda_{\mathbf{F},n}(T)} \wedge M\}G) = Z_\emptyset^{-1} E_P(\frac{\lambda_{\mathbf{F},n}(T)}{\lambda_{\mathbf{F},n}(T) \vee \frac{1}{M}}G) \leq E_P G, \qquad n \geq N. \tag{3.3}$$

Let $n \to \infty$ and apply martingale and dominated convergence theorems to get

$$\begin{aligned} &Z_\emptyset E_{\mathcal{Q}_{\mathbf{F},\infty}}(\{A \wedge M\}G) \\ &= E_P(\mathbf{1}[\lambda_{\mathbf{F},\infty}(T) > \frac{1}{M}]G) + E_P(M\mathbf{1}[\lambda_{\mathbf{F},\infty}(T) \leq \frac{1}{M}]\lambda_{\mathbf{F},\infty}(T)G). \end{aligned} \tag{3.4}$$

Now let $M \to \infty$ to get

$$Z_\emptyset E_{\mathcal{Q}_{\mathbf{F},\infty}}(A \cdot G) = E_P(\mathbf{1}[\lambda_{\mathbf{F},\infty}(T) > 0]G). \tag{3.5}$$

Now $A = \frac{1}{\lambda_{\mathbf{F},\infty}(T)}\mathcal{Q}_{\mathbf{F}}$-a.s. In particular $A > 0$ on $[\lambda_{\mathbf{F},\infty}(T) < \infty]$ and

$$\begin{aligned} \frac{d\mathcal{Q}_{\mathbf{F},\infty}}{dP} &= \frac{Z_\emptyset^{-1}}{A}\mathbf{1}[0 < \lambda_{\mathbf{F},\infty}(T) < \infty] \\ &= Z_\emptyset^{-1}\lambda_{\mathbf{F},\infty}(T) \quad \text{on } [\lambda_{\mathbf{F},\infty}(T) < \infty]. \qquad \square \end{aligned} \tag{3.6}$$

Proof of Corollary 2.1. Note that $\lambda_{\mathbf{F},n}(T) \to \lambda_{\mathbf{F},\infty}(T)$ $P - a.s.$ by the martingale convergence theorem. Now, by positivity, L^1- convergence implies $E\lambda_{\mathbf{F},\infty}(T) = \lim_{n\to\infty} E\lambda_{\mathbf{F},n}(T) = Z_\emptyset$ and in particular, $[\lambda_{\mathbf{F},\infty}(T) < \infty]$ $\mathcal{Q}_{\mathbf{F},\infty} - a.s.$ Conversely, by the Lebesgue decomposition theorem $\mathcal{Q}_{\mathbf{F},\infty} \circ \pi_\Omega^{-1}(dw) = Z_\emptyset^{-1}\lambda_{\mathbf{F},\infty}(T)P(dw)$. Thus $1 = \int_\Omega \mathcal{Q}_{\mathbf{F},\infty} \circ \pi_\Omega^{-1}(dw) = Z_\emptyset^{-1}E_P\lambda_{\mathbf{F},\infty}(T)$ and, therefore, $Z_\emptyset = E_P\lambda_{\mathbf{F},\infty}(T)$, i.e. $\lim_{n\to\infty} E_P\lambda_{\mathbf{F},n}(T) = E_P\lambda_{\mathbf{F},\infty}(T)$. The L^1-convergence follows from this, positivity, and the martingale convergence theorem using Scheffe's theorem [BW, p. 635]. $\square$

Proof of Theorem 2.3. Define $P_{\mathbf{F},t} << P$ on $\mathcal{F}_n$ as indicated and then apply Kolmogorov's extension theorem to get $P_{\mathbf{F},t}$ on $\mathcal{F}$ for each $t \in T$. Observe that $t \to P_{\mathbf{F},t}(A)$ is measurable for all $A \in \mathcal{F}_n, n \geq 1$. For a bounded $\mathcal{F}_n-$measurable function G and a bounded Borel function g on T, one has

$$\begin{aligned} \int_T g(t)E_{P_{\mathbf{F},t}}G\lambda(dt) &= \int_T g(t)E_P(Z_\emptyset^{-1}\prod_{i=0}^{n} W_{t|i}F_{t|n}G)\lambda(dt) \\ &= E_P\int_T g(t)Z_\emptyset^{-1}\prod_{i=0}^{n} W_{t|i}F_{t|n}G\lambda(dt) \\ &= E_{\mathcal{Q}_{\mathbf{F},\infty}}(gG). \end{aligned} \tag{3.7}$$

The idea for the second formula is to decompose the total mass $\lambda_{\mathbf{F},n}(T)$ generated from $\emptyset$ according to the mass generated from each node $t|i, i = 0, 1, \ldots,$ down the $t-$path. Remember that there will be a slight irregularity in the trees off the $t-$path; namely $b-1$ branches off the initial node and b branches at the subsequent nodes. That is,

$$
\begin{aligned}
\lambda_{\mathbf{F},n}(T) &= \sum_{|\gamma|=n} b^{-n} \prod_{i=0}^{n} W_{\gamma|i} F_{\gamma|n} \\
&= b^{-n} \prod_{i=0}^{n} W_{t|i} F_{t|n} + \sum_{\substack{|\gamma|=n \\ \gamma \neq t|n}} b^{-n} \prod_{i=0}^{n} W_{\gamma|i} F_{\gamma|n} \\
&= b^{-n} \prod_{i=0}^{n} W_{t|i} F_{t|n} + \sum_{j=0}^{n-1} \sum_{\substack{|\gamma|=n-j \\ \gamma|1 \neq t|j+1}} b^{-n} \prod_{i=0}^{j} W_{t|i} \prod_{i=1}^{n-j} W_{t|j*\gamma|i} F_{t|j*\gamma} \\
&= \sum_{j=0}^{n} b^{-j} \prod_{i=0}^{j} W_{t|i} M_{n,j}(t),
\end{aligned}
\tag{3.8}
$$

where

$$
M_{n,j}(t) := \sum_{\substack{|\gamma|=n-j \\ \gamma|1 \neq t|j+1}} b^{-(n-j)} \prod_{i=1}^{n-j} W_{t|j*\gamma|i} F_{t|j*\gamma}.
$$

Observe that, apart from the $b-1$ irregularity in the tree, given $\mathcal{F}_{t,0}, M_{n,j}(t)$ is distributed as a total mass. Taking into account this irregularity one gets, using the martingale property of the weight system and the conditional mean one property of generators,

$$
E_P[M_{n,j}(t)|\mathcal{F}_{t,0}] = \begin{cases} \frac{b-1}{b} F_{t|j}, & \text{if } j \leq n-1, \\ F_{t|n} & \text{if } j = n. \end{cases}
\tag{3.9}
$$

□

Proof of Corollary 2.2. For a bounded $\mathcal{F}_n-$measurable function $H_n : \Omega \to \mathbf{R}$ and a continuous function g on T,

$$
\begin{aligned}
E_{\mathcal{Q}_{\mathbf{F},\infty}}[H_n g] &= E_P[H_n \frac{\int g(t) \lambda_{\mathbf{F},n}(dt)}{\lambda_{\mathbf{F},n}(T)} \lambda_{\mathbf{F},n}(T)] \\
&= E_{\mathcal{Q}_{\mathbf{F},\infty} \circ \pi_\Omega^{-1}}[H_n \frac{\int g(t) \lambda_{\mathbf{F},n}(dt)}{\lambda_{\mathbf{F},n}(T)}].
\end{aligned}
$$

By the usual appeal to countable convergence determining classes, $\mathcal{Q}_{\mathbf{F},\infty} \circ \pi_\Omega^{-1}-$ a.s.,

$$
\frac{\lambda_{\mathbf{F},n}(dt)}{\lambda_{\mathbf{F},n}(T)} \Rightarrow \mu(dt),
\tag{3.10}
$$

where $\mu(\omega, dt) \mathcal{Q}_{\mathbf{F},\infty} \circ \pi_\Omega^{-1}(d\omega) = \mathcal{Q}_{\mathbf{F},\infty}(d\omega \times dt)$. For $\mathcal{Q}_{\mathbf{F},\infty} \circ \pi_\Omega^{-1}-$a.e. $\omega \in [\lambda_{\mathbf{F},\infty}(T) < \infty]$, $\frac{\lambda_{\mathbf{F},n}(dt)}{\lambda_{\mathbf{F},n}(T)} \Rightarrow \frac{\lambda_{\mathbf{F},\infty}(dt)}{\lambda_{\mathbf{F},\infty}(T)}$, since $\lambda_{\mathbf{F},n}$ converges vaguely to $\lambda_{\mathbf{F},\infty}$ and $0 < \lambda_{\mathbf{F},\infty}(T) < \infty$. So,

$$
\begin{aligned}
&\mathbf{1}[\lambda_{\mathbf{F},\infty}(T) < \infty] \mathcal{Q}_{\mathbf{F},\infty}(d\omega \times dt) \\
&\quad = Z_{\mathbf{F},\infty}^{-1} \frac{\lambda_{\mathbf{F},n}(dt)}{\lambda_{\mathbf{F},n}(T)} \lambda_{\mathbf{F},n}(T)(\omega) P(d\omega) = Z_{\mathbf{F},\infty}^{-1} \lambda_{\mathbf{F},n}(dt) P(d\omega).
\end{aligned}
$$

For the last term of the formula of Corollary 2.2, Theorem 2.2 implies for any bounded measurable function $H : \Omega \to \mathbf{R}$, that

$$\begin{aligned}&\int H(\omega)\mathbf{1}[\lambda_{\mathbf{F},\infty}(T) = \infty]\mathcal{Q}_{\mathbf{F},\infty} \circ \pi_\Omega^{-1}(d\omega)\\ &\qquad = \int\int H(\omega)\mathbf{1}[\lambda_{\mathbf{F},\infty}(T) = \infty](\omega)P_{\mathbf{F},s}(d\omega)\lambda(ds).\end{aligned} \tag{3.11}$$

For λ-a.e. s, $P_{\mathbf{F},s}$-a.s. ω for $\gamma \in \{s|n\}_{n=0}^\infty$, $\lambda_{\mathbf{F},n}(\Delta_\gamma) \le b^{-1}\prod_{j\le m} W_{s|j}\times M_{n,m}(s)$ where here $m := \max\{j : s|j = \gamma|j\}$ and $M_{n,m}(s)$ is the positive $P_{\mathbf{F},s}-$martingale in the statement of Theorem 2.3. Thus for λ-a.e. s, $P_{\mathbf{F},s}$-a.e. ω, for $\gamma \in \{s|n\}_{n=0}^\infty$, one has $\lim_{n\to\infty}\lambda_{\mathbf{F},n}(\Delta_\gamma) < \infty$. Since the limit of a sum is the sum of limits and $T = \bigcup_{|\gamma|=n}\Delta_\gamma$, for λ-a.e. s, $P_{\mathbf{F},s}$-a.e. $\omega \in [\lambda_{\mathbf{F},\infty}(T) = \infty]$, for $\gamma \notin \{s|n\}_{n=0}^\infty$, $\lim_{n\to\infty}\lambda_{\mathbf{F},n}(\Delta_\gamma) < \infty$ and for every n, $\lambda_{\mathbf{F},\infty}(\Delta_{s|n}) = \infty$. That is, for λ-a.e. s, $P_{\mathbf{F},s}$-a.e. $\omega \in [\lambda_{\mathbf{F},\infty}(T) = \infty]$, "there exists a unique $t \in T$ such that $\lim_n \lambda_{\mathbf{F},n}(\Delta_\gamma) < \infty \iff \gamma \notin \{s|n\}_{n=0}^\infty$". Since the statement in quotes is only a function of ω and not s, (3.11) implies $\mathcal{Q}_{\mathbf{F},\infty}\circ\pi_\Omega^{-1}-$a.e. $\omega \in [\lambda_{\mathbf{F},\infty} = \infty]$, "there is a unique $t \in T$ such that $\lim \lambda_{\mathbf{F},n}(\Delta_\gamma) < \infty \iff \gamma \notin \{t|n\}_{n=0}^\infty$". Using this statement along with (3.10), $\mathcal{Q}_{\mathbf{F},\infty}\circ\pi_\Omega^{-1}-$a.e. $\omega \in [\lambda_{\mathbf{F},\infty}(T) = \infty]$, there is a unique $t \in T$ such that if $\gamma \notin \{t|n\}_{n=0}^\infty$ then $\mu_\omega(\Delta_\gamma) = \lim_n \frac{\lambda_{\mathbf{F},n}(\Delta_\gamma)}{\lambda_{\mathbf{F},n}(T)} = 0$ and so $\mu_\omega = \delta_t$.□

Proof of Theorem 2.4. The lower bound is obvious. For the upper bound use (3.8) to write after introducing the factor $c_j^{-1}c_j$,

$$\begin{aligned}\lambda_{\mathbf{F},n}(T) &= b^{-n}\prod_{i=0}^n W_{t|i}F_{t|n}\\ &+ \sum_{j=0}^{n-1}\sum_{\substack{|\gamma|=n-j\\ \gamma|1\neq t|j+1}} c_j^{-1}b^{-j}\prod_{i=0}^j W_{t|i}F_{t|j}\sum_{\substack{|\gamma|=n-j\\ \gamma|1\neq t|j+1}} F_{t|j}^{-1}c_j b^{-(n-j)}\prod_{i=1}^{n-j} W_{t|j*\gamma|i}F_{t|j*\gamma}\\ &\le b^{-n}\prod_{i=0}^n W_{t|i}F_{t|n} + (\sup_j c_j^{-1}b^{-j}\prod_{i=0}^j W_{t|i}F_{t|j})M_n.\end{aligned}$$

Since, similarly to (3.9), M_n is a positive $\{\mathcal{F}_n\}-$submartingale having a predictable part bounded in expectation, i.e. $\sup_n E_{\mathcal{Q}_{\mathbf{F},\infty}}A_n = E_{\mathcal{Q}_{\mathbf{F},\infty}}\sum_j c_j < \infty$, it follows that M_n is a.s. convergent. □

Proof of Corollary 2.3. Fix $t \in T$. For each N, let

$$A_N := [\sum_{j=0}^\infty \frac{1}{b^j}\prod_{i\le j} W_{t|i}F_{t|j} \le N] \tag{3.12}$$

and define

$$c_j(N) := \frac{1}{b^j}\prod_{i\le j} W_{t|i}F_{t|j}\mathbf{1}[A_N]. \tag{3.13}$$

The $c_j(N), j \ge 1$, are clearly admissible, in particular $E_{\mathcal{Q}_{\mathbf{F},t}}\sum_{j=0}^\infty c_j(N) \le N < \infty$. Thus the $P_{\mathbf{F},t}-$martingale, $M_n(N)$ associated with the $c_j(N), j \ge 1$, sequence

converges. But $\lambda_{\mathbf{F},n}(T)\mathbf{1}[A_N] = (M_n(N) + c_n)\mathbf{1}[A_N]$. Since $\mathbf{1}[A_N]c_n \to 0$ and $\mathbf{1}[A_N]M_n(N)$ converges, $P_{\mathbf{F},t}(A_N) \leq P_{\mathbf{F},t}(\lambda_{\mathbf{F},\infty}(T) < \infty)$. Thus, $P_{\mathbf{F},t}(\bigcup A_N) \leq P_{\mathbf{F},t}(\lambda_{\mathbf{F},\infty}(T) < \infty) = \mathcal{Q}_{\mathbf{F},\infty}(\lambda_{\mathbf{F},\infty}(T) < \infty)$. If $P_{\mathbf{F},t}(\bigcup A_N) > 0$, then $E_P\lambda_{\mathbf{F},\infty}(T) > 0$ follows from the Lebesgue decomposition. If $P_{\mathbf{F},t}(\bigcup A_N) = 1$, then apply Corollary 2.1 to see $\mathcal{Q}_{\mathbf{F},\infty}(\lambda_{\mathbf{F},\infty}(T) < \infty) = 1$. □

The proof of the Composition Theorem 2.5 is given in [WW2]. The proof of Theorem 2.6 is an elementary computation and will be omitted.

For the proof of Theorem 2.7 we will need a lemma.

Lemma 3.2. *For $\gamma \in T^*$ let $\mathcal{F}_{\gamma,\infty} := \sigma\{W_{\gamma|i} : i \leq |\gamma|, W_{\gamma*\tau}, \tau \in T^*\}$. If X is $\mathcal{F}_{\gamma,\infty}$-measurable, then $E[X|\mathcal{F}_{|\gamma|}]$ is $\mathcal{F}_{\gamma,0}-$measurable.*

Proof. The proof is by induction after defining

$$\mathcal{F}_{\gamma,n} := \sigma\{W_{\gamma|i} : i \leq |\gamma|, W_{\gamma*\tau}, |\tau| \leq n\}. \tag{3.14}$$

Note that $\mathcal{F}_{\gamma,\infty} = \sigma\{\bigcup \mathcal{F}_{\gamma,n}\}$. Note that the lemma is true if X is $\mathcal{F}_{\gamma,0}$-measurable and suppose that it is true if X is $\mathcal{F}_{\gamma,n}-$measurable, say $X = \prod_{i=1}^{|\gamma|} G_{\gamma|i} \prod_{0<|\tau|\leq n+1} G_{\gamma*\tau}$, where G_η is $\sigma\{W_\eta\}-$measurable. Then, using conditional independence,

$$\begin{aligned} E[X|\mathcal{F}_{|\gamma|}] &= \prod_{i=1}^{|\gamma|} G_{\gamma|i} E[\prod_{0<|\tau|\leq n+1} G_{\gamma*\tau}|\mathcal{F}_{|\gamma|}] \\ &= \prod_{i=1}^{|\gamma|} G_{\gamma|i} \prod_{i=0}^{b-1} E[\prod_{|\zeta|\leq n} G_{\gamma*i*\zeta}|\mathcal{F}_{|\gamma|}] \\ &= \prod_{i=1}^{|\gamma|} G_{\gamma|i} \prod_{i=0}^{b-1} E[E[\prod_{|\zeta|\leq n} G_{\gamma*i*\zeta}|\mathcal{F}_{|\gamma|}]|\mathcal{F}_{|\gamma|}], \end{aligned} \tag{3.15}$$

since $\mathcal{F}_{|\gamma|} \subset \mathcal{F}_{|\gamma|+1}$. Now apply the induction hypothesis to get for each i that $H_i = E[\prod_{|\zeta|\leq n} G_{\gamma*i*\zeta}|\mathcal{F}_{|\gamma|+1}]$ is $\mathcal{F}_{\gamma*i,0}$-measurable.

$$E[X|\mathcal{F}_{|\gamma|}] = \prod_{i=1}^{|\gamma|} G_{\gamma|i} \prod_{i=0}^{b-1} E[H_i|\mathcal{F}_{|\gamma|+1}]$$

is then $\mathcal{F}_{\gamma*i,0}-$measurable. Now, by definition of the general multiplicative cascade, the conditional distribution of $W_{\gamma*i}$ depends only on $W_{\gamma|k}, k = 0, 1, \ldots n$. □

Lemma 3.3. *Let $\alpha, \beta, \gamma \in T^*$, with $|\alpha| = |\beta|$ and let $\phi_\alpha : T \to T, \phi_\beta : T \to T$ be the continuous one to one maps defined by $\phi_\alpha(t) := \gamma * \alpha * t, \phi_\beta(t) = \gamma * \beta * t$. If $G : M^+(T) \to R$ is a Borel measurable function such that $G(\lambda_{\mathbf{F},\infty}|_{\Delta_{\gamma*\alpha}} \circ \phi_\alpha) \in L^1(P)$, then $G(\lambda_{\mathbf{F},\infty}|_{\Delta_{\gamma*\beta}} \circ \phi_\beta) \in L^1(P)$ and one has*

$$E_P[G(\lambda_{\mathbf{F},\infty}|_{\Delta_{\gamma*\alpha}} \circ \phi_\alpha)|\mathcal{F}_{|\gamma|}] = E_P[G(\lambda_{\mathbf{F},\infty}|_{\Delta_{\gamma*\beta}} \circ \phi_\beta)|\mathcal{F}_{|\gamma|}]$$

and the conditional probability is $\mathcal{F}_{\gamma,0}-$measurable.

Proof. Let $I = \{\gamma|i\}_{i=0}^{|\gamma|} \cup \{\gamma * (\alpha|i)\}_{i=1}^{|\alpha|-1} \cup \{\gamma * \alpha * \tau : \tau \in T^*\}$ and let $J = \{\gamma|i\}_{i=0}^{|\gamma|} \cup \{\gamma * (\beta|i)\}_{i=1}^{|\beta|-1} \cup \{\gamma * \beta * \tau : \tau \in T^*\}$. Also let $\phi : I \to J$ be the obvious

map for which $\phi(\gamma|i) = \gamma|i, i \leq |\gamma|, \phi(\gamma * (\alpha|i)) = \gamma * (\beta|i), i = 1, \ldots, |\alpha| - 1$, and $\phi(\gamma*\alpha*\tau) = \gamma*\beta*\tau, \tau \in T^*$. For $n \in \mathbf{Z}^+$, let $N(n) = \#\{\tau \in I : |\tau| \leq n\}$. Then by a standard induction on n one has for bounded Borel functions $H : [0,\infty)^{N(n)} \to \mathbf{R}$,

$$E_P[H(\{W_\tau\}_{\tau\in I,|\tau|\leq n})] = E_P[H(\{W_{\phi(\tau)}\}_{\tau\in I,|\tau|\leq n})]. \tag{3.16}$$

Therefore for any n, and any bounded Borel $G : M^+(T) \to \mathbf{R}$, applying this to $H \times G(\lambda_{\mathbf{F},n}|_{\Delta_{\gamma*\alpha}} \circ \phi_\alpha)$, where H is $\mathcal{F}_{|\gamma|}-$measurable, and then differentiating out H,

$$E_P[G(\lambda_{\mathbf{F},\infty}|_{\Delta_{\gamma*\alpha}} \circ \phi_\alpha)|\mathcal{F}_{|\gamma|}] = E_P[G(\lambda_{\mathbf{F},\infty}|_{\Delta_{\gamma*\beta}} \circ \phi_\beta)|\mathcal{F}_{|\gamma|}] \tag{3.17}$$

and the conditional probability is $\mathcal{F}_{\gamma,0}-$measurable by Lemma 3.2. Now, P-a.s., $\lambda_{\mathbf{F},n}|_{\Delta_{\gamma*\alpha}} \circ \phi_\alpha \to \lambda_{\mathbf{F},\infty}|_{\Delta_{\gamma*\alpha}} \circ \phi_\alpha$, and similarly for β in place of α. So for bounded vaguely continuous functions $G : M^+(T) \to \mathbf{R}$, by bounded convergence one has (3.17). Extension to bounded Borel G then follows using the dominated convergence theorem. Extension to positive Borel measurable G such that $G(\lambda_{\mathbf{F},\infty}|_{\Delta_{\gamma*\alpha}} \circ \phi_\alpha) \in L^1(P)$ follows from the monotone convergence theorem, and then consider positive and negative parts for the final extension as asserted. □

Proof of Theorem 2.7. Since $\lambda_{\mathbf{F},\infty}(\Delta_{\gamma*\alpha}) \in L^1(P)$, by Lemma 3.3, it follows that $E[\lambda_{\mathbf{F},\infty}(\Delta_{\gamma*\alpha})|\mathcal{F}_{|\gamma|}]$ is $\mathcal{F}_{\gamma,0}-$measurable and for $|\beta| = |\alpha|$,

$$E[\lambda_{\mathbf{F},\infty}(\Delta_{\gamma*\alpha})|\mathcal{F}_{|\gamma|}] = E[\lambda_{\mathbf{F},\infty}(\Delta_{\gamma*\beta})|\mathcal{F}_{|\gamma|}]. \tag{3.18}$$

Let $H_\gamma = E[\lambda_{\mathbf{F},\infty}(\Delta_\gamma)|\mathcal{F}_{|\gamma|}]$. By additivity and (3.18),

$$E[\lambda_{\mathbf{F},\infty}(\Delta_{\gamma*\alpha})|\mathcal{F}_{|\gamma|}] = b^{-|\alpha|} \sum_{|\beta|=|\alpha|} E[\lambda_{\mathbf{F},\infty}(\Delta_{\gamma*\beta})|\mathcal{F}_{|\gamma|}] = \frac{H_\gamma \lambda(\Delta_{\gamma*\alpha})}{\lambda(\Delta_\gamma)} \tag{3.19}$$

for $\alpha \in T^*$. Thus, $Q'_n(t) = \sum_{|\gamma|=n} \frac{H_\gamma}{\lambda(\Delta_\gamma)} \mathbf{1}_{\Delta_\gamma}(t)$ and H_γ is $\mathcal{F}_{\gamma,0}-$measurable. Now, $Q'_n \leq Q_n$ and therefore $F_{t|n} = \frac{H_{t|n}}{\lambda(\Delta_{t|n})} = \frac{Q'_n(t)}{Q_n(t)} \leq 1$, $(\frac{0}{0} = 1)$. So $\{F_\gamma\}$ is a weight decomposition. □

Proof of Theorem 2.8. For $\mu \in M^+(T)$, let $\int_{[0,1]} \mu_v \nu_\mu(dv)$ denote the dimension spectral disintegration of μ. The set $A_{s,r} = \{\mu \in M^+(T) : \nu_\mu[0,s] > r\}$ is a Borel subset of $M^+(T)$ for $s \in [0,1], r$. Thus $\psi_s(\mu) = \nu_\mu[0,s]$ is positive and Borel measurable. Since $\int_{[0,s]} \mu_\beta(\Delta_{\gamma*\alpha})\nu_\mu(d\beta) = \psi_s(\lambda_\infty|_{\Delta_{\gamma*\alpha}} \circ \phi_\alpha)$, one has by Lemma 3.3 for $|\alpha| = |\beta|$

$$E[\nu_{\gamma*\alpha}[0,s]|\mathcal{F}_{|\gamma|}] = E[\nu_{\gamma*\beta}[0,s]|\mathcal{F}_{|\gamma|}] \tag{3.20}$$

and is $\mathcal{F}_{\gamma,0}-$measurable. As in the proof of Theorem 2.7, one has $\{F_\gamma(s) = E[\nu_\gamma[0,s]|\mathcal{F}_n]\}$ is a weight decomposition and, moreover,

$$\lambda_{\mathbf{F}(s),n} = E[\int_{[0,s]} \mu_\beta \nu(d\beta)|\mathcal{F}_n].$$

Thus (i) and (ii) are now immediate from this. □

Proof of Theorem 2.9. Let us first show that the spectral weights satisfy properties $(0)-(2)$. By definition we have

$$F_\gamma(s) = \frac{E[\nu_\gamma[0,s]|\mathcal{F}_n]}{\lambda_n(\Delta_n(\gamma))}, \quad n = |\gamma|. \tag{3.22}$$

Thus $F_\gamma(1) = \frac{E[\lambda_\infty(\Delta_n(\gamma))|\mathcal{F}_n]}{\lambda_n(\Delta_n(\gamma))} = \frac{\lambda'_n(\Delta_n(\gamma))}{\lambda_n(\Delta_n(\gamma))}$ is the strong weight of the T-martingale decomposition by Theorem 2.8. Also right continuity follows from Theorem 2.8 (ii) and the Dominated convergence theorem. Finally $F_\gamma(s) \le F_\gamma(1)$ by monotonicity of ν_γ. For (1) we apply the percolation theorem to get for $\alpha > s$, using [K1],

$$Q^{(\alpha)} \int_{[0,s]} \mu_\beta \nu(d\beta) = 0, \tag{3.23}$$

and for $\alpha \le s$, letting $\mathcal{F}_\infty$ denote the Borel sigma-field of Ω independent of the defining sigma-field of $Q^{(\alpha)}$,

$$E[Q^{(\alpha)} \int_{(s,1]} \mu_\beta \nu(d\beta)|\mathcal{F}_\infty] = \int_{(s,1]} \mu_\beta \nu(d\beta) \quad P\text{-}a.s. \tag{3.24}$$

Properties (1) and (2) follow from Theorem 2.9 (i), noting that $\lambda_{\mathbf{F}(1),\infty} - \lambda_{\mathbf{F}(s),\infty} = \lambda_{\mathbf{F}(1)-\mathbf{F}(s),\infty}$. For the converse suppose that $\mathbf{G}(s) = \{G_\gamma(s) : \gamma \in T^*\}$ is a weight decomposition satisfying (0), (1), (2). Fix $s \in T$. Then a.s.,

$$\lambda_{\mathbf{G}(s),n}(T) \to \lambda_{\mathbf{G}(s),\infty}(T), \tag{3.25}$$

and

$$0 \le \lambda_{\mathbf{G}(s),n}(T) \le \lambda_{\mathbf{G}(1),n}(T) \to \lambda_{\mathbf{G}(1),\infty}(T) \tag{3.26}$$

in L^1. Thus, $\lambda_{\mathbf{G}(s),n}(T) \to \lambda_{\mathbf{G}(s),\infty}(T)$ in L^1. Also,

$$\lambda_{\mathbf{G}(s),\infty} + \lambda_{\mathbf{G}(1)-\mathbf{G}(s),\infty} = \lambda_{\mathbf{G}(1),\infty} \tag{3.27}$$

since $\mathbf{G}(s)$ and $\mathbf{G}(1) - \mathbf{G}(s)$ are weight decompositions and $\mathbf{G}(s) + \mathbf{G}(1) - \mathbf{G}(s) = \mathbf{G}(1)$. Thus,

$$\lambda_{\mathbf{G}(s),\infty} + \lambda_{\mathbf{G}(1)-\mathbf{G}(s),\infty} = \lambda_\infty = \int_{[0,1]} \mu_\beta \nu(d\beta). \tag{3.28}$$

For $\alpha > s$,

$$Q^{(\alpha)} \lambda_{\mathbf{G}(1)-\mathbf{G}(s),\infty} = Q^{(\alpha)} \int_{[\alpha,1]} \mu_\beta \nu(d\beta) \ge Q^{(\alpha)} \int_{(\alpha,1]} \mu_\beta \nu(d\beta), \tag{3.29}$$

where $Q^{(\alpha)}$ is a β-model ($\beta = \alpha$) independent of $\{\lambda_{\mathbf{G}(s),n}\}$. Take expectations with respect to $Q^{(\alpha)}$, i.e. integrate out $Q^{(\alpha)}$, to get for $\alpha > s$, P-a.s.,

$$\lambda_{\mathbf{G}(1)-\mathbf{G}(s),\infty} \ge \int_{[\alpha,1]} \mu_\beta \nu(d\beta) \tag{3.30}$$

and

$$\lambda_{\mathbf{G}(s),\infty} \leq \int_{[0,\alpha]} \mu_\beta \nu(d\beta). \tag{3.31}$$

Let $\alpha \to s$, to get P-a.s.

$$\lambda_{\mathbf{G}(s),\infty} \leq \int_{[0,s]} \mu_\beta \nu(d\beta). \tag{3.32}$$

On the other hand, if $s \geq \alpha$, then

$$Q^{(\alpha)} \lambda_{\mathbf{G}(1)-\mathbf{G}(s),\infty} \leq Q^{(\alpha)} \lambda_\infty = Q^{(\alpha)} \int_{[\alpha,1]} \mu_\beta \nu(d\beta). \tag{3.33}$$

So, taking expectations with respect to P and using independence, we have a.s.,

$$\lambda_{\mathbf{G}(1)-\mathbf{G}(s),\infty} \leq \int_{[\alpha,1]} \mu_\beta \nu(d\beta) \tag{3.34}$$

and therefore for $s \geq \alpha$, a.s.,

$$\int_{[0,\alpha)} \mu_\beta \nu(d\beta) \leq \lambda_{\mathbf{G}(s),\infty}. \tag{3.35}$$

Thus, (3.35) and (3.32) provide

$$\int_{[0,s)} \mu_\beta \nu(d\beta) \leq \lambda_{\mathbf{G}(s),\infty} \leq \int_{[0,s]} \mu_\beta \nu(d\beta). \tag{3.36}$$

Fix n and take conditional expectations to get

$$E[\int_{[0,s)} \mu_\beta \nu(d\beta) | \mathcal{F}_n] \leq \lambda_{\mathbf{G}(s),n} \leq E[\int_{[0,s]} \mu_\beta \nu(d\beta) | \mathcal{F}_n]. \tag{3.37}$$

One has for all α, P-a.s., for any Δ_n,

$$\lim_{\epsilon \to 0^+} \lambda_{\mathbf{F}(\alpha-\epsilon),n}(\Delta_n) \leq \lambda_{\mathbf{G}(s),n}(\Delta_n) \leq \lambda_{\mathbf{F}(\alpha),n}(\Delta_n). \tag{3.38}$$

Choose a countable dense subset D of $[0,1]$ to get (3.38) P-a.s., $\forall \alpha \in D$. Then by the right continuity of both $G_\gamma(\alpha)$ and $F_\gamma(\alpha), \gamma \in T^*$, we have $\lambda_{\mathbf{G}(s),n}(\Delta_n) = \lambda_{\mathbf{F},n}(\Delta_n)$ for all s, and for all Δ_n $P - a.s.$ Thus, $P - a.s.$, if $\prod_{i \leq n} W_{\gamma|i} \neq 0$, then $F_\gamma(s) = G_\gamma(s)$. In particular, $\mathbf{G}(s), 0 \leq s \leq 1$, provide the spectral weights. □

The Corollary 2.4 follows directly from Theorem 2.9 using Theorem 2.6 and Corollary 2.3.

Proof of Theorem 2.10. For $A \in \mathcal{B}$, one has for $\gamma \in T^*$,

$$\begin{aligned}
&\int_A E[\prod_{i\le n} W_{\gamma|i}\frac{dF_\gamma}{d\sigma}(x)|\mathcal{F}_{n-1}]\sigma(dx)\\
&= E[\prod_{i\le n} W_{\gamma|i}\int_A \frac{dF_\gamma}{d\sigma}(x)|\mathcal{F}_{n-1}]\sigma(dx)\\
&= E[\prod_{i\le n} W_{\gamma|i}F_\gamma(A)|\mathcal{F}_{n-1}]\\
&= \prod_{i\le n} W_{\gamma|i}F_{\gamma|n-1}(A)\\
&= \int_A \prod_{i\le n-1} W_{\gamma|i}\frac{dF_{\gamma|n-1}}{d\sigma}(x)\sigma(dx). \qquad (3.39)
\end{aligned}$$

Thus,

$$E[\prod_{i\le n} W_{\gamma|i}\partial F_\gamma(x)|\mathcal{F}_{n-1}] = \prod_{i\le n-1} W_{\gamma|i}\partial F_{\gamma|n-1}(x), \quad \sigma\text{-}a.s. \qquad (3.40)$$

This and the obvious measurability property inherited from the weight system $\mathbf{F}(A)$ completes the proof. □

Proof of Theorem 2.11. For $\sigma - a.e.x, Z_{\emptyset,\partial\mathbf{F}(x)} = E[W_\emptyset \partial F_\emptyset(x)] > 0$ yields by our Lebesque Decomposition Theorem 2.2 that

$$Z_{\emptyset,\partial\mathbf{F}(x)}\mathcal{Q}_{\partial\mathbf{F}(x),\infty}(d\omega \times dt) = \lambda_{\partial\mathbf{F}(x),\infty}(dt)P(d\omega). \qquad (3.41)$$

Thus for σ-a.e. x, $Z_{\emptyset,\partial\mathbf{F}(x)} = E\lambda_{\partial\mathbf{F}(x),\infty}(T)$. Now by the Fubini-Tonelli theorem one has $P-a.s$ for $\sigma-$a.e. x and for every cylinder Δ_γ of T, $\lambda_{\partial\mathbf{F}(x),n}(\Delta_\gamma)$ converges as $n \to \infty$. Now, by definition, for every $A \in \mathcal{B}(X)$,

$$F(A) = \int_A \partial F(x)\sigma(dx). \qquad (3.42)$$

So, in particular,

$$\lambda_{\mathbf{F}(T),n}(T) = \int_T \lambda_{\partial\mathbf{F}(x),n}(T)\sigma(dx). \qquad (3.43)$$

Letting $n \to \infty$ one has by Fatou's lemma that

$$\lambda_{\mathbf{F}(T),\infty}(T) \ge \int_T \lambda_{\partial\mathbf{F}(x),\infty}(T)\sigma(dx). \qquad (3.44)$$

Taking P-expectations on both sides one gets

$$\begin{aligned}
Z_{\emptyset,\mathbf{F}(T)} &\ge E\lambda_{\mathbf{F}(T),\infty}(T)\\
&\ge E\int_T \lambda_{\partial\mathbf{F}(x),\infty}(T)\sigma(dx)\\
&= \int_T E[\lambda_{\partial\mathbf{F}(x),\infty}(T)]\sigma(dx)\\
&= \int_T Z_{\emptyset,\partial F(x)}\sigma(dx) = Z_{\emptyset,\mathbf{F}(T)}. \qquad (3.45)
\end{aligned}$$

Thus, one has P-a.s.

$$\lambda_{\mathbf{F}(T),\infty}(T) \geq \int_T \lambda_{\partial\mathbf{F}(x),\infty}(T)\sigma(dx), \tag{3.46}$$

So there is a P-null set $\mathcal{N}$ such that off this null set:

(i) For σ-a.e. x and for every cylinder Δ_γ of T, $\lambda_{\partial F(x),n}(\Delta_\gamma)$ converges as $n \to \infty$ and

(ii) $\int_T \lambda_{\partial\mathbf{F}(x),n}(T)\sigma(dx) \to \int_T \lambda_{\partial\mathbf{F}(x),\infty}(T)\sigma(dx)$ as $n \to \infty$,

(iii) For any $A \in \mathcal{B}(X)$, and any cylinder Δ, $\lambda_{\partial\mathbf{F}(x),n}(\Delta) \leq \lambda_{\partial\mathbf{F}(x),n}(T)$ σ-a.e. $x \in A$, and

Applying the Dominated convergence theorem to (3.42) now yields

$$\lambda_{F(A),n}(\Delta) = \int_A \lambda_{\partial F(x),n}(\Delta) \to \int_A \lambda_{\partial F(x),\infty}(\Delta). \tag{3.47}$$

□

The proof of Corollary 2.5 is simply a matter of definitions once Theorem 2.11 is obtained.

Proof of Theorem 2.12. The first bound follows directly from Theorem 2.3 and sublinearity for $0 < \epsilon < 1$, using the basic x-transform property that

$$E_P\lambda^{1+\epsilon}_{\mathbf{F},n}(T) = E_{P_{\mathbf{F},t}}\lambda^{\epsilon}_{\mathbf{F},n}(T), \quad t \in T. \tag{3.48}$$

Now the convergence criterion yields $\sup_n E_{P_{\mathbf{F},t}}\lambda^h_{\mathbf{F},n}(T) < \infty$. Now apply Corollary 2.1. □

Proof of Corollary 2.6. Under (i) one obtains the convergence condition of Theorem 2.12 and hence $E_P\lambda^h_\infty(T) < \infty$. For part (ii) simply note that for $t \in T$,

$$E_P\lambda^h_n(T) = E_{P_t}\lambda^{h-1}_n(T) \geq E_{P_t}(b^{-n}\prod_{i\leq n} W_{t|i})^{h-1} \to \infty \tag{3.49}$$

by the divergence condition. Now if $\{\lambda_n\}$ lives, then $E[\lambda_\infty(T)|\mathcal{F}_n] = \lambda_n(T)$, and therefore $\lambda_\infty(T) \in L^h(P) \iff \sup_n E_P\lambda^h_n(T) < \infty$. □

4. Applications to Independent Cascades

In this section we consider homogeneous independent cascades $\lambda_\infty = Q_\infty\lambda$. As an illustration of the general decomposition theory we first give a new treatment of the Kahane-Peyrière theorem. We give here the original results from [KP].

Theorem 4.1 (Kahane-Peyrière). *Let λ denote normalized Haar measure on T and let $\lambda_\infty := Q_\infty\lambda$. Then,*

(i) (*Non-degeneracy*). $\chi'_b(1^-) < 0 \iff E\lambda_\infty(T) > 0$.

(ii) (*Support Size*) *Assume that $E(\lambda_\infty(T)\log\lambda_\infty(T)) < \infty$. Then there is a Borel subset S of T a.s. having Hausdorff dimension $D = -\chi'_b(1) = 1 - EW\log_b W$ such that $\lambda_\infty(S) = \lambda_\infty(T)$, and if B is a Borel subset of T of Hausdorff dimension less than D, then $\lambda_\infty(B) = 0$.*

(iii) (*Divergence of Moments*) *Let $h > 1$. Then $\lambda_\infty(T)$ has a finite moment of order $h \iff h < h_c = \sup\{h \geq 1 : \chi_b(h) < 0\} \iff EW^h < b^{h-1}$.*

Proof. For (i) suppose that $EW\log_b W - 1 < 0$. Fix $c \in (1, b^{1-EW\log_b W})$ and take $c_j = c^{-j}$ in Corollary 2.3. Use the strong law of large numbers on the random walk $\frac{1}{j}\sum_{k=1}^{j}\log_b W_{t|k}$ under P_t together with the size-bias transform property $E_{P_t}\log_b W = E_P W\log_b W$. Conversely, if $EW\log_b W - 1 > 0$, then, again using the strong law of large numbers, $b^{-n}\prod_{k\le n} W_{t|k} \to \infty \quad P_t-$a.s. for $t \in T$. Use the lower submartingale bound of Theorem 2.4 and Lebesgue decomposition Theorem 2.2 to conclude $\lambda_\infty(T) = 0$ P-a.s. For the critical case in which $EW\log_b W - 1 = 0$, but $P_t(\log_b W = 0) < 1$, use the Chung-Fuchs recurrence criterion [CF] in place of the strong law of large numbers to get that $\limsup b^{-n}\prod_{k\le n} W_{t|k} = \infty$ P_t-a.s. By the size-bias transform property, the very special case $EW\log_b W - 1 = 0$, $P_t(\log_b W = 0) = 1$ occurs in and only in the case of the $\beta-$model with $\beta = 1$. This case is covered by the extinction of a Galton-Watson branching process at criticality.

To prove (ii) simply note that the percolated cascade $Q^{(\beta)}Q_\infty\lambda$ has a shift in the parameter $D = 1 - EW\log_b W$ by the amount β, i.e. to $D+\beta$ as explained in the more general Proposition 4.3 below. The carrying dimension then follows from (i) and Corollary 2.1 of [WW2].

Finally to prove (iii), assume the nondegeneracy criterion $EW\log_b W - 1 < 0$ from (i) and let $1 < h \le 2$ such that $0 > \chi_b(h) = \log_b E_P W^h - (h-1)$. Then the conclusion $E_P\lambda_\infty^h < \infty$ follows immediately from Theorem 2.12. To bootstrap to $2 < h \le 3$ and $0 > \chi_b(h) = \log_b E_P W^h - (h-1)$, we use the submartingale upper bound of Theorem 2.4 as follows. Fix arbitrary $t \in T$. Let $c \equiv c(h) \in (b^{\frac{1}{h-1}\chi_b(h)}, 1)$. Using independence of generators along the t-path with the (martingale) subtree masses and Jensen's inequality, we have

$$
\begin{aligned}
&E_{P_t}[\lambda_n^{h-1}(T)] \\
&= (\sum_{j=0}^{n}(cb)^{-j}\prod_{i\le j} W_{t|i}c^j M_{j,n}(t))^{h-1} \\
&\le E_{P_t}(\max_{0\le j\le n}(cb)^{-j}\prod_{i\le j} W_{t|i})^{h-1} E_{P_t}(\sum_{j=0}^{n} c^j M_{j,n}(t))^{h-1} \\
&\le E_{P_t}(\max_{0\le j\le n}(cb)^{-j(h-1)}\prod_{i\le j} W_{t|i}^{h-1})(1-c)^{-(h-1)} E_{P_t}\sum_{j=0}^{n} c^j M_{j,n}(t)^{h-1}
\end{aligned}
\tag{4.1}
$$

Now, by positivity, the "max"in the first factor in (4.1) is dominated by

$$
\sum_{j=0}^{\infty} E_{P_t}[(cb)^{-j}\prod_{i\le j} W_{t|i})^{h-1}] < \infty
$$

by our choice of c = c(h). Since the P_t-distribution of the masses $M_{n,j}$ equals the P-distribution, the "sum"in the second factor of (4.1) is $\sum_{j=0}^{n} c^j E_P M_{n,j}^{h-1}$. Each of the terms $M_{n,j}^{h-1}$ is a submartingale and $EM_{n,j}^{h-1} < EM_{n,\infty}^{h-1} < \infty$ by previous condition since $1 < h-1 \le 2$ and the $M_{n,\infty}$ terms are identically distributed as a total cascade mass (up to the slight initial irregularity in the subtree). Proceed inductively to larger values of $m < h \le m+1, m = 1, 2, \ldots$. Finally, if $0 \le \chi_b(h) = \log_b E_P W^h - (h-1)$, then one may show $E_P\lambda_\infty^h(T) = \infty$ as follows. Fix $t \in T$. Let

let $c_j = b^{-j}\prod_{i\le j} W_{t|i}$ and let $M_{k,j}$ be as defined in (3.8) of the proof of Theorem 2.3. Since $EW\log_b W - 1 < 0$, one has $P_t - a.s\ c_j \to 0$ as $j\to\infty$. Thus the random integer $N := \inf\{n : c_n = \sup c_j\}$ is a.s. finite. Now, $c_{N\wedge n}M_{n,N\wedge n} \le \lambda_n(T), \forall n$ and therefore, $c_N M_{\infty,N} \le \lambda_\infty(T)$ P_t-a.s. Also, c_N and $M_{\infty,N}$ are P_t-independent. Thus,

$$E_{P_t} c_N^{h-1} E_{P_t} M_{\infty,N}^{h-1} \le E_P \lambda_\infty^h(T), \tag{4.2}$$

since $E_{P_t}\lambda_\infty^{h-1}(T) = E_P\lambda_\infty^h(T)$ as λ_∞ lives. Now, $E_{P_t}M_{\infty,N}^{h-1} > 0$ since λ_∞ lives. So, by (4.2) it is sufficient to show $E_{P_t}c_N^{h-1} = \infty$. But $E_{P_t}c_j^{h-1} = E_P b^{j\chi_b(h)} \ge 1$ $\forall j$ and $c_j \to 0$ P_t-a.s. Thus, by a contrapositive argument using the dominated convergence theorem, $E_{P_t}\sup_j c_j^{h-1} = \infty$. □

Remark 4.1. In view of its triviality under the size-bias transform it is often the case that the $\beta-$model requires special consideration as in the proof of (i) above. Another illustration is the estimation of the gauge function for exact Hausdorff dimension based on the law of the iterated logarithm in [WW2].

Homogeneous independent cascades may be zero with positive probability. However in these cases there is a distinct finite interface between zero and nonzero masses, i.e. in view of the next proposition, homogeneous independent cascades cannot just fade away.

Proposition 4.2. *Consider a homogeneous independent cascade* λ_∞ *such that* $EW\log_b W < 1$, *one has*

$$P(\lambda_n(T) \ne 0\ \forall n | \lambda_\infty(T) = 0) = 0.$$

Proof. First observe that the number $Z_n = \sum_{|\gamma|=n}\mathbf{1}[\lambda_n(\Delta_\gamma)\ne 0]$ of nonzero nth generation cylinder masses is a supercritical Galton-Watson branching process with Binomial offspring distribution. In particular, therefore $Z_n \to \infty$ on $[\lambda_\infty(T)\ne 0]$. Now

$$P(\exists n, Z_n \ge M, \lambda_\infty(T) = 0) \le P(\lambda_\infty(T) = 0)^M.$$

Also,

$$P(\exists M, Z_n \le M, \lambda_n(T) \ne 0, \forall n, \lambda_\infty(T) = 0) \le P(\exists M, 1 \le Z_n \le M\ \forall n) = 0.$$

Therefore

$$\begin{aligned} P(\forall n, \lambda_n(T)\ne 0, \lambda_\infty(T) = 0) &\le P(\exists n, Z_n \ge M, \lambda_\infty(T) = 0) \\ &\le P(\lambda_n(T) = 0)^M \to 0, \text{ as } M\to\infty. \end{aligned} \tag{4.3}$$

This proves the assertion.

In addition to its utility in the form of the percolation method as a tool for analyzing dimension spectra, the general composition theorem also has a number of "non-percolative"consequences. We illustrate this with the following general result. The special case of percolation is used in the previous theorem to compute the carrying dimension (ii) from the non-degeneracy criterion (i).

Proposition 4.3 (A Codimension Formula). *Suppose that $R_\infty\lambda, S_\infty\lambda$ are mutually independent homogeneous independent cascades with carrying dimensions D_R, D_S respectively. Then $R_\infty(S_\infty\lambda)$ is an independent cascade with carrying dimension D_{RS} and*

$$(1-D_R)+(1-D_S)=1-D_{RS}.$$

Proof. This is a direct application of the composition theorem and the simple fact that for independent mean one non-negative random variables U, V one has $E[UV\log UV]=EU\log U+EV\log V$. □

One may note that the moment condition $E\lambda_\infty(T)\log\lambda_\infty(T)<\infty$ included in the Kahane-Peyrière theorem is not necessary using the percolation method. This was observed in [K] without proof of the percolation theorem. The following Proposition 4.4 shows that this is indeed an improvement over the original result.

Proposition 4.4. *Assume $EW\log_b W<1$. Then $E\lambda_\infty(T)\log\lambda_\infty(T)<\infty$ if and only if $EW(\log W)^2<\infty$.*

Proof. First let us note that

$$-e^{-1}\le E_P[\lambda_\infty(T)\log_b\lambda_\infty(T)]=E_{\mathcal{Q}}\log_b\lambda_\infty(T). \tag{4.4}$$

The final expectation in (4.4) exists since by Fatou's lemma

$$\begin{aligned} E_{\mathcal{Q}}[\log_b\lambda_\infty(T)\wedge 0] &= -E_{\mathcal{Q}}[-\log_b\lambda_\infty(T)\vee 0]\\ &\ge -\liminf E_{\mathcal{Q}}[-\log_b\lambda_n(T)\vee 0]\\ &= -\liminf E_P[(-\log_b\lambda_n(T)\vee 0)\lambda_n(T)]\ge -e^{-1}. \end{aligned} \tag{4.5}$$

Let $c\in(b^{E[W\log_b W-1},1)$. Then, using Theorem 2.3,

$$\log_b\lambda_n(T)\le\log_b[\sup_{j\le n}(cb)^{-j}\prod_{i\le j}W_{t|i}]+\log_b(\sum_{j=0}^{n-1}c^jM_{n,j}+c^n). \tag{4.6}$$

The last term in (4.6) is the logarithm of a martingale and therefore a supermartingale term. Take expectations with respect to P_t in (4.6) and use the supermartingale (or Jensen's inequality) to bound the expectation of that term by the initial expectation $\frac{b-1}{b}$. Then,

$$\begin{aligned} &E_P[\lambda_n(T)\log_b\lambda_n(T)]=E_{P_t}[\log_b\lambda_n(T)]\\ &\le E_{P_t}[\sup_{j\le n}\log_b(cb)^{-j}\prod_{i\le j}W_{t|i}]+\log_b(\sum_{j=0}^{n-1}c^j(\frac{b-1}{b})+c^n). \end{aligned} \tag{4.7}$$

Using Fatou's lemma,

$$\lambda_n(T)\log_b\lambda_n(T)\ge -e^{-1},\quad \lambda_n(T)\log_b\lambda_n(T)\to\lambda_\infty(T)\log_b\lambda_\infty(T),$$

(4.8)

$$E_P[\lambda_\infty(T)\log_b\lambda_\infty(T)]\le\log(\frac{b-1}{b}\frac{1}{1-c})+\liminf_n E_{P_t}[\sup_{j\le n}(\log_b(cb)^j\prod_{i\le j}W_{t|i})].$$

Using the monotone convergence theorem in the second expression,

$$E_P[\lambda_\infty(T)\log_b\lambda_\infty(T)] \le \log(\frac{b-1}{b}\frac{1}{1-c}) + E_{P_t}\sup_{j\ge 1}(\log_b(cj)^{-j}\prod_{i\le j}W_{t|i}). \tag{4.9}$$

Now observe that under $P_t, \log_b(cb)^{-j}\prod_{i\le j}W_{t|i}$ is a one-dimensional random walk with independent increments having mean displacement

$$\mu = E_{P_t}[\log_b W - 1 - \log_b c] = E_P[W\log_b W] - 1 - \log_b c < 0. \tag{4.10}$$

According to a theorem variously attributed to Kiefer, Wolfowitz, Darling, Erdös, and Kakutani, the random variable $\sup_{j\ge 1} S_j$ has a finite first moment if and only if

$$E_{P_t}[(\log_b W_{t|i} - 1 - \log_b c)\vee 0]^2 < \infty, \tag{4.11}$$

see [KW]. But (4.11) is equivalent to the condition that $E_P[W(\log_b W)^2] < \infty$ in the present application. For the converse, fix $t \in T$. As in the proof of Theorem 4.1 (iii), P_t-a.s. $c_N M_{\infty,N} \le \lambda_\infty(T)$ and c_N and $M_{\infty,N}$ are independent. So,

$$\log c_N + \log M_{\infty,N} \le \log\lambda_\infty(T). \tag{4.12}$$

Since $\lambda_\infty(T)$ lives, there is an $\alpha > 0$ such that $\beta = P(M_{\infty,N} \ge \alpha) > 0$. Multiply (4.12) by $\mathbf{1}[M_{\infty,N} \ge \alpha]$ and take expectations to get

$$\beta(E_{P_t}\log c_N + \log\alpha) \le E_{P_t}\mathbf{1}[M_{\infty,N} \ge \alpha]\log\lambda_\infty(T). \tag{4.13}$$

However, for $E[W(\log_b W)^2] = \infty$, the theorem of Kiefer et al. [KW], implies $E_{P_t}\mathbf{1}[M_{\infty,N} \ge \alpha]\log\lambda_\infty(T) = \infty$. But (4.4) now yields $E_{P_t}\log\lambda_\infty(T) = \infty$. □

Remark 4.2. There are other instances, for example where standard limit theorems under applications of the size-bias transform need moment conditions such as $E\lambda_\infty(T)(\log\lambda_\infty(T))^\theta < \infty$; another example is the use of the law of the iterated logarithms in the estimates on the exact Hausdorff dimension given in [WW2].

We close this section with an application of the composition theorem to homogeneous independent cascades having a *log-infinitely divisible* generator distribution. In particular, in the spirit of the recent activity in superprocesses we introduce a notion of a *supercascade* which seems to share in some of the qualitative properties. We shall see that the supercascades provide a natural class of examples from the point of view of the dimension spectral theory.

We proceed from the simple observation that in view of the general composition theorem, a homogeneous independent cascade having a log-infinitely divisible generator distribution has a natural "divisibility" property as well; namely, for each natural number n one may write the cascade as a random composition of n independent cascades. Let$\{X_s^\emptyset : s \ge 0\}, X_0^\emptyset = 0$ a.s., be a process with independent increments having right continuous samples paths. Assume $E_P e^{X_s^\emptyset} < \infty$, and let $c_s = \log E_P e^{X_s^\emptyset}$. Also let $\beta(s) = E[\frac{e^{X_s^\emptyset - c_s}(X_s^\emptyset - c_s)}{\log b}]$. Now let $\{\{X_s^\gamma : s \ge 0\} : \gamma \in T^*\}$

be a countable family of i.i.d. independent increment processes distributed as $\{X_s^{\emptyset} : s \geq 0\}, X_0^{\emptyset} = 0$. Define

$$Q_n(s,t) = \prod_{i \leq n} e^{X_s^{t|i} - c_s}, \qquad s \geq 0, t \in T. \tag{4.14}$$

Then for each fixed $s \geq 0$, $\{Q_n(s,t)\}$ defines a homogeneous independent cascade. Suppose we are given a finite measure σ. Then by the Fubini-Tonelli theorem it follows that P-a.s., for σ-a.e. s, $\lim Q_n(s,\cdot)\lambda$ exists vaguely. Thus, P-a.s., for σ-a.e. s, $Q_\infty(s)\lambda$ is well-defined.

Remark 4.3. It is an interesting problem to try to define $Q_\infty(s)\lambda$ P-a.s. for every s. However this is not required for the results presented here.

In addition to the obvious ones, let us see that some familiar generators are log-infinitely divisible. It should be noted that the first example is made possible by considering probability distributions on the extended real number system $\mathbf{R} \cup \{-\infty\}$. This is a semigroup on which infinite divisibility may be defined probabilistically in terms of sums of independent random variables, but not in terms of convolution powers.

Example 4.1. (a) ($\beta-$model) One has $W = e^X$ where $P(X = \beta \log b) = b^{-\beta} = 1 - P(X = -\infty)$.

(b) (Uniform generator) Let W be uniformly distributed on [0,2]. One has $W = e^X$ where $\log 2 - X$ has an Exponential distribution. For this example it is interesting to note that the total cascade mass has a Gamma distribution. The same is also true for a class of Beta distributed generators.

In view of Example 4.1(a), one has by the composition theorem and the fact that the class of infinitely divisible distributions is closed under convolution that log-infinite divisibility is preserved under percolation.

Here is an analytic approach to supercascades. Let $\mathcal{M}^+$ denote the space of finite positive measures with the vague topology. Let $\mathbf{B}(\mathcal{M}^+)$ be the space of bounded Borel functions on $\mathcal{M}^+$. Then the *supercascade* is the Markov process defined by the contraction semigroup $\{T_s : s \geq 0\}$ given by

$$T_s G(\sigma) = E_P G(Q_\infty(s)\sigma), s \geq 0, G \in \mathbf{B}(\mathcal{M}^+). \tag{4.15}$$

To prove the semigroup property simply observe that i.i.d. compositions are Markov with stationary transition probabilities.

The following theorem contains the example from [KK, Example 2] as a special case.

Theorem 4.5. *Let σ be a given finite measure. Let $D(s) := 1 - \beta(s)$, with $\beta(s)$ assumed to be increasing. Then*

$$\mu = \int_{[0,\infty)} \mu_x \nu(dx),$$

where $\nu << \sigma$ is given by

$$\frac{d\nu}{d\sigma}(x) = Q_\infty(D^{-1}(x), \cdot)\lambda(T),$$

and

$$\mu_x = \frac{Q_\infty(D^{-1}(x),\cdot)\lambda}{Q_\infty(D^{-1}(x),\cdot)\lambda(T)}$$

is a dimension spectral disintegration having a absolutely continuous dimension spectrum with respect to the prescribed measure σ.

The proof of Theorem 4.5 consists of a few mostly obvious lemmas to check that quantities are well defined and computable.

Lemma 4.1. *$P-$a.s., for $\sigma-$a.e. s, on $[Q_\infty(s)\lambda(T) > 0]$ $Q_\infty(s)\lambda$ is unidimensional with carrying dimension $D(s) = 1 - \beta(s)$.*

Proof. This is another direct application of the Fubini-Tonelli theorem along the lines above. □

Lemma 4.2. *Fix $\alpha \in [0,\infty)$. For $\epsilon \geq 0$ define*

$$\tilde{Q}_n(\epsilon,t) := \prod_{i\leq n} e^{X^{(t|i)}_{\alpha+\epsilon} - X^{(t|i)}_{\alpha} - c_{\alpha+\epsilon} + c_\alpha}.$$

Then $\{\tilde{Q}_n(\epsilon,t)\}\epsilon \geq 0$ is well-defined and independent of $\{Q(\alpha)\}$.

Proof. Well-definedness is as before, and independence is by the independence of increments. □

Lemma 4.3. *If Q is an arbitrary positive T-martingale and σ is a finite measure carried by a Borel set B, then a.s. $Q\sigma$ is carried by B.*

Proof. $Q(\sigma_1 + \sigma_2) = Q\sigma_1 + Q\sigma_2$ simply because a limit of a finite sum is the sum of the limits. Moreover, if $\sigma = \sum_{i=1}^\infty \sigma_i$, then $Q\sigma = \sum_{i=1}^\infty Q\sigma_i$ may be seen as follows. $EQ\sum_{i=n+1}^\infty \leq \int q(t)(\sum_{i=1}^\infty \sigma_i(dt))$ by the T-martingale decomposition. Therefore, one may choose an increasing sequence $\{n_j\}$ of positive integers such that $Q\sum_{i=n_j}^\infty \sigma_i \to 0$ a.s. Now, $Q\sigma = Q\sum_{i=1}^{n_j}\sigma_i + o(1)$ as $j \to \infty$. □

Lemma 4.4. *Fix $\alpha \in [0,\infty)$. Then for σ-a.e. $\beta \geq \alpha$, P-a.s.*

$$\tilde{Q}_\infty(\beta-\alpha)Q_\infty(\alpha)\lambda = Q_\infty(\beta)\lambda,$$

and if B_α is a Borel set carrying $Q_\infty(\alpha)\lambda$, then $Q_\infty(\beta)\lambda$ is carried on B_α.

Proof. Apply the composition theorem and use the previous lemma under independence. □

Lemma 4.5. *Fix $\alpha > 0$. Then P-a.s. there is a Borel set B_α such that*
(i) *$dim B_\alpha = 1 - \beta(\alpha)$ if $Q_\infty(\alpha)\lambda(T) > 0$.*
(ii) *For σ-a.e. $\beta \geq \alpha, Q_\infty(\beta)\lambda(T - B_\alpha) = 0$.*

Proof of Theorem 4.5. First let us observe that if $\int_{1-\beta(s)} > \alpha Q_\infty(B)\lambda\sigma(ds) > 0$ then $Q_\infty(s)\lambda(B) > 0$ for some set of positive σ measure. So by Lemma 4.1, $\dim(B) > \alpha$; i.e. $\int_{1-\beta(s)>\alpha} Q_\infty\lambda\sigma(ds)$ is P-a.s. $\alpha-$regular in the potential theory language. On the other hand, Lemma 4.5 can be used to get that $\int_{1-\beta(s)} < \alpha Q_\infty\lambda\sigma(ds)$ is $\alpha-$singular as follows. Let s_α denote the unique α such that $1 - \beta(s_\alpha) = \alpha$. Let $s_{\alpha_i} \downarrow s_\alpha$. For each i, there is a Borel set B_i of dimension $1 - \beta(s_{\alpha_i})$ which is σ-a.s. a carrying set for $Q(s)\lambda$.

If $s \geq s_i$ then the $\beta-$cascade, $(\beta = \alpha)$, $Q^{(\alpha)}$ kills $\int_{E_i} Q_\infty(s)\lambda\sigma(ds)$, where $E_i = \{s : 1 - \beta(s_{i-1}) \leq 1 - \beta(s) \leq 1 - \beta(s_i)\}$. This completes the proof as the rest of the statement simply involves normalizing the integrands. □

5. Applications to Markov Cascades

In this section we consider the cascade of a Markov generator process with state space $S \in \mathcal{B}[0,\infty)$. In [WW1] the Kahane-Peyriére theory was extended to finite state Markov generators by applying the Perron-Frobenius theory to the recursion on the total mass; i.e. by essentially finite dimensional techniques. Neither this approach nor that in [KP] appears to be suitable for more general state spaces when there is dependence. However, we will see here that the theory given in section 2 readily applies.

For the finite state cases there is a natural decomposition of the state space according to classes of states which are survival transient/recurrent or failure transient/recurrent. Even in the case of a finite state single ergodic irreducible class one obtains interesting critical phenomena; see section 5 of [WW1]. In the present paper we will not restrict S, however we will make a $\psi-$recurrence hypothesis which will allow us to define an entropy parameter. In this way our results and methods on problems (i) and (ii) will completely subsume the Kahane-Peyriére theory of homogeneous independent cascades. Partial results are given on problem (iii) in this generality. Examples will be given to illustrate certain critical aspects of the null-recurrent cases.

Let $q(dy|x)$ be a mean one transition probability kernel on the state space $S \in \mathcal{B}[0,\infty)$. Then $\tilde{q}(dy|x) := yq(dy|x), x, y \in S^+ = S \cap (0,\infty)$ is a transition probability kernel. Unless indicated otherwise, we will assume an initialization of the form $P(W_\emptyset = x) = 1$. We will indicate the corresponding distribution of either the Markov chain or the Markov cascade determined by this initialization and the transition law $q(dy|x)$ as $(\delta_x - q(dy|x))$. Let $\tilde{X}_0, \tilde{X}_1, \ldots$ denote a generic stochastic process with distribution governed by $(\delta_x - \tilde{q}(dy|x))$. Here is the relationship between this and the $\mathcal{Q}_\infty$ probability distribution.

Proposition 5.1. *Consider a dependent cascade for which the generator path process is a stationary Markov chain with transition probabilities $q(dy|x)$. Then $\{\tilde{W}_n\}$ is a stationary Markov chain with transition probabilities $\tilde{q}(dy|x) = yq(dy|x)$ under $\mathcal{Q}_\infty$.*

Proof. This is a simple computation of the $\mathcal{Q}_\infty$ distribution of $\tilde{W}_n, n \geq 0$, using the definition of $\mathcal{Q}_\infty$. □

We define a (possibly random) "entropy parameter" by

$$H = \limsup \frac{1}{n+1} \sum_{i=0}^{n} \log_b X_i. \tag{5.1}$$

With the exception of a class of transient examples discussed at the end of this section, we will suppose that the size-bias transform $\tilde{q}(dy|x)$ is $\psi-$recurrent. Then, up to positive scalar multiples, there is a unique $\tilde{q}-$invariant sigma-finite measure $\tilde{\pi}$. Moreover, the tail sigma-field is finitely atomic with respect to $\tilde{q}$, see [OR]. So in this case the parameter H is a.s. a constant; note that the tail atoms are averaged out in the formula (5.1).

Theorem 5.2. *If $\tilde{q}(dy|x)$ is $\psi-$recurrent, then*

(i) *If $H < 1$, then $E\lambda_\infty(T) = 1$.*

(ii) *If* $H > 1$, *then* $E\lambda_\infty(T) = 0$.

Proof. The proof of Theorem 4.1(i) given in the independent case applies here with the constant parameter H replacing the strong law of large numbers in the computation of $\sqrt[n]{b^{-n}\prod_{i\le n} W_{t|i}}$ for large n. □

Corollary 5.1. *If* $H < 1$ *then* λ_∞ *is unidimensional with carrying dimension* $1-H$ *on* $[\lambda_\infty(T) > 0]$.

Proof. In view of Theorem 2.6 one has that the percolated Markov cascade is a Markov cascade and the parameter H is merely shifted to $H+\beta$ under percolation. From here the proof is exactly as in for Theorem 4.1(ii). □

Remark 5.1. As part of Corollary 5.1 it is proven that

$$H \ge 0. \tag{5.2}$$

That is, since $Q^{(\beta)}\lambda_n$ lives for $H+\beta < 1$, the carrying dimension of λ_∞ is $1-H \le 1$.

In the positive recurrent case one has $\tilde\pi(S^+) < \infty$ and this makes the Markov chain on S^+ with transition law $\tilde q(dy|x)$ ergodic. In particular, one has a strong law of large numbers of the following form (see [BW, p.229] and references therein):

$$\frac{1}{n}\sum_{i=0}^{n-1} f(X_i) \to E_{\tilde\pi} f, \quad f \in L^1(\pi). \tag{5.3}$$

Now, if $\tilde\pi$ is a probability, then

$$0 \ge \int \log_b(x)\mathbf{1}_{\{x\le 1\}}\tilde\pi(dx) = \int\int_{\{x\le 1\}} \tilde\pi(dx) \ge -\frac{e^{-1}}{\log b}. \tag{5.4}$$

In particular $\log_b \in L^1(\tilde\pi)$ and

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}\log_b \tilde X_k = \int \log_b(x)\tilde\pi(dx). \tag{5.5}$$

Note that from the point of view of cascade theory, the transition law $\tilde q(dy|x)$ on S defines an equivalence class of transition kernels $q(dy|x)$ on S obtained by moding out transitions from zero. Nonetheless, it is natural to ask for conditions on q which guarantee ergodicity of the transform $\tilde q$. According to the following example some conditions will be required.

Example 5.1 (Birth-Collapse Generator). We note here that an irreducible ergodic Markov generator on S may be transformed to a transient kernel $\tilde q(dy|x)$. Take $S = \{0, 1, 2, \dots\}$ and define $q_{i,i+1} = \frac{1}{i+1} = 1 - q_{i,0}$. Then under $\tilde q$ the motion is clearly transient.

Following [WW1] we will say that the transition law is $\pi-$*mean reversible* if the mean is preserved under the adjoint with respect to π. The following is a simple extension of a finite state case in [WW1].

Proposition 5.3. *If $q(dy|x)$ has mean one for each $x \in S^+$ and has an invariant probability π such that $\pi(S^+) = 1$, then $\int_S y\pi(dy) = 1$. If, in addition, $q(dy|x)$ is $\pi-$mean reversible, then $\tilde{q}(dy|x)$ has an invariant probability $\tilde{\pi} << \pi$ such that*

$$\frac{d\tilde{\pi}}{d\pi}(x) = x.$$

Moreover, in this case if π is a unique invariant probability for $q(dy|x)$, then the size-bias transform process is ergodic under $\tilde{q}(dy|x)$.

Proof. That $\int_S y\pi(dy) = 1$ follows immediately from invariance and Fubini-Tonelli. Similarly, the reversibility condition and definition of the adjoint with respect to π implies that, for a bounded Borel measurable function f,

$$\begin{aligned} \int_{S^+}\int_{S^+} f(y)\tilde{q}(dy|x)\tilde{\pi}(dx) &= \int_{S^+}\int_{S^+} f(y)yq(dy|x)x\pi(dx) \\ &= \int_{S^+}\int_{S^+} f(y)yxq^*(dx|y)\pi(dy) \\ &= \int_{S^+} f(y)y\pi(dy) = \int_{S^+} f(y)\tilde{\pi}(dy). \end{aligned} \tag{5.6}$$

Thus $\tilde{\pi}$ is an invariant probability for $\tilde{q}(dy|x)$. Let f be a bounded Borel function on S and let

$$A_f = [\lim_n \frac{1}{n+1}\sum_{i=0}^{n} f(X_i) \in A], \qquad A \in \mathcal{B}. \tag{5.7}$$

Then there is a Borel subset B_f of $\mathbf{R}$ such that $P_{\tilde{\pi}}-$a.s.,

$$A_f = B_f^{\mathbf{N}}, \quad A_f^c = (B_f^c)^{\mathbf{N}}. \tag{5.8}$$

Moreover, $P_{\tilde{\pi}}(A_f \cup A_f^c) = 1$. For $\tilde{\pi}$-a.e. x, $X_1 \in B_f \Rightarrow X_n \in B_f, \forall n$. Define $\tilde{S} = \{x \in S^+ : q^{(n)}(B_f|x) = \mathbf{1}_{B_f}(x)\forall n\}$. Then $\tilde{\pi}(\tilde{S}) = 1$. Now, $\frac{d\tilde{\pi}}{d\pi}(y) = y$ and $\pi(\{0\}) = 0$ and hence $\pi << \tilde{\pi}$. Therefore, since $(\tilde{\pi} - \tilde{q})-$a.s. $[X_n \in B_f] \Rightarrow [X_{n+1} \in B_f]$ and $[X_n \in B_f^c] \Rightarrow [X_{n+1} \in B_f^c]$, one also has $(\pi - q)-$a.s. $[X_n \in B_f] \Rightarrow [X_{n+1} \in B_f]$. It follows from this, using the dominated convergence theorem, that

$$(\pi - q)(B_f^{\mathbf{N}}) = \lim(\pi - q)[X_i \in B_f \forall i \leq n] = \pi(B_f). \tag{5.9}$$

Now $(\pi - q)(B_f^{\mathbf{N}}) \in \{0, 1\}$ by ergodicity. In particular, $\pi(B_f) = 1$ and hence $\tilde{\pi}(B_f) \in \{0, 1\}$ since $\tilde{\pi} << \pi$. Since f is arbitrary this proves ergodicity of $(\tilde{\pi} - \tilde{q})$.□

Proposition 5.4. *If $\tilde{q}(dy|x)$ is positive recurrent with countable state space, then $E\lambda_\infty(T) = 0$ in the case H=1.*

Proof. Fix $t \in T$, and initial $r \in S$ such that $\tilde{\pi}(\{r\}) > 0$. Let T_k denote the kth hitting time of the state r and let $\tau_k = T_k - T_{k-1}, T_0 = 0$. By the Strong Markov property $Y_k := \sum_{j=T_{k-1}+1}^{T_k}(\log_b X_j - 1)$ is an i.i.d. sequence with mean zero (by Wald's formula). By the Chung-Fuchs criterion,

$$\limsup_{n\to\infty} \sum_{k=1}^{n} Y_k \geq 0. \tag{5.10}$$

It follows that there is an increasing sequence of stopping times $\{\tilde{T}_i\}_{i=0}^{\infty}$ which comprise an $\mathcal{F}_{t,0}-$ measurable sequence such that $b^{-\tilde{T}_i}\prod_{j\leq\tilde{T}_i}\tilde{X}_j \geq .5\ \forall i$, a.s., and $\tilde{X}_{\tilde{T}_i} = r\ \forall i$ a.s. Using Theorem 2.3, one has

$$\lambda_\infty(T) \geq .5\sum_{j=1}^{\infty} M_{\tilde{T}_j}, \tag{5.11}$$

where conditionally given $\tilde{T}_1, \tilde{T}_2, \ldots,\ M_{\tilde{T}_j}, j = 1, 2, \ldots,$ is an i.i.d. sequence (distributed as a total mass cascade up to the usual initial irregularity associated with Theorem 2.3). In particular, there is an $\epsilon > 0$ such that $P(M_{\tilde{T}_j} \geq \epsilon) > 0 \iff \exists\delta > 0$ such that $P(\lambda_\infty(T) \geq \delta) > 0$. So if $E_P\lambda_\infty(T) > 0$, then this implies in (5.11) that under P_t, $\lambda_\infty(T) = \infty$, which is a contradiction since t is arbitrary. □

The proposition below provides an alternative representation of the entropy parameter which is valid under certain "boundedness" conditions. To make the formulation we need a notion of *cluster measure* for $\psi-$recurrent Markov processes. Let $\mathcal{A} = \{A_1, \ldots, A_m\}$ be an arbitrary finite partition of S^+.

Proposition 5.5. *There is an a.s. unique closed subset $\mathcal{K}_\mathcal{A}$ of*

$$\mathcal{P}_n := \{(\lambda_1, \ldots, \lambda_m) \in [0,1]^m : \sum_{i=1}^{m}\lambda_i = 1\}$$

such that $(\delta_x - \tilde{q})-$a.s.

$$cl\{(\frac{1}{n}\sum_{j=1}^{n}\mathbf{1}_{A_i}(X_j))_{i=1}^{m}\}_{n=1}^{\infty} = \mathcal{K}_\mathcal{A},$$

where "cl(F)" denotes the set of cluster points of a set F.

Proof. Observe that for $\mathcal{K} \subset \mathcal{P}_n$, the event [$\mathcal{K}$ contains a cluster point sequence] is a tail event. Since under $\psi-$recurrence the tail is finitely atomic, the Cesàro average makes this event a zero-one event; see [OR]. For each $l = 1, 2, \ldots$ let $\{J_{l,i}\}_{i=1}^{l}$ be a non-overlapping partition of $[0,\infty)$ into closed intervals such that J_{l+1} refines J_l for each $l \geq 1$, and $\bigcup J_l$ separates points of $[0,\infty)$. Let

$$\mathcal{K}_l := \bigcup_i\{J_{l,i} : J_{l,i}\text{a.s. contains a cluster point of } \{(\frac{1}{n}\sum_{j=1}^{n}\mathbf{1}_{A_i}(X_j))_{i=1}^{m}\}_{n=1}^{\infty}\}, \tag{5.12}$$

and

$$\mathcal{K}_\mathcal{A} := \bigcap_l \mathcal{K}_l. \tag{5.13}$$

This solves the problem. □

For an algebra $\mathcal{G}$ define

$$\mathcal{K}_\mathcal{G} = \bigcap_\mathcal{A}\mathcal{K}_{\mathcal{A},G}, \tag{5.14}$$

where the intersection is over finite partitions $\mathcal{A}$ and

$$\mathcal{K}_{\mathcal{A},G} = \{\nu \in \mathcal{M}^+(\mathcal{G}) : \nu|_{\mathcal{A}} \in \mathcal{K}_{\mathcal{A}}\}. \tag{5.15}$$

Suppose now that S^+ is bounded. We will call an algebra $\mathcal{G}$ of subsets of S^+ *good* if for every $\epsilon > 0$, there is a partition $0 = a_0 < a_1 < \cdots < a_n$ such that $a_n > \sup S^+, [a_{i-1}, a_i) \cap S^+ \in \mathcal{G}$ for each i and $|a_i - a_{i-1}| < \epsilon$. If X consists of the uniform limits of $\mathcal{G}-$simple functions and $\nu \in \mathcal{K}_{\mathcal{G}}$, then one has the obvious functional $\int f d\nu$ defined on X; e.g. see [DUNSCH]. If $\mathcal{G}$ is good then X contains a copy of the continuous functions on the closure of S^+. Therefore for every $\nu \in \mathcal{K}_{\mathcal{G}}$ there is a countably additive (regular) probability measure $\tilde{\nu}$ on Borel sets such that $\int f d\nu = \int f d\tilde{\nu}$ for continuous f. Under the boundedness condition on S^+, if $\mathcal{G}$ is good, then the improper integral $\int \log_b(s)\nu(ds)$ can be defined as

$$\int \log_b(s)\nu(ds) = \lim_{a \to 0^+} \int \log_b(s) \mathbf{1}_{[a,\infty)\cap S^+}(s)\nu(ds), \tag{5.16}$$

where $a \in \{a : [a, \infty) \cap S^+ \in \mathcal{G}\}$ in the range of this limit; let us refer to such a as *good for* $\mathcal{G}$. It is not hard to see that under these conditions the improper finitely additive integral is the same as the countably additive integral $\int \log_b(s)\tilde{\nu}(ds)$.

Proposition 5.6. *If S^+ is bounded and if $\mathcal{G}$ is a good countable algebra of subsets of S^+, then*

$$H = \sup_{\nu \in K_{\mathcal{G}}} \int \log_b x \nu(dx).$$

Proof. Observe first that for any $s \in S^+, r \in (0, 1]$, one has

$$\tilde{q}([0, r]|s) = \int_{[0,r]} x q(dx|s) \leq r. \tag{5.17}$$

Let $s_0 \in S^+$, then by a standard coupling argument (e.g. [LIND]) one may construct of probability space and stochastic processes $\{\tilde{X}_n\}_{n=0}^{\infty}, \{U_n\}_{n=1}^{\infty}$ on the probability space such that

(i) $\{\tilde{X}_n\}_{n=0}^{\infty}$ is distributed as the $\delta_{s_0} - \tilde{q}$ Markov chain.
(ii) $\{U_n\}_{n=1}^{\infty}$ is i.i.d. uniform on [0,1].
(iii) $U_n \leq \tilde{X}_n$, $n \geq 1$.

It follows that a.s. for any $r \in (0, 1]$

$$0 \geq \liminf \frac{1}{n} \sum_{i=0}^{n} \log_b \tilde{X}_i \mathbf{1}[\tilde{X}_i \leq r] \geq \int_{(0,r]} \log_b(x)\lambda(dx). \tag{5.18}$$

Fix $\nu \in \mathcal{K}_{\mathcal{G}}$ and let D_ν be the random set for which

$$\lim_{\substack{n \to \infty \\ n \in D_\nu}} \frac{1}{n} \sum_{i=0}^{n-1} \mathbf{1}[\tilde{X}_i \in A] = \nu(A), \quad \forall A \in \mathcal{G}. \tag{5.19}$$

Then for any finite sequence $0 < a_1 < \cdots < a_m$ of good points for $\mathcal{G}$ such that $a_n > \sup S^+$,

$$\begin{aligned}
&\sum_{i=2}^{m-1} \log_b a_i \nu([a_{i-1}, a_i) \cap S^+) \\
&\geq \limsup_{\substack{n\to\infty \\ n\in D_\nu}} \frac{1}{n} \sum_{i=0}^{n-1} \log_b(\tilde{X}_i) \\
&\geq \liminf_{\substack{n\to\infty \\ n\in D_\nu}} \frac{1}{n} \sum_{i=0}^{n-1} \mathbf{1}[\tilde{X}_i \geq a_1] \log_b(\tilde{X}_i) + \int_{[0,a_1)} \log_b(x)\lambda(dx) \\
&\geq \sum_{i=2}^{m-1} \log_b a_{i-1} \nu([a_{i-1}, a_i) \cap S^+) + \int_{[0,a_1)} \log_b(x)\lambda(dx)
\end{aligned} \tag{5.20}$$

By squeezing points between a_1 and a_m one now obtains

$$|\int_{[a_1,a_n)} \log_b s\nu(ds) - \limsup_{\substack{n\to\infty \\ n\in D_\nu}} \frac{1}{n} \sum_{i=0}^{n-1} \log_b(\tilde{X}_i)| \leq \int_{[0,a_1)} \log_b(x)\lambda(dx). \tag{5.21}$$

Therefore,

$$\int \log_b s\nu(ds) = \limsup_{\substack{n\to\infty \\ n\in D_\nu}} \frac{1}{n} \sum_{i=0}^{n-1} \log_b(\tilde{X}_i) \leq H. \tag{5.22}$$

Thus,

$$\sup_{\nu\in\mathcal{K}_\mathcal{G}} \int \log_b s\nu(ds) \leq H.$$

But there is a random set D_0 such that

$$\lim_{\substack{n\to\infty \\ n\in D_0}} \frac{1}{n} \sum_{i=0}^{n-1} \log_b(\tilde{X}_i) = H.$$

Further, there is a $D_1 \subset D_0$ such that for every $A \in \mathcal{G}$,

$$\lim_{\substack{n\to\infty \\ n\in D_1}} \frac{1}{n} \sum_{i=0}^{n-1} \mathbf{1}[\tilde{X}_i \in A] \tag{5.23}$$

exists. Letting ν_1 be the element of $\mathcal{K}_\mathcal{G}$ associated with (5.23), one sees that (5.22) implies

$$H = \limsup_{\substack{n\to\infty \\ n\in D_1}} \frac{1}{n} \sum_{i=0}^{n-1} \log_b(\tilde{X}_i) = \int \log_b s\nu_1(ds).$$

□

Example 5.2 (Null-recurrent with death at $H = 1$).

Take $b = 2, S = \{s_n\}_{n=0}^{\infty}$, with

$$s_0 = 0, \quad s_n = 2 + \frac{1}{n}, \qquad n \geq 1.$$

Define

$$q_{s_i,s_j} = \begin{cases} 1 & \text{if } i = j = 0, \\ \frac{1}{s_2} & \text{if } i = 1, j = 2, \\ 1 - \frac{1}{s_2} & \text{if } i = 1, j = 0, \\ \frac{1}{2s_{i+1}} & \text{if } i \geq 2, j = i + 1, \\ \frac{1}{2s_{i-1}} & \text{if } i \geq 2, j = i - 1, \\ 1 - \frac{1}{2s_{i+1}} - \frac{1}{2s_{i-1}} & \text{if } i \geq 2, j = 0. \end{cases} \tag{5.24}$$

Then $\tilde{q}_{s_i,s_j}$ is the transition law of the simple symmetric random walk on the half line $\{1, 2, \dots\}$ with reflecting boundary at 1. In particular this motion is null-recurrent. To obtain $H = 1$ one may apply the Proposition 5.4 with $\mathcal{G}$ the finite-cofinite algebra of subsets of $\{1, 2, \dots\}$, noting that for any finite subset F, the asymptotic proportion of time spent in F is 0, and is 1 in F^c. Thus,

$$H = \int \log_b s_n \nu(ds_n) = \lim_{n\to\infty} \log_2 s_n = 1. \tag{5.25}$$

Alternatively, one may compute H from (5.1) as well. In any case, suppose that $\tilde{X}_0 = s_k$. Then $\mathcal{Q}_\infty-$a.s.,

$$\lambda_n(T) \geq \frac{1}{2^n} \prod_{i\leq n} \tilde{X}_i \geq \frac{1}{2^n} s_k s_{k+1} \cdots s_{k+n} = \prod_{j=0}^{n} (1 + \frac{1}{2k + 2j}) \to \infty. \tag{5.26}$$

Thus $\lambda_\infty(T) = \infty, \mathcal{Q}_\infty-$a.s. and hence $\mathcal{Q}_\infty$ is singular with respect to P so that $\lambda_\infty(T) = 0, P-$a.s. by Theorem 2.2.

Example 5.3 (Null-recurrent with survival at $H = 1$).

Modify the previous Example 5.3 by taking $s_0 = 0, s_n = 2(1 - \frac{1}{n})^2, n \geq 1$. The null-recurrence remains but now

$$\frac{1}{2^n} \prod_{i\leq n} \tilde{X}_i \leq \frac{1}{2^n} s_k s_{k+1} \cdots s_{k+n} = \prod_{j=0}^{n} (1 - \frac{1}{k + j})^2 = (\frac{k - 1}{k + n})^2. \tag{5.27}$$

Thus

$$\sum_n \frac{1}{2^n} \prod_{i\leq n} \tilde{X}_i \leq \sum_n (\frac{k - 1}{k + n})^2 < \infty. \tag{5.28}$$

Let us now consider the divergence of moment problems.

Theorem 5.7. *Let $1 < h \leq 2$. Assume that there exist natural numbers $1 \leq l \leq N$ such that*

$$\tilde{q}^l(B|x) \leq \frac{M}{N} \sum_{m=1}^{N} \tilde{q}^m(B|y), \forall x, y \in S^+, \qquad B \in \mathcal{B}.$$

Assume further that for arbitrary $t \in T$,

$$\lim_{L\to\infty} \limsup_{n\to\infty} \frac{1}{n} \log E_{P_t}(\mathbf{1}_{[\frac{L}{h-1},\infty)}(S_{t,n}) e^{(h-1)S_{t,n}}) = -\infty,$$

where

$$S_{t,n} := \frac{1}{n} \sum_{i=0}^{n-1} \log W_{t|i}.$$

Then the rate function I exists,

$$\tilde{I}^*(h-1) = \lim_{n\to\infty} \frac{1}{n} \log E_{P_t} e^{n\phi_h(S_{t,n})},$$

and one has:

(a) *If $\frac{\tilde{I}^*(h-1)}{\log b} - (h-1) < 0$ then $E_P \lambda^h(T) < \infty$.*

(b) *If $\frac{\tilde{I}^*(h-1)}{\log b} - (h-1) > 0$ then $E_P \lambda^h(T) = \infty$.*

Proof. This is merely an application of Corollary 2.6 under conditions on the Markov generator to assure a good large deviation rate and Varadahn's formula.□

Remark 5.2. The uniformity condition in the previous theorem is clearly satisfied for the i.i.d. generator and for the finite state Markov generator studied in [WW1]. In general this uniformity is more than enough to insure the uniqueness of an invariant probability $\tilde{\pi}$, see [DEUST]. While complete results are known for the divergence of moments in the cases of finite state Markov chains and homogeneous independent cascades, to bootstrap Corollary 2.5 to higher moments in general is plausible under enough uniformity conditions on the transition probabilities; see the proof of Theorem 4.1(iii). So it is natural to conjecture that under sufficient uniformity, the critical parameter for h_c for the divergence of moments is given by

$$h_c = \inf\{h > 1 : \frac{\tilde{I}^*(h-1)}{\log b} - (h-1)\} > 0. \tag{5.29}$$

As already noted, this conjecture is known to be valid for finite state Markov cascades and for general homogeneous independent cascades. This issue is not pursued any further here.

Theorem 5.8 (Dimension Spectra: Finite State Case). *Consider a Markov cascade having a finite state generator process. Let*

$$S^+ = \bigcup_{i=1}^{m} \tilde{R}_i \cup \tilde{T}$$

denote the decomposition with respect to $\tilde{q}(dy|x)$ into irreducible recurrent classes $\tilde{R}_i$ and transient class $\tilde{T}$. Let $\tilde{H}_i$ denote the event that the process eventually hits $\tilde{R}_i$ for $i \leq m$.

$$h(i,r) = P_{(\delta_r - \tilde{q})}(\tilde{H}_i).$$

Define

$$\partial F_\gamma(i) = h(i, W_\gamma), \qquad \gamma \in T^*.$$

Let $D_i = [1 - E_{\tilde{\pi}_i}(\log_b W)] \vee 0$ *where* $\tilde{\pi}_i$ *is the invariant probability on* R_i. *Then* λ_∞ *has a spectral disintegration given by*

$$\lambda_\infty = \sum_{i=1}^m \lambda_{\partial F(i),\infty} = \int \sum_{i:D_i=\beta} \frac{\lambda_{\partial F(i),\infty}}{\sum_{j:D_j=\beta} \lambda_{\partial F(j),\infty}(T)} \nu(d\beta),$$

where the spectral measure ν *is absolutely continuous with respect to counting measure with density* $\nu(\{\beta\}) = \sum_{j:D_j=\beta} \lambda_{\partial F(j),\infty}(T)$.

Proof. This is a straightforward application of Theorem 2.11 and Corollary 2.5. The details are left to the reader. □

6. Applications to Exchangeable Cascades

Let $\mathbf{Prob}[0,\infty)$ denote the space of probability measures on $[0,\infty)$, and let $\mathbf{Prob}_1[0,\infty)$ be the space of probability measures on $[0,\infty)$ having mean one. We assume a deFinneti measure F on $\mathbf{Prob}_1[0,\infty)$ to be given. By an *exchangeable cascade with deFinetti measure* F we are referring to the multiplicative cascade defined by the exchangeable generator process with deFinetti measure F. The following fact is simply a matter of checking permutation invariance and is left to the reader.

Lemma 6.1. *Let* $\{X_n\}$ *be an exchangeable sequence of random variables with deFinetti measure F. Then for each n, given* $X_0 = x_0, X_1 = x_1, \ldots, X_n = x_n$, (*conditionally*) $X_1, X_2, \ldots$ *is an exchangeable sequence.*

We will denote the conditional deFinetti measure by $F_{||x_0,x_1,\ldots,x_n}$. That is,

$$\begin{aligned} &P(X_{n+1} \in B_1, \ldots, X_{n+m} \in B_m | X_0, \ldots, X_n) \\ &\quad = \int_{\mathbf{P}rob[0,\infty)} \prod_{i=1}^m \mu(B_i) F_{||X_0,\ldots,X_n}(d\mu). \end{aligned} \tag{6.1}$$

Proposition 6.1. *Let* $\pi_1 := \int_{\mathbf{Prob}[0,\infty)} \mu F(d\mu)$. *Then the following are equivalent*:

(a) $F_{||x} << F$.

(b) $\mu << \pi_1$ *for* F*-a.e.* μ.

(c) F *is supported on a strongly* σ*-compact subset of* $\mathbf{Prob}[0,\infty)$.

Moreover, in case (a) *one has*

$$\frac{dF_{||x}}{dF}(\mu) = \frac{d\mu}{d\pi_1}(x).$$

Proof. (a) $\Rightarrow$ (b) : For Borel sets $A \subset [0,\infty), B \subset [0,\infty)^{\mathbf{N}}$ one has

$$\begin{aligned}
&\int_{\mathbf{P}rob[0,\infty)} \mu(A)\mu^{\mathbf{N}}(B)F(d\mu) \\
&= P(X_0 \in A \cap (X_1, X_2, \dots) \in B) \\
&= E[\mathbf{1}[X_0 \in A]P((X_1, X_2, \dots) \in B|X_0)] \\
&= \int_A \int_{\mathbf{P}rob[0,\infty)} \mu^{\mathbf{N}}(B)F_{||x}(d\mu)\pi_1(dx) \qquad (6.2)\\
&= \int_A \int_{\mathbf{P}rob[0,\infty)} \mu^{\mathbf{N}}(B)\frac{dF_{||x}}{dF}(\mu)F(d\mu)\pi_1(dx) \\
&= \int_{\mathbf{P}rob[0,\infty)} \int_A \frac{dF_{||x}}{dF}(\mu)\pi_1(dx)\mu^{\mathbf{N}}(B)F(d\mu).
\end{aligned}$$

Therefore for $F-$a.e. μ,

$$\mu(A) = \int_A \frac{dF_{||x}}{dF}(\mu)\pi_1(dx). \qquad (6.3)$$

(b) $\Rightarrow$ (c) : Define $S : L^1_{+,1}(\pi_1) \to \{\mu : \mu << \pi_1\}$ by $S(f) = fd\pi_1$. Then since S is continuous from the strong to vague topologies, $F \circ S^{-1}$ is a Borel measure on the Polish space $L^1_{+,1}(\pi_1)$.

(c) $\Rightarrow$ (a) : Let $A = \cup K_j, F(A) = 1$ where each K_j is strongly compact. In particular, for each j there is a ν_j such that $\mu << \nu_j, \forall \mu \in K_j$. Define $\nu_A = \sum_j \frac{1}{2^j}\nu_j$. Then $\mu << \nu_A$ $\forall \mu \in A$. Then for bounded Borel $G(x,\mu)$,

$$\begin{aligned}
\int\int G(x,\mu)\mu(dx)F(d\mu) &= \int\int G(x,\mu)\frac{d\mu}{d\nu_A}(x)\nu_A(dx)F(d\mu) \\
&= \int\int G(x,\mu)\frac{d\mu}{d\nu_A}(x)F(d\mu)\nu_A(dx). \qquad (6.4)
\end{aligned}$$

Letting $G(x,\mu) = 1$, $\pi << \nu_A$ and $\int \frac{d\mu}{d\nu_A}(x)F(d\mu) = \frac{d\pi}{d\nu_A}(x)$. Thus, continuing (6.4), and using the fact that π is the marginal,

$$\int\int G(x,\mu)\mu(dx)F(d\mu) = \int \frac{\int G(x,\mu)\frac{d\mu}{d\nu_A}(x)F(d\mu)}{\int \frac{d\mu}{d\nu_A}(x)F(d\mu)}\pi(dx) \qquad (6.5)$$

and therefore $F_{||x} << F, \pi$-a.e. x and

$$\frac{dF_{||x}}{dF}(\mu) = \frac{\frac{d\mu}{d\nu_A}(x)}{\int \frac{d\mu}{d\nu_A}(x)F(d\mu)}.$$

□

Proposition 6.2. *Consider a dependent cascade for which the generator path process is an exchangeable process with deFinetti measure F such that one has*

$F(\mathbf{Prob}_1[0,\infty)) = 1$. *Then under* $\mathcal{Q}_\infty, \{\tilde{W}_n\}$ *is an exchangeable process with de Finetti measure* $\tilde{F} := F \circ S^{-1}$, *where* $S : \mathbf{Prob}_1[0,\infty) \to \mathbf{Prob}[0,\infty)$ *by* $\frac{dS(\mu)}{d\mu}(x) = x$.

Proof. Fix an arbitrary $t \in T$. For Borel sets $B_0, \ldots, B_n$,

$$\begin{aligned} &\mathcal{Q}_\infty(\tilde{W}_{t|0} \in B_0, \ldots, \tilde{W}_{t|n} \in B_n) \\ &= E[\mathbf{1}(W_\emptyset \in B_0, \ldots, W_{t|n} \in B_n) \prod_{i \le n} W_{t|i}] \\ &= \int_{\mathbf{Prob}[0,\infty)} \int_{B_0} \cdots \int_{B_n} \prod_{i \le n} x_i \mu^{n+1}(dx_0 \times \cdots \times dx_n) F(d\mu). \end{aligned} \tag{6.7}$$

Now apply the change of variable formula. □

Proposition 6.3. *Consider an arbitrary exchangeable cascade with deFinetti measure F. Let A be a measurable subset of* $\mathbf{Prob}_1[0,\infty)$. *Then*

$$\mathbf{F}(A) := \{F_{||W_\emptyset,\ldots,W_{\gamma|n}}(A) : \gamma \in T^*\}$$

is a weight decomposition, where $F_{||W_\emptyset,\ldots,W_{\gamma|n}}$ *is the conditional deFinetti measure given* $(W_\emptyset, \ldots, W_{\gamma|n})$.

Proof. We have

$$\begin{aligned} &E_P[\prod_{i=0}^{n+1} W_{t|i} F_{||W_{t|0},\ldots,W_{t|n+1}}(A)|\mathcal{F}_n] \\ &= \prod_{i=0}^{n} W_{t|i} E_P[W_{t|n+1} F_{||W_{t|0},\ldots,W_{t|n+1}}(A)|\mathcal{F}_n]. \end{aligned} \tag{6.8}$$

Note that

$$\begin{aligned} &E_P[W_{t|n+1} F_{||W_{t|0},\ldots,W_{t|n+1}}(A)|\mathcal{F}_n] = \int_{[0,\infty)} x_{n+1} F_{||W_{t|0},\ldots,W_{t|n+1}}(A) \\ &\times q_{n+1}(dx_{n+1}|x_0, \ldots, x_n)|_{x_0 = W_{t|0},\ldots,x_{n+1} = W_{t|n+1}}. \end{aligned} \tag{6.9}$$

Now we will show the right hand side of (6.9) is a.s. $F_{||W_{t|0},\ldots,W_{t|n+1}}(A)$, which will complete the proof in view of (6.8). Let $g_i, 0 \le i \le n$, be bounded Borel functions.

(6.10)

$$\begin{aligned} &\int \prod_{i \le n} g_i(x_i) \int x_{n+1} F_{||x_0,\ldots,x_{n+1}}(A) q_{n+1}(dx_{n+1}|x_0, \ldots, x_n) p_{n+1}(dx_0, \ldots, dx_n) \\ &= \int \int \prod_{i \le n} g_i(x_i) x_{n+1} \mathbf{1}_A(\mu) F_{||x_0,\ldots,x_{n+1}}(d\mu) p_{n+1}(dx_0, \ldots, dx_n) \\ &= \int_A \int \prod_{i \le n} g_i(x_i) \mu^{n+1}(dx_0 \times \cdots \times dx_n) \int x_{n+1} \mu(dx_{n+1}) F(d\mu) \\ &= \int \prod_{i \le n} g_i(x_i) F_{||x_0,\ldots,x_{n+1}}(A) p_{n+1}(dx_0, \ldots, dx_n), \end{aligned}$$

where the last two lines follow from the mean one condition and Bayes theorem, respectively. This yields the desired result. □

Proposition 6.4. *Consider an arbitrary exchangeable cascade with deFinetti measure F. Let A be a measurable subset of* $\mathbf{Prob}_1[0,\infty)$. *Then*

$$P_{\mathbf{F}(A),t} - a.s., \quad \lim_{n\to\infty} \sqrt[n]{\frac{1}{b^n}\prod_{i=0}^{n} W_{t|i}} \in \{e^{E_\mu W \log W - \log b} : \mu \in A\}.$$

If $\mathbf{F}(A)$ *is differentiable with respect to F, then*

$$P_{\partial F(\mu),t} - a.s., \quad \lim_{n\to\infty} \sqrt[n]{\frac{1}{b^n}\prod_{i=0}^{n} W_{t|i}} = e^{E_\mu W \log W - \log b}.$$

Proof. The first limit follows directly from Proposition 6.2, the Lebesgue decomposition Theorem 2.2, and Birkhoff's ergodic theorem. The second limit is computed in the same way, noting in this case that by Proposition 6.1

$$\begin{aligned} E_{P_{\partial F(\mu),t}} \log W &= E_P[W \frac{dF_{||W}}{dF}(\mu) \log W] \\ &= E_P[W \frac{d\mu}{d\pi_1}(W) \log W] \\ &= \int\int x \log x \frac{d\mu}{d\pi_1}(x)\nu(dx)F(d\nu) \\ &= \int x \log x \mu(dx). \end{aligned}$$

□

Theorem 6.5. *Let* $\{Q_n\}$ *be an arbitrary exchangeable cascade. Then the strong weights of the T-martingale decomposition are given by* $\mathbf{F}(A)$ *where*

$$A := \{\mu \in \mathbf{Prob}_1[0,\infty) : \int_{[0,\infty)} x \log_b x \mu(dx) < 1\}.$$

The proof requires computations furnished by the following lemmas.

Lemma 6.2. *Let A be a measurable subset of* $\mathbf{Prob}[0,\infty)$ *with* $F(A) > 0$. *For arbitrary* $t \in T$, *the* $P_{\mathbf{F}(A),t}$*−distribution of* $\{W_{t|i}\}_{i=1}^{\infty}$ *is exchangeable with deFinetti measure* $\frac{\mathbf{F}(A\cap\cdot)\circ S^{-1}}{F(A)}$, *where S is defined at Proposition* 6.2.

Proof. Let G be an arbitrary bounded $\sigma\{W_{t|i} : i \le n\}$-measurable function. Then,

(6.12)

$$\begin{aligned} &E_{\mathbf{F}(A),t} G \\ &= E_P[\prod_{i=0}^{n} W_{t|i} F_{||W_{t|0},\ldots,W_{t|n+1}}(A) G] F(A)^{-1} \\ &= E_P[\prod_{i=0}^{n} W_{t|i} \int \mathbf{1}_A(\mu) F_{||W_{t|0},\ldots,W_{t|n+1}}(d\mu) G] F(A)^{-1} \\ &= F(A)^{-1} \int\int \mathbf{1}_A(\mu) \prod_{i=0}^{n} x_i F_{||x_|0,\ldots,x_{n+1}}(d\mu) G(x_0,\ldots,x_n) dp_{n+1}(dx_0,\ldots,dx_n) \\ &= F(A)^{-1} \int_A \int G(x_0,\ldots,x_n) \prod_{i=0}^{n} x_i \mu^{n+1}(dx_0 \times \cdots \times dx_n) F(d\mu). \qquad \square \end{aligned}$$

Now apply the change of variable.

Lemma 6.3. *Let S be the transformation defined in Proposition* 6.2. *Then*

$$F \circ S^{-1}_{||(x_0,\ldots,x_n)} = F_{||x_0,\ldots,x_n} \circ S^{-1}.$$

Proof. This will follow from Bayes' theorem. Let $\mathbf{x} = (x_0, \ldots, x_n)$. Let G be an arbitrary bounded $\sigma\{W_{t|i} : i \leq n\}$-measurable function. Also let H be a bounded measurable function on $\mathbf{Prob}[0, \infty)$. Then,

$$\begin{aligned}
&\int\int G(\mathbf{x})H(\mu)\prod_{i=0}^{n} x_i F_{\mathbf{x}} \circ S^{-1}(d\mu)p_{n+1}(d\mathbf{x}) \\
&= \int\int G(\mathbf{x})H \circ S(\rho)\prod_{i\leq n} x_i F_{\mathbf{x}}(d\rho)p_{n+1}(d\mathbf{x}) \\
&= \int\int G(\mathbf{x})H \circ S(\rho)\prod_{i\leq n} x_i \rho^{n+1}(d\mathbf{x})F(d\rho) \\
&= \int\int G(\mathbf{x})H \circ S(\rho)S(\rho)^{n+1}(d\mathbf{x})F(d\rho) \\
&= \int\int G(\mathbf{x})H(\mu)\mu^{n+1}(d\mathbf{x})F \circ S^{-1}(d\mu) \\
&= \int\int\int G(\mathbf{x})H(\mu)F \circ S^{-1}_{\mathbf{x}}(d\mu)\sigma^{n+1}(d\mathbf{x})F \circ S^{-1}(d\sigma) \\
&= \int\int\int G(\mathbf{x})H(\mu)F \circ S^{-1}_{\mathbf{x}}(d\mu)\prod_{i\leq n} x_i\sigma^{n+1}(d\mathbf{x})F(d\sigma) \\
&= \int\int G(\mathbf{x})H(\mu)\prod_{i\leq n} x_i F \circ S^{-1}_{\mathbf{x}}(d\mu)p_{n+1}(d\mathbf{x}).
\end{aligned} \tag{6.13}$$

This proves the lemma. □

Proof of Theorem 6.5. Without loss of generality first assume $F(A) > 0$. Since $\frac{\mathbf{F}(A\cap\cdot)\circ S^{-1}}{F(A)}$−a.s., $\mu \in S^{-1}(A)$, we have $\int \log_b x\mu(dx) < 1$. Therefore,

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n}(\log_b W_{t|i} - 1) < 0 \tag{6.14}$$

and hence

$$\limsup_{n\to\infty} \sqrt[n]{b^{-n}\prod_{i\leq n} W_{t|i}} < 1. \tag{6.15}$$

Thus,

$$P_{\mathbf{F}(A),t}(\sum_{j=0}^{\infty} b^{-j}\prod_{i\leq j} W_{t|i}F_{||W_{t|0,\ldots,t|j}}(A) < \infty) = 1. \tag{6.16}$$

Thus $\mathbf{F}(A)$ provides the strong weights if it can be shown that $\lambda_{\mathbf{F}(A^c),\infty}(T) = 0$ P-a.s. (i.e. Corollary 2.3). Write $A^c = B_0 \cup B_1, B_0 := A^c - \{\mu_b\}, B_1 := \{\mu_b\}$, where $\mu_b := (1-b^{-1})\delta_0 + b^{-1}\delta_b$. Now assume that $F(B_0) > 0$. Then using Lemma 6.2, the $P_{\mathbf{F}(B_0),t}-$distribution of $\{W_{t|i}\}_{i=1}^{\infty}$ is exchangeable with deFinetti measure $\frac{\mathbf{F}(B_0\cap\cdot)\circ S^{-1}}{F(B_0)}$. By Lemma 6.3,

$$\begin{aligned}
&\frac{1}{b^n}\prod_{i\le n} W_{t|i}F_{||W_{t|0},\dots,W_{t|n+1}}(B_0)\\
&= \frac{1}{b^n}\prod_{i\le n} W_{t|i}F_{||W_{t|0},\dots,W_{t|n+1}}\circ S^{-1}(S(B_0))\\
&= \frac{1}{b^n}\prod_{i\le n} W_{t|i}F\circ S^{-1}_{||W_{t|0},\dots,W_{t|n+1}}(S(B_0)). \qquad (6.17)
\end{aligned}$$

Now,

$$\begin{aligned}
&\int \mu^{\mathbf{N}}(F\circ S^{-1}_{||W_{t|0},\dots,W_{t|n+1}}(S(B_0)) \to 1)\frac{F(d\mu)}{F(B_0)}\\
&= \int \mathbf{1}_{B_0}(\mu)(S\mu)^{\mathbf{N}}(F\circ S^{-1}_{||W_{t|0},\dots,W_{t|n+1}}(S(B_0)) \to 1)\frac{F(d\mu)}{F(B_0)}\\
&= \int_{S(B_0)} \mu^{\mathbf{N}}(F\circ S^{-1}_{||W_{t|0},\dots,W_{t|n+1}}(S(B_0)) \to 1)\frac{F(d\mu)}{F(B_0)}\\
&= \frac{1}{F(B_0)}\int_B \mu^{\mathbf{N}}(F_{||W_{t|0},\dots,W_{t|n+1}}(B) \to 1)F(d\mu), \qquad (6.18)
\end{aligned}$$

where

$$F_{||W_{t|0},\dots,W_{t|n+1}}(B) = E[\mathbf{1}_B(\mu)|W_{t|0},\dots,W_{t|n+1}]. \qquad (6.19)$$

In particular, (6.18) is a uniformly integrable martingale and therefore

$$\mu^{\mathbf{N}}(F_{||W_{t|0},\dots,W_{t|n+1}}(B) \to \mathbf{1}_B(\mu)) = 1, \qquad (6.20)$$

for $F-$a.s. μ. Since for the random walk $\limsup \sum_{i=0}^{n}\log_b(W_{t|i}) - 1 = \infty$ if the W's are i.i.d. distributed on B_0 by the Chung-Fuchs criterion, one has by the exchangeability of the $P_{\mathbf{F}(B_0),t}-$distribution of $\{W_{t|i}\}_{i=0}^{\infty}$ that (6.20) implies for arbitrary $t \in T$, $P_{\mathbf{F},t}(B_0)-$a.s.,

$$\limsup_n \frac{1}{b^n}\prod_{i\le n} W_{t|i}F\circ S^{-1}_{||W_{t|0},\dots,W_{t|n+1}}(S(A^c)) = \infty. \qquad (6.21)$$

This implies $\lambda_{\mathbf{F}(B_0),\infty}(T) = \infty$ $\mathcal{Q}_{\mathbf{F}(B_0),\infty}-$a.s. and therefore $\lambda_{\mathbf{F}(B_0),\infty}(T) = 0$, P-a.s. by Theorem 2.2. Now it is left to check $\lambda_{\mathbf{F}(B_1),\infty}(T) = 0$ P-a.s. and hence $\mathbf{F}(A^c)$ consists of the weak weights. For each n, $\lambda_{\mathbf{F}(B_1),n}(T) = \alpha_n Z_n$ where $\alpha_n = F_{||b,\dots,b}(B_1)$ for $n+1$ b's and $Z_n = \#\{\gamma \in T^*||\gamma| = n, W_{\gamma|i} = b, i = 0,\dots,n\}$. Since $\sup\{\mu(\{b\})|\mu \in \mathbf{Prob}_1\} = b^{-1}$, and $Z_{n+1} = \sum_{i=1}^{Z_n} X_i$, where X_i is binomial with parameters $b, \mu_{n+1}(\{b\})$ for some $\mu_{n+1} \in \mathbf{Prob}_1$, it follows that $\{Z_n\}$ is dominated by a critical branching process and therefore dies. Thus, $\lambda_{\mathbf{F}(B_1),\infty}(T) = 0$, P-a.s.□

Corollary 6.1. *For any exchangeable cascade with the deFinetti measure F, one has $E\lambda_\infty(T) > 0$ if and only if $F(A) > 0$, where A is defined in Theorem 6.5. In fact, $E\lambda_{\mathbf{F}(A),\infty}(T) = F(A)$.*

Proof. This follows from Theorem 6.5 and the definition of strong weights. □

Theorem 6.6. *For any exchangeable cascade with deFinetti measure F, the spectral s-weights defined at Theorem 2.8 are given by $\mathbf{F}(A_s)$ for*

$$A_s := \{\mu \in \mathbf{Prob}_1[0,\infty) : 1 - \int_{[0,\infty)} x \log_b x \mu(dx) \leq s\}.$$

In particular, the dimension spectrum is given by

$$\nu([0,s]) = \lim_{n\to\infty} \sum_{|\gamma|=n} b^{-n} \prod_{j=0}^{n} W_{\gamma|j} F_{||W_\emptyset,\ldots,W_{\gamma|n}}(A_s).$$

Proof. The proof is made by checking the conditions of Theorem 2.9. That $\mathbf{F}(A_s)$ is a weight decomposition follows from Proposition 6.3. Right continuity is obvious from monotonicity of the deFinetti measure. That $s = 1$ is the T-martingale decomposition follows by Theorem 6.5. The conditions (1) and (2) follow from Corollary 6.1 applied to the percolated cascade, since an inspection of Theorem 2.6 shows the percolated cascade to have an exchangeable generator. Apply Lemmas 6.2 and 6.3 to see the resulting shift in dimension. □

Theorem 6.7. *Assume that $F_{||x} << F$. Then the spectral s-weights have a differentiable local structure with respect to $F \circ \phi^{-1}$ where $\phi : A \to (0,1]$ by $\phi(\mu) = 1 - \int x \log_b(x)\mu(dx)$.*

Proof. For all s,

$$F(A_s) = F \circ \phi^{-1}([0,s]). \tag{6.22}$$

Now, by hypothesis, π_1-a.s., $F_{||x} << F$. So F is supported on some subset B of $\mathbf{Prob}_1$ and therefore, B is strongly sigma-compact. Thus, P-a.s., for any $t \in T$, $F_{W_{t|0},\ldots,W_{t|n}}(B) = 1$ and therefore

$$F_{W_{t|0},\ldots,W_{t|n+1}} << F_{W_{t|0},\ldots,W_{t|n}}. \tag{6.23}$$

It follows that P-a.s. for all $\gamma \in T^*$,

$$F_{W_{t|0},\ldots,W_{t|n}} << F \tag{6.24}$$

where $|\gamma| = n$. Thus, P-a.s. for all $\gamma \in T^*$, $|\gamma| = n$,

$$F_{W_{t|0},\ldots,W_{t|n}} \circ \phi^{-1} << F \circ \phi^{-1}.$$

□

Example 6.1 (A Polya Urn Cascade). This is the Example 1.2 described in the introduction. More formally, let

$$\mu_p := p\delta_1 + (1-p)(\frac{1}{2}\delta_a + \frac{1}{2}\delta_{2-a}), \qquad 0 \le p \le 1. \tag{6.26}$$

Define a deFinetti measure by

$$F(A) := \lambda(\{p \in [0,1] : \mu_p \in A\}), \qquad A \in \mathcal{B}(\mathbf{Prob}_1[0,\infty)). \tag{6.27}$$

Then for $\mathbf{c} = (c_0, \ldots, c_n) \in \{1, a, 2-a\}^{n+1}$, letting $\#1(\mathbf{c}) = card\{i \le n : c_i = 1\})$, in the notation of Theorem 2.6,

$$q_n(\mathbf{c}) = \int \mu_p^{n+1}(\{\mathbf{c}\})\lambda(dp) = B(\#1(\mathbf{c})+1, n+2-\#1(\mathbf{c}))2^{-n-1+\#1(\mathbf{c})}, \tag{6.28}$$

where $B(x,y)$ is the Beta function. In particular one may check that

$$F_{||\mathbf{c}}(A) = \beta_{\#1(\mathbf{c})+1, n+2-\#1(\mathbf{c})}(\{p : \mu_p \in A\}), \tag{6.29}$$

where $\beta_{\alpha_1,\alpha_2}$ is the Beta-probability distribution with parameters α_1, α_2. In particular it follows that (cf. Prop. 6.1),

$$F_{||\mathbf{c}} << F, \quad \frac{dF_{||\mathbf{c}}}{dF}(\mu_p) = \frac{p^{\#1(\mathbf{c})}(1-p)^{n+1-\#1(\mathbf{c})}}{B(\#1(\mathbf{c})+1, n+2-\#1(\mathbf{c}))}. \tag{6.30}$$

Lemma 6.4. *For F-a.e. μ_p,*

$$\lambda_{\partial F(\mu_p),n}(T) \to \lambda_{\partial F(\mu_p),\infty}(T) \text{ in } L^1.$$

Moreover, $\dim \lambda_{\partial F(\mu_p),\infty} \ge 1 - h(a)(1-p)$, *where*

$$h(a) = \frac{1}{2}[a\log_2 a + (2-a)\log_2(2-a)].$$

Proof. We will show that $\lambda_{\partial F(\mu_p),n}(T)$ is a strong weight of the T-martingale decomposition to get $P_{\partial F(\mu_p),t}(\sum_{n=0}^{\infty} \frac{1}{2^n}\prod_{j\le n} W_{t|j}\partial F_{t|n}(\mu_p) < \infty) = 1$. Note that for $0 < p < 1$, $\partial F_{t|n}(\mu_p) = \frac{p^{\#}(1-p)^{n+1-\#}\Gamma(n+3)}{\Gamma(\#+1)\Gamma(n+2-\#)} = \binom{n+1}{\#}p^{\#}(1-p)^{n+1-\#} \times (n+2) \le n+2$. Secondly, $\sqrt[n]{\frac{1}{2^n}\prod_{i\le n} W_{t|i}} \to e^{E_{\mu_p}W\log W - \log 2} < 1, P_{\partial F(\mu_p),t} - a.s.$ Thus,

$$P_{\partial F(\mu_p),t}(\sum_{n=0}^{\infty}\frac{1}{2^n}\prod_{j\le n} W_{t|j}\partial F_{t|n}(\mu_p) < \infty) = 1.$$

The L^1-convergence follows by Corollary 2.3. Now, for $p \in [0,1]$, noting by Theorem 2.6 that the percolated cascade $Q^{(\alpha)}\lambda_{\partial F(\mu_p),\infty}$ has an exchangeable generator with deFinetti measure $F^{(\alpha)} \equiv F \circ \tau_\alpha^{-1}$, one has that $Q^{(\alpha)}\lambda_{\partial F(\mu_p),\infty}$ is distributed as $\lambda_{\partial F^{(\alpha)}(\mu_p),\infty}$. If $\alpha + E_{\mu_p}W\log W - 1 < 0$, then as above, $\partial F^{(\alpha)}_{t|n}(\mu_p) < n+2$ and $P_{\partial F^{(\alpha)}(\mu_p),t} - a.s.$ $\sqrt[n]{\frac{1}{2^n}\prod_{i\le n} W_{t|i}} \to e^{\log 2(\alpha + E_P W\log_2 W - 1)}$ and, therefore, $\lambda_{\partial F^{(\alpha)}(\mu_p),\infty}$ lives $(a, p > 0.)$ Thus, P-a.s., λ-a.e. $p \in [0,1]$, if $\lambda_{\partial F,\infty}(T) > 0$ then $\dim \lambda_{\partial F,\infty} > 1 - (1-p)h(a)$. □

The following result now follows from the general theory (Corollary 2.5) and a change of variables.

Proposition 6.8. *The Polya urn cascade has a T-martingale decomposition with an absolutely continuous dimension spectra $\nu << \lambda$ and given by*

$$\frac{d\nu}{d\lambda}(s) = \frac{1}{h(a)}\mathbf{1}_{[1-h(a),1]}(s)\lambda_{\partial F(\mu_{1+\frac{s-1}{h(a)}}),\infty}(T).$$

In addition,

$$\mu_\beta = \frac{\lambda_{\partial F(\mu_{1+\frac{\beta-1}{h(a)}}),\infty}}{\lambda_{\partial F(\mu_{1+\frac{\beta-1}{h(a)}}),\infty}(T)},$$

and μ_β is unidimensional of dimension β.

One may obtain the following polynomials as the "best"least squares approximations to the spectral density. The interested reader may wish to compare this prediction to Monte Carlo urn simulations. For $1 - h(a) < s < 1$,

$$\begin{aligned} &E[\frac{d\nu}{d\lambda}(s)|\mathcal{F}_n](\omega) \\ &= \frac{(n+2)}{2^n h(a)^{n+1}} \sum_{\#1=0}^{n} \sum_{\#a=0}^{n+1-\#1} H_n(\omega, \#1, \#a) a^{\#a}(2-a)^{n+1-\#1-\#a}\binom{n+1}{\#1} \\ &\times (h(a)+s-1)^{\#1(\omega)}(1-s)^{n+1-\#1(\omega)}, \end{aligned} \tag{6.31}$$

where $H_n(\omega, i, j) = card\{\gamma : |\gamma| = n, \#\{k : w_{\gamma|k} = 1\} = i, \#\{k : w_{\gamma|k} = a\} = j\}$.

We conclude this section with an analysis of the moment problem for exchangeable cascades. The following example serves to illustrate the delicacy of the moment problem in this setting.

Example 6.2. Let ρ be an arbitrary probability measure on (0,1) and consider the exchangeable cascade with deFinetti measure defined by

$$F = \int_{(0,1)} (b^{-\alpha}\delta_{b^\alpha} + (1-b^{-\alpha})\delta_0)\rho(d\alpha). \tag{6.32}$$

Let

$$A := \{b^{-\alpha}\delta_{b^\alpha} + (1-b^{-\alpha})\delta_0 : 0 < \alpha < 1\}. \tag{6.33}$$

Then $F(A) = 1$, and $\int x^h \mu(dx) < b^{h-1}$ for all $h > 1$. However choosing ρ such that

$$\int_{(0,1)} \frac{1}{1-b^{\alpha-1}}\rho(d\alpha) = \infty, \tag{6.34}$$

one gets

$$E\lambda_\infty^2(T) = \infty. \tag{6.35}$$

To see this simply note that given α, $\lambda_n(T)$ is distributed as $b^{n(1-\alpha)}Z_n(\alpha)$ where $\{Z_n(\alpha)\}$ is a supercritical Galton-Watson branching process with binomial offspring distribution. Using Theorem 8.1 from [HAR, p.13], the equation (6.35) then follows.

Theorem 6.7. *For any exchangeable cascade with deFinetti measure F,*

$$\sup_n E_P \lambda_n^{1+\epsilon}(T) \leq (\frac{b-1}{b})^\epsilon \int \frac{1}{1-\gamma} dF \circ \chi_h^{-1}(\gamma),$$

where

$$\chi_h(\mu) := \frac{\int x^h \mu(dx)}{b^{h-1}}.$$

Proof. Fix $t \in T$. Using Theorem 2.3,

$$\begin{aligned} E_{P_{\mathbf{F}(A),t}}[\lambda^\epsilon_{\mathbf{F}(A),n}(T)] &\leq (\sum_{j=0}^{n-1} b^{-j} \prod_{i \leq j} W_{t|i} F_{t|j} \frac{b-1}{b} + b^{-n} \prod_{i \leq n} W_{t|i} F_{t|n})^\epsilon \\ &\leq \sum_{j=0}^{n} b^{-j\epsilon} \prod_{i \leq j} W^\epsilon_{t|i} F^\epsilon_{t|j}. \end{aligned} \tag{6.36}$$

Now take expectations to get

$$\begin{aligned} &E_P[\lambda^{1+\epsilon}_{\mathbf{F}(A),n}(T)] \\ &= E_{P_{\mathbf{F}(A),t}}[\lambda^\epsilon_{\mathbf{F}(A),n}(T)] \\ &\leq \frac{1}{\mathbf{F}(A)} \int \sum_{j=0}^{n} b^{-j\epsilon} E_P[\prod_{i \leq j} W^{1+\epsilon}_{t|i} F^{1+\epsilon}_{t|j}] \\ &= \int_A \sum_{j=0}^{n} b^{-j\epsilon} [\int x^{1+\epsilon} \mu(dx)]^j F(d\mu) \\ &\leq \int_A \frac{1}{1 - \frac{\int x^{1+\epsilon}\mu(dx)}{b^\epsilon}} F(d\mu). \end{aligned} \tag{6.37}$$

Now make a change of variables. □

Corollary 6.2. *For $0 < \delta < 1, 0 < h < 1$, let*

$$A_{\delta,h} := \{\mu \in \mathbf{P}rob_1[0,\infty) : \int x^h \mu(dx) < \delta b^{h-1}\}.$$

If $F(A_{\delta,h}) = 1$, then $\sup_n E\lambda_n^h(T) < \infty$.

Proof. Observe that

$$\gamma := \chi_{1+\epsilon}(\mu) = \frac{\int x^{1+\epsilon}\mu(dx)}{b^\epsilon} \leq \delta \tag{6.38}$$

implies that

$$\frac{1}{1-\gamma} \leq \frac{1}{1-\delta}. \tag{6.39}$$

Therefore, by the martingale convergence theorem, $\lambda_n(T)$ converges a.s. and in $L^{1+\epsilon}$ and therefore in L^1. □

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