

PARTIALLY HYPERBOLIC FIXED POINTS WITH CONSTRAINTS

PATRICK BONCKAERT

ABSTRACT. We investigate the local conjugacy, at a partially hyperbolic fixed point, of a diffeomorphism (vector field) to its normally linear part in the presence of constraints, where the change of variables also must satisfy the constraints. The main result is applied to vector fields respecting a singular foliation, encountered, by F. Dumortier and R. Roussarie, in the desingularization of families of vector fields.

1. INTRODUCTION AND MOTIVATION

Let x denote the variable in the center manifold near a partially hyperbolic singularity 0 of a vector field X and let y be the normal variable; then we can write $X = X_x \frac{\partial}{\partial x} + X_y \frac{\partial}{\partial y}$ and define its normally linear part by $NX := X_x(x, 0) \frac{\partial}{\partial x} + \frac{\partial X_y}{\partial y}(x, 0) \cdot y \frac{\partial}{\partial y}$. We study the question about conjugating X to NX near the singularity, i.e. about finding new coordinates in which the vector field is normally linear. Remark that NX is linear in its y variable. If there are no further requirements on this new coordinates then there is a theorem of F. Takens [T] stating that, under certain nonresonance conditions on the eigenvalues at the singularity, one can find C^k coordinates near 0 for any $k \in \mathbb{N}$ putting the vector field in normally linear form.

However in many problems extra constraints show up like e.g. the fact that a vector field can be a family of lower dimensional vector fields and that at each stage (normal forms, coordinate changes) we want to preserve this family character. In their paper about a geometric description and explanation of the canard phenomenon [DR2] F. Dumortier and R. Roussarie encounter vector fields respecting a singular foliation and need to use normal forms and coordinate changes respecting this extra structure. More precisely they need the following result that I propose to prove in this paper, as a consequence of a more general theorem. It concerns families of vector fields on \mathbb{R}^3 having a partially hyperbolic fixed point at the origin and which have $\mathbb{R}^2 \times \{0\}$ as a center manifold. Let $x = (x_1, x_2)$ denote the variables in this center manifold.

Received by the editors March 8, 1994.

1991 *Mathematics Subject Classification*. Primary 58F14, 58F36, 34C20.

Key words and phrases. Partially hyperbolic fixed point, normal form, constraint, structure preserving.

Theorem 1. *Let $p, q \in \mathbb{N}$, and let $k \in \mathbb{N}$ be given. There exists $N \in \mathbb{N}$ such that the following holds. Let (x_1, x_2, y) denote variables of \mathbb{R}^3 and let $X_\lambda = X_\lambda(x_1, x_2, y)$ be a C^N P -parameter family of vector fields defined near $0 \in \mathbb{R}^3$ having the following properties*

- (i) X_λ is tangent to the level surfaces $x_1^p x_2^q = \text{constant}$.
- (ii) the linear part $dX_0(0)$ of X_0 at the origin has exactly one non-zero eigenvalue and $y = 0$ is the related center manifold.

Then there exists a local family of C^k changes of variables H_λ near the origin $(\lambda, x_1, x_2, y) = 0$ such that in the new coordinates

$$\begin{aligned} & (H_\lambda)_* X_\lambda(x_1, x_2, y) \\ &= X_{\lambda,1}(x_1, x_2, 0) \frac{\partial}{\partial x_1} + X_{\lambda,2}(x_1, x_2, 0) \frac{\partial}{\partial x_2} + \frac{\partial X_{\lambda,3}}{\partial y}(x_1, x_2, 0) \cdot y \frac{\partial}{\partial y} \end{aligned}$$

and such that moreover H_λ leaves the level surfaces $x_1^p x_2^q = \text{constant}$ invariant.

The vector field like in the last expression is thus the normally linear part of X_λ . So the change of variables H_λ has to satisfy a certain constraint.

In particular one also asks that a family of vector fields is conjugated to its normally linear part using a family of changes of variables. This is also a type of constraint if we consider the parameter direction as an extra direction in the center manifold: from the theorem of Takens it does not follow immediately that the change of variables $H(\lambda, x, y)$ is equal to λ in the parameter direction. As a matter of fact this result about families is often used [BP,CP,IY,NPT,PT], since the adaptations in the proof of [T] are minor; but it will follow as a trivial application of our methods. Special thanks to Freddy Dumortier and Robert Roussarie for inspiring discussions.

2. THE RESULT

We will give the theorem (and its proof) for diffeomorphisms, but, as usual, a similar result holds for vector fields (cf. the adaptations in [T]). For the sake of readability of the paper we will assume that the codimension of the centermanifold is equal to one: then there are no nonresonance conditions on the eigenvalues and the essential ideas are maybe more clear.

In order to fix the notations we consider diffeomorphisms $f : \mathbb{R}^c \times \mathbb{R} \rightarrow \mathbb{R}^c \times \mathbb{R} : (x, y) \mapsto (f_x, f_y)(x, y)$ having a partially hyperbolic fixed point at the origin 0 ; \mathbb{R}^c is the centermanifold. Mostly we will write $v = (x, y) \in \mathbb{R}^c \times \mathbb{R}$ and $f = (f_x, f_y)$.

We only aim for results on a neighbourhood of $(x, y) = (0, 0)$, so everything is local, except for some constructions mentioned further on with cut off functions. All germs are taken at the origin.

Definition. If f is a diffeomorphism of $\mathbb{R}^c \times \mathbb{R}$ having \mathbb{R}^c as a centermanifold, then we call

$$Nf(x, y) = (f_x(x, 0), \frac{\partial f_y}{\partial y}(x, 0) \cdot y)$$

its *normally linear part*.

(2.1) The formal, the semiformal and the path condition. Let D_0 be a group of germs in $0 \in \mathbb{R}^c \times \mathbb{R}$ of diffeomorphisms fixing 0.

The formal condition. First of all we have a bunch of notations about multilinear maps coming from Taylor developments. Let $i, j \in \mathbb{N}$. If A is an $i + j$ -linear operator we will write shortly $Ax^i y^j := A(x, \dots, x, y, \dots, y)$ (i times x and j times y). This will be used in the Taylor development in the origin of a map $f(x, y)$, namely $A = \frac{1}{(i+j)!} \frac{\partial^{i+j} f}{\partial x^i \partial y^j}(0, 0)$. If moreover L and M are linear operators in x resp. y we will abbreviate

$$AL^i M^j .x^i y^j := A(Lx, \dots, Lx, My, \dots, My)$$

(i times Lx and j times My). The space of $i + j$ -linear maps $(\mathbb{R}^c)^i \times \mathbb{R}^j \rightarrow \mathbb{R}^c \times \mathbb{R}$ will be denoted by I_{ij} . Let us denote, for $i, j \in \mathbb{N}$

$$I_{ij}(D_0) = \{C \in I_{ij} \mid \exists H \in D_0 : H(x, y) = (x, y) + Cx^i y^j + O(|(x, y)|^{i+j+1})\}.$$

We say that D_0 satisfies the formal N -condition if this is a vector space for all (i, j) with $2 \leq i + j \leq N$.

The semiformal condition. Let us denote, for $i \in \mathbb{N}$, $i \geq 1$,

$$J_i(D_0) = \{A \mid A \text{ is a germ of a map } \mathbb{R}^c \rightarrow \mathbb{R}^c \times \mathbb{R} \\ \text{and } \exists H \in D_0 : H(x, y) = (x, y) + A(x)y^i + O(y^{i+1})\}.$$

We say that D_0 satisfies the semiformal N -condition if $J_i(D_0)$ is a vector space for all $i \leq N$ and if the limit of a pointwise convergent sequence A_n in $J_i(D_0)$, defined on some fixed neighbourhood of 0, is again in $J_i(D_0)$

The path condition. Let D_1 be the set of germs of time τ -dependent vector fields Z_τ , $\tau \in [0, 1]$, for which there exists a C^1 path of diffeomorphisms $(g_\tau)_{\tau \in [0, 1]}$ in D_0 such that

$$Z_\tau(v) = \frac{\partial g_\tau}{\partial \tau}(g_\tau^{-1}(v))$$

or equivalently

$$Z_\tau(g_\tau(v)) = \frac{\partial g_\tau}{\partial \tau}(v).$$

We say that D_0 satisfies the path N -condition if D_1 is a vector space and if the following three statements are true.

Statement 1. If $f \in D_0$ is of the form

$$f(x, y) = Nf(x, y) + A(x, y)y^N$$

for some A of class C^N , then there must exist a C^1 path $(A_\tau)_{\tau \in [0, 1]}$ of C^N maps such that $A_0 = A$, $A_1 = 0$ and such that the diffeomorphism f_τ defined by $f_\tau(x, y) := Nf(x, y) + A_\tau(x, y)y^N$ is an element of D_0 . So this means that the path f_τ connects f with Nf within D_0 .

Statement 2. For each element $(Z_\tau)_{\tau \in [0, 1]}$ of D_1 the flow starting at the identity is in D_0 , more precisely the solution f_τ of

$$\begin{cases} Z_\tau(f_\tau(v)) = \frac{\partial f_\tau}{\partial \tau}(v) \\ f_0 = Id \end{cases}$$

must be in D_0 .

Statement 3. The pointwise limit of a sequence in D_1 , defined on some fixed neighbourhood of the origin, is again in D_1 .

Theorem 2. Let $N \in \mathbb{N}$, $N \geq 1$. Let D_0 be a group of germs in $0 \in \mathbb{R}^c \times \mathbb{R}$ of C^N diffeomorphisms fixing the origin and satisfying the formal $2N$, the semiformal N - and the path N -condition. Let f be a C^{2N} element of D_0 satisfying the following properties:

- (i) $df(0) = \begin{bmatrix} A_c & 0 \\ 0 & A_s \end{bmatrix}$ where the eigenvalues of A_c have modulus equal to 1 and $|A_s| < 1$;
- (ii) $\mathbb{R}^c \times \{0\}$ is a centermanifold for f ;
- (iii) $df(0) \in D_0$.

Then there exists a C^N element H of D_0 such that

$$H \circ f = Nf \circ H.$$

(2.2) Proof of Theorem 2.

(A) The formal part.

Proposition 1. There exists a C^{2N} element $H \in D_0$ such that

$$j_{2N}(H \circ f \circ H^{-1})(0) = j_{2N}(Nf)(0).$$

(j_{2N} denotes the Taylor polynomial of order $2N$.)

Proof. This follows from the formal $2N$ condition and from Broer's methods in [Br]. For completeness sake we sketch the big steps. The elimination of higher order terms goes inductively, and if $g = f - Nf$ takes the form

$$g(x, y) = Ax^i y^j + \sum_{\substack{p+q=i+j \\ (p,q) \neq (i,j)}} a_{pq} x^p y^q + O(|(x, y)|^{i+j+1}),$$

then we can eliminate the term in $x^i y^j$ by a change of variables

$$H(x, y) = (x, y) + Cx^i y^j + O(|(x, y)|^{i+j+1})$$

not affecting the other terms of order $i+j$, as can be seen by equating and computing a bit, and where $C = (c, d)$ is a solution of

$$(\times) \quad (c, d) - (A_c^{-1}c.A_c^i A_s^j, A_s^{-1}d.A_c^i A_s^j) = (A_c^{-1}a, A_s^{-1}b).$$

The fact that this equation actually has a solution follows from the assumption about the eigenvalues of $df(0)$ in Theorem 2; more concretely the linear map $(c, d) \mapsto (A_c c, A_s d) - (c.A_c^i A_s^j, d.A_c^i A_s^j)$ is invertible. The proposition immediately follows from the next lemma:

Lemma 1. The solution $C = (c, d)$ of the foregoing equation is an element of $I_{ij}(D_0)$.

Proof. Let $C' = (c', d') \in I_{ij}(D_0)$. By definition there exists $H' \in D_0$ such that $H'(x, y) = (x, y) + C'x^i y^j + O(|(x, y)|^{i+j+1})$. Put $H_1 = df(0)^{-1} \circ H' \circ f$. Then $H_1 \in D_0$. So $H_1(x, y) = (x, y) + (c_1, d_1)x^i y^j + O(|(x, y)|^{i+j+1})$ with $C_1 = (c_1, d_1) \in I_{ij}(D_0)$. Equating terms of order $x^i y^j$ in the equality $df(0) \circ H_1 = H' \circ f$ gives

$$(c_1, d_1) - (A_c^{-1}c', A_s^{-1}d') = (A_c^{-1}c'.A_c^i A_s^j, A_s^{-1}d'.A_c^i A_s^j).$$

Expressing the fact that $df(0)^{-1} \circ f \in D_0$ quickly yields $(A_c^{-1}a, A_s^{-1}b) \in I_{ij}(D_0)$. This last space is a vector space. From this we infer that the operator T defined by $T.(c', d') = (A_c^{-1}c'.A_c^i.A_s^j, A_s^{-1}d'.A_c^i.A_s^j)$ maps $I_{ij}(D_0)$ into itself. From the solution of (\times) we see that $I - T$ is invertible. Since we just obtained that $(I - T)I_{ij}(D_0) \subset I_{ij}(D_0)$ we know from linear algebra that necessarily $I_{ij}(D_0) = (I - T)^{-1}I_{ij}(D_0)$. This proves the lemma and Proposition 1. \square

NOTE. The invertibility of the linear map just mentioned reflects the fact that there are some so called nonresonance conditions between the eigenvalues. More specifically let μ_1, \dots, μ_c be the eigenvalues of A_c . Then all terms, in the Taylor series of f_x , with at least one factor y can be eliminated since for all $k \in \{1, \dots, c\}$, all j_1, \dots, j_c and all $j \geq 1$

$$\mu_k \neq \mu_1^{j_1} \dots \mu_c^{j_c} A_s^j$$

(take the modulus to see this). For a similar reason the terms, in the Taylor series of f_y , with at least a factor y^2 can be eliminated.

(B) *The semiformal part.*

From the 'formal part' (A) it follows that we may assume that f takes the following form:

$$f(x, y) = (f_x(x, 0) + ya(x, y), \frac{\partial f_y}{\partial y}(x, 0).y + y^2b(x, y))$$

where $a(x, y) = O(|(x, y)|^{2N-1})$ is of class C^{2N-1} and $yb(x, y) = O(|(x, y)|^{2N-1})$ is of class C^{2N-1} .

We try to eliminate terms with 'low powers of y '. This cannot be done on the formal level, and we have to use changes of variables H of finite order of differentiability:

Proposition 2. *There exist a C^N germ of a diffeomorphism H in D_0 (a change of variables) such that $\tilde{f} = H \circ f \circ H^{-1}$ is of the form*

$$\tilde{f}(x, y) = (f_x(x, 0) + y^N \tilde{a}(x, y), \frac{\partial f_y}{\partial y}(x, 0).y + y^N \tilde{b}(x, y))$$

with $(\tilde{a}, \tilde{b})(x, y) = O(|(x, y)|^N)$ of class C^N .

Proof. We proceed by induction on i , $1 \leq i \leq N - 1$, and assume that f takes the form

$$f(x, y) = (f_x(x, 0) + y^i a(x, y), \frac{\partial f_y}{\partial y}(x, 0).y + y^i b(x, y)) =: (u, v)$$

where $(a, b)(x, y) = O(|(x, y)|^{2N-i})$ is of class C^{2N-i} . The case $i = 1$ is done. Let us abbreviate $M = 2N - i$. We can write $a(x, y) = a(x, 0) + O(y)$. We try to find a change of variables H of the form

$$H(x, y) = (x + c(x)y^i, y + d(x)y^i) + O(y^{i+1}) =: (x', y')$$

such that $(c, d)(x) = O(|x|^M)$ is of class C^M as well as H . We want to have that $H \circ f = \tilde{f} \circ H$ where

$$\tilde{f}(x, y) = (f_x(x, 0), \frac{\partial f_y}{\partial y}(x, 0).y) + O(y^{i+1}).$$

This gives an equation of the form

$$\begin{cases} u + c(u)v^i = f_x(x', 0) + O(y^{i+1}) \\ v + d(u)v^i = \frac{\partial f_y}{\partial y}(x', 0) \cdot y' + O(y^{i+1}) \end{cases}$$

so we have to solve in (c, d) (a few calculations):

$$(2) \quad \begin{cases} a(x, 0) + c(f_x(x, 0)) \cdot \frac{\partial f_y}{\partial y}(x, 0)^i = \frac{\partial f_x}{\partial x}(x, 0)c(x) \\ b(x, 0) + d(f_x(x, 0)) \frac{\partial f_y}{\partial y}(x, 0)^i = \frac{\partial f_y}{\partial y}(x, 0) \cdot d(x). \end{cases}$$

Let us denote

$$\bar{A}(x) = (\bar{a}, \bar{b})(x) := \left(\frac{\partial f_x}{\partial x}(x, 0)^{-1} \cdot a(x), \frac{\partial f_y}{\partial y}(x, 0)^{-1} \cdot b(x) \right)$$

and let T be the operator acting on germs of functions $A = (a, b) : \mathbb{R}^c \rightarrow \mathbb{R}^c \times \mathbb{R}$ defined by

$$TA(x) = \left(\frac{\partial f_x}{\partial x}(x, 0)^{-1} \cdot a(f_x(x, 0)), \frac{\partial f_y}{\partial y}(x, 0)^i, \frac{\partial f_y}{\partial y}(x, 0)^{-1} b(f_x(x, 0)) \frac{\partial f_y}{\partial y}(x, 0)^i \right).$$

Then the foregoing becomes

$$\bar{A}(x) + T(c, d)(x) = (c, d).$$

In the sequel we will assume that the derivative of f satisfies globally certain inequalities, based on the assumption (i) about the eigenvalues of $df(0)$ in Theorem 2. We can obtain them by using ‘cut off’ functions, i.e. functions being 1 on a neighbourhood of the origin and being 0 outside some bigger neighbourhood. This construction is described in detail in [Bo] and will not be repeated here for shortness sake.

Lemma 2. *There exists $\beta < 1$ and a neighbourhood U of $x = 0$ in \mathbb{R}^c such that, assuming a decent ‘cut off’ construction for f , the following statement holds.*

Let $E > 0$. If A is a C^M map $U \rightarrow \mathbb{R}^c \times \mathbb{R}$ satisfying $|A(x)| \leq E|x|^M, \dots, |d^j A(x)| \leq E|x|^{M-j}, \dots, |d^M A(x)| \leq E$ for all $x \in U$ then TA is of class C^M and $|TA(x)| \leq \beta E|x|^M, \dots, |d^j TA(x)| \leq \beta E|x|^{M-j}, \dots, |d^M TA(x)| \leq \beta E$ for all $x \in U$.

Proof. Let us write $A = (a, b)$ with respect to the product $\mathbb{R}^c \times \mathbb{R}$ and let us treat the components of TA separately. In terms of the abbreviations

$$L(x) = \frac{\partial f_x}{\partial x}(x, 0)^{-1}, \quad B(x) = \frac{\partial f_y}{\partial y}(x, 0)^i, \quad \bar{f}(x) = f_x(x, 0)$$

we can write the \mathbb{R}^c component of $TA(x)$ as

$$T_1 a(x) := L(x) \cdot a(\bar{f}(x)) \cdot B(x).$$

Let $1 \leq j \leq M$. From a straightforward use of Leibnitz' rule for differentiation of a product (or in general: of a bilinear operation ' \cdot ') and from the higher order chain rule we can derive an expression of the form

$$|d^j(T_1a)(x)| \leq |L(x).d^j a(\bar{f}(x)).(d\bar{f}(x))^j.B(x)| + C(f, j) \sum_{k=0}^{j-1} |d^k a(\bar{f}(x))|$$

where $C(f, j)$ is some constant depending in a polynomial way on the bounds of the derivatives of order 1 to j of f (remember: a decent 'cut off' is assumed here). One can be more explicit about this constant, but we don't need this here. The first term in the right hand side can be estimated as follows ($|\cdot|_\infty$ denotes the supremum norm)

$$|L(x).d^j a(\bar{f}(x)).(d\bar{f}(x))^j.B(x)| \leq |L|_\infty E(|d\bar{f}|_\infty |x|)^{M-j} |d\bar{f}|_\infty^j |B|_\infty.$$

Because of assumption (i) in Theorem 2 about the eigenvalues we can arrange that

$$\alpha := |L|_\infty (|d\bar{f}|_\infty)^M |B|_\infty < 1;$$

so

$$|L(x).d^j a(\bar{f}(x)).(d\bar{f}(x))^j.B(x)| \leq \alpha E|x|^{M-j}.$$

The remaining terms can be estimated by

$$C(f, j) E(|d\bar{f}|_\infty |x|)^{M-j+1}.$$

We can summarize this by

$$|d^j(T_1a)(x)| \leq E|x|^{M-j} [\alpha + C'(f, j)|x|]$$

where C' is some other constant.

The second (\mathbb{R} -) component of $TA(x)$ is estimated in a completely similar way; this time the key inequality is

$$\left| \frac{\partial f_y}{\partial y}(x, 0)^{-1} \right| \cdot \left| \frac{\partial f_y}{\partial y}(x, 0)^i \right| < 1;$$

we obtain an $\alpha' < 1$ and an estimate of the form

$$|d^j(TA)(x)| \leq E|x|^{M-j} [\alpha' + C''(f, j)|x|].$$

If we choose $\alpha' < \beta < 1$, Lemma 2 quickly follows from this.

We proceed with the proof of Proposition 2. From Lemma 2 it follows that the series $\sum_{s=0}^{\infty} T^s \bar{A}$ as well as its derivatives up to order M converge uniformly on U , since they are dominated by the geometric series $E \sum_{s=0}^{\infty} \beta^s$. We conclude that $(c, d) := \sum_{s=0}^{\infty} T^s \bar{A}$ is a C^M solution of (2), satisfying moreover $|d^j(c, d)(x)| \leq \frac{1}{1-\beta} E|x|^{M-j}$.

Lemma 3. (i) $\bar{A} \in J_i(D_0)$ and
(ii) $A \in J_i(D_0) \Rightarrow TA \in J_i(D_0)$.

Proof. (i) Consider $H_1 := Nf^{-1} \circ f$. By the path condition surely $Nf \in D_0$ and hence $H_1 \in D_0$. By the definition of $J_i(D_0)$ the coefficient of y^i in the development of H_1 is an element of it. More concretely let us write $H_1(x, y) = (x, y) + (a_1(x), b_1(x))y^i + O(y^{i+1})$. Then $(a_1, b_1) \in J_i(D_0)$. On the other hand we express the fact that $Nf \circ H_1 = f$. After calculating a bit we can equate terms in y^i and obtain that (a_1, b_1) is in fact equal to \bar{A} , so $\bar{A} \in J_i(D_0)$.

(ii) Let $A \in J_i(D_0)$ and let $H \in D_0$ be such that $H(x, y) = (x, y) + A(x)y^i + O(y^{i+1})$. Like in (i) we have that $H_1 := Nf^{-1} \circ H \circ f \in D_0$, $H_1(x, y) = (x, y) + (c_1(x), d_1(x))y^i + O(y^{i+1})$ with $(c_1, d_1) \in J_i(D_0)$. Again, equating terms in y^i in the equation $Nf \circ H_1 = H \circ f$ and calculating gives $(c_1, d_1) = TA + \bar{A}$. Using part (i) we infer that $TA \in J_i(D_0)$. \square

Now we can finish the proof of Proposition 2. We have $\bar{A} \in J_i(D_0)$. By induction $T^s \bar{A} \in J_i(D_0), \forall s \in \mathbb{N}$. Since $J_i(D_0)$ is a vector space every partial sum $\sum_{s=0}^n T^s \bar{A}$ is in $J_i(D_0)$. Finally the condition about pointwise convergence implies that $(c, d) \in J_N(D_0)$, so that there exists $H \in D_0$ of the form

$$H(x, y) = (x, y) + (c, d)(x)y^i + O(y^{i+1})$$

such that $H \circ f = \tilde{f} \circ H$. \square

(C) *The ‘path’ part.*

We continue with the proof of Theorem 2. Because Nf is not necessarily linear (giving a nonlinear equation for H) we want to use the ‘linearization’ method of the path. We describe it briefly, and refer to [DRR, R, M] for more details.

The ideas of the sequel are similar to those in [Bo]. For the sake of being self contained we copy some of them briefly and give the main steps in the estimates. All the occurring objects can be defined globally on $\mathbb{R}^c \times \mathbb{R}$ by using cut off functions: for details we refer to [Bo].

Because of Proposition 2 we may assume that f has the form

$$f(x, y) = (f_x(x, 0) + y^N a(x, y), \frac{\partial f_y}{\partial y}(x, 0) \cdot y + y^N b(x, y))$$

with $A(x, y) := (a, b)(x, y) = O(|(x, y)|^N)$ of class C^N . By the path N -condition there exists a C^1 path $(A_\tau)_{\tau \in [0, 1]}$ connecting A with 0 such that

$$f_\tau(x, y) := Nf(x, y) + A_\tau(x, y)y^N$$

is a path in D_0 ; this last path connects f with Nf . We look for a C^1 path of diffeomorphisms Ψ_τ in D_0 such that

$$(3) \quad \begin{cases} \Psi_\tau \circ f = f_\tau \circ \Psi_\tau \\ \Psi_0 = \text{Identity.} \end{cases}$$

The variables are abbreviated by $(x, y) =: v$ and $(\tau, v) =: w$; we also denote $|w|_\Sigma = |y|$, this is the distance to $\Sigma := [0, 1] \times \mathbb{R}^c \times \{0\}$.

Putting $G(\tau, v) = f_\tau(v)$ and $\Omega(\tau, v) = \Psi_\tau(v)$ this means that we have to solve (in Ω):

$$\begin{cases} G(\tau, \Omega(\tau, f^{-1}(v))) = \Omega(\tau, v) \\ \Omega(0, \cdot) = \text{Identity.} \end{cases}$$

We differentiate the first equation with respect to τ , then we replace v by $\Psi_\tau^{-1}(v)$. After a calculation this gives, using $\Psi_\tau \circ f^{-1} \circ \Psi_\tau^{-1} = f_\tau^{-1}$ and denoting

$$(4) \quad \begin{aligned} Z(\tau, v) &= \frac{\partial \Omega}{\partial \tau}(\tau, \Psi_\tau^{-1}(v)) \text{ and} \\ X(\tau, v) &= (df_\tau(v))^{-1} \cdot \frac{\partial G}{\partial \tau}(\tau, v), \end{aligned}$$

the following linear equation in Z :

$$(5) \quad Z(\tau, v) - df_\tau^{-1}(f_\tau(v)) \cdot Z(\tau, f_\tau(v)) = X(\tau, v).$$

Lemma 4. *One has*

$$X(\tau, v) = -\frac{\partial f_\tau^{-1}}{\partial \tau}(f_\tau(v)).$$

Proof. Write $h_\tau = f_\tau^{-1}$, $G(\tau, \cdot) = f_\tau$ and $H(\tau, \cdot) = h_\tau$. First

$$h_\tau \circ f_\tau = Id$$

so

$$dh_\tau(f_\tau(v)) \circ df_\tau(v) = Id.$$

Second

$$H(\tau, G(\tau, v)) = v$$

so differentiating with respect to τ gives

$$\frac{\partial H}{\partial \tau}(\tau, G(\tau, v)) + \partial_2 H(\tau, G(\tau, v)) \cdot \frac{\partial f_\tau}{\partial \tau}(v) = 0$$

and since

$$\partial_2 H(\tau, G(\tau, x)) = dh_\tau(f_\tau(v)) = df_\tau(v)^{-1}$$

we find

$$\begin{aligned} (df_\tau(v))^{-1} \cdot \frac{\partial f_\tau}{\partial \tau}(v) &= \partial_2 H(\tau, G(\tau, x)) \cdot \frac{\partial f_\tau}{\partial \tau}(v) \\ &= -\frac{\partial H}{\partial \tau}(\tau, G(\tau, v)) = -\frac{\partial f_\tau^{-1}}{\partial \tau}(f_\tau(v)). \quad \square \end{aligned}$$

We consider $Z_\tau = Z(\tau, \cdot)$ and $X_\tau = X(\tau, \cdot)$ as time τ -dependent vector fields. By Lemma 4 we see that $X_\tau \in D_1$ (cf. the path condition). We try to find a solution Z of (5) such that Z_τ is also in D_1 . Because then the solution $\Psi_\tau = \Omega(\tau, \cdot)$ of the initial value problem

$$(6) \quad \begin{cases} \frac{\partial \Omega}{\partial \tau}(\tau, v) = Z(\tau, \Omega(\tau, v)) \\ \Omega(0, \cdot) = \text{Identity} \end{cases}$$

is a solution for (3), as was checked in [M], so Ψ_1 will solve our original problem and will be in D_0 .

Solution of equation (5). The idea is similar to the one in the proof of Proposition 2. This time we consider the operator L defined by

$$(LX)_\tau(v) := df_\tau^{-1}(f_\tau(v)) \cdot X_\tau(f_\tau(v)).$$

Let us put

$$\mathcal{X} := \{X : [0, 1] \times \mathbb{R}^c \times \mathbb{R} \rightarrow \mathbb{R}^c \times \mathbb{R} : X(\tau, x, 0) \equiv 0\}$$

and for $N \in \mathbb{N}$ and $E > 0$:

$$\begin{aligned} \mathcal{X}(N, E) = \{X : X \in \mathcal{X} \text{ and } X \text{ is of class } C^N \text{ and} \\ X(\tau, x, 0) = \frac{\partial X}{\partial y}(\tau, x, 0) = \dots = \frac{\partial^N X}{\partial y^N}(\tau, x, 0) = 0 \\ \text{and } \left| \frac{\partial^N X}{\partial y^N}(\tau, x, y) \right| \leq E\}. \end{aligned}$$

Observe that X obtained by the method of the path is an element of $\mathcal{X}(N, |d^N X|_\infty)$, always assuming a decent cut off for f .

Lemma 5. *There exists $\beta < 1$ such that for all $E > 0$, all $X \in \mathcal{X}(N, E)$ and all $j \in \mathbb{N}$ the following estimate holds:*

$$|L^j X(w)| \leq \beta^j E |w|_\Sigma^N$$

for all $w \in [0, 1] \times \mathbb{R}^c \times \mathbb{R}$.

Proof. As

$$|L^j X(\tau, v)| \leq |df_\tau^{-1}|_\infty^j |X(\tau, f_\tau^j(v))|$$

and as, by the invariance of $\mathbb{R}^c \times \{0\}$ for f_τ ,

$$|\pi_y \circ f_\tau(x, y)| \leq \left| \frac{\partial(\pi_y \circ f_\tau)}{\partial y} \right|_\infty |y|$$

(π_y is the projection on the \mathbb{R} component) we get

$$|L^j X(\tau, v)| \leq \left[|df_\tau^{-1}|_\infty \cdot \left| \frac{\partial(\pi_y \circ f_\tau)}{\partial y} \right|_\infty \right]^j E |(\tau, v)|_\Sigma^N$$

It is not difficult to see that Lemma 5 now follows from assumption (i) about the eigenvalues in Theorem 2 if we use cut off functions (cf. [Bo]). \square

Consequently, for each $X \in \mathcal{X}(N, E)$ the series $\sum_{j=0}^\infty L^j X$ converges locally uniformly to a continuous solution Z of (5).

Now we show that if X_τ is in D_1 then also $(LX)_\tau$ is. Let

$$X_\tau(v) = \frac{\partial g_\tau}{\partial \tau}(g_\tau^{-1}(v))$$

with $g_\tau \in D_0$. Put $h_\tau = f_\tau^{-1} \circ g_\tau$. Then

$$\frac{\partial h_\tau}{\partial \tau}(u) = \frac{\partial f_\tau^{-1}}{\partial \tau}(g_\tau(u)) + df_\tau^{-1}(g_\tau(u)) \cdot \frac{\partial g_\tau}{\partial \tau}(u);$$

if we replace

$$u = h_\tau^{-1}(v) = g_\tau^{-1} \circ f_\tau(v)$$

then we obtain

$$\begin{aligned} \frac{\partial h_\tau}{\partial \tau}(h_\tau^{-1}(v)) &= \frac{\partial f_\tau^{-1}}{\partial \tau}(f_\tau(v)) + df_\tau^{-1}(f_\tau(v)) \cdot \frac{\partial g_\tau}{\partial \tau}(g_\tau^{-1} \circ f_\tau(v)) \\ &= \frac{\partial f_\tau^{-1}}{\partial \tau}(f_\tau(v)) + df_\tau^{-1}(f_\tau(v)) \cdot X_\tau(f_\tau(v)) \\ &= \frac{\partial f_\tau^{-1}}{\partial \tau}(f_\tau(v)) + (LX)_\tau(v). \end{aligned}$$

As h_τ and f_τ^{-1} are both paths in D_0 and as D_1 is a vector space, we see that the path $(LX)_\tau$ is again in D_1 .

By induction every partial sum of the series $\sum_{j \geq 0} L^j X$ is in D_1 , and by the third statement in the path condition we infer that our solution Z of (5) is in D_1 , as desired.

Let us finally show that Z is of class C^N .

Lemma 6. *There exist $\beta < 1$ and $\varepsilon > 0$ with the following property. Let $E > 0$. If $X \in \mathcal{X}$ is of class C^N and if for all $|w|_\Sigma \leq \varepsilon$ and $0 \leq i \leq N$*

$$|d^i X(w)| \leq E|w|_\Sigma^{N-i}$$

then for all $|w|_\Sigma \leq \varepsilon$ and $0 \leq i \leq N$

$$|d^i(LX)(w)| \leq \beta E|w|_\Sigma^{N-i}.$$

Proof. Let us put shortly $w' = (\tau, v') = (\tau, f_\tau(v))$. Just like in Lemma 2 we can make an estimate

$$|d^i(LX)(w)| \leq |df_\tau^{-1}(v') \cdot d^i X(w') \cdot (df_\tau(v))^i| + C(f, i) \sum_{k=0}^{i-1} |d^k X(w')|$$

where $C(f, i)$ is some constant depending in a polynomial way on the bounds of the derivatives of order 1 to $i+1$ of f . The first term in the latter right hand side is estimated by

$$|df_\tau^{-1}(v') \cdot d^i X(w') \cdot (df_\tau(v))^i| \leq |df_\tau^{-1}| E \left| \frac{\partial(\pi_y \circ f_\tau)}{\partial y} \right|_\infty^{N-i} |w|_\Sigma^{N-i} |df_\tau|_\infty^i$$

and for the second term in it we observe that

$$|d^k X(w')| \leq E|w'|_\Sigma^{N-k} \leq E|w|_\Sigma^{N-k}$$

so we conclude that

$$|d^i(LX)(w)| \leq E|w|_{\Sigma}^{N-i} \left[|df_{\tau}^{-1}| \left| \frac{\partial(\pi_y \circ f_{\tau})}{\partial y} \right|_{\infty}^{N-i} |df_{\tau}|_{\infty}^i + iC(f, i)|w|_{\Sigma} \right].$$

The lemma now follows from assumption (i) about the eigenvalues in Theorem 2. \square

From this lemma we derive that the i -th derivative of the series $\sum_{j=0}^{\infty} L^j X$ converges uniformly on the strip $|w|_{\Sigma} \leq \varepsilon$, since it is dominated by the geometric series $\sum_{j=0}^{\infty} \beta^j E|w|_{\Sigma}^{N-i}$. Consequently the limit Z is of class C^N on that strip; because of the dynamical behavior in the \mathbb{R} direction it is C^N everywhere.

Let us finally come to the solution of (6). From the foregoing lemma it follows that $|Z(w)| \leq \frac{1}{1-\beta} E|w|_{\Sigma}^N$ on some strip $|w|_{\Sigma} \leq \varepsilon$. Consider the differential equation $\frac{du}{dt} = \frac{1}{1-\beta} E u^N$, $u \in \mathbb{R}^+$. The solution of this with initial value $u(0) = u_0$ will not

blow up for $t \in [0, 1]$ provided that $u_0 < \left(\frac{E(N-1)}{1-\beta}\right)^{-\frac{1}{N+1}}$. Consequently the solution

$\Psi_1 = \Omega(1, \cdot)$ will exist on the region $|w|_{\Sigma} < \left(\frac{E(N-1)}{1-\beta}\right)^{-\frac{1}{N+1}}$.

This finishes the proof of Theorem 2. \square

3. EXAMPLES

Now we show that the motivating example (Theorem 1) of section 1 satisfies the formal, semiformal and path conditions for any $N \geq 1$. So write, for the centermanifold, $\mathbb{R}^c = \mathbb{R}^P \times \mathbb{R}^2$ and $x = (\lambda, x_1, x_2) \in \mathbb{R}^P \times \mathbb{R}^2$. λ is to be interpreted as a parameter, and (x_1, x_2) are the ‘actual’ center variables.

Let $p, q \in \mathbb{N}$ be given, $p, q \geq 1$. Let D_0 consist of those germs of diffeomorphisms f of $\mathbb{R}^c \times \mathbb{R}$ which leave the level sets of the function

$$F(\lambda, x_1, x_2, y) = (\lambda, x_1^p x_2^q)$$

invariant. Let us also write f in components as $f = (f_x, f_y) = (f_{\lambda}, f_1, f_2, f_y) \in \mathbb{R}^P \times \mathbb{R}^2 \times \mathbb{R}$. Invariance means that $F(f(\lambda, x_1, x_2, y)) = F(\lambda, x_1, x_2, y)$. For the first component of F this gives $f_{\lambda}(\lambda, x_1, x_2, y) = \lambda$. This reflects the fact that λ is a parameter.

Necessarily the $x_k = 0$ hyperplanes, $k = 1, 2$, are invariant. Hence we can write

$$f_k(\lambda, x_1, x_2, y) = g_k(\lambda, x_1, x_2, y) \cdot x_k$$

for some g_k . Invariance of the level sets of F then implies that

$$(i) \quad g_1^p g_2^q = 1.$$

Let us now check the formal, semiformal and path conditions for D_0 for any $N \in \mathbb{N}$.

(A) *The formal condition.* We describe the elements of $I_{ij}(D_0)$. If $H \in D_0$ with $H = (\lambda, H_1, H_2, H_y)$ then we write, for the Taylor expansion, $H_k = (1 + h_k + \dots)x_k$, $k = 1, 2$, and one must have

$$(1 + h_1 + \dots)^p = (1 + h_2 + \dots)^{-q}$$

(dots mean irrelevant higher order terms) so $ph_1 = -qh_2$. We see that for an element $C = (c, d) \in I_{ij}(D_0)$ the term $cx^i y^j$ always is a multiple of $(0, qx_1, -px_2)$ (the 0-entry in the first component comes from the parameter $\lambda \in \mathbb{R}^P$; this will remain so in the sequel). Hence $I_{ij}(D_0)$ is a vector space.

(B) *The semiformal condition.* Let us describe the elements of $J_i(D_0)$. Let $H = (\lambda, H_1, H_2, H_y) \in D_0$. We consider its development in powers of y and write $H_k = (1 + a_k(x)y^N + \dots)x_k$. In the same way as in (A) the invariance of the level sets for H implies that $pa_1 = -qa_2$ and for an element $A = (a, b) \in J_i(D_0)$ its a -component must be a multiple of $(0, qx_1, -px_2)$. Conversely if $a(x) = \bar{a}(x)(0, qx_1, -px_2)$ for some function \bar{a} the map

$$H(x, y) = (\lambda, (1 + q\bar{a}(x)y^N).x_1, (1 + q\bar{a}(x))^{-p/q}x_2, H_y(x, y))$$

is an element of D_0 . We conclude that $A = (a, b) \in J_i(D_0)$ if and only if $a(x)$ is a multiple of $(0, qx_1, -px_2)$. Hence $J_i(D_0)$ is a vector space and the limit of a pointwise convergent sequence in it, defined on some fixed neighbourhood of 0, is again in it.

(C) *The path condition.* As before we can write, for $x = (\lambda, x_1, x_2)$, $f(x, y) = (\lambda, g_1(x, y).x_1, g_2(x, y).x_2, f_y(x, y))$ with $g_1^p g_2^q = 1$. Hence

$$Nf(x, y) = (\lambda, g_1(x, 0).x_1, g_2(x, 0).x_2, \frac{\partial f_y}{\partial y}(x, 0).y).$$

Let us come to Statement 1 in the path condition. So write $f(x, y) = Nf(x, y) + A(x, y)y^N$, where $A(x, y) = (0, a_1(x, y).x_1, a_2(x, y).x_2, b(x, y)) \in \mathbb{R}^p \times \mathbb{R}^3$. A direct calculation shows that the path

$$\begin{aligned} f_\tau(x, y) &= (\lambda, g_1(x, 0)(1 + \tau g_1(x, 0)^{-1}a_1(x, y)y^N).x_1, \\ &g_2(x, 0).(1 + \tau g_1(x, 0)^{-1}a_1(x, y)y^N)^{-p/q}.x_2, \frac{\partial f_y}{\partial y}(x, 0).y + \tau b(x, y)y^N) \end{aligned}$$

connects f with Nf **within** D_0 (i.e. one verifies that f_τ leaves the level sets of F invariant). The coefficient of y^N is the wanted A_τ in Statement 1.

Secondly, a time-dependent vector field Z_τ in D_1 is tangent to the level sets of F , so its flow will leave the level sets of F invariant. This is Statement 2. Finally, Statement 3 of the path condition is trivial since a pointwise limit of such vector fields remains tangent to the level sets of F .

4. GENERALISATIONS

Let us indicate how to derive, in a standard way [DR1], the similar result for vector fields thanks to the uniqueness of the solutions of the equations in the proof of Theorem 2. Suppose that X is a vector field near $0 \in \mathbb{R}^c \times \mathbb{R}$ such that its flow X_t is in D_0 . Suppose that $dX(0) = \begin{bmatrix} A_c & 0 \\ 0 & A_s \end{bmatrix}$ where

(i) the eigenvalues of A_c are on the imaginary axis and those of A_s have real part < 0 ,

(ii) $\mathbb{R}^c \times \{0\}$ is a centermanifold for X ;

(iii) $d(X_t)(0) \in D_0$ for every $t \in [0, 1]$.

Let us in fact show that the diffeomorphism $H \in D_0$, obtained from the Main Theorem 2, conjugating the time one of X to the time one of NX , actually conjugates the flow for all times $t \in [0, 1]$. We claim in fact that $\Psi_t := X_{-t} \circ H^{-1} \circ NX_t \circ H$ equals the identity. A straightforward calculation [DR1] shows that $NX_1 \circ \Psi_t =$

$\Psi_t \circ NX_1$ (index 1 means time one). Now we are in the situation of the ‘path part’ (C) of the proof of Theorem 2, this time with $f = f_t = NX_1$. We obtain an equation

$$Z(t, v) - df^{-1}(f(v)) \cdot Z(t, f(v)) = 0.$$

with $Z(t, v) = \frac{\partial \Psi_t}{\partial t}(t, \Psi_t^{-1}(v))$. From the estimates in Lemma 5 it follows that this equation has a unique solution, thanks to the fact that df satisfies certain global estimates. We conclude that $Z = 0$ and Ψ_t is the identity.

Question. Suppose that we would like to replace the group of diffeomorphisms D_0 in the result by some Lie subalgebra L_0 of vector fields. One could ask that the change of variables H conjugating X to NX be the time t of some element of L_0 . The methods we use do not imply this result, e.g. the change of variables Ψ_1 obtained by the method of the path in part (C) of the proof is the time one of a time-dependent vector field.

Higher codimensions. In case that the codimension of the centermanifold is more than one, some nonresonance conditions on the eigenvalues of $df(0)$ have to be satisfied in order to have a sufficiently differentiable change of coordinates. Suppose that f satisfies the Sternberg $\alpha(df(0), k)$ -condition as defined by Takens in [T]. Then we expect that one can obtain a C^k element H of D_0 if D_0 satisfies the formal, semiformal and path conditions for N sufficiently large with respect to k . We believe that the proof may follow the same lines but part (B) of it (the semiformal part) will however be considerably more technical, cf. Takens’ original proof in [T]. Also part (C) will have to be split up in stable and unstable directions.

The methods we developed here can be applied to other types of normal form theorems. For example we can prove in a very comparable way the following result about hyperbolic singularities in the plane, suggested by F. Dumortier:

Theorem 3. *Let X be a C^∞ vector field near $0 \in \mathbb{R}^2$, $dX(0)$ hyperbolic. Suppose that $X \cdot F = 0$ where $F(x, y) = x^p y^q$ with $(p, q) = 1$. Then there exists a C^∞ change of variables φ near 0 satisfying*

- (i) $\varphi_* X(x, y) = g(x^p y^q) \left[qx \frac{\partial}{\partial x} - py \frac{\partial}{\partial y} \right]$ and
- (ii) $F(\varphi(x, y)) = F(x, y)$ i.e. φ respects the level sets of F .

REFERENCES

- [BP] Beloqui, J. and Pacifico, M.J., *Quasi-transversal saddle-node bifurcation on surfaces*, Ergod. Th. & Dynam. Sys. **10** (1990), 63–88. MR **92a**:58104
- [Bo] Bonckaert, P., *On the continuous dependence of the smooth change of coordinates in parametrized normal form theorems*, Journal of Differential Equations **106** (1993), 107–120. MR **94j**:58150
- [Br] Broer, H., *Formal normal form theorems for vector fields and some consequences in the volume preserving case*, Dynamical systems and Turbulence, Warwick, Lecture Notes in Math. **898**, Springer-Verlag, Berlin-Heidelberg-New York, 1981. MR **83j**:58085
- [CP] Carneiro, M. and Palis, J., *Bifurcations and global stability of families of gradients*, Publ. Math. IHES **70** (1989), 103–168. MR **91f**:58048
- [DRR] Dumortier, F. D., Rodrigues, P. and Roussarie, R., *Germes of diffeomorphisms in the plane*, Lecture Notes in Mathematics **902**, Springer-Verlag, 1981. MR **83f**:58008
- [DR1] Dumortier, F. and Roussarie, R., *Germes de difféomorphismes et de champs de vecteurs en classe de différentiabilité finie*, Ann. Inst. Fourier **33** (1983), 195–267. MR **84m**:58114

- [DR2] Dumortier, F. and Roussarie, R., *Canard cycles and center manifolds*, Memoirs of the AMS (to appear).
- [IY] Il'yashenko, Y. and Yakovenko, S., *Finitely smooth normal forms of local families of diffeomorphisms and vector fields*, Russian Math. Surveys **46:1** (1991), 1–43. MR **92i**:58165
- [M] Mather, J., *Stability of C^∞ mappings: II Infinitesimal stability implies stability*, Ann. of Math. **89** (1969), 254–291. MR **41**:4582
- [NPT] Newhouse, S., Palis, J. and Takens, F., *Bifurcation and stability of families of diffeomorphisms*, Publ. Math. IHES **57** (1983), 5–71. MR **84g**:58080
- [PT] Palis, J. and Takens, F., *Stability of parametrized families of gradient vector fields*, Annals of Math. **118** (1983), 383–42. MR **85i**:58093
- [R] Roussarie, R., *Modèles locaux de champs et de formes*, Astérisque **30** (1975), 1–181. MR **55**:13444
- [T] Takens, F., *Partially hyperbolic fixed points*, Topology **10** (1971), 133–147. MR **46**:6399

LIMBURGS UNIVERSITAIR CENTRUM, B-3590 DIEPENBEEK, BELGIUM
E-mail address: pbonckae@luc.ac.be