FURTHER NICE EQUATIONS FOR NICE GROUPS

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Abstract. Nice sextinomial equations are given for unramified coverings of the affine line in nonzero characteristic \( p \) with \( \Omega^-(2m,q) \) and \( \Omega^-(2m,q) \) as Galois groups where \( m > 3 \) is any integer and \( q > 1 \) is any power of \( p > 2 \).

1. Introduction

Let \( m > 3 \) be any integer, let \( q > 1 \) be any power of a prime \( p > 2 \), consider the polynomials \( F^- = F^-(Y) = Y^n + T^n Y^{2m} + X^n Y^{u'} - TY^{w'} - 1 \) and \( F^* = F^*(Y) = Y^n + XY + 1 \) in indeterminates \( T, X, Y \) over an algebraically closed field \( k \) of characteristic \( p \), where \( n = 1 + q + \cdots + q^{2m-1} \), \( u' = 1 + q + \cdots + q^{m+1} \), \( n = 1 + q + \cdots + q^m \), \( w = 1 + q + \cdots + q^{m-1} \), \( n^* = 1 + q + \cdots + q^{m+1} \), and consider their respective Galois groups \( \text{Gal}(F^-, k(X, T)) \) and \( \text{Gal}(F^*, k(X)) \).

Both these are special cases of the families of polynomials giving unramified coverings of the affine line in nonzero characteristic which were written down in my 1957 paper [A01]. In my “Nice Equations” paper [A04], as a consequence of Cameron-Kantor Theorem I [CaK] on antiflag transitive collineation groups, I proved that \( \text{Gal}(F^*, k(X)) = \text{PSL}(m, q) \). In the present paper, as a consequence of Kantor’s characterization of Rank 3 groups in terms of their subdegrees [Kan], supplemented by Cameron-Kantor Theorem IV [CaK], I shall show that \( \text{Gal}(F^-, k(X, T)) = \text{PΩ}^-(2m,q) \).

1 Note that Kantor’s Rank 3 characterization depends on the Buekenhout-Shult characterization of polar spaces [BuS] which itself depends on Tits’ classification of spherical buildings [Tit]. Recall that the Rank of a transitive permutation group is the number of orbits of its 1-point stabilizer, and the sizes of these orbits are called subdegrees.

As a corollary of the above theorem that the Galois group of \( F^- \) is \( \text{PΩ}^-(2m,q) \), I shall show that the Galois group of a more general polynomial \( f^- \) is also \( \text{PΩ}^-(2m,q) \).

Moreover, by slightly changing \( f^- \) and \( F^- \), I shall show that we get polynomials \( \phi^- \) and \( \phi^-_2 \) whose Galois group is the negative orthogonal group \( \Omega^- \). The polynomials \( f^-, \phi^- \) and \( \phi^-_2 \) are also special cases of the families of polynomials giving unramified coverings of the affine line in nonzero characteristic written down in [A01].

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\(^1\)The projective negative (resp: positive) orthogonal group \( \text{PΩ}^-(2m,q) \) (resp: \( \text{PΩ}^+(2m,q) \)) is also called the projective elliptic (resp: hyperbolic) orthogonal group.

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As in [A03] and [A04], here the basic techniques will be MTR (the Method of Throwing away Roots) and FTP (Factorization of Polynomials).

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2. Notation and Outline

Let \( k_p \) be a field of characteristic \( p > 0 \), let \( q > 1 \) be any power of \( p \), and let \( m > 1 \) be any integer.\(^2\) To abbreviate frequently occurring expressions, for every integer \( i \geq -1 \) we put

\[
\langle i \rangle = 1 + q + q^2 + \cdots + q^i \quad \text{(convention: \( \langle 0 \rangle = 1 \) and \( \langle -1 \rangle = 0 \)).}
\]

We shall frequently use the geometric series identity

\[
1 + Z + Z^2 + \cdots + Z^i = \frac{Z^{i+1} - 1}{Z - 1}
\]

and its corollary

\[
\langle i \rangle = 1 + q + q^2 + \cdots + q^i = \frac{q^{i+1} - 1}{q - 1}.
\]

Let

\[
f^-(Y) = Y^{(2m-1)} - 1 + \sum_{i=1}^{m-1} \left(T^i_Y Y^{(m-1+i)} - T^i_Y Y^{(m-1-i)}\right)
\]

and note that then \( f^- \) is a monic polynomial of degree \( (2m - 1) = 1 + q + q^2 + \cdots + q^{2m-1} \) in \( Y \) with coefficients in the polynomial ring \( k_p[T_1, \ldots, T_{m-1}] \). Now the constant term of \( f^- \) is \(-1\) and the \( Y \)-exponent of every other term in \( f^- \) is 1 modulo \( p \), and hence \( f^- - Yf^- Y' = -1 \) where \( f^- Y' \) is the \( Y \)-derivative of \( f^- \). Therefore \( \text{Disc}_Y(f^-) = -1 \) where \( \text{Disc}_Y(f^-) \) is the \( Y \)-discriminant of \( f^- \), and hence the Galois group \( \text{Gal}(f^-, k_p(T_1, \ldots, T_{m-1})) \) is well-defined as a subgroup of the symmetric group \( \text{Sym}(2m-1) \).

For \( 1 \leq e \leq m - 1 \), let \( f_e^- \) be obtained by substituting \( T_i = 0 \) for all \( i > e \) in \( f^- \), i.e., let

\[
f_e^- = f_e^-(Y) = Y^{(2m-1)} - 1 + \sum_{i=1}^{e} \left(T^i_Y Y^{(m-1+i)} - T^i_Y Y^{(m-1-i)}\right)
\]

and note that then \( f_e^- \) is a monic polynomial of degree \( (2m - 1) = 1 + q + q^2 + \cdots + q^{2m-1} \) in \( Y \) with coefficients in the polynomial ring \( k_p[T_1, \ldots, T_{e}] \) and, as above, \( \text{Disc}_Y(f_e^-) = -1 \) and the Galois group \( \text{Gal}(f_e^-, k_p(T_1, \ldots, T_{e})) \) is a subgroup of \( \text{Sym}(2m-1) \). Note that if \( m > 2 \) and \( k = k_p = \) an algebraically closed field (of characteristic \( p > 0 \)), then \( F^- \) is obtained by substituting \( X, T \) for \( T_1, T_2 \) in \( f_e^- \) and hence \( \text{Gal}(F^-, k(X, T)) = \text{Gal}(f_e^-, k_p(T_1, T_2)) \).

\(^2\)In the Abstract and the Introduction we assumed \( p > 2 \) and \( m > 3 \). But in the rest of the paper, unless stated otherwise, we only assume \( p > 0 \) and \( m > 1 \).
In Section 3, we factor $f^-$ as $f^- = \overline{f}f^*$ where $\overline{f} = \overline{f}(Y)$ and $f^* = f^*(Y)$ are monic polynomials of degrees $(q^m+1)(m-2)$ and $q^{m-1}(q^m+1)$ in $Y$ with coefficients in $k_p[T_1, \ldots, T_{m-1}]$, respectively, and in case of $p \neq 2$ we factor $f^*$ further as $f^* = f^{**}f^{***}$ where $f^{**} = f^{**}(Y)$ and $f^{***} = f^{***}(Y)$ are both monic polynomials of degree $q^{m-1}(q^m+1)/2$ in $Y$ with coefficients in $k_p[T_1, \ldots, T_{m-1}]$. In Section 3, we show that if $p = 2$ then $\overline{f}$ and $f^*$ are irreducible in $k_p(T_1, \ldots, T_{m-1})[Y]$, and if $p \neq 2$ then $\overline{f}$, $f^{**}$ and $f^{***}$ are irreducible in $k_p(T_1, \ldots, T_{m-1})[Y]$. Given any $e$ with $1 \leq e \leq m - 1$, by putting $T_i = 0$ for all $i > e$ in $\overline{f}$ and $f^*$ we get $f_e^- = T_{e'}f_e^*$ where $T_{e'}$ and $f_e^*$ are monic polynomials of degrees $(q^m+1)(m-2)$ and $q^{m-1}(q^m+1)$ in $Y$ with coefficients in $k_p[T_1, \ldots, T_e]$ respectively. Likewise, if $p \neq 2$ then by putting $T_i = 0$ for all $i > e$ in $f^{**}$ and $f^{***}$ we get $f_e^{**} = f_e^{**}f_e^{***}$ where $f_e^{**}$ and $f_e^{***}$ are both monic polynomials of degree $q^{m-1}(q^m+1)/2$ in $Y$ with coefficients in $k_p[T_1, \ldots, T_{m-1}]$. In Section 3, we also show that if $p = 2$ then $\overline{f}$ and $f_e^*$ are irreducible in $k_p(T_1, \ldots, T_e)[Y]$, and if $p \neq 2$ then $\overline{f}$, $f_e^{**}$ and $f_e^{***}$ are irreducible in $k_p(T_1, \ldots, T_{m-1})[Y]$.

In Section 4, we throw away a root of $\overline{f}$ to get its twisted derivative $f'(Y, Z)$, and we let $g(Y, Z)$ be the polynomial obtained by first dividing the $Z$-roots of $f'(Y, Z)$ by $Y$ and then changing $Y$ to $1/Y$. Assuming $m > 2$, in Section 4, we factor $g(Y, Z)$ into two factors; to motivate the calculations, we first do this for $m = 3$. The $Z$-degrees of these factors turn out to be $q(q^{m-1} + 1)(m - 3)$ and $q^{2m-2}$. In Section 4, assuming $m > 2$, we show that these factors are irreducible in case of $L_2$ and hence also in case of $\overline{f}$ and $f_e$ for $2 \leq e \leq m - 1$, and therefore $\text{Gal}(L_2, k_p(T_1, \ldots, T_{m-1}))$ and $\text{Gal}(f_e, k_p(T_1, \ldots, T_e))$ for $2 \leq e \leq m - 1$ are Rank 3 groups with subdegrees $1, q(q^{m-1} + 1)(m - 3)$ and $q^{2m-2}$. In Section 6, from this Rank 3 description, we deduce the result that if $m > 3 \leq p$ and $k_p$ is algebraically closed then $\text{Gal}(f^-, k_p(T_1, \ldots, T_{m-1})) = \text{Gal}(f_e^-, k_p(T_1, \ldots, T_e)) = \Omega^-(2m, q)$ for $2 \leq e \leq m - 1$.

Consider the monic polynomials

$$\phi^- = \phi^-(Y) = Y^{q^m-1} - 1 + \sum_{i=1}^{m-1} \left( T_i Y^{q^{m+i-1}} - T_i Y^{q^{m-1}} \right)$$

and

$$\phi_e^- = \phi_e^-(Y) = Y^{q^{m-1}} - 1 + \sum_{i=1}^{e} \left( T_i Y^{q^{m+i-1}} - T_i Y^{q^{m-1}} \right) \quad \text{for} \, 1 \leq e \leq m - 1$$

of degree $q^{2m-1}$ in $Y$ with coefficients in $k_p[T_1, \ldots, T_{m-1}]$ and $k_p[T_1, \ldots, T_e]$, respectively, and note that, as before, Disc$_Y(\phi^-) = \text{Disc}_Y(\phi_e^-) = -1$. In Section 6, as a consequence of the above result about the Galois groups of $f^-$ and $f_e^-$, we show that if $m > 3 \leq p$ and $k_p$ is algebraically closed then $\text{Gal}(\phi^-, k_p(T_1, \ldots, T_{m-1})) = \text{Gal}(\phi_e^-, k_p(T_1, \ldots, T_e)) = \Omega^-(2m, q)$ for $2 \leq e \leq m - 1$.

In Section 5, we give a review of linear algebra including definitions of $\Omega^-(2m, q)$ and $\Omega^-(2m, q)$.

### 3. Factorization of the Basic Equation

We find a root $h_m(Y) \in \text{GF}(p)[Y]$ of the polynomial

$$Y^{q^{m+1}R^2} - R - \left( Y^{2(m-1)} - 1 \right)$$
by telescopically putting
\[ h_m(Y) = \sum_{\mu=0}^{m-1} Y^{(q^m+1)(m-2-\mu)} \]
and checking that then
\[ Y^{q^m+1}h_m(Y)^q - h_m(Y) - \left( Y^{(2m-1)} - 1 \right) = 0 \]
and, for any integer \( 0 < i < m \), we find a root \( h_i(Y,T_i) \in \text{GF}(p)[Y,T_i] \) of the polynomial
\[ Y^{q^m+1}R_i^q - R_i - \left( T_i Y^{(m-1+i)} - T_i Y^{(m-1-i)} \right) \]
by telescopically putting
\[ h_i(Y,T_i) = \sum_{\mu=0}^{i-1} T_i^{\mu-1-n} Y^{q^m(i-2-\mu) + (m-2-\mu)} \]
and checking that then
\[ Y^{q^m+1}h_i(Y,T_i)^q - h_i(Y,T_i) - \left( T_i Y^{(m-1+i)} - T_i Y^{(m-1-i)} \right) = 0. \]
By summing the above equations, upon letting
\[ \mathcal{J} = \mathcal{J}(Y) = \sum_{\mu=0}^{m-1} Y^{(q^m+1)(m-2-\mu)} + \sum_{i=1}^{m-1} \sum_{\mu=0}^{i-1} T_i^{\mu-1-n} Y^{q^m(i-2-\mu) + (m-2-\mu)}, \]
we get
\[ Y^{q^m+1}\mathcal{J}(Y)^q - \mathcal{J}(Y) - f^{-}(Y) = 0. \]
From the above equation it follows that
\[ f^{-} = \mathcal{J} f^{*} \quad \text{where} \quad f^{*} = f^{*}(Y) = Y^{q^m+1}\mathcal{J}(Y)^{q-1} - 1 \]
and
\[ \text{if } p \neq 2 \text{ then } f^{*} = f^{**} f^{***} \]
where
\[ f^{**} = f^{**}(Y) = Y^{(q^m+1)/2}\mathcal{J}(Y)^{(q-1)/2} - 1 \]
and
\[ f^{***} = f^{***}(Y) = Y^{(q^m+1)/2}\mathcal{J}(Y)^{(q-1)/2} + 1. \]
Note that the \( (\mu = 0) \) term in the above first summation is \( Y^{(q^m+1)(m-2)} \) and its exponent \( (q^m+1)(m-2) \) is strictly greater than the \( Y \)-exponent of every other term in the above two summations. Hence \( \mathcal{J} \) is a monic polynomial of degree \( (q^m+1)(m-2) \) in \( Y \) with coefficients in \( k_p[T_1, \ldots, T_{m-1}] \). Therefore \( f^{*} \) is a monic polynomial of degree \( (q^m+1)[1 + (q-1)(m-2)] = q^{m-1}(q^m+1) \) in \( Y \) with coefficients in \( k_p[T_1, \ldots, T_{m-1}] \), and if \( p \neq 2 \) then \( f^{**} \) and \( f^{***} \) are both monic.
polynomials of degree $q^{m-1}(q^m + 1)/2$ in $Y$ with coefficients in $k_p[T_1, \ldots, T_{m-1}]$. Thus

$$f^* = \overline{f}^*$$

where $\overline{f}$ and $f^*$ are monic polynomials of degrees $(q^m + 1)(m - 2)$ and $q^{m-1}(q^m + 1)$ in $Y$ with coefficients in $k_p[T_1, \ldots, T_{m-1}]$ respectively,

and if $p \neq 2$ then $f^* = f^{**}f^{**}$ where $f^{**}$ and $f^{***}$ are both monic polynomials of degree $q^{m-1}(q^m + 1)/2$ in $Y$ with coefficients in $k_p[T_1, \ldots, T_{m-1}]$.

For $1 \leq e \leq m - 1$, let $\overline{f}_e = \overline{f}_e(Y)$ and $f^*_e = f^*_e(Y)$ be obtained by putting $T_i = 0$ for all $i > e$ in $\overline{f}$ and $f^*$, respectively, and if $p \neq 2$ then let $f^{**}_e = f^{**}_e(Y)$ and $f^{***}_e = f^{***}_e(Y)$ be obtained by putting $T_i = 0$ for all $i > e$ in $f^{**}$ and $f^{***}$, respectively. Then by (3.0),

(3.1)

for $1 \leq e \leq m - 1$ we have:

$$f^*_e = \overline{f}_e f^*_e$$

where $\overline{f}_e$ and $f^*_e$ are monic polynomials of degrees $(q^m + 1)(m - 2)$ and $q^{m-1}(q^m + 1)$ in $Y$ with coefficients in $k_p[T_1, \ldots, T_{e}]$, respectively,

and if $p \neq 2$ then $f^*_e = f^{**}_e f^{**}_e$ where $f^{**}_e$ and $f^{***}_e$ are both monic polynomials of degree $q^{m-1}(q^m + 1)/2$ in $Y$ with coefficients in $k_p[T_1, \ldots, T_{e}]$.

Now

$$f^*_e = A_e T_1^q - B_e T_1 + C_e$$

where

$$0 \neq A_e = Y^{(m)} \in k_p[Y]$$

and

$$B_e = Y^{(m-2)} \in k_p[Y]$$

and

$$C_e = Y^{(2m-1)} - 1 + \sum_{i=2}^e \left( T_i^q Y^{(m-1+i)} - T_i Y^{(m-1-1)} \right) \in k_p[Y, T_1, \ldots, T_{e}]$$

and hence in particular $\deg f^*_e = q$. Also clearly $\deg f^*_e = 1$ and hence $\deg f^*_e = q - 1$ and if $p \neq 2$ then $\deg f^*_e = (q - 1)/2 = \deg f^{**}_e$.

In case of $p = 2$, the irreducibility of $\overline{f}_e$ and $f^*_e$ will follow from Lemmas (4.2) and (4.3) of [A05]. In case of $p \neq 2$, for establishing the irreducibility of $\overline{f}_e$, $f^{**}_e$ and $f^{***}_e$ we now prove the following lemma.

Lemma (3.2). Let $Q$ be a field of characteristic $p$ and consider a univariate polynomial $g_0 = A_0 Y - B_0 + C_0$ with $A_0, B_0, C_0$ in $Q$ such that $A_0 \neq 0 \neq B_0$. Assume that $g_0 = g_0''g_0'''$ in $Q[T]$ with $\deg g_0 = 1$ and $\deg g_0'' > 0 < \deg g_0'''$. Also assume that for some real discrete valuation $I$ of $Q$ (whose value group is the group of all integers) we have $\gcd(q - 1, I(B_0/A_0)) = 2$. Then $g_0''$ and $g_0'''$ are irreducible in $Q[I]$. To see this, we note that by assumption $g_0'' = A_0'' T + B_0''$ with $0 \neq A_0'' \in Q$ and $B_0'' \in Q$. Now $-B_0'/A_0'$ is a root of $g_0/A_0 = T^q - (B_0/A_0) T + (C_0/A_0)$ and hence

$$[T - (B_0'/A_0')]^q - (B_0/A_0)[T - (B_0'/A_0')] + (C_0/A_0) = T^q - (B_0/A_0)].$$
Therefore, in view of the $Q$-automorphism $T \rightarrow T - (B_0/A_0)$ of $Q[T]$, we see that $g_0/A_0$ factors into exactly one more nonconstant monic irreducible factor in $Q[T]$ as $T^{q-1} - (B_0/A_0)$, i.e., upon writing $g_0/A_0 = \theta_1\theta_2\ldots\theta_p$ and $T^{q-1} - (B_0/A_0) = \theta'_1\theta'_2\ldots\theta'_p$, where $\theta_1, \theta_2, \ldots, \theta_p, \theta'_1, \theta'_2, \ldots, \theta'_p$ are nonconstant monic irreducible polynomials in $Q[t]$, we have $p = 1 + \rho'$. By assumption 2 divides $q - 1$ and hence we must have $p \neq 2$. Also 2 divides $I(B_0/A_0)$ and hence $I(B_0/A_0) = 2s$ where $s$ is an integer. We can take an element $\Lambda$ in $Q$ with $I(\Lambda) = 1$, and then we can take an element $\Delta$ in an algebraic closure $Q'$ of $Q$ with $B_0/A_0 = (\Delta^2)^2$. Now $I((B_0/A_0)/\Lambda^{2s}) = 0$ and hence by the Discriminant Criterion we see that $I$ is unramified in $Q(\Delta)$. Therefore upon taking an extension $I'$ of $I$ to $Q(\Delta)$ we have $I'(\Delta^2)^2 = s$ and hence $\text{GCD}((q-1)/2, I'(\Delta^2)^2) = 1 = \text{GCD}((q-1)/2, I'(\Delta^2)^2)$. In $Q(\Delta)[T]$ we have $T^{q-1} - (B_0/A_0) = [T^{q-1/2} - \Delta^2]/[T^{q-1/2} + \Delta^2]$. By taking $\Delta' \in Q'$ with $\Delta'^{(q-1)/2} = \Delta^2$ and then taking an extension $I'$ of $I$ to $Q(\Delta, \Delta')$ and letting $r$ be the reduced ramification exponent of $I'$ over $I'$, we have $T^r = (\Delta^2)/((q-1)/2) = rF(\Delta^2)/((q-1)/2) = rs/(q-1)/2$. Consequently $rs/(q-1)/2$ must be an integer and hence, because $\text{GCD}((q-1)/2, I'(\Delta^2)^2) = 1$, it follows that $r$ divides $(q-1)/2$. Since the field degree $[Q(\Delta, \Delta') : Q(\Delta)]$ is at least $r$, we conclude that $[Q(\Delta, \Delta') : Q(\Delta)] \geq (q-1)/2$. Since $\Delta'$ is a root of the polynomial $T^{(q-1)/2} - \Delta^2$, this polynomial must be irreducible in $Q(\Delta)[T]$. Similarly the polynomial $T^{(q-1)/2} + \Delta^2$ is also irreducible in $Q(\Delta)[T]$. Consequently $\rho' \leq 2$ and hence $\rho \leq 3$. Therefore the polynomials $g_0^0$ and $g_0^0$ must be irreducible in $Q[T]$.

The following lemma is an easy consequence of the Gauss Lemma.

**Lemma (3.3).** Let $\kappa$ be a field, and let $g_0 = g_0^0g_0^0g_0^0$ where $g_0^0, g_0^0, g_0^0$ are monic polynomials of positive degrees in $Z$ with coefficients in the $(d + 1)$-variable polynomial ring $\kappa[x_1, \ldots, x_d, T]$. Assume that the polynomials $g_0^0, g_0^0, g_0^0$ have positive $T$-degrees and are irreducible in the ring $\kappa(x_1, \ldots, x_d, Z)[T]$. Also assume that the coefficients of $g_0$ as a polynomial in $T$ have no nonconstant common factor in $\kappa(x_1, \ldots, x_d, Z)$. Then the polynomials $g_0^0, g_0^0, g_0^0$ are irreducible in the ring $\kappa(x_1, \ldots, x_d, T)[Z]$.

By letting $I$ be the $Y$-adic valuation of $Q = k_0(Y, T_1, \ldots, T_e)$, i.e., the real discrete valuation whose valuation ring is the localization of $k_0[Y, T_1, \ldots, T_e]$ at the principal prime ideal generated by $Y$, we see that $I(A_0) = \langle m \rangle$ and $I(B_0) = \langle m - 2 \rangle$, and hence $I(B_0/A_0) = \langle m - 2 \rangle - \langle m \rangle = -q^{-1}(1 + g)$. Therefore $\text{GCD}(q - 1, I(B_0/A_0)) = 1$ or 2 according as $p = 2$ or $p \neq 2$. Also obviously $A_0$ and $C_0$ have no nonconstant common factors in $k_0[Y, T_1, \ldots, T_e]$. Therefore, if $p = 2$ then by Lemmas (4.2) and (4.3) of [A05], and if $p \neq 2$ then by the above Lemmas (3.2) and (3.3), for $1 \leq e \leq m - 1$ we have that

\begin{equation}
\text{(3.4) } \begin{cases}
\text{if } p = 2 \text{ then } T_e^* \text{ and } f_e^* \text{ are irreducible in } k_p(T_1, \ldots, T_e[Y], \\
\text{if } p \neq 2 \text{ then } T_e^* f_e^{**} \text{ and } f_e^{***} \text{ are irreducible in } k_p(T_1, \ldots, T_e[Y].)
\end{cases}
\end{equation}

By taking $e = m - 1$ in (3.4) we see that

\begin{equation}
\text{(3.5) } \begin{cases}
\text{if } p = 2 \text{ then } T \text{ and } f^* \text{ are irreducible in } k_p(T_1, \ldots, T_{m-1})[Y], \\
\text{if } p \neq 2 \text{ then } T, f^{**} \text{ and } f^{***} \text{ are irreducible in } k_p(T_1, \ldots, T_{m-1})[Y].
\end{cases}
\end{equation}
4. Twisted Derivative and its Factorization

Recall that
\[ \bar{f} = \bar{f}(Y) = \sum_{\mu=0}^{m-1} Y^{(q^m+1)(m-2-\mu)} + \sum_{i=1}^{m-1} \sum_{\mu=0}^{i-1} T_i^{q^{m-1}-\mu} Y^{q^m(i-2-\mu)+(m-2-\mu)}. \]

Solving the equation \( \bar{f} = 0 \), we get
\[ T_i = \frac{\sum_{\mu=0}^{m-1} Y^{(q^m+1)(m-2-\mu)} + \sum_{i=2}^{m-1} \sum_{\mu=0}^{i-2} T_i^{q^{m-1}-\mu} Y^{q^m(i-2-\mu)+(m-2-\mu)}}{-Y^{(m-2)}} \]
and hence
\[ f'(Y,Z) = \frac{\bar{f}(Z) - \bar{f}(Y)}{Z - Y} \quad (\text{def of the twisted derivative } f' \text{ of } \bar{f}) \]
\[ = \frac{\sum_{\mu=0}^{m-2} (Z^{(q^m+1)(m-2-\mu)} - Y^{(q^m+1)(m-2-\mu)})}{Z - Y} + \frac{\sum_{\mu=0}^{m-1} Y^{(q^m+1)(m-2-\mu)} Z^{(m-2)} - Y^{(m-2)}}{Z - Y} \]
\[ + \frac{\sum_{i=2}^{m-1} \sum_{\mu=0}^{i-2} T_i^{q^{m-1}-\mu} Y^{q^m(i-2-\mu)+(m-2-\mu)} Z^{(m-2)} - Y^{(m-2)}}{Z - Y} \]
\[ + \frac{\sum_{i=2}^{m-1} \sum_{\mu=0}^{i-2} T_i^{q^{m-1}-\mu} (Z^{q^m(i-2-\mu)+(m-2-\mu)} - Y^{q^m(i-2-\mu)+(m-2-\mu)})}{Z - Y}. \]

Therefore
\[ g = g(Y,Z) \]
\[ = Y^{(q^m+1)(m-2)-1} f'(1/Y, Z/Y) \quad (\text{def of polynomial } g \text{ obtained by dividing roots of } f' \text{ by } Y \text{ and then changing } Y \text{ to } 1/Y) \]
\[ = \sum_{\mu=0}^{m-2} (Z^{(q^m+1)(m-2-\mu) - 1}) Y^{(q^m+1)q^{m-1}-\mu} \]
\[ + \frac{\sum_{\mu=0}^{m-1} Y^{(q^m+1)q^{m-1}-\mu}}{Z - 1} \times \frac{Z^{(m-2)} - 1}{Z - 1} \]
\[ + \frac{\sum_{i=2}^{m-1} \sum_{\mu=0}^{i-2} T_i^{q^{m-1}-\mu} Y^{q^m(i-2-\mu)+(m-2-\mu) + q^{m-1}-\mu}}{Z - 1} \times \frac{Z^{(m-2)} - 1}{Z - 1} \]
\[ + \frac{\sum_{i=2}^{m-1} \sum_{\mu=0}^{i-2} T_i^{q^{m-1}-\mu} (Z^{q^m(i-2-\mu)+(m-2-\mu) - 1}) Y^{q^m(i-2-\mu)+(m-2-\mu) + q^{m-1}-\mu}}{Z - 1}. \]

For \( i = m \), the powers of \( Z \) in the last summation coincide with the corresponding powers of \( Z \) in the first summation; moreover, for \( \mu = m - 1 \), by convention \( (Z^{(q^m+1)(m-2-\mu) - 1}) = 0 \), and hence the first summation can be extended to \( m - 1 \). Consequently, upon letting
\[ D_{i\mu} = \frac{Z^{q^m(i-2-\mu)+(m-2-\mu) - 1}}{Z - 1} \times \frac{Z^{(m-2)} - 1}{Z - 1} \quad \text{for } 2 \leq i \leq m \text{ and } 0 \leq \mu \leq i - 1. \]
we get
\[
g = \sum_{\mu=0}^{m-1} D_{m\mu} Y(q^{m+1})q^{m-1-\mu}(\mu-1)
\]
\[
+ \sum_{\mu=0}^{m-1} \sum_{i=2}^{m-1} D_{i\mu} Y(q^{m-1-i+1})(m-1+\mu-i+q^{m-1-i}(\mu-1)) T_{i}^{\mu-1-\mu}.
\]

It follows that if \( m = 2 \) then
\[
g = \frac{Z(Z^2 - 1)}{Z - 1} - Y^{q^2 + 1} \quad \text{with} \quad \frac{Z(Z^2 - 1)}{Z - 1} \in (k_{p}(Z)) \setminus \{ Z^{2}k_{p}[Z] \}
\]
and hence \( g \) is irreducible in \( k_{p}(Z)[Y] \) and therefore by the Gauss Lemma \( g \) is irreducible in \( k_{p}(Y)[Z] \). Thus

(4.0) \[
\begin{cases}
\text{if } m = 2, \text{ then } g \text{ is a monic polynomial of degree } q^{2} \text{ in } Z \\
\text{with coefficients in } k_{p}[Y], \text{ and } g \text{ is irreducible in } k_{p}(Y)[Z].
\end{cases}
\]

Henceforth assuming \( m > 2 \), and displaying dependence on \( T_{2} \), we get
\[
g = D_{20} Y^{q^{m+1}(m-3)} T_{2}^{q^{m}} + D_{21} Y^{q^{m}(m-2)+q^{m-2}} T_{2}
\]
\[
+ \sum_{\mu=0}^{m-1} D_{m\mu} Y^{q^{m-1-\mu}(\mu-1)}
\]
\[
+ \sum_{i=3}^{m-1} \sum_{\mu=0}^{m-1} D_{i\mu} Y^{q^{m-1-i+1}(m-1+\mu-i)+q^{m-1-i}(\mu-1)} T_{i}^{\mu-1-\mu}.
\]

Now upon letting
\[
\tilde{T}_{i} = Y^{q^{m}(m-1-i)} T_{i} \quad \text{for } 2 \leq i \leq m - 1
\]
we get
\[
g = D_{20} \tilde{T}_{2}^{q} + D_{21} Y^{q^{m+1}(m-3)} \tilde{T}_{2}
\]
\[
+ \sum_{\mu=0}^{m-1} D_{m\mu} Y^{q^{m-1-\mu}(\mu-1)}
\]
\[
+ \sum_{i=3}^{m-1} \sum_{\mu=0}^{m-1} D_{i\mu} Y^{q^{m-1-i+1}(m-1+\mu-i)+q^{m-1-i}(\mu-1)} \tilde{T}_{i}^{\mu-1-\mu}.
\]

Hence upon letting
\[
\tilde{Y} = Y^{q^{m+1}}
\]
and
\[
\tilde{T}_{i} = \begin{cases}
\tilde{T}_{i} & \text{for } 2 \leq i \leq m - 1, \\
1 & \text{for } i = m,
\end{cases}
\]
we get
\[ g = D_{20} \tilde{T}_2^q + D_{21} \tilde{Y} q^{m-2} \tilde{T}_2 + \sum_{i=3}^{m} \sum_{\mu=0}^{i-1} D_{i\mu} \tilde{Y} q^{m-1-\mu} \tilde{T}_i^{1-\mu}. \]

Expanding the exponents of \( \tilde{Y} \) we get
\[ g = D_{20} \tilde{T}_2^q + D_{21} \tilde{Y} q^{m-2} \tilde{T}_2 + \sum_{i=3}^{m} \sum_{\mu=0}^{i-1} D_{i\mu} \tilde{Y} q^{m-1-\mu} \tilde{T}_i^{1-\mu} \]

where the dots indicate geometric series with ratio \( q \). Upon letting
\[ \tilde{D}_{i\mu} = D_{i,i-1-\mu} \quad \text{for} \quad 2 \leq i \leq m \quad \text{and} \quad 0 \leq \mu \leq i-1 \]

we get
\[ \tilde{D}_{i\mu} = \frac{Z^m (\mu-1) + (m-1-i+\mu)}{Z-1} \quad \text{for} \quad 2 \leq i \leq m \quad \text{and} \quad 0 \leq \mu \leq i-1 \]

and arranging the terms according to descending powers of \( \tilde{Y} \) we get
\[ g = \tilde{D}_{20} \tilde{Y} q^{m-2} \tilde{T}_2 + \tilde{D}_{21} \tilde{T}_2^q + \sum_{i=3}^{m} \sum_{\mu=0}^{i-1} \tilde{D}_{i\mu} \tilde{Y} q^{m-1-\mu} \tilde{T}_i^{1-\mu} \]

and simplifying the expression of \( \tilde{D}_{20} \) and \( \tilde{D}_{21} \) we have
\[ \tilde{D}_{20} = -\frac{Z^{(m-3)} (Z^q - 1)}{Z-1} \quad \text{and} \quad \tilde{D}_{21} = \frac{Z^{(m-2)} (Z^q - 1)}{Z-1}. \]

For a moment, assuming \( m = 3 \), we note that
\[ g = \tilde{D}_{20} \tilde{Y} q \tilde{T}_2 + \tilde{D}_{21} \tilde{T}_2 + \tilde{D}_{30} \tilde{Y}^{1+q} + \tilde{D}_{31} \tilde{Y}^q + \tilde{D}_{32} \]

where
\[ \tilde{D}_{20} = -\frac{Z (Z^q - 1)}{Z-1} \quad \text{and} \quad \tilde{D}_{21} = \frac{Z^{1+q} (Z^{q^2} - 1)}{Z-1} \quad \text{and} \quad \tilde{D}_{30} = -\frac{(Z^{1+q} - 1)}{Z-1} \]

and
\[ \tilde{D}_{31} = \frac{Z^{1+q} (Z^{q^2} - 1)}{Z-1} \quad \text{and} \quad \tilde{D}_{32} = \frac{Z^{1+q} (Z^{q^2+q^4} - 1)}{Z-1} \]

and to factor \( g \) we try to find a \( \tilde{T}_2 \)-root \( E_{30} \tilde{Y} + E_{31} \) of \( g \). To do this we first put
\[ E_{30} = \frac{\tilde{D}_{30}}{-\tilde{D}_{20}} = \frac{(Z^{1+q} - 1) Z^q (Z^q - 1)}{Z-1} = \frac{(Z^q - 1) (Z^{1+q} - 1)}{Z-1}. \]
then we put
\[ E_{31} = \frac{\tilde{D}_{31} + \tilde{D}_{21}E_{30}^g}{-\tilde{D}_{20}} \]
\[ = \frac{Z^{1+q}(Z^{q^3-1})}{Z^{q^3-1}} + \frac{Z^{1+q}(Z^{q^3-1})}{Z^{q^3-1}} \left( \frac{(Z^{1+q}-1)^q}{Z(Z^{q^3-1})} \right) \]
\[ = \frac{Z^q(Z^{q^3-1})}{Z^{q^3-1}} \left( Z^{q^3-1} - (Z^{q^3-1})(Z^{q^3} - 1) \right) \]
\[ = \frac{(Z^{q^3} - Z^{q^3+q} + Z^q)}{(Z^q - 1)(Z^{q^3} - 1)} \]
\[ = \frac{(Z^{q^3} - Z^{q^3+q} + Z^q)}{(Z^q - 1)(Z^{q^3} - 1)} \]
\[ = \frac{(Z^{q^3} + q^2 + 1)}{(Z^{q^3} - 1)} , \]

and finally we calculate the term free of $\tilde{Y}$ to be
\[ \tilde{D}_{32} + \tilde{D}_{21}E_{31}^g = \frac{Z^{1+q}(Z^{q^3+q^4} - 1)}{Z^{q^3-1}} + \left( \frac{Z^{1+q}(Z^{q^3} - 1)}{Z^{q^3-1}} \right) \left( \frac{(Z^{q^3}+q^2+1)}{(Z^{q^3} - 1)} \right) \]
\[ = \frac{Z^{1+q}(Z^{q^3+q^4} - 1)}{Z^{q^3-1}} + \frac{Z^{1+q}(-Z^{q^3}+q^4+1)}{Z^{q^3-1}} \]
\[ = 0 . \]

Alternatively, for “the fictitious term” $E_{32}$, we have
\[ E_{32} = \frac{\tilde{D}_{32} + \tilde{D}_{21}E_{31}^g}{-\tilde{D}_{20}} = \frac{\tilde{D}_{32} + \tilde{D}_{21}E_{31}^g}{-\tilde{D}_{20}} \left( \frac{\tilde{D}_{31} + \tilde{D}_{21}E_{20}^g}{-\tilde{D}_{20}} \right) \]
\[ = \frac{\tilde{D}_{32}}{D_{20}} + \frac{\tilde{D}_{21}\tilde{D}_{31}}{D_{20}} + \frac{\tilde{D}_{21}E_{30}^g}{D_{20}} \]
\[ = \frac{\tilde{D}_{32}}{D_{20}} + \frac{\tilde{D}_{21}\tilde{D}_{31}}{D_{20}} + \frac{\tilde{D}_{21}E_{30}^g}{D_{20}} \]
\[ = \frac{\tilde{D}_{32}}{D_{20}} + \frac{\tilde{D}_{21}\tilde{D}_{31}}{D_{20}} + \frac{\tilde{D}_{21}E_{30}^g}{D_{20}} \]

and by substituting the values of $\tilde{D}_{20}, \tilde{D}_{21}, \tilde{D}_{30}, \tilde{D}_{31}, \tilde{D}_{32}$, we see this to be 0.

Now, without assuming $m = 3$, but henceforth again assuming $m > 2$, to factor $g$, for any $3 \leq i \leq m$, we try to find a $T_i$-root
\[ \sum_{\mu=0}^{i-2} E_{\mu} \tilde{Y}^{q^{m-i+n} + \cdots + q^{m-3} + \tilde{g}^\mu} T_i \]

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of
\[ \hat{D}_{20} \hat{Y}^{q^{-1}} \hat{T}_2 + \hat{D}_{21} \hat{T}_2 + \sum_{\mu=0}^{i-1} \hat{D}_{i\mu} \hat{Y}^{q^{-1}+\mu} \hat{T}_2^\mu, \]
i.e., we try to find \( E_{i\mu} \) in \( \text{GF}(p)(Z) \) such that
\[ \sum_{\mu=0}^{i-1} \hat{D}_{i\mu} \hat{Y}^{q^{-1}+\mu} \hat{T}_2^\mu = -\hat{D}_{20} \hat{Y}^{q-1} \left( \sum_{\mu=0}^{i-2} E_{i\mu} \hat{Y}^{q^{-1}+\mu} \hat{T}_2^\mu \right) \]
\[ - \hat{D}_{21} \left( \sum_{\mu=0}^{i-2} E_{i\mu} \hat{Y}^{q^{-1}+\mu} \hat{T}_2^\mu \right)^q. \]
Equating coefficients of
\[ \hat{Y}^{q^{-1}+\mu} \hat{T}_2^\mu \]
to zero, we try to find \( E_{i\mu} \) in \( \text{GF}(p)(Z) \) such that
\[ \hat{D}_{i\mu} = \begin{cases} -\hat{D}_{20} E_{i\mu} & \text{for } \mu = 0, \\ -\hat{D}_{20} E_{i\mu} - \hat{D}_{21} E_{i-1\mu-1} & \text{for } 1 \leq \mu \leq i-2, \\ -\hat{D}_{21} E_{i-1\mu-1} & \text{for } \mu = i-1. \end{cases} \]
Since \( \hat{D}_{20} \neq 0 \), we can successively find the values of \( E_{i\mu} \) for \( 0 \leq \mu \leq i-2 \) by solving all except the last equation, and then get a condition by substituting these in the last equation. Upon letting
\[ J_{i\mu} = \sum_{j=0}^{\mu} (-1)^{(\mu-j)} \frac{\hat{D}_{21}^{(\mu-j-1)} \hat{D}_{20}^{(\mu-j)} \hat{D}_{i\mu}^{(\mu-j)}}{\hat{D}_{20}^{(\mu-j)}} \quad \text{for } 0 \leq \mu \leq i-1 \]
these values are
\[ E_{i\mu} = J_{i\mu} \quad \text{for } 0 \leq \mu \leq i-2 \]
and the condition is
\[ J_{i,i-1} = 0. \]
Substituting the simplified expressions of \( \hat{D}_{20} \) and \( \hat{D}_{21} \), for \( 0 \leq \mu \leq i-1 \) and \( 0 \leq j \leq \mu \) we get
\[ \frac{\hat{D}_{21}^{(\mu-j-1)}}{\hat{D}_{20}^{(\mu-j)}} = \left[ \frac{Z^{(m-2)} (Z^m - 1)}{Z - 1} \right]^{(\mu-j-1)} \frac{Z - 1}{-Z^{(m-3)} (Z^{m-2} - 1)}^{(\mu-j)} \]
\[ = \frac{Z^{(m-2)(\mu-j-1)-(m-3)(\mu-j)}}{(-1)^{(\mu-j)} (Z - 1)^{-\mu-j} \prod_{l=0}^{\mu-j} (Z^l - 1)^{\mu-j}} \]
\[ \times \frac{Z}{(m-2)(\mu-j-1)-(m-3)(\mu-j)} \prod_{l=0}^{\mu-j} (Z^{m-2+l} - 1) \]
\[ = \frac{(-1)^{(\mu-j)} (Z - 1)^{-\mu-j-1} \prod_{l=0}^{\mu-j} (Z^{m-2+l} - 1)}{Z^{(m-2)(\mu-j-1)-(m-3)(\mu-j)} \prod_{l=0}^{\mu-j} (Z^{m-2+l} - 1)} \]
\[ \times \frac{Z}{(m-2)(\mu-j-1)-(m-3)(\mu-j)} \prod_{l=0}^{\mu-j} (Z^{m-2+l} - 1) \]
\[ = \frac{(-1)^{(\mu-j)} (Z - 1)^{-\mu-j-1} \prod_{l=m-1}^{\mu-j} (Z^{l+1} - 1)}{Z^{(m-2)(\mu-j-1)-(m-3)(\mu-j)} \prod_{l=m-1}^{\mu-j} (Z^{l+1} - 1)} \]
\[ \times \frac{Z}{(m-2)(\mu-j-1)-(m-3)(\mu-j)} \prod_{l=m-1}^{\mu-j} (Z^{l+1} - 1) \]
\[ = \frac{(-1)^{(\mu-j)} (Z - 1)^{-\mu-j-1} (Z^{m-2} - 1) (Z^{m-1} - 1)}{Z^{(m-2)(\mu-j-1)-(m-3)(\mu-j)} \prod_{l=m-1}^{\mu-j} (Z^{l+1} - 1)} \]
where, for the last equation, a separate but trivial argument may be made in the case of \( j = \mu \) by noting that then the extra (purposefully inserted) term \( (Z^{m+\mu-j-1} - 1) \) in the numerator equals the extra term \( (Z^{m-1} - 1) \) in the denominator. Therefore by substituting the values of \( \tilde{D}_{ij} \), for \( 0 \leq \mu \leq i - 1 \) we get

\[
J_{ij} = \sum_{j=0}^{\mu} (-1)^{j-i} \left[ \frac{Z^{(m-2)(\mu-j-1)-(m-3)(\mu-j)} (Z^{m+\mu-j-1} - 1)}{(-1)^{\mu-j} (Z - 1)^{-q^{\mu-j}} (Z^{m-2} - 1) (Z^{m-1} - 1)} \right] \times
\]

\[
\times \left[ \frac{Z^m j - (m-1+i+j)}{Z - 1} \right] ^{m-i-1}
\]

\[
= \sum_{j=0}^{\mu} \frac{Z^{(m-2)(\mu-j-1)-(m-3)(\mu-j)+q^{m+\mu-j}(m-2)}}{(Z^{m-2} - 1) (Z^{m-1} - 1)} \left( Z^{m+\mu-j-1} - 1 \right)
\]

\[
- \sum_{j=0}^{\mu} \frac{Z^{(m-2)(\mu-j-1)-(m-3)(\mu-j)+q^{m-2}(m-2)}}{(Z^{m-2} - 1) (Z^{m-1} - 1)} \left( Z^{m+\mu-j-1} - 1 \right)
\]

where

the first exponent of \( Z \) in the last summation

\[
= (m-2)(\mu-j-1) - (m-3)(\mu-j) + q^{m+\mu-j}(m-2)
\]

\[
= [(m-2)(\mu-j) - q^{m-1} - (m-2)q^{m-2}] q^{m-2} q^{m-j}(m-2)
\]

\[
= q^{m-2} q^{m-j}
\]

and

the first exponent of \( Z \) in the last but one summation

\[
= (m-2)(\mu-j-1) - (m-3)(\mu-j) + q^{m+\mu-j}(j-1) + q^{m-j}(m-1-i+j)
\]

\[
= [(m-2)(\mu-j) - q^{m-1} - (m-2)q^{m-2}] q^{m-1} q^{m-j}(m-1-i+j)
\]

\[
= q^{m-2} q^{m-j} - q^{m-j}(m-2) + q^{m+\mu-j}(j-1) + q^{m-j}(m-1-i+j)
\]

\[
= q^{m-2} q^{m-j} + q^{m+\mu-j-1} + q^{m+\mu-j-1} - q^{m-j}[(m-2) - (m-1-i+j)]
\]

\[
= q^{m-2} q^{m-j-1} - q^{m+\mu-j-1}.
\]
Therefore

\[ J_{j,i} = \frac{Z_{q^{m-2} \mu (\mu+1)} - Z_{q^{m-2}} (Z_{q^{m-1} \mu} - 1)}{(Z_{q^{m-2}} - 1) (Z_{q^{m-1}} - 1)} \]

Now by putting \( \mu = i - 1 \) we see that

\[ J_{i,i-1} = 0. \]

It follows that, upon letting

\[ E_{i} = \frac{Z_{q^{m-2} \mu (i-2-\mu)} - Z_{q^{m-2}} (Z_{q^{m-1} \mu} - 1)}{Z_{q^{m-2} \mu (i-3-\mu)} (Z_{q^{m-2}} - 1) (Z_{q^{m-1}} - 1)} \]

for \( 3 \leq i \leq m \) and \( 0 \leq \mu \leq i - 1 \).
we have
\[
\sum_{\mu=0}^{i-1} \hat{D}_{\mu} \hat{Y}^{q^{m-i+i'\ldots+i+q^{m-2}}} \hat{T}_i^\mu \\
= -\hat{D}_{20} \hat{Y}^{q^{m-2}} \left( \sum_{\mu=0}^{i-2} E_{1\mu} \hat{Y}^{q^{m-i+i'\ldots+i+q^{m-3}}} \hat{T}_i^\mu \right) \\
- \hat{D}_{21} \left( \sum_{\mu=0}^{i-2} E_{1\mu} \hat{Y}^{q^{m-i+i'\ldots+i+q^{m-3}}} \hat{T}_i^\mu \right)^q 
\]
for 3 \leq i \leq m.

By \( q \)-linearity, summing the above equations we get
\[
\sum_{i=3}^m \sum_{\mu=0}^{i-1} \hat{D}_{\mu} \hat{Y}^{q^{m-i+i'\ldots+i+q^{m-2}}} \hat{T}_i^\mu \\
= -\hat{D}_{20} \hat{Y}^{q^{m-2}} \left( \sum_{i=3}^m \sum_{\mu=0}^{i-2} E_{1\mu} \hat{Y}^{q^{m-i+i'\ldots+i+q^{m-3}}} \hat{T}_i^\mu \right) \\
- \hat{D}_{21} \left( \sum_{i=3}^m \sum_{\mu=0}^{i-2} E_{1\mu} \hat{Y}^{q^{m-i+i'\ldots+i+q^{m-3}}} \hat{T}_i^\mu \right)^q .
\]

Therefore recalling that
\[
\hat{D}_{20} = -\frac{Z^{(m-3)} (Z q^{m-2} - 1)}{Z - 1} \quad \text{and} \quad \hat{D}_{21} = \frac{Z^{(m-2)} (Z q^m - 1)}{Z - 1}
\]
and letting
\[
D = -\hat{D}_{21}/\hat{D}_{20} \quad \text{and} \quad E = \hat{D}_{20} \sum_{i=3}^m \sum_{\mu=0}^{i-2} E_{1\mu} \hat{Y}^{q^{m-i+i'\ldots+i+q^{m-3}}} \hat{T}_i^\mu
\]
we get
\[
D = Z (Z - 1) (q^{m-1} + 1) (q^m - 1)
\]
and
\[
E = \sum_{i=3}^m \sum_{\mu=0}^{i-1} \left( \frac{Z^{q^{m+i'+\ldots+i+q^{m-1}}} - 1}{Z - 1} \right) \left( \frac{Z^{(\mu)} - 1}{Z - 1} \right)^{q^{m-1}} Z^{(m+\mu-3)} \hat{Y}^{q^{m+i'+\ldots+i+q^{m-3}}} \hat{T}_i^\mu 
\]
and
\[
-DE^q + \hat{Y}^{q^{m-2}} \hat{T}_i^q + \sum_{i=3}^m \sum_{\mu=0}^{i-1} \hat{D}_{\mu} \hat{Y}^{q^{m-i+i'\ldots+i+q^{m-2}}} \hat{T}_i^\mu = 0 .
\]

The above equation says that \( E/\hat{D}_{20} \) is a \( \hat{T}_2 \)-root of
\[
g = \hat{D}_{21} \hat{T}_2^q + \hat{D}_{20} \hat{Y}^{q^{m-2}} \hat{T}_2 + \sum_{i=3}^m \sum_{\mu=0}^{i-1} \hat{D}_{\mu} \hat{Y}^{q^{m-i+i'\ldots+i+q^{m-2}}} \hat{T}_i^\mu .
\]
Hence upon letting
\[ g' = E - \tilde{D}_{20} \tilde{T}_2 \quad \text{and} \quad g'' = DE^{q-1} - \hat{Y}^{m-2} + \sum_{l=1}^{q-1} D \tilde{D}_{20}^{l} E^{q-1-l} \tilde{T}_2 \]
we obtain
\[
g' g'' = \left( DE^{q-1} - \hat{Y}^{m-2} E \right) + \sum_{l=1}^{q-1} D \tilde{D}_{20}^{l} E^{q-1-l} \tilde{T}_2
- \left( D \tilde{D}_{20} E^{q-1} - \tilde{D}_{20} \hat{Y}^{m-2} \right) \tilde{T}_2 - \sum_{l=2}^{q-1} D \tilde{D}_{20}^{l} E^{q-1-l} \tilde{T}_2
= \left( DE^{q-1} - \hat{Y}^{m-2} E \right) + \left( D \tilde{D}_{20} E^{q-1} \right) \tilde{T}_2 + \sum_{l=2}^{q-1} D \tilde{D}_{20}^{l} E^{q-1-l} \tilde{T}_2
- \left( \sum_{l=1}^{q-1} D \tilde{D}_{20}^{l} E^{q-1-l} \tilde{T}_2 \right) - D \tilde{D}_{20}^{q} \tilde{T}_2
= \tilde{D}_{21} \tilde{T}_2^{q} + \tilde{D}_{20} \hat{Y}^{m-2} \tilde{T}_2 + \left( DE^{q-1} - \hat{Y}^{m-2} E \right)
= \tilde{D}_{21} \tilde{T}_2^{q} + \tilde{D}_{20} \hat{Y}^{m-2} \tilde{T}_2 + \sum_{i=3}^{m-1} \sum_{\mu=0}^{i-1} \tilde{D}_{\mu} \hat{Y}^{q^{m-i+\mu} + \cdots + q^{m-2}} \tilde{T}_i^{q^{\mu}}
= g.
\]
Thus we get the factorization
\[ g = g' g'' \]
where by substituting the values of \( \hat{Y} \) and \( \tilde{T}_1 \) we have
\[ g = \tilde{D}_{21} Y^{q^{m+1}(m-3)} T_2 + \tilde{D}_{20} Y^{q^{m-2}+q^m(m-2)} T_2 + \sum_{\mu=0}^{m-1} \tilde{D}_{\mu} Y^{q^{m+1}(m-2-\mu)}
+ \sum_{i=3}^{m-1} \sum_{\mu=0}^{i-1} \tilde{D}_{\mu} Y^{q^{m-i+\mu} + \cdots + q^{m-2}} T_i^{q^{\mu}} \]
and
\[ g' = E - \tilde{D}_{20} Y^{q^m(m-3)} T_2 \]
and
\[ g'' = DE^{q-1} - Y^{(q^{m+1})q^{m-2}} + \sum_{l=1}^{q-1} D \tilde{D}_{20}^{l} E^{q-1-l} Y^{q^m(m-3)} T_2 \]
and
\[ E = \sum_{\mu=0}^{m-2} \tilde{E}_{\mu} Y^{q^{m+1}(m-3-\mu)} + \sum_{i=3}^{m-1} \sum_{\mu=0}^{i-2} \tilde{E}_{\mu} Y^{q^{m-i+\mu} + \cdots + q^{m-2}} T_i^{q^{\mu}} \]
with

$$\hat{E}_{i\mu} = \left(\frac{Z^{q^m+i-1}(q^m+i-1) - 1}{Z-1}\right) \left(\frac{Z^{(\mu)} - 1}{Z-1}\right) Z^{(m+i-1)}$$

for $3 \leq i \leq m$ and $0 \leq \mu \leq i-2$,

and where we recall that

$$\hat{D}_{i\mu} = \frac{Z^{q^m(\mu+1)+m+i-\mu} - Z^{(m-2)}}{Z-1}$$

for $3 \leq i \leq m$ and $0 \leq \mu \leq i-1$

and

$$\hat{D}_{20} = \frac{Z^{(m-3)}}{Z-1} \left(\frac{Z^{q^m} - 1}{Z-1}\right)$$

and

$$\hat{D}_{21} = \frac{Z^{(m-2)}}{Z-1} \left(\frac{Z^{q^m} - 1}{Z-1}\right)$$

and

$$D = \frac{-\hat{D}_{21}}{\hat{D}_{20}} = Z(Z-1)^{(q^m+1)(q-1)}.$$  

By (4.6) we see that, for $3 \leq i \leq m$ and $0 \leq \mu \leq i-2$, $\hat{E}_{i\mu}$ is a monic polynomial of degree

$q^m+i-1 - 1 + q^{m-1}i(\mu-1) + (m+i-\mu-1) = q(m-3) + q^m(\mu-1)$

in $Z$ with coefficients on GF$(p)$. Therefore, since $Y^{q^m+1}x^{(m-3)\mu} = 1$ for $\mu = m-2$, by (4.5) we see that $E$ is a monic polynomial of degree

$q(m-3) + q^m(m-2-1) = q(q^{m-1}+1)(m-3)$

in $Z$ with coefficients in GF$(p)[Y,T_2,\ldots,T_{m-1}]$. Consequently, in view of (4.3) and (4.8) we conclude that $g'$ is a monic polynomial of degree $q(q^{m-1}+1)(m-3)$ in $Z$ with coefficients in GF$(p)[Y,T_2,\ldots,T_{m-1}]$. Obviously $g$ is a monic polynomial of degree

$(\deg T) - 1 = (q^m+1)(m-2-1) = q^m(m-2) + q(m-3)$

in $Z$ with coefficients in GF$(p)[Y,T_2,\ldots,T_{m-1}]$. Hence in view of (4.1), (4.4), (4.8) and (4.9) we see that $g''$ is a monic polynomial of degree

$q^m(m-2) + q(m-3) - q(q^{m-1}+1)(m-3) = q^{2m-2}$

in $Z$ with coefficients in GF$(p)[Y,T_2,\ldots,T_{m-1}]$. Thus

(4.10) \( \{ \)

$g'$ and $g''$ are monic polynomials of degrees $q(q^{m-1}+1)(m-3)$ and $q^{2m-2}$

in $Z$ with coefficients in GF$(p)[Y,T_2,\ldots,T_{m-1}]$ respectively.

Without assuming $m > 2$, for $1 \leq e \leq m-1$, let $f'_e$ and $g_e$ denote the members of GF$(p)[Y,Z,T_2,\ldots,T_e]$ obtained by putting $T_i = 0$ for all $i > e$ in $f'$ and $g$.
respectively. Then \( f'_e \) is the twisted derivative of \( f_e \), and dividing the \( Z \)-roots of \( f'_e \) by \( Y \) and afterwards changing \( Y \) to \( 1/Y \) we get \( g_e \) which is a monic polynomial of degree \( q^{m} (m - 2) + q(m - 3) \) in \( Z \) with coefficients in \( \text{GF}(p)[Y,T_2,\ldots,T_e] \).

Again henceforth assuming \( m > 2 \), for \( 1 \leq e \leq m - 1 \), let \( g'_e \) and \( g''_e \) denote the members of \( \text{GF}(p)[Y,Z,T_2,\ldots,T_e] \) obtained by putting \( T_i = 0 \) for all \( i > e \) in \( g' \) and \( g'' \) respectively. Then in view of (4.1) and (4.10),

\[
\begin{align*}
\text{for } 1 \leq e \leq m - 1 \text{ we have } g_e &= g'_e g''_e \text{ where } g'_e \text{ and } g''_e \text{ are monic polynomials of degrees } q(q^{m-1} + 1)(m - 3) \text{ and } q^{2m-2} \text{ in } Z \text{ with coefficients in } \text{GF}(p)[Y,T_2,\ldots,T_e] \text{ respectively. }
\end{align*}
\]

By (4.2), (4.3), (4.5), (4.6), (4.7) and (4.8) we have

\[
g_2 = A_2 T_2^3 - B_2 T_2 + C_2 \quad \text{and} \quad g'_2 = A'_2 T_2 + B'_2
\]

where \( A_2, B_2, C_2, A'_2, B'_2 \) are the nonzero elements in \( \text{GF}(p)[Y,Z] \) given by

\[
A_2 = \hat{D}_{21} Y^{q^{m+1}(m-3)} \quad \text{and} \quad B_2 = -\hat{D}_{20} Y q^{2m^{-2}} q^{m}(m-2)
\]

and

\[
C_2 = \sum_{\mu=0}^{m-1} \hat{D}_{m\mu} Y^{(q^{m+1})q^{\mu}(m-2-\mu)}
\]

and

\[
A'_2 = -\hat{D}_{20} Y^{q^{m}(m-3)} \quad \text{and} \quad B'_2 = \sum_{\mu=0}^{m-2} \hat{E}_{m\mu} Y^{(q^{m+1})q^{\mu}(m-3-\mu)}
\]

By letting \( I \) to be the \( Z \)-adic valuation of \( Q = k_p(Y,Z) \), i.e., the real discrete valuation whose valuation ring is the localization of \( k_p(Y,Z) \) at the principal prime ideal generated by \( Z \), we see that \( I(A_2) = (m - 2) \) and \( I(B_2) = (m - 3) \) and hence

\[
I(B_2/A_2) = (m - 3) - (m - 2) = -q^{m-2} \text{ and therefore } \text{GCD}(q - 1, I(B_2/A_2)) = 1.
\]

In view of (4.7) and (4.8) we also see that \( A_2 \) and \( C_2 \) have no nonconstant common factor in \( k_p(Y,Z) \), because \( \mu = m - 1 \) gives the nonzero term \( \hat{D}_{m,m-1} \) of \( C_2 \) which is independent of \( Y \), and \( \mu = 0 \) gives the highest \( Y \)-degree term of \( C_2 \) and its coefficient is

\[
\hat{D}_{m0} = \frac{1 - Z^{(m-2)}}{Z - 1}.
\]

Therefore by Lemmas (4.2) and (4.3) of [A05] we conclude that

\[
\text{(4.12) the polynomials } g'_2 \text{ and } g''_2 \text{ are irreducible in } k_p(Y,T_2)[Z].
\]

As an immediate consequence of (4.12) we see that

\[
\text{(4.13) the polynomials } g'_e \text{ and } g''_e \text{ are irreducible in } k_p(Y,T_2,\ldots,T_{m-1})[Z]
\]

and, for \( 2 \leq e \leq m - 1 \),

\[
\text{(4.14) the polynomials } g'_e \text{ and } g''_e \text{ are irreducible in } k_p(Y,T_2,\ldots,T_e)[Z].
\]

Note that

\[
\text{(4.14) in (4.1) to (4.13) we assumed } m > 2.
\]

Recall that \( f_e \) is irreducible in \( k_p(T_1,\ldots,T_e)[Y] \), its twisted derivative is \( f'_e(Y,Z) \), and \( g_e \) is obtained by dividing the \( Z \)-roots of \( f'_e(Y,Z) \) by \( Y \) and then changing \( Y \) to \( 1/Y \); therefore by (4.1), (4.10), (4.11), (4.13) and (4.14) we get the following
Theorem (4.15). If \( m = 2 \) then \( \text{Gal}(\overline{f}, k_p(T_1)) = \text{Gal}(\overline{f_1}, k_p(T_1)) \) is a 2-transitive permutation group of degree \( q^m + 1 \). If \( m > 2 \) and \( 2 \leq e \leq m - 1 \) then \( \text{Gal}(\overline{f_e}, k_p(T_1, \ldots, T_e)) \) is a transitive permutation group of Rank 3 with subdegrees \( 1, q(q^{m-1} + 1)(m-3) \) and \( q^{2m-2} \). Hence in particular, if \( m > 2 \) then \( \text{Gal}(\overline{f}, k_p(T_1, \ldots, T_{m-1})) \) is a transitive permutation group of Rank 3 with subdegrees \( 1, q(q^{m-1} + 1)(m-3) \) and \( q^{2m-2} \).

Notation. Recall that \( < \) denotes a subgroup, and \( \triangleleft \) denotes a normal subgroup. Let the groups \( \text{SL}(m, q) \triangleleft \text{GL}(m, q) \triangleleft \text{TL}(m, q) \) and \( \text{PSL}(m, q) \triangleleft \text{PGL}(m, q) \triangleleft \text{PTL}(m, q) \) and their actions on \( \text{GF}(q)^m \) and \( \mathcal{P}(\text{GF}(q)^m) \) be as on pages 78-80 of [A03]. Let

\[
\Theta_m : \text{TL}(m, q) \rightarrow \text{PTL}(m, q) = \text{TL}(m, q)/\text{GF}(q)^*
\]

be the canonical epimorphism where we identify the multiplicative group \( \text{GF}(q)^* \) with scalar matrices, which constitute the center of \( \text{GL}(m, q) \).

Now in view of Proposition 3.1 of [A04], by (3.0), (3.1), (3.4) and (3.5) we get the following

Theorem (4.16). Assuming \( \text{GF}(q) \subset k_p \), for \( 1 \leq e \leq m - 1 \), in a natural manner we may regard

\[
\text{Gal}(\phi_e^-, k_p(T_1, \ldots, T_e)) < \text{GL}(2m, q) \quad \text{and} \quad \text{Gal}(f_e^-, k_p(T_1, \ldots, T_e)) < \text{PGL}(2m, q)
\]

and then

\[
\Theta_{2m}(\text{Gal}(\phi_e^-, k_p(T_1, \ldots, T_e))) = \text{Gal}(f_e^-, k_p(T_1, \ldots, T_e))
\]

and \( \text{Gal}(f_e^-, k_p(T_1, \ldots, T_e)) \) has two or three orbits on \( \mathcal{P}(\text{GF}(q)^{2m}) \) of sizes \( (q^m + 1)(m-2), q^{m-1}(q^m + 1) \) or \( (q^m + 1)(m-2), q^{m-1}(q^m + 1)/2, q^{m-1}(q^m + 1)/2 \) according as \( p = 2 \) or \( p \neq 2 \). In particular, again assuming \( \text{GF}(q) \subset k_p \), in a natural manner we may regard

\[
\text{Gal}(\phi^-, k_p(T_1, \ldots, T_{m-1})) < \text{GL}(2m, q)
\]

and

\[
\text{Gal}(f^-, k_p(T_1, \ldots, T_{m-1})) < \text{PGL}(2m, q)
\]

and then

\[
\Theta_{2m}(\text{Gal}(\phi^-, k_p(T_1, \ldots, T_{m-1}))) = \text{Gal}(f^-, k_p(T_1, \ldots, T_{m-1}))
\]

and \( \text{Gal}(f^-, k_p(T_1, \ldots, T_e)) \) has two or three orbits on \( \mathcal{P}(\text{GF}(q)^{2m}) \) of sizes \( (q^m + 1)(m-2), q^{m-1}(q^m + 1) \) or \( (q^m + 1)(m-2), q^{m-1}(q^m + 1)/2, q^{m-1}(q^m + 1)/2 \) according as \( p = 2 \) or \( p \neq 2 \).

Recall that a quasi-\( p \) group is a finite group which is generated by its \( p \)-Sylow subgroups. Since \( \text{Discy}_e f_e^- = -1 = \text{Discy}_e \phi_e^- \) for \( 1 \leq e \leq m - 1 \), by the techniques of the proofs of Proposition 6 of [A01] and Lemma 34 of [A02] we get the following

Theorem (4.17). If \( k_p \) is algebraically closed then, \( \text{Gal}(f_e^-, k_p(T_1, \ldots, T_e)) \) and \( \text{Gal}(\phi_e^-, k_p(T_1, \ldots, T_e)) \) for \( 1 \leq e \leq m - 1 \), are quasi-\( p \) groups. In particular, if \( k_p \) is algebraically closed then, \( \text{Gal}(f^-, k_p(T_1, \ldots, T_{m-1})) \) and \( \text{Gal}(\phi^-, k_p(T_1, \ldots, T_{m-1})) \) are quasi-\( p \) groups.
5. Review of Linear Algebra

Recall that we are assuming \( m > 1 \). Let \( \epsilon \in \{+,-\} \). Let \( \epsilon' = (1 - \epsilon)/2 \) and note that then \( \epsilon' = 0 \) or 1 according as \( \epsilon = + \) or \( - \) respectively.

Fix \( \nu \in \text{GF}(q) \) such that \( T^2 + T + \nu \) is irreducible in \( \text{GF}(q)[T] \). Consider the quadratic forms \( \psi^+(x) = x_1x_{m+1} + \cdots + x_mx_{2m} \) and \( \psi^-(x) = x_1x_{m+1} + \cdots + x_{m-1}x_{2m-1} + x_2^2 + x_{m+2}x_{3m} + \nu x_{2m}^2 \). Define the orthogonal group \( O^*(2m,q) \) as the group of all \( \epsilon \in \text{GL}(2m,q) \) which leave the quadratic form \( \psi^\epsilon \) unchanged, i.e., \( \psi^\epsilon(x) = \psi^\epsilon(x) \). Let the general orthogonal group \( GO^*(2m,q) \) be defined as the group of all \( \epsilon \in \text{GL}(2m,q) \) such that for some \( \lambda(\epsilon) \in \text{GF}(q) \) we have \( \psi^\epsilon(\xi) = \lambda(\epsilon)\psi^\epsilon(\xi) \) for all \( \xi \in \text{GF}(q)^{2m} \). Let the semilinear orthogonal group \( \Gamma O^*(2m,q) \) be defined as the group of all \( \epsilon \in \text{GL}(2m,q) \) and \( \epsilon \in \text{GL}(2m,q) \) such that for some \( \lambda(\tau,e) \in \text{GF}(q)^{2m} \) we have \( \psi^\epsilon(\xi,e) = \lambda(\tau,e)\psi^\epsilon(\xi)^e \) for all \( \xi \in \text{GF}(q)^{2m} \). Define the special orthogonal group \( SO^*(2m,q) = SL(2m,q) \cap O^*(2m,q) \). Let \( O^*(2m,q) = O^*(2m,q) \) if \( (m,q,e) \neq (2,2,+) \), and let \( O^+(4,2) \) be the subgroup of \( SO^*(4,2) \) containing \( O^+(4,2) \), as defined in Definition 4 on page 30 of [LiK], such that \( [SO^+(4,2) : O^+(4,2)] = 2 = [O^+(4,2) : O^+(4,2)] \). Thus we get the sequence \( O^*(2m,q) < O^+(2m,q) < SO^*(2m,q) < O^*(2m,q) < GO^*(2m,q) < GO^*(2m,q) \) of orthogonal groups and by applying \( \Theta_{2m} \) to them we get the corresponding sequence \( PO^*(2m,q) < PH^*(2m,q) < PSO^*(2m,q) < PO^*(2m,q) < PGO^*(2m,q) < PGO^*(2m,q) \) of projective orthogonal groups.3

Note that for any \( H < GL(2m,q) \) we have

\[
\Omega^*(2m,q) < H \iff \Pi^*(2m,q) \subset \Theta_{2m}(H).
\]

In case \( (m,q,e) \neq (2,2,+) \), this follows exactly as in the proof of Lemma 2.3 of [A04] because then by Theorem 11.46 of [Tay] \( \Omega^*(2m,q) \) is generated by Siegel transformations. By the definition of a Siegel transformation (11.17 of [Tay]) we see that its order is \( p \) or 1, and the said proof is based on the fact that the group is generated by elements of \( p \)-power order, i.e., equivalently the fact that it is a quasi-\( p \) group. So (5.1) holds also for \( (m,q,e) = (2,2,+) \) because by Proposition 2.9.1(iv) of [LiK] \( \Omega^*(4,2) \) is a quasi-\( 2 \)-group.

3Instead of taking the specific quadratic form \( \psi^\epsilon \), in [LiK] these groups are defined for each quadratic form of “Witt defect \( \epsilon' \).” Dickson [Dic] defines these groups for \( p \neq 2 \) by taking a different set of specific quadratic forms thus: if either \( \epsilon = + \) and \( q \equiv 1 \) (mod 4) or \( \epsilon = + \) and \( q \equiv 3 \) (mod 4) with \( m \) even or \( \epsilon = - \) and \( q \equiv 3 \) (mod 4) with \( m \) odd then take the quadratic form to be \( x_1^2 + \cdots + x_{2m}^2 \); if either \( \epsilon = + \) and \( q \equiv 3 \) (mod 4) with \( m \) odd or \( \epsilon = - \) and \( q \equiv 3 \) (mod 4) with \( m \) even then take the quadratic form to be \( x_1^2 + \cdots + x_{2m-1}^2 - x_{2m}^2; \) and finally if \( \epsilon = - \) and \( q \equiv 3 \) (mod 4) then take the quadratic form to be \( x_1^2 + \cdots + x_{2m-1}^2 - \nu x_{2m}^2 \) with \( \mu \in \text{GF}(q) \setminus \text{GF}(q)^2 \). By the singular points of \( \Pi^*(2m,q) \) we mean the images in \( \Pi^*(\text{GF}(q)^{2m}) \) of the nonzero \( \xi \in \text{GF}(q)^{2m} \) at which the quadratic form vanishes. By Exercise 11.3 on page 174 of [Tay] we see that the cardinality of the singular points of \( \Pi^*(2m,q) \) is \( (q^{m-1+\epsilon'} + 1)(m-1-\epsilon') \), and hence the cardinality of the nonsingular points of \( \Pi^*(2m,q) \) is \( q^{m-1}(q^m - 1 + 2\epsilon') \). By 11.24 and 11.27 on pages 150-151 of [Tay] we see that \( \Pi^*(2m,q) \) acts transitively on its singular points, and by using Witt’s Lemma (page 81 of [Asc]) we see that if \( p = 2 \), then \( \Pi^*(2m,q) \) acts transitively on its nonsingular points, whereas if \( p \neq 2 \), then \( \Pi^*(2m,q) \) has two equal size orbits of nonsingular points. Finally, by the sixth line of Table 5.4.C on page 200 of [LiK] which starts with \( D_2^2(q) \), we see that if \( m > 3 \) and \( \Phi < \text{PGL}(2m,q) \) is isomorphic to \( \Pi^*(2m,q) \), then \( \Pi^*(2m,q) = \delta^{-1}\Phi \Phi \) for some \( \delta \in \text{PGL}(2m,q) \).
By 2.1.B, 2.10.4(ii) and 2.10.6(i) of [LiK], for any $H < \text{GL}(2m, q)$ we have

$$\Omega'(2m, q) < H \iff \Omega'(2m, q) < \text{GO}'(2m, q)$$

and by 2.1.C of [LiK] we have

$$[\text{GO}'(2m, q) : \Omega'(2m, q)] \begin{cases} \not\equiv 0 \pmod{p} & \text{if } p > 2, \\ = 2 & \text{if } p = 2. \end{cases}$$

Since $\Omega'(2m, q)$ is quasi-$p$, it is generated by the $p$-power elements of $\Omega'(2m, q)\text{GF}(q)^*$, and hence these two subgroups have the same normalizer in $\text{GL}(2m, q)$. Also clearly $\text{GF}(q)^* < \text{GO}'(2m, q)$. Therefore by (5.2), for any $G < \text{PGL}(2m, q)$ we have

$$\text{P}\Omega'(2m, q) < G \iff \text{P}\Omega'(2m, q) < \text{PGO}'(2m, q)$$

and by (5.3) we get

$$[\text{PGO}'(2m, q) : \text{P}\Omega'(2m, q)] \begin{cases} \not\equiv 0 \pmod{p} & \text{if } p > 2, \\ = 2 & \text{if } p = 2. \end{cases}$$

Finally, since $\text{GF}(q)^* < \text{GO}'(2m, q)$, for any $H < \text{GL}(2m, q)$ we have

$$H < \text{GO}'(2m, q) \iff \Theta_{2m}(H) < \text{PGO}'(2m, q).$$

In view of Theorem IV of [CaK], by Corollary 1(iii) of Kantor [Kan] we get the following:

**Theorem (5.7)** [KANTOR]. Assume that $m > 3$. Let $G$ be a transitive permutation group of Rank 3 with subdegrees $1, q(q^{m-2}+1)(m-2) - 1$ and $q^{m-2}$. Then the permuted set can be identified with the singular points of $\text{P}\Omega'(2m, q)$ so that $\text{P}\Omega'(2m, q)_1 < G < \text{PGO}'(2m, q)_1$ where $\text{P}\Omega'(2m, q)_1$ and $\text{PGO}'(2m, q)_1$ denote the permutation groups on the said singular points induced by $\text{P}\Omega'(2m, q)$ and $\text{PGO}'(2m, q)$ respectively.

For applying (5.7), we first prove the following

**Lemma (5.8).** Let $G < \text{PGL}(m, q)$ have orbits $\Delta_1, \ldots, \Delta_r$ of sizes $d_1, \ldots, d_r$ on $\mathcal{P}(\text{GF}(q)^m)$, and note that then $\sum_{i=1}^r d_i = (m-1)$. Assume that there is no positive integer $r < m$ together with a proper subset $\rho$ of $\{1, \ldots, r\}$ such that $\sum_{i \in \rho} d_i = (r-1)$. Also assume that there is no integral divisor $s > 1$ of $m$ together with a disjoint partition $\sigma(1) \cup \cdots \cup \sigma(s) = \{1, \ldots, \rho\}$ of $\{1, \ldots, r\}$ into pairwise disjoint nonempty subsets $\sigma(1), \ldots, \sigma(s)$ such that for $1 \leq j \leq s$ we have $\sum_{i \in \sigma(j)} d_i = \binom{s}{j}(q-1)^{s-j}(m/s - 1)$. Then $G$ acts faithfully on each of its orbits.

Namely, the first assumption implies that $\Theta_{m}^{-1}(G)$ does not map any proper subspace of $\text{GF}(q)^m$ (of positive dimension $r < m$) onto itself.\footnote{In view of this observation, by the last line of Table 5.4.A on page 199 of [LiK] which starts with $B_2^2(q)$, we see that if $m = 3$ and $\Phi < \text{PGL}(2m, q)$ is isomorphic to and has the same size orbits as $\text{P}\Omega'(2m, q)$, then $\text{P}\Omega'(2m, q) = \delta^{-1}\Phi\delta$ for some $\delta \in \text{PGL}(2m, q)$.} Therefore, regarding
\( \mathcal{P}(\text{GF}(q)^m) \) as the set of all 1-dimensional subspaces of \( \text{GF}(q)^m \), it follows that \( \Delta_1 \) spans \( \text{GF}(q)^m \). Let \( \Psi = \{ \gamma \in \Theta_m^{-1}(G) : \gamma(M) = M \text{ for all } M \in \Delta_1 \} \). Then \( \Psi \circ \Theta_m^{-1}(G) \). Recall that a maximal eigenspace of \( \Psi \) is a maximal subspace \( L \) of \( \text{GF}(q)^m \) such that for some homomorphism \( \alpha_L : \Psi \to \text{GF}(q)^* \) we have \( \gamma(z) = \alpha_L(\gamma)z \) for all \( \gamma \in \Theta_m(\Psi) \) and \( z \in L \). Since \( \Delta_1 \) spans \( \text{GF}(q)^m \), we get a direct sum decomposition \( \text{GF}(q)^m = L_1 + \cdots + L_s \) where \( L_1, \ldots, L_s \) are maximal eigenspaces of \( \Psi \). Since \( \Psi \circ \Theta_m^{-1}(G) \), it follows that \( \Theta_m^{-1}(G) \) acts transitively on this decomposition, and hence \( \dim L_i = m/s \) for \( 1 \leq i \leq s \). For \( 1 \leq j \leq s \) let \( \Lambda_j \) be the set of all \( M \in \mathcal{P}(\text{GF}(q)^m) \) such that, for every \( 0 \neq z \in M \), the cardinality of \( \{1 \leq i \leq s : \text{proj}_i(z) \neq 0 \} \) is \( j \) where \( \text{proj}_i : L_1 + \cdots + L_s \to L_i \) is the natural projection. Then the cardinality of \( \Lambda_j \) is \( (\binom{m}{j})(q-1)^{j-1}((m/s)-1)! \). Since \( \Theta_m^{-1}(G) \) acts transitively on the above decomposition, there is a disjoint partition \( \sigma(1) \cup \cdots \cup \sigma(s) = \{1, \ldots, e\} \) of \( \{1, \ldots, e\} \) such that for \( 1 \leq j \leq s \) we have \( \Lambda_j = \cup_{i \in \sigma(j)} \Delta_i \). Therefore for \( 1 \leq j \leq s \) we have \( \sum_{i \in \sigma(j)} d_i = (\binom{m}{j})(q-1)^{j-1}((m/s)-1)! \). Consequently by the second assumption we must have \( s = 1 \). Therefore \( \Psi = \text{GF}(q)^* \) and hence \( G \) acts faithfully on \( \Delta_1 \). Similarly \( G \) acts faithfully on each of its orbits.

In view of (5.8) and the previous two footnotes, we get the following corollary of (5.7):

**Corollary (5.9).** Assume that \( m > 3 \). Let \( G < \text{PGL}(2m,q) \) have 2 or 3 orbits on \( \mathcal{P}(\text{GF}(q)^{2m}) \) of sizes \( (q^m+1)(m-2), q^{m-1}(q^m+1) \) or \( (q^m+1)(m-2), q^{m-1}(q^m+1)/2 \) according as \( p = 2 \) or \( p \neq 2 \). Assume that \( G \) is Rank \( 3 \) with subdegrees \( 1, q(q^2+2e+1)(m-2-e) \) and \( q^{m-2} \) on the orbit of size \( (q^m+1)(m-2) \). Then \( \text{PGL}(2m,q) < \delta^{-1} \text{G} \delta \) for some \( \delta \in \text{PGL}(2m,q) \).

As in (5.7), let \( \text{PGL}(2m,q)_1 \) denote the permutation group induced by \( \text{PGL}(2m,q) \) on its singular points (whose cardinality is \( (q^m+1)(m-2) \)). In case of \( p = 2 \), let \( \text{PGL}(2m,q)_2 \) denote the permutation group induced by \( \text{PGL}(2m,q) \) on its nonsingular points (whose cardinality is \( q^{m-1}(q^m+1) \)). In case of \( p \neq 2 \), the permutation groups induced by \( \text{PGL}(2m,q) \) on its two nonsingular orbits (whose common cardinality is \( q^{m-1}(q^m+1)/2 \)) are easily seen to be equivalent and we denote them by \( \text{PGL}(2m,q)_2 \). Now by (5.8) we see that

\[
\text{PGL}(2m,q)_1 \cong \text{PGL}(2m,q) \cong \text{PGL}(2m,q)_2
\]

where \( \cong \) denotes isomorphism as abstract groups.

### 6. Galois Groups

By (4.15), (4.16), (5.1), (5.6) and (5.9) we get the following

**Theorem (6.1).** If \( m > 3 \) and \( \text{GF}(q) \subset k_p \), then, for \( 2 \leq e \leq m-1 \), in a natural manner, we have

\[
\Omega^-(2m,q) < \text{Gal}(\phi_e^-, k_p(T_1, \ldots, T_e)) < \text{GO}^-(2m,q)
\]

and

\[
\text{PGL}^-(2m,q) < \text{Gal}(\phi_e^-, k_p(T_1, \ldots, T_e)) < \text{PGO}^-(2m,q).
\]

Hence in particular, if \( m > 3 \) and \( \text{GF}(q) \subset k_p \), then, in a natural manner we have

\[
\Omega^-(2m,q) < \text{Gal}(\phi^-, k_p(T_1, \ldots, T_{m-1})) < \text{GO}^-(2m,q)
\]
and 

\[ P\Omega^{-}(2m, q) < \text{Gal}(f^{-}, k_{p}(T_{1}, \ldots, T_{m-1})) < P\Omega^{-}(2m, q) \].

By (3.0), (3.1), (3.4), (3.5), (4.17), (5.2), (5.3), (5.4), (5.5), (5.10) and (6.1) we get the following

**Theorem (6.2).** If \( m > 3 \leq p \) and \( k_{p} \) is algebraically closed, then, for \( 2 \leq e \leq m - 1 \), in a natural manner we have 

\[ \text{Gal}(\phi^{-}, k_{p}(T_{1}, \ldots, T_{m-1})) = \text{Gal}(\phi_{e}^{-}, k_{p}(T_{1}, \ldots, T_{e})) = \Omega^{-}(2m, q) \]

and 

\[ \text{Gal}(f^{-}, k_{p}(T_{1}, \ldots, T_{m-1})) = \text{Gal}(f_{e}^{-}, k_{p}(T_{1}, \ldots, T_{e})) = P\Omega^{-}(2m, q) \]

and 

\[ \text{Gal}(f_{e}^{-}, k_{p}(T_{1}, \ldots, T_{m-1})) = \text{Gal}(f_{e}^{-}, k_{p}(T_{1}, \ldots, T_{e})) = P\Omega^{-}(2m, q) \]

and 

\[ \text{Gal}(f^{**}, k_{p}(T_{1}, \ldots, T_{m-1})) = \text{Gal}(f^{**}_{e}, k_{p}(T_{1}, \ldots, T_{e})) = P\Omega^{-}(2m, q) \]

and 

\[ \text{Gal}(f_{e}^{**}, k_{p}(T_{1}, \ldots, T_{m-1})) = \text{Gal}(f_{e}^{**}, k_{p}(T_{1}, \ldots, T_{e})) = P\Omega^{-}(2m, q) \]

and 

\[ \text{Gal}(f_{e}^{***}, k_{p}(T_{1}, \ldots, T_{m-1})) = \text{Gal}(f_{e}^{***}, k_{p}(T_{1}, \ldots, T_{e})) = P\Omega^{-}(2m, q) \]

and 

\[ \text{Gal}(f_{e}^{***}, k_{p}(T_{1}, \ldots, T_{m-1})) = \text{Gal}(f_{e}^{***}, k_{p}(T_{1}, \ldots, T_{e})) = P\Omega^{-}(2m, q) \].

**Remark (6.3).** We shall discuss the \( m \leq 3 \) or \( p = 2 \) case elsewhere.

**References**


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