

ANALYSIS OF THE WU METRIC. I: THE CASE OF CONVEX THULLEN DOMAINS

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ABSTRACT. We present an explicit description of the Wu metric on the convex Thullen domains which turns out to be the first natural example of a purely Hermitian, non-Kählerian invariant metric. Also, we show that the Wu metric on these Thullen domains is in fact real analytic everywhere except along a lower dimensional subvariety, and is C^1 smooth overall. Finally, we show that the holomorphic curvature of the Wu metric on these Thullen domains is strictly negative where the Wu metric is real analytic, and is strictly negative everywhere in the sense of current.

1. INTRODUCTION

1.1. General. One of the main results of this article is that there exists a natural class of invariant non-Kähler Hermitian metrics.

Traditionally, invariant metrics have played an important central role in the study of both differential geometry and several complex variables. A list of the widely studied well-known examples to date consists of the Poincaré-Bergman metric, the Einstein-Kähler metric, the Carathéodory metric and the Kobayashi metric. ([4], [8], [14] to list only a few.) None of these invariant metrics belong to Hermitian geometry properly, as the first two are Kählerian, while the next two are only Finslerian. Although it is agreed in principle that Hermitian geometry is the most natural and general geometry to study in conjunction with the theory of complex analytic functions, the study of this geometry received relatively little attention partly due to the lack of natural examples of non-Kähler Hermitian metrics. We show in this paper that the metric introduced by H. Wu in his article [20] turns out to be the first natural example of non-Kähler Hermitian metrics. Throughout this article, we call this metric the *Wu metric*. A description of this metric is presented in Section 2 of this article.

The Wu metric is an upper semicontinuous invariant pseudo-Hermitian metric defined on every complex manifold. It is a complete, continuous Hermitian metric whenever the Kobayashi metric is complete. It is Hermitian in the sense that it is a Hermitian inner product on each tangent space. But, above all, this metric is special since it possesses the distance decreasing property up to a fixed constant factor for any holomorphic mapping between two complex manifolds. This distance decreasing property plays a major role in applying the Kobayashi or Carathéodory

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metric in the study of analytic function theory, and this property is not shared by the aforementioned invariant Kählerian metrics.

Several fundamental questions arise immediately. The first question is whether the Wu metric is smooth and, if it is, whether it is in fact Kählerian. Then another question is whether the Wu metric possesses negative holomorphic curvature, since it satisfies an analogue of the distance decreasing property. (See for instance [1], [17], [22], for backgrounds.)

In case the manifold under consideration happens to be homogeneous, in the sense that the group of holomorphic automorphisms acts transitively, the above questions are very easy to answer, as remarked in [20]. However, no further results were known beyond the homogeneous cases. Thus in this article, we explore the above questions on a bounded domain in \mathbb{C}^2 which is not homogeneous but possesses a noncompact automorphism group. In the light of earlier studies on the bounded domains with noncompact automorphism groups ([3], [11], [18]), the Thullen domains provide the first nontrivial case for the above questions. The Thullen domains in question are defined by

$$E_m = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^{2m} < 1\}$$

where m is a real number not smaller than $1/2$.

It is striking to see that in such a “simple” case the Wu metric already begins to develop a special behavior such as being real analytic outside an explicit singular set. Moreover it shows a well controlled strict negative holomorphic curvature behavior. As a consequence, our results in particular imply a non-Kähler invariant solution of a conjecture of Kobayashi on the Thullen domains which is different from the earlier work of K. Azukawa and M. Suzuki ([2]). We will explain our results more precisely in what follows.

1.2. Main theorems: Smoothness and curvature of the Wu metric. Our first two theorems concern the smoothness of the Wu metric.

Theorem 1. *For $1/2 \leq m < 1$, the Wu metric h_m is real analytic on $\{(z_1, z_2) \in E_m \mid z_2 \neq 0\}$. At the point $(a, 0) \in E_m$ for any a with $|a| < 1$, the Wu metric is in the Hölder class $C^{1,2m-1}$, but not in the class C^2 . Moreover, the Wu metric is Hermitian but not Kähler.*

For the case $m > 1$, we need to decompose E_m into the following four disjoint sets:

$$\begin{aligned} M^+ &= \{f(0, b) \mid f \in \text{Aut } E_m, 2^{-1/2m} < b < 1\}, \\ M^0 &= \{f(0, 2^{-1/2m}) \mid f \in \text{Aut } E_m\}, \\ M^- &= \{f(0, b) \mid f \in \text{Aut } E_m, 0 < b < 2^{-1/2m}\}, \\ Z &= \{f(0, 0) \mid f \in \text{Aut } E_m\}. \end{aligned}$$

Here, $\text{Aut } E_m$ is the real four dimensional group of biholomorphic automorphisms generated by the complex rotations and the Möbius transformations of the following form:

$$(z_1, z_2) \mapsto \left(\frac{z_1 + a}{1 + \bar{a}z_1}, \frac{(1 - |a|^2)^{1/2m}}{(1 + \bar{a}z_1)^{1/m}} z_2 \right)$$

for $|a| < 1$, by taking any branch of the m -th root.

Notice that the sets M^0 and Z are real submanifolds of codimension 1 and 2, respectively, and hence they are “thin” subsets of the domain E_m . For $m > 1$ we have

Theorem 2. *For $m > 1$, the Wu metric is real analytic everywhere except on a subset of real codimension at least one, while it is at least C^1 smooth everywhere. More precisely, when m is an integer, the Wu metric h_m is real analytic on the open set $E_m \setminus M^0$, whereas it is only C^1 smooth at the points in M^0 . If m is not an integer, h_m is real analytic on the set $M^+ \cup M^-$, while it is $C^{[2m]-1}$ smooth at the points in Z , and of class C^1 at the points in M^0 . Also, the Wu metric is Kähler on M^+ but non-Kähler on M^- .*

Explicit expressions of the Wu metric in these cases are given in Theorems 6, 7 and 9. This seems to be the first natural explicit example of an invariant Hermitian metric that is neither entirely C^∞ smooth, nor Kähler. Also, we would like to remark that the proofs of Theorems 1 and 2 do not depend on brutal calculation. The technical devices that we use to overcome the computational difficulties, *square transform* and *square convexity*, could conceivably be useful in other similar situations.

Concerning the holomorphic curvature of the Wu metric of the Thullen domains, we have the following.

Theorem 3. *For $1/2 \leq m < 1$, at every point $(z_1, z_2) \in E_m$ with $z_2 \neq 0$, the holomorphic curvature of the Wu metric is bounded from above by $-1/2$.*

A striking behavior of the holomorphic curvature is observed in case $m > 1$, which is

Theorem 4. *For every $m > 1$, the Wu metric of E_m at every point of M^+ is a Kähler metric with constant holomorphic curvature -2 . At every point of M^- , the Wu metric is a non-Kähler Hermitian metric with the holomorphic curvature bounded from above and below by negative constants.*

1.3. Negative holomorphic curvature current. A central theme of this article is the study of the following conjecture of Kobayashi.

Conjecture (Kobayashi [14]). *On every complex manifold whose Kobayashi metric is complete and proper, there exists a complete Hermitian metric with its holomorphic curvature bounded from above by a negative constant.*

It is worth mentioning that Wu ([20]) has observed that the following modified conjecture will be equally effective in application.

Conjecture. *On every complex manifold whose Kobayashi metric is complete and proper, there exists a C^k smooth (for some $k \geq 0$) complete Hermitian metric with its holomorphic curvature bounded above by a negative constant in the sense of current.*

In this context, we present

Theorem 5. *The holomorphic curvature of the Wu metric on the Thullen domain E_m is bounded above by a negative constant in the sense of current, for every $m > 1/2$.*

The derivation of this theorem from Theorems 1 through 4 is rather elementary. We choose to include the proof in an appendix at the end of this article. Notice that Theorem 5 provides a natural answer to Kobayashi's conjecture for the Thullen domains E_m with $m \geq 1/2$. It suggests that the Wu metric (or its modification) may be a natural answer to the aforementioned conjecture of Kobayashi.

It is even more striking to see that, for every real number $m > 1$, the domain E_m admits an explicitly described open neighborhood (M^+ as described above) of the strongly pseudoconvex boundary points in which the holomorphic curvature of the Wu metric is identically -2 . This result is quite impressive compared with the mere *asymptotic* constancy of the holomorphic curvature of the Bergman and Einstein-Kähler metrics near the strongly pseudoconvex boundary. (See [5], [12], [13].)

The properties and behavior of the Wu metric in the cases of non-convex Thullen domains and strongly convex smooth domains will be discussed in separate articles.

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1.4. Organization of this paper. The organization of this paper from here on is as follows. In Section 2, the construction of the Wu metric is presented, with a list of several important properties. Section 3 is about the smoothness of the Wu metric. Finally, Section 4 is about the curvature analysis of the Wu metric of Thullen domains. Along the way, we actually give explicit formulas for the Wu metric on Thullen domains.

2. FUNDAMENTALS OF THE WU METRIC

Throughout this section without exception, β_0 denotes the Poincaré-Bergman metric of the unit ball B^n in \mathbb{C}^n with the constant holomorphic curvature -4 . Also, a complex manifold M is hyperbolic (in the sense of Kobayashi), if the Kobayashi distance is a proper distance on M .

2.1. Construction of the Wu metric. Let M be a complex manifold of complex dimension n and let $x \in M$. Define the set \mathcal{F}_x by

$$\mathcal{F}_x := \{f : B^n \rightarrow M \mid f \text{ holomorphic, } f(0) = x, f_* : T_0B \rightarrow T_xM \text{ nonsingular}\}.$$

Then, for each $f \in \mathcal{F}_x$ we define a Hermitian inner product $f_!\beta_0$ on the tangent space T_xM of M by

$$f_!\beta_0(u, v) := \beta_0(f_*^{-1}u, f_*^{-1}v), \quad \forall u, v \in T_xM.$$

Now, consider the set

$$\Psi_x = \{f_!\beta_0 \mid f \in \mathcal{F}_x\}$$

for each $x \in M$, and the set

$$\Psi = \bigcup_{x \in M} \Psi_x,$$

which is of course a biholomorphic invariant of M .

Let \mathcal{Q}_x be the set of all positive semi-definite Hermitian inner products on T_xM . In \mathcal{Q}_x , define the relation \ll by

$$\alpha \ll \beta \text{ if and only if } \alpha(u, u) \leq \beta(u, u), \forall u \in T_xM.$$

Also define the relation \prec by

$$\alpha \prec \beta \text{ if and only if } \det[\alpha(u_i, u_j)] \leq \det[\beta(u_i, u_j)]$$

for any choice of a basis $\{u_1, \dots, u_n\}$ for T_xM . As discussed in [20], the comparison between the determinants is independent of the choice of basis even though the determinant itself is not. Finally, let $\ell(\Psi_x)$ be the set of lower bounds of Ψ_x with respect to the partial ordering \ll . Then, the following hold:

Lemma 1. $\ell(\Psi_x)$ is a compact subset of \mathcal{Q}_x equipped with the usual topology of the space of all sesquilinear forms.

Lemma 2. For every x , there exists a unique element h_x in $\ell(\Psi_x)$ which satisfies

$$g_x \prec h_x, \forall g_x \in \ell(\Psi_x).$$

Therefore, we arrive at

Definition 1. The *Wu metric* on a complex manifold M is defined by the assignment

$$x \mapsto h_x : M \rightarrow \mathcal{Q} = \bigcup_{x \in M} \mathcal{Q}_x$$

satisfying $h_x \in \mathcal{Q}_x$ for each $x \in M$.

2.2. Fundamental properties. The preceding construction shows that the Wu metric depends only on the complex linear structure of each tangent space. Therefore, it follows immediately that the Wu metric is an invariant metric, in the sense that every biholomorphic mapping is an isometry with respect to the Wu metrics of the domain and target complex manifolds.

A more intuitive description of the Wu metric is also available, which will play an important role in the later sections of this article. For this, note first the fact that each Hermitian metric is completely determined by its unit ball in each tangent space. Then we arrive at the following geometric description, which is also discussed in [20].

Fix $x \in M$. Endow any Hermitian inner product on T_xM , and introduce the volume form accordingly. Let k_M denote the Kobayashi-Royden infinitesimal metric on M . Then, for each positive definite Hermitian inner product α on T_xM , consider its unit ball

$$B_\alpha = \{v \in T_xM \mid \alpha(v, v) \leq 1\}$$

and the set

$$\mathcal{K}_x = \{v \in T_xM \mid k_M(x; v) \leq 1\},$$

commonly called the Kobayashi indicatrix of M at x . Denote by P_x the set of all positive definite Hermitian inner products on T_xM . Then, we have

Proposition 1 (Wu). *The Wu metric h_x on T_xM satisfies*

1. $\mathcal{K}_x \subseteq B_{h_x}$, and
2. *volume of $B_{h_x} \leq$ volume of B_g for every $g \in \mathcal{P}_x$ that satisfies $\mathcal{K}_x \subseteq B_g$, where the volume is measured by any Hermitian inner product on T_xM .*

In other words, the Wu metric is obtained pointwise by considering the complex ellipsoid of minimum “volume” centered at the origin containing the Kobayashi indicatrix in each tangent plane. The following results are given in [20] without a detailed proof. A proof is given in [21] based upon the arithmetic-geometric mean inequality. We remark that an alternative proof can be obtained by a variation of Fritz John’s work [10].

Proposition 2 (Wu). *Let M, N be Kobayashi hyperbolic complex manifolds with the Wu metrics h_M and h_N respectively. The following hold:*

1. *The Wu metric is an invariant metric, in the sense that $F^*h_N = h_M$ for any biholomorphic mapping $F : M \rightarrow N$.*
2. *The Wu metric is a continuous positive definite Hermitian metric.*
3. *The Wu metric is complete whenever the Kobayashi distance is a complete distance.*
4. *If $f : M \rightarrow N$ is a holomorphic mapping and $\dim M = n$, then $f^*h_N \leq \sqrt{n}h_M$.*

We remark that Lempert’s work [16] on convex domains shows that the Kobayashi metric is represented in each complex direction by the push-forward of the Poincaré metric through extremal maps. However, for the Wu metric, such a pointwise representation fails in general. In fact, for any holomorphic mapping $f : B^n \rightarrow M$ with $f(0) = x$ and df_0 nonsingular, we have

$$k_M(u)^2 = k_M(f_*w)^2 = f^*k_M(w)^2 \leq \beta_0(w, w) = \beta_0(f_*^{-1}u, f_*^{-1}u) = f_!\beta_0(u, u)$$

for any $u \in T_xM$ and $w \in T_0B^n$ with $f_*w = u$. This shows that the unit ball for the Hermitian inner product $f_!\beta_0$ in T_xM is always a subset of the Kobayashi indicatrix. Consequently, the Wu metric at $x \in M$ cannot coincide with $f_!\beta_0$ for any holomorphic function $f : B^n \rightarrow M$ with $f(0) = x$, unless the Kobayashi-Royden metric at x itself is already Hermitian, which seldom occurs. See, for example, ([7]) and ([16]). Note for instance that the Kobayashi-Royden metric on a convex domain is never Hermitian unless the convex domain itself is biholomorphic to the unit open ball ([16]). This already provides an infinite dimensional family of holomorphically inequivalent convex domains ([7]) on which the Kobayashi-Royden metric is not Hermitian, consequently on which the Wu metric cannot admit pointwise representation as explained above.

3. EXPLICIT DESCRIPTION AND SMOOTHNESS ON THULLEN DOMAINS

3.1. Preliminaries for the Kobayashi-Royden metric. We first introduce some basic facts about the Kobayashi-Royden metric in the Thullen domain $E_m = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^{2m} < 1\}$ where m is a real number $> 1/2$. Since the holomorphic tangent bundle of the Thullen domain E_m is trivial, we identify from now on the tangent bundle TE_m of E_m with $E_m \times \mathbb{C}^2$ and use the standard Euclidean coordinates $(z; v) = (z_1, z_2; v_1, v_2)$ throughout this article.

According to [6] and [9], we have the following explicit expression for the *square* of the Kobayashi-Royden metric at $(0, b)$ in the direction $v = (v_1, v_2)$:

$$(2) \quad k^2((0, b); v) = \frac{|v_1|^2}{1 - |b|^{2m}} + \frac{m^2 |b|^{2m-2} |v_2|^2}{(1 - |b|^{2m})^2}, \text{ for } u = m|v_2/v_1| \leq |b|,$$

$$(3) \quad k^2((0, b); v) = \frac{|v_2|^2 t^2 m^2 (1 - a^2)^2}{|b|^2 (t^2 - 1)^2 (m(1 - a^2) + a^2)^2}$$

$$(3) \quad = \frac{a^2 t^2 |v_1|^2}{(1 - t^{2m} |b|^{2m})^2}, \text{ for } u = m|v_2/v_1| > |b|,$$

where a and t are defined implicitly as follows: the term a by

$$(4) \quad a = \frac{2m|b|}{u + (u^2 + 4m(m-1)|b|^2)^{1/2}}$$

and the term t by the unique positive solution of the equation

$$(5) \quad t^2 a^2 + t^{2m} (1 - a^2) |b|^{2m} = 1.$$

Here, notice that one obtains

$$(6) \quad 0 < a < 1, \text{ and } 1 < t < 1/|b|.$$

It is relatively straightforward to confirm that the above formulae for the square of the Kobayashi metric have natural continuation to the cases such as $b = 0$ and $v_1 = 0$. See [6] for details. Furthermore, away from the zero section of the tangent bundle of the Thullen domain E_m , both expressions (2) and (3) are smooth in b , separately, as b ranges over the open unit disk in \mathbb{C} . In the tangent space at $(0, b)$, it is shown in [6] that the Kobayashi metric is C^1 smooth at the point $v = (v_1, v_2) \neq (0, 0)$ with $m|v_2| = |b||v_1|$. Since the action of the automorphism group of biholomorphisms of the Thullen domain E_m is real analytic and every point of E_m is in the orbit of the point $(0, b)$ for some b with $0 \leq b < 1$, these establish the C^1 smoothness of the Kobayashi-Royden metric off the zero section of the tangent bundle. However, no further investigation on further differentiability was attempted in [6], perhaps due to the complexity of the computation. We will show in later sections by a simpler method, which is a by-product of our methods, that at the point $v = (v_1, v_2) \neq (0, 0)$ in the tangent space $T_{(0,b)}E_m = \mathbb{C}^2$ with $m|v_2| = |b||v_1|$ for every $(0, b) \in E_m$ the Kobayashi-Royden metric is in fact C^2 but not C^3 smooth. For the proof, see Sections 3.2.2 and 4.2.2.

It follows by the explicit expression of the Kobayashi-Royden metric $k^2(v)$ of the Thullen domain above ([6]) that at each reference point $(0, b) \in E_m$ the Kobayashi metric is a function of $|v_1|$ and $|v_2|$. Therefore, the Wu metric constructed from the Kobayashi metric takes a very special form at the reference point $(0, b)$ in this coordinate system. To observe this, recall first that the Wu metric in our coordinate system above is given by the ellipsoid of the minimum Euclidean volume containing the Kobayashi indicatrix in the tangent space at each reference point in the domain E_m . We will call such an ellipsoid the *best fitting (complex) ellipsoid*. Fix $z \in E_m$ and let the unit sphere of the Wu metric in $T_z E_m$ be represented by

$$r_1 |v_1|^2 + r_2 |v_2|^2 + 2\operatorname{Re} r_3 v_1 \bar{v}_2 = 1,$$

where $r_1, r_2 > 0$ and $r_3 \in \mathbb{C}$. Notice that this ellipsoid is *not* invariant under the action of the group

$$T^2 = \{(v_1, v_2) \mapsto (e^{i\theta}v_1, e^{i\psi}v_2) \mid \theta, \psi \in \mathbb{R}\}$$

unless $r_3 = 0$. Therefore, if $r_3 \neq 0$, the Kobayashi indicatrix in $T_{(0,b)}E_m$ would then admit more than one best fitting ellipsoid, because the Kobayashi indicatrix $\mathcal{K}_{(0,b)}$ is invariant under the action by T^2 and this action is Euclidean volume preserving. This contradicts Lemma 2 on the uniqueness of the Wu metric, which is in this context the uniqueness of the best fitting ellipsoid. (See also [10].) Therefore, combining this with the basic results on the Wu metric of the preceding section, we arrive at

Proposition 3. *In terms of Euclidean coordinates on the tangent bundle $E_m \times \mathbb{C}^2 = TE_m$ of the Thullen domain E_m , for every $(0, b) \in E_m$ the unit sphere of the Wu metric in $T_{(0,b)}E_m$ is of the form*

$$r_1|v_1|^2 + r_2|v_2|^2 = 1,$$

where r_1 and r_2 are positive real-valued continuous functions of b .

At this point, it is convenient to introduce the following terminology and notation for later purposes.

Definition 2. Let $(0, b) \in E_m$. Restrict v_1, v_2 to the positive real axis. Then, let C_{low} denote the curve $k^2 = 1$ in the first quadrant of \mathbb{R}^2 , with k^2 given by (2) above, which we call the *lower K-curve*. In turn, we define the *upper K-curve* C_{up} to be the curve $k^2 = 1$ where k^2 in this case is given by (3).

Notice in particular that the expressions of the upper curve C_{up} and the lower curve C_{low} are C^∞ smooth function in the variable b , for all b with $0 < |b| < 1$. In case m is a positive integer, they extend smoothly across $b = 0$. We shall show later how this leads to the C^∞ smoothness of the Wu metric of the Thullen domains.

3.2. Square convexity of the K-curves. Throughout this section, we assume

$$(7) \quad 0 < b < 1.$$

To investigate the differentiability of the Wu metric which is now the solutions of the extremal problem with respect to finding a parameterized family of best fitting regular ellipsoids, we introduce the concept of *square convexity* as follows:

Definition 3. Let $f : (t_0, t_1) \rightarrow \mathbb{R}$ be a strictly decreasing positive real-valued function defined on the interval (t_0, t_1) with $t_0 \geq 0$. The function f is said to be *square-concave* if

$$\frac{d^2(f(\sqrt{s})^2)}{ds^2} \leq 0 \text{ for all } s \in (t_0^2, t_1^2).$$

If the second derivative is strictly negative, we call f *strictly square-concave*. Likewise, we may define the *square-convexity* and *strict square-convexity* if the second derivative above is positive.

Now we state the main result of this section. Let us denote by $v_2 = f_{up}(v_1)$ the function representing the graph of the upper K-curve C_{up} in the first quadrant of \mathbb{R}^2 , and by $v_2 = f_{low}(v_1)$ the function representing the lower K-curve, respectively. (See Definition 2 for the terminology.) Then we have

Proposition 4. f_{low} is always both square-convex and square-concave. f_{up} is strictly square-concave if $m > 1$, and strictly square-convex if $1/2 \leq m < 1$.

Proof. The first statement is obvious, since C_{low} is part of an ellipse centered at the origin. Notice that under the coordinate change $(v_1, v_2) \mapsto (v_1^2, v_2^2)$ all regular ellipses centered at the origin become straight lines in the first quadrant. This implies that

$$\frac{d^2(f_{low}(\sqrt{s})^2)}{ds^2} = 0.$$

To prove the second and third statements, we will derive a parametric equation for f_{up} in terms of t from (3), (4) and (5). Here we use the notation in 3.1. Rewrite (5) as

$$a^2 = \frac{1 - t^{2m}|b|^{2m}}{t^2 - t^{2m}|b|^{2m}}.$$

Then,

$$1 - a^2 = \frac{t^2 - 1}{t^2 - t^{2m}|b|^{2m}}.$$

To describe f_{up} from the curve $C_{up} : k^2 = 1$ with k^2 as in (3), we have

$$\frac{at|v_1|}{1 - t^{2m}|b|^{2m}} = 1$$

and

$$\frac{|v_2|tm(1 - a^2)}{|b|(t^2 - 1)(m(1 - a^2) + a^2)} = 1.$$

After substituting the expressions of a^2 and $1 - a^2$ into these equations, we arrive at the parametric equation for $v_2 = f_{up}(v_1)$ representing the upper K-curve C_{up} at $(0, b) \in E_m$ in terms of t with $b > 0$ as follows:

$$(8) \quad \begin{cases} v_1 = \frac{1}{t} \sqrt{1 - t^{2m}b^{2m}} \sqrt{t^2 - t^{2m}b^{2m}}, \\ v_2 = \frac{b}{mt} (mt^2 - m + 1 - t^{2m}b^{2m}), \quad 1 < t < 1/b. \end{cases}$$

To show the square-convexity or concavity of the function $v_2 = f_{up}(v_1)$, it suffices to check the sign of the expression

$$(9) \quad A = \frac{x'(t)y''(t) - x''(t)y'(t)}{(x'(t))^3}, \quad 1 < t < 1/b,$$

where

$$(10) \quad \begin{cases} x(t) = (v_1(t))^2 = (1 - t^{2m}b^{2m})(1 - t^{2m-2}b^{2m}), \\ y(t) = (v_2(t))^2 = \frac{b^2}{m^2t^2} (mt^2 - m + 1 - t^{2m}b^{2m})^2 \end{cases}$$

for all $m \geq 1/2$.

First we show $x'(t)$ is never zero for all $t \in (1, 1/b)$. A direct computation yields

$$(11) \quad \begin{aligned} x'(t) &= -2t^{2m-3}b^{2m}(mt^2 + m - 1 - (2m - 1)t^{2m}b^{2m}) \\ &< -2mt^{2m-3}b^{2m}(t^2 - 1) \\ &< 0. \end{aligned}$$

Now, we investigate the sign of A from calculating N , the denominator of A for $1 < t < 1/b$, explicitly. First,

$$x'(t) = 2t^{2m-3}b^{2m} \cdot E,$$

where

$$E = (2m - 1)t^{2m}b^{2m} - m + 1 - mt^2.$$

Before computing the derivatives of $y(t)$, let

$$F = mt^2 - m + 1 - t^{2m}b^{2m}.$$

Then,

$$y'(t) = -2b^2m^{-2}t^{-3} \cdot F \cdot E.$$

For the second derivative, we get

$$y''(t) = -2b^2m^{-2}t^{-4}[(tF' - 3F)E + tFE'].$$

Combining all these, we obtain

$$(12) \quad \begin{aligned} N &= x'(t)y''(t) - x''(t)y'(t) \\ &= 8b^{2m+2}t^{2m-7}m^{-1} \cdot E^2 \cdot (t^2 - 1)(m - 1). \end{aligned}$$

In fact, there are simple geometric reasons why the preceding factorization should occur. When $m = 1$, the Thullen domain E_1 is the unit ball, whose Kobayashi metric is indeed Hermitian. In such a case N vanishes, since every regular ellipsoid centered at the origin is square-convex and concave at the same time. This suggests that $m - 1$ is a factor of N . The reason why $t^2 - 1$ should be a factor is that $t^2 = 1$ corresponds to the case when $b = 1$. As $b \rightarrow 1^-$, it turns out that the whole Kobayashi indicatrix “converges” to the extension of the ellipsoid defined by equating the expression (2) with 1. Again, the lower K-curve is square-convex and concave at the same time, and so one may expect that $t^2 - 1$ should be a factor.

Now, combining (9), (10), (11) and (12), we deduce that the function f_{up} is strictly square-concave if $m > 1$, and strictly square-convex if $1/2 \leq m < 1$. Therefore, the proof of the proposition is now complete. \square

3.3. Wu metric on E_m for $1/2 \leq m < 1$. We will first compute the Wu metric tensor at the point $(0, b) \in E_m$, where $0 < b < 1$. Recall that under the coordinate transform $(x, y) \mapsto (x^2, y^2)$ all the regular ellipses in \mathbb{R}^2 centered at the origin become a straight line segment in the first quadrant. Since the upper K-curve C_{up} and the lower K-curve C_{low} form together a curve which is at least C^1 smooth ([6]), obviously this curve will stay C^1 smooth even after the coordinate transform $(x, y) \mapsto (x^2, y^2)$ of the first quadrant. Thus, Proposition 4 implies that the curves transformed from C_{up} and C_{low} , under the aforementioned coordinate transform, together form a C^1 smooth convex curve. Therefore, the straight line segment in the first quadrant which encloses the smallest area with the two axes containing this C^1 smooth convex curve in its interior has to be the line segment joining the v_2 -intercept of the transformed upper K-curve C_{up} and the v_1 -intercept of the transformed lower K-curve C_{low} . Consequently, by Proposition 3 it is easy

to see that the best fitting complex ellipsoid, which is the Wu metric sphere at $(0, b) \in E_m$, is given by

$$\begin{aligned}
 (13) \quad h_{1\bar{1}}(0, b) &= \frac{1}{1 - b^{2m}}, \\
 h_{1\bar{2}}(0, b) &= \bar{h}_{2\bar{1}}(0, b) = 0, \\
 h_{2\bar{2}}(0, b) &= \frac{1}{(1 - b^2)^2}.
 \end{aligned}$$

Now, using the invariance of the Wu metric under the action of the automorphisms of E_m such as

$$(14) \quad (z_1, z_2) \rightarrow \left(\frac{z_1 - p_1}{1 - \bar{p}_1 z_1}, \frac{|p_2|}{p_2} \cdot \frac{(1 - |p_1|^2)^{1/2m} z_2}{(1 - \bar{p}_1 z_1)^{1/m}} \right),$$

which maps (p_1, p_2) with $p_2 \neq 0$ to $(0, |p_2|(1 - |p_1|^2)^{-1/2m})$, and using the continuity of the Wu metric (Proposition 2), we arrive at the following explicit formula for the Wu metric tensor.

Theorem 6. *The Wu metric of the Thullen domain E_m with $1/2 \leq m < 1$ in the coordinates $(p_1, p_2; v_1, v_2)$ for $E_m \times \mathbb{C}^2 = TE_m$ is given by*

$$\begin{aligned}
 (15) \quad h_m &= \left(\frac{1}{(1 - |p_1|^2)(1 - |p_1|^2 - |p_2|^{2m})} + \frac{(1 - |p_1|^2)^{-2+1/m} |p_1|^2 |p_2|^2}{m^2((1 - |p_1|^2)^{1/m} - |p_2|^2)^2} \right) dp_1 \otimes d\bar{p}_1 \\
 &+ \frac{(1 - |p_1|^2)^{-1+1/m} \bar{p}_1 p_2}{m((1 - |p_1|^2)^{1/m} - |p_2|^2)^2} dp_1 \otimes d\bar{p}_2 \\
 &+ \frac{(1 - |p_1|^2)^{-1+1/m} p_1 \bar{p}_2}{m((1 - |p_1|^2)^{1/m} - |p_2|^2)^2} dp_2 \otimes d\bar{p}_1 \\
 &+ \frac{(1 - |p_1|^2)^{1/m}}{((1 - |p_1|^2)^{1/m} - |p_2|^2)^2} dp_2 \otimes d\bar{p}_2
 \end{aligned}$$

From the explicit description of the Wu metric in the above, it is easy to see that the Wu metric is real analytic except when $p_2 = 0$. In particular, the proof of Theorem 1 follows immediately. It is also simple to observe that the Wu metric is not C^2 smooth at the points $(p_1, 0) \in E_m$ for any p_1 with $|p_1| < 1$, although it is of Hölder class $C^{1,2m-1}$. Moreover, a direct computation shows that the Wu metric given above is not Kähler. This fact is also observed in the analysis of the curvature tensor in Section 4 of this article.

3.4. Wu metric of E_m for $m > 1$. For the description of the Wu metric tensor in this case, we need to decompose the Thullen domain E_m into the four sets M^+, M^-, M^0 and Z as described in Section 1.2. Notice that E_m is the disjoint union of these four sets, and that M^0 and Z are low dimensional submanifolds of codimension 1 and 2 respectively, while M^+ and M^- are open subsets. Thus, we will first give an explicit description of the Wu metric and its smoothness on M^+ and M^- . Then, we will discuss how the Wu metric extends across the thin sets M^0 and Z .

3.4.1. *Wu metric at $(p_1, p_2) \in M^+$.* Keeping in mind the definition of M^+ in the above, we will first concentrate on the Wu metric at $(0, b) \in E_m$ with $2^{-1/2m} < b < 1$ and its smooth dependence on b . In fact, we are going to show that the unit ball of the Wu metric in $T_{(0,b)}E_m$ in this case is given by the extension of the lower K-curve defined by

$$\frac{|v_1|^2}{1-b^{2m}} + \frac{m^2 b^{2m-2} |v_2|^2}{(1-b^{2m})^2} = 1.$$

Recall that the Kobayashi-Royden metric in each tangent space of E_m is given in a two piece formula, as in (2) and (3) of Section 3.1. In order to obtain an explicit formula for the Wu metric, we first compute the “*joining point*” of the upper and lower K-curves C_{up} and C_{low} (see Definition 2 in Section 3.1), which is the point $v = (v_1, v_2)$ satisfying

$$k(v) = 1 \quad \text{and} \quad m|v_2| = b|v_1|.$$

A direct calculation shows that the joining point in the first quadrant of \mathbb{R}^2 is given by $(1-b^{2m}, b(1-b^{2m})/m)$. Now consider the following affine rescaling in the tangent space $\mathbb{C}^2 = T_{(0,b)}E_m$, defined by

$$(16) \quad (v_1, v_2) \mapsto \left(\frac{v_1}{\sqrt{1-b^{2m}}}, \frac{mb^{m-1}v_2}{1-b^{2m}} \right).$$

Under this rescaling, the lower K-curve transforms to part of the unit circle centered at the origin. Notice that the minimal ellipsoid stays minimal through affine rescaling. When we pass to the coordinate transform $(v_1, v_2) \mapsto (v_1^2, v_2^2)$ of the first quadrant of \mathbb{R}^2 , the whole K-curve (upper and lower curves joined at the joining point) becomes a graph of a concave function defined on part of the positive v_1 -axis bounding a convex region together with the coordinate axes. More importantly, the lower K-curve now becomes a straight line segment of slope -1 . Furthermore, the joining point between upper and lower curves now transforms to the point $((1-b^{2m})^{1/4}, \sqrt{b^m})$. Since $1/2 < b^{2m} < 1$ in our case, it is obvious that the best fitting ellipsoid corresponding to the unit sphere of the Wu metric is now the straight line segment of slope -1 which is the extension of the low K-curve transformed by the aforementioned coordinate change $(v_1, v_2) \mapsto (v_1^2, v_2^2)$ in the first quadrant of \mathbb{R}^2 . Thus we obtain the following key lemma:

Lemma 3. *The minimum ellipsoid which gives the Wu metric is given by the extension of the lower K-curve if $b \geq 2^{-1/2m}$.*

Consequently, at $(0, b) \in M^+$ with $b > 2^{-1/2m}$, the Wu metric is given by

$$(17) \quad \begin{aligned} g_{1\bar{1}}(0, b) &= \frac{1}{1-b^{2m}}, \\ g_{1\bar{2}}(0, b) &= 0, \\ g_{2\bar{1}}(0, b) &= 0, \\ g_{2\bar{2}}(0, b) &= \frac{m^2 b^{2m-2}}{(1-b^{2m})^2}. \end{aligned}$$

Due to the real analytic action of $\text{Aut } E_m$ on M^+ , this already proves that the Wu metric is real analytic at every point $p = (p_1, p_2) \in M^+$. However, for the

sake of completeness and for later purposes, we present in the following an explicit expression of the Wu metric tensor using the automorphism

$$f : (z_1, z_2) \mapsto \left(\frac{z_1 - p_1}{1 - \bar{p}_1 z_1}, \frac{|p_2|}{p_2} \cdot \frac{(1 - |p_1|^2)^{1/2m} z_2}{(1 - \bar{p}_1 z_1)^{1/m}} \right),$$

which maps (p_1, p_2) with $p_2 \neq 0$ to $(0, |p_2|(1 - |p_1|^2)^{-1/2m})$.

Theorem 7. *The Wu metric tensor of the Thullen domain E_m for $m > 1$ at $(p_1, p_2) \in M^+$ is given by*

$$\sum_{\alpha, \beta=1}^2 g_{\alpha\bar{\beta}}(p_1, p_2) dp_1 \otimes \overline{dp_2},$$

where

$$\begin{aligned} g_{1\bar{1}}(p_1, p_2) &= \frac{1 - |p_2|^{2m}}{(1 - |p_1|^2 - |p_2|^{2m})^2}, \\ g_{1\bar{2}}(p_1, p_2) &= \frac{m|p_2|^{2m-2}\bar{p}_1 p_2}{(1 - |p_1|^2 - |p_2|^{2m})^2}, \\ g_{2\bar{1}}(p_1, p_2) &= \overline{g_{1\bar{2}}(p_1, p_2)}, \\ g_{2\bar{2}}(p_1, p_2) &= \frac{m^2|p_2|^{2m-2}(1 - |p_1|^2)}{(1 - |p_1|^2 - |p_2|^{2m})^2}. \end{aligned}$$

Later in Section 4.3.1, we will show that the holomorphic curvature of the above tensor is in fact the constant -2 .

3.4.2. Wu metric at $(p_1, p_2) \in M^-$. We are now interested in the smooth dependence of the Wu metric tensor at $(0, b)$ on the real variable b when $0 < b < 2^{-1/2m}$. In this case, it is also possible to obtain an explicit expression for the Wu metric, which we give at the end of this section. However, we will take a more general approach based upon the study of extremal problems by Fritz John in [10]. We take such an approach here because this method can also be applied to the thin sets M^0 and Z .

Since the volume of the unit ball of the Wu metric in Proposition 3 is $\pi/(2r_1 r_2)$, we may reformulate the regularity problem of the Wu metric of a Thullen domain E_m as follows:

- (*) Investigate the smooth dependence on the real variable b of the minimum point of the function

$$F(r_1, r_2) = -r_1 r_2 \text{ for } r_1, r_2 > 0$$

under the constraint

$$G_b(r_1, r_2; v_1, v_2) = \{k_{(0,b)}(v_1, v_2)\}^2 - r_1|v_1|^2 - r_2|v_2|^2 \geq 0, \quad \forall v \in S_b,$$

where $k_{(0,b)}(v_1, v_2)$ denotes the Kobayashi-Royden metric of the Thullen domain E_m evaluated at $(0, b; v_1, v_2) \in TE_m = E_m \times \mathbb{C}^2$, and where S_b is the boundary of a certain compact neighborhood of the origin in \mathbb{C}^2 .

In fact, the constraint in the above should be given for all $v \in \mathbb{C}^2$, but the homogeneity of the function G_b in v enables us to replace \mathbb{C}^2 in the constraint equation by a compact set such as S_b . Incidentally, note also that due to the same reasoning, we have tremendous freedom in choosing an appropriate S_b because the minimum point of the problem (*) above stays invariant even if we change S_b for every b .

In the following, we will follow Fritz John's method ([10]), with a subtle difference that our choice for S_b will also vary with b . This becomes important in the later analysis of the regularity of $r(b)$.

Suppose that $r = (r_1, r_2)$ is a solution of the extremal problem (*) above. Note that by (2) and (3) of Section 3.1, the functions $k_{(0,b)}(v_1, v_2)$ and $G_b(r_1, r_2; v_1, v_2)$ depend only on $|v_1|$ and $|v_2|$. Hence, instead of the problem (*) above, we may consider the following simplified but equivalent problem:

(**) Investigate the regularity of the \mathbb{R}^2 -valued function $r = r(b)$ which yields the minimum of the function

$$F(r_1, r_2) = -r_1 r_2 \text{ for } r_1, r_2 > 0$$

under the constraint

$$G_b(r, v) = K(b, v) - (r_1 v_1^2 + r_2 v_2^2) \geq 0,$$

for all $v = (v_1, v_2) \in S = S_b \subset \mathbb{R}^2$, where $S = S_b$ is a simple closed curve (depending upon the parameter b in general) in \mathbb{R}^2 containing the origin inside, and where

$$K(b, v) = K(b, |v|) = k_{(0,b)}(v)^2, \text{ for } v = (v_1, v_2) \in \mathbb{C}^2.$$

The most crucial step in Fritz John's analysis of the necessary conditions on the above extremal problem is the following:

Lemma 4. *Let \hat{r} be a solution of (**) above, and let $S' = \{v \in S_b \mid G_b(\hat{r}, v) = 0\}$. Then, the intersection of the two sets*

$$\left\{ (u_1, u_2) \in \mathbb{R}^2 \mid \frac{\partial F}{\partial r_1}(\hat{r})u_1 + \frac{\partial F}{\partial r_2}(\hat{r})u_2 < 0 \right\},$$

and

$$\left\{ (u_1, u_2) \in \mathbb{R}^2 \mid \frac{\partial G_b}{\partial r_1}(\hat{r}, v)u_1 + \frac{\partial G_b}{\partial r_2}(\hat{r}, v)u_2 > 0, \forall v \in S' \right\}$$

is empty.

While referring to [10] for a precise proof of this lemma, we give a rough but more intuitive recapitulation of the actual proof. In the statement of the lemma above, the vectors in the first set represent the directions along which the directional derivative is negative. Thus they represent the directions along which the value of the function F is decreasing. On the other hand, those in the second set represent the directions along which the value of G is increasing. Hence, any common element of the two sets above would represent a direction which will further decrease the

value of F while satisfying the condition $G > 0$. Since this obviously cannot happen to the solution of the minimum problem (**), the conclusion of the lemma follows.

Now, if one interprets the conclusion of Lemma 4 above again, it simply says that the (possibly infinitely many) gradient vectors $-\nabla_r F(\hat{r})$ and $\nabla_r G_b(\hat{r}, v)$ for all $v \in S'$ cannot be contained in a single open half plane in \mathbb{R}^2 . Thus, the conclusion of Lemma 4 is equivalent to there being $\hat{v}, \hat{w} \in S'$ and constants $\lambda_0 \geq 0, \lambda_1 > 0, \lambda_2 > 0$ such that $\lambda_0 \nabla_r F(\hat{r}) - \lambda_1 \nabla_r G_b(\hat{r}, \hat{v}) - \lambda_2 \nabla_r G_b(\hat{r}, \hat{w}) = 0$. Therefore, we arrive at the following special case of a theorem of F. John:

Theorem 8 (F. John). *Let $(0, b) \in E_m$ and let $\hat{r} = \hat{r}(b)$ be a solution of the extremal problem (**) above. Then, there exist points $\hat{v}, \hat{w} \in S_b$ and real numbers $\lambda_0 \geq 0, \lambda_1 > 0, \lambda_2 > 0$ satisfying the following two conditions:*

1. $G_b(\hat{r}, \hat{v}) = 0, G_b(\hat{r}, \hat{w}) = 0$.
2. *The function $\Phi_b(r) = \lambda_0 F(r) - \lambda_1 G_b(r, \hat{v}) - \lambda_2 G_b(r, \hat{w})$ has a critical point at \hat{r} .*

Now, we apply this theorem to the Thullen domain E_m with $m > 1$ at the reference point $(0, b)$ with $0 < b < 2^{-1/2m}$. Recalling the preceding discussions on the (strong) square-concavity of the Kobayashi curves, we obtain the following key lemma, which is complementary to Lemma 3:

Lemma 5. *For every $b \in \mathbb{C}$ with $|b| < 2^{-1/2m}$, the unit sphere of the Wu metric in $T_{(0,b)}E_m$ does not meet the lower curve of the Kobayashi indicatrix. Moreover, the contact point between the Wu ellipse and the upper K-curve C_{up} in the first quadrant of \mathbb{R}^2 is unique and is not on the v_2 -axis.*

Proof. We use the square-concavity of the K-curves here. After the affine rescaling as in the paragraph preceding Lemma 3 followed by the change of variables to squares as before, the first claim reduces to proving the following simple statement:

In the first quadrant of the xy -plane, consider the continuous function defined by

$$g(x) = \begin{cases} g_1(x) & \text{if } 0 < x < 1 - e_1, \\ -x + 1 & \text{if } 1 - e_1 < x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

where e_1 is a constant such that $0 < e_1 < 1/2$. Assume also that g is C^2 smooth and $g_1''(x) < 0$ for all $0 < x < 1 - e_1$. Then the smallest possible area of the triangle bounded by the axes and a straight line which stays above the graph of $y = g(x)$ is smaller than $1/2$.

In fact, a triangular area smaller than $1/2$ containing the area under the graph of $y = g(x)$ is easily attained by a straight line passing through the point $(1 - g(1/2), g(1/2))$ with slope slightly larger than -1 , since $g(1/2) < 1/2$. This shows that the minimum triangular area enclosed by a straight line segment together with the coordinate axes, which corresponds to the volume of the best fitting ellipsoid, is achieved by a straight line different from the one obtained by extending the straight line corresponding to the lower K-curve. We now use the fact that the whole curve corresponding to the Kobayashi indicatrix is C^1 smooth and square-concave (Proposition 4) to conclude that the line segment minimizing the triangular area must in fact be part of the tangent line to the upper K-curve at a certain point \hat{v} . The strong square-concavity of the upper K-curve then implies that \hat{v} is unique and does not lie on the v_2 -axis. \square

Now we are ready to prove the real analytic dependence of the Wu metric in the variable b in the range $0 < b < 2^{-1/2m}$.

First we consider the two conditions in the conclusion of Theorem 8 in our special setting. Let us denote by $\hat{r} = \hat{r}(b)$ the solution of extremal problem (**). Then, we set

$$S_b = \{(v_1, v_2) \in \mathbb{R}^2 \mid K(b, v) = 1\}.$$

Suppose that $\hat{v} = (\hat{v}_1, \hat{v}_2)$ and $\hat{w} = (\hat{w}_1, \hat{w}_2)$ are points satisfying the conditions in Theorem 8; then by Lemma 5 and the symmetry of $G_b(r, v)$ we have $\hat{v}_1^2 = \hat{w}_1^2$ and $\hat{v}_2^2 = \hat{w}_2^2$. By condition (2) of Theorem 8, this implies that

$$(18) \quad \hat{\lambda} \hat{r}_2 = \hat{v}_1^2,$$

$$(19) \quad \hat{\lambda} \hat{r}_1 = \hat{v}_2^2,$$

where $\hat{\lambda} > 0$. (Although Theorem 8 only guarantees that $\hat{\lambda} \geq 0$, it is easy to see that $\hat{\lambda}$ cannot be zero by Lemma 5.) Also by Lemma 5, we see that at the point \hat{v} the Kobayashi indicatrix is real analytic and the tangent line of the upper K-curve coincides with the tangent line of the Wu ellipse. Thus, if we denote

$$(20) \quad E(r, v) = r_1 v_1^2 + r_2 v_2^2 - 1,$$

we have

$$(21) \quad E(\hat{r}, \hat{v}) = 0,$$

$$(22) \quad K(b, \hat{v}) = 1,$$

$$(23) \quad \hat{\mu} \frac{\partial K}{\partial v_j}(b, \hat{v}) = \frac{\partial E}{\partial v_j}(\hat{r}, \hat{v}) \quad (j = 1, 2),$$

where $\hat{\mu} > 0$.

Note that (18), (19) and (21) imply

$$\hat{r}_1 \hat{v}_1^2 = \hat{r}_2 \hat{v}_2^2 = \frac{1}{2}.$$

In turn, (23) is equivalent to the pair of equations

$$\hat{\mu} \hat{v}_j K_{v_j}(b, \hat{v}) = 2 \hat{r}_j \hat{v}_j^2 = 1 \quad (j = 1, 2),$$

where the variable subscript v_j represents the partial derivative with respect to v_j . In summary, we now arrive at

$$\begin{cases} 2\hat{r}_1 \hat{v}_1^2 - 1 = 0, \\ 2\hat{r}_2 \hat{v}_2^2 - 1 = 0, \\ \hat{\mu} \hat{v}_1 K_{v_1}(b, \hat{v}) - 1 = 0, \\ \hat{\mu} \hat{v}_2 K_{v_2}(b, \hat{v}) - 1 = 0, \\ K(b, \hat{v}) - 1 = 0. \end{cases}$$

Let $V = \{b \in \mathbb{R} \mid 0 < b < 2^{-1/2m}\}$ and define the mapping $H : V \times (0, \infty)^5 \rightarrow \mathbb{R}^5$ by

$$(24) \quad H(b, r_1, r_2, v_1, v_2, \mu) = (2r_1v_1^2 - 1, 2r_2v_2^2 - 1, \mu v_1K_{v_1} - 1, \mu v_2K_{v_2} - 1, K - 1).$$

Notice that H is real analytic near the point $(b, \hat{r}, \hat{v}, \hat{\mu})$. Now, a direct computation shows that the Jacobian minor of H with respect to the variables r_1, r_2, v_1, v_2, μ of the mapping is nonzero at $(\hat{r}_1, \hat{r}_2, \hat{v}_1, \hat{v}_2, \hat{\mu})$ if and only if

$$(25) \quad \det \left\{ \begin{pmatrix} \hat{v}_1^2 K_{v_1 v_1} & \hat{v}_1 \hat{v}_2 K_{v_1 v_2} \\ \hat{v}_1 \hat{v}_2 K_{v_2 v_1} & \hat{v}_2^2 K_{v_2 v_2} \end{pmatrix} + \begin{pmatrix} \hat{v}_1 K_{v_1} & 0 \\ 0 & \hat{v}_2 K_{v_2} \end{pmatrix} \right\} \neq 0$$

at the point $(b, \hat{r}, \hat{v}, \hat{\mu}) \in V \times (0, \infty)^5$. Using the expression (3) in Section 3.1, we can conclude that the Kobayashi metric is in fact strongly convex and $K_{v_j} > 0$ for $j = 1, 2$. This, together with the fact that \hat{v}_1 and \hat{v}_2 are non-zero (by Lemma 5), shows us that (25) indeed holds. Since (***) implies that $H(b, \hat{r}, \hat{v}, \hat{\mu}) = 0$, by the Implicit Function Theorem, locally there exists a uniquely determined triple of real analytic functions $\hat{r}(b), \hat{v}(b)$ and $\hat{\mu}(b)$ such that $H(b, \hat{r}(b), \hat{v}(b), \hat{\mu}(b)) = 0$. In particular, $\hat{r} = \hat{r}(b)$ is real analytic for $0 < b < 2^{-1/2m}$. This proves the real analyticity of the Wu metric along the line $L = \{(0, b) \mid 0 < b < 2^{-1/2m}\}$.

Now, recalling the definition of the open region M^- which admits a real analytic action of $\text{Aut } E_m$ whose orbits are transverse to the line L , and citing the invariance of the Wu metric under the action of automorphisms of E_m , we can immediately conclude that the Wu metric tensor is real analytic on M^- . This concludes the discussion on the smoothness of the Wu metric on M^- .

As mentioned in the beginning of the section, we now present an explicit description of the Wu metric tensor of E_m at the points of M^- exploiting Lemma 5 in Section 3.4.2 together with Proposition 4 of Section 3.2.

Theorem 9. *The Wu metric tensor of the Thullen domain E_m for $m > 1$ at the point $(0, b)$ with $0 < b < 2^{-1/2m}$ is given by*

$$(26) \quad \frac{t^2}{2t^{2m}b^{2m}((-m+1) + mt^2 - t^{2m}b^{2m})} dz^1 \otimes \overline{dz^1} + \frac{m^2 t^2}{2b^2((-m+1) + mt^2 - t^{2m}b^{2m})^2} dz^2 \otimes \overline{dz^2},$$

where t is the unique solution to the equation

$$(27) \quad t^2 + (m-2)t^{2m}b^{2m} - (m+1)t^{2m+2}b^{2m} + 2t^{4m}b^{4m} = 0$$

satisfying $1 < t < 1/b$.

Proof. Fix a point $(0, b)$, with $0 < b < 2^{-1/2m}$. By Lemma 5, the ellipse in \mathbb{R}^2 , which corresponds to the unit sphere of the Wu metric, is tangent to the upper K-curve at exactly one point in the first quadrant of \mathbb{R}^2 . Under the change of coordinates $(v_1, v_2) \mapsto (v_1^2, v_2^2)$ of \mathbb{R}^2 , this ellipse becomes a straight line, which we call ℓ momentarily, as observed earlier. This line ℓ is tangential to the transformation of the upper K-curve whose parametric equation $(x(t), y(t))$ is given in (10) of Section 3.2, and of course the line ℓ and the curve $(x(t), y(t))$ share no other intersection

than the common tangent point, which we denote by $(x(t_1), y(t_1))$. Then, the v_1 - and v_2 -intercepts of the line ℓ are

$$\frac{-y(t_1)x'(t_1) + y'(t_1)x(t_1)}{y'(t_1)}$$

and

$$\frac{y(t_1)x'(t_1) - y'(t_1)x(t_1)}{x'(t_1)},$$

respectively. Since the line ℓ together with the coordinate axes encloses the smallest triangular area in the first quadrant when compared to other tangent lines of $(x(t), y(t))$, by considering the derivative of the area functional we can conclude that $t = t_1$ satisfies the following equation:

$$(28) \quad [y(t)x'(t) - y'(t)x(t)][y(t)x'(t) + y'(t)x(t)][x'(t)y''(t) - x''(t)y'(t)] = 0.$$

In Section 3.2, we have shown that

$$x'(t) < 0, \quad y'(t) > 0, \quad y(t)x'(t) - y'(t)x(t) < 0,$$

and

$$x'(t)y''(t) - x''(t)y'(t) > 0$$

for all $1 < t < 1/b$. Therefore, the condition for t_1 now becomes

$$y(t_1)x'(t_1) + y'(t_1)x(t_1) = 0,$$

or equivalently, using the parameterization (10),

$$t_1^2 + (m-2)t_1^{2m}b^{2m} - (m+1)t_1^{2m+2}b^{2m} + 2t_1^{4m}b^{4m} = 0.$$

The intersections of L with the coordinate axes in the first quadrant can now be expressed in terms of t_1 and b , and the coefficients of the Wu metric tensor are then easily computed as asserted. \square

Again, we remark that the smoothness of the Wu metric tensor on M^- can also be obtained by applying the Implicit Function Theorem to the conclusion of Theorem 9 above. Such a proof could be slightly simpler than using the functional H , but it is not helpful when it comes to the investigation of the smoothness of the Wu metric on the thin sets M^0 and Z .

3.4.3. Wu metric at $(p_1, p_2) \in Z$. First, notice that the K-curve in this case is solely given by a natural extension of the formula (3) in Section 3.1, since the lower K-curve represented by (2) shrinks to a single point. Furthermore, at $(0, 0)$ the formula (3) yields that the set $k(v) = 1$ is represented by the equation

$$(29) \quad v_1^2 + v_2^{2m} = 1$$

in the first quadrant of \mathbb{R}^2 . Notice that this curve is clearly square-concave, and is strictly square-concave away from the v_2 -axis. Following [6], one can deduce that the lower K-curve at $(0, b) \in E_m$ with $-2^{-1/2m} < b < 2^{-1/2m}$ depends on the real

variable b , is real analytic if $m > 1$ is an integer, and $C^{[2m]}$ smooth if $m > 1$ is not an integer. Here, $[2m]$ represents the largest integer which is not larger than $2m$. Also, it is easy to verify that the contact point between the K-curve described by (29) and the ellipse representing the Wu metric sphere is unique and stays away from the v_2 -axis. By the continuity of the Wu metric (Proposition 2), and the smoothness (at least C^2) of the upper K-curve there exists a constant $\epsilon > 0$ such that the contact point between the K-curves and the Wu ellipse is unique and is on the upper K-curve at which the upper K-curve is strictly square-concave at every reference point $(0, b)$ with $-\epsilon < b < \epsilon$.

Therefore, we may now apply the Implicit Function Theorem to the function H defined in (24), to obtain

Proposition 6. *The Wu metric tensor of E_m with $m > 1$ is real analytic if m is an integer, and $C^{[2m]-1}$ smooth otherwise, at every point $(z, 0) \in E_m$ with $|z| < 1$.*

Proof. The proof of appropriate smoothness of the Wu metric tensor on the line $\{(0, b) \mid -\epsilon < b < \epsilon\}$ is given above. Notice that the proof and the conclusion are the same at points on the disk $U = \{(0, b) \mid b \in \mathbb{C}, |b| < \epsilon\}$. Now the assertion of the proposition follows from the real analytic action by $\text{Aut } E_m$ across U along Z , similarly to the preceding cases. \square

3.4.4. *Wu metric at $(p_1, p_2) \in M^0$.* The primary goal of this section is to prove that the Wu metric is C^1 smooth at the point $(0, 2^{-1/2m})$, which together with the preceding three sections will finally conclude the proof of Theorem 2.

We begin by showing that the Kobayashi indicatrix at every point is indeed C^2 smooth in each tangent plane. The C^1 smoothness is shown in [6], but no investigation on further differentiability was given, mainly due to the complexity of their computation. Here, we present a thorough conclusion.

Proposition 7. *Let $m > 1$ and let $(0, b) \in E_m$ be fixed, for some $b > 0$. Then the Kobayashi indicatrix is globally C^2 smooth, but not C^3 smooth.*

Proof. The proof follows from a simple implicit differentiation, using our parametric equation (10) introduced earlier. To avoid stating trivialities, we will simply sketch the proof here. First, write as $v_2 = f_{up}(v_1)$ and $v_2 = f_{low}(v_1)$ respectively the equations for the upper and lower K-curves in \mathbb{R}^2 for the Kobayashi indicatrix in the tangent space $\mathbb{C}^2 = T_{(0,b)}E_m$ as before. Again it is enough to check the smoothness of the whole K-curve (upper and lower curves combined) at the “joining point.” (See the paragraph preceding Lemma 3 in Section 3.4.1 for this terminology.) Now consider the parametric equation (10) considered earlier in the context of the square-concavity of the Kobayashi curves. It is elementary to see that the following statement is valid:

When evaluated at the joining point of the upper and the lower K-curves, we have $d^\ell f_{up}/dv_1^\ell = d^\ell f_{low}/dv_1^\ell$ for $\ell = 1, \dots, p$ if and only if $(d/dx_1)^\ell x_2$ for the upper K-curve coincides with that for the lower K-curve for every $\ell = 1, \dots, p$.

Notice that the value of $d^\ell x_2/dx_1^\ell$ for the lower K-curve is identically zero for every $\ell \geq 2$. But it is easy to check that for the upper K-curve, the value of $d^\ell x_2/dx_1^\ell$ is zero for $\ell = 2$ while nonzero for $\ell = 3$. Therefore, the Kobayashi indicatrix at the joining point between the upper and lower pieces is C^2 smooth but not C^3 smooth. \square

Now we are ready to establish

Proposition 8. *The Wu metric for the Thullen domain E_m with $m > 1$ is C^1 smooth at each point $z \in M^0 = \{f(0, 2^{-1/2m}) \mid f \in \text{Aut } E_m\}$.*

Proof. This proposition follows again from the arguments above involving the Implicit Function Theorem on Fritz John's method with a few adjustments: First, the contact points between the K-curve and the best fitting ellipse may no longer be unique. But, the set of contact points is always connected due to the square-concavity of the K-curve. By a careful application of the Intermediate Value Theorem on the gradients of the volume function and the constraints, one can choose a new contact point so that the same functional H defined in (24) for the Implicit Function Theorem arguments can be used as before. Notice however that the functional involves the first derivative of the K-curve. Therefore, the Implicit Function Theorem yields the C^1 smoothness of the Wu metric at the points in M^0 . \square

We remark that the choice of the special contact points in the proof can also be made to be a C^1 smooth function of the reference points, as a consequence of the Implicit Function Theorem. More importantly, we remark that in general C^2 smoothness of the Wu metric at points of M^0 cannot be obtained. As an application of our analysis of the holomorphic curvature of the Wu metric in the following section, we will explicitly show in Appendix A at the end of the paper that C^2 smoothness of the Wu metric is unattainable across M^0 in E_2 .

4. HOLOMORPHIC CURVATURE OF THE WU METRIC

4.1. Notation and generalities. Let g be a C^2 Hermitian metric on a complex manifold M with $g = \sum g_{i\bar{j}} dz^i d\bar{z}^j$ in local coordinates. Then the coefficients of the curvature tensor are given by :

$$R_{i\bar{j}k\bar{\ell}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^\ell} + \sum g^{\alpha\bar{\beta}} \frac{\partial g_{i\bar{\beta}}}{\partial z^k} \frac{\partial g_{\alpha\bar{j}}}{\partial \bar{z}^\ell}.$$

The holomorphic curvature at a point p in the direction $(\zeta^1, \zeta^2, \dots, \zeta^n)$ is

$$\frac{\sum R_{i\bar{j}k\bar{\ell}}(p) \zeta^i \bar{\zeta}^j \zeta^k \bar{\zeta}^\ell}{\sum g_{i\bar{j}}(p) g_{k\bar{\ell}}(p) \zeta^i \bar{\zeta}^j \zeta^k \bar{\zeta}^\ell}.$$

The holomorphic curvature is said to be bounded from above by a negative constant c , if

$$\sum R_{i\bar{j}k\bar{\ell}}(p) \zeta^i \bar{\zeta}^j \zeta^k \bar{\zeta}^\ell < c \sum g_{i\bar{j}}(p) g_{k\bar{\ell}}(p) \zeta^i \bar{\zeta}^j \zeta^k \bar{\zeta}^\ell$$

for every vector $(\zeta^1, \zeta^2, \dots, \zeta^n)$. (See [15] for instance.) If g is only continuous, the above definition can still have a well-defined meaning as explained in [19] and [20]: the holomorphic curvature of g is *bounded from above by a negative constant c in the sense of current*, if for every embedded Riemann surface $S \subset M$ the inequality $\partial\bar{\partial} \log G \geq -(c/2)Gdz \wedge d\bar{z}$ holds in the sense of current, where $g|_S = G dz d\bar{z}$. If g is C^2 , the two notions of negativity introduced here obviously coincide.

It is not known in general what bounded domains admit an invariant Hermitian metric whose holomorphic curvature is negative everywhere. Partial answers are known for the Einstein-Kähler metric ([5]) and for the Bergman metric ([2]) of a

Thullen domain E_m with $m > 1$. In this section, we shall prove Theorems 3 and 4, which show comprehensively the behavior of the holomorphic curvature of the Wu metric of the Thullen domains E_m for all real numbers $m \geq 1/2$. Again, the case $m = 1$ is excluded since the Thullen domain E_1 is the unit ball in \mathbb{C}^2 and the Wu metric then coincides with the normalized Poincaré-Bergman metric. Now we proceed with a detailed analysis of the holomorphic curvature of the Wu metric.

4.2. Holomorphic curvature on E_m with $1/2 < m < 1$. Due to the isometric action by the automorphism group for the domain E_m as pointed out in the preceding sections, it suffices to investigate the properties of the holomorphic curvature tensor at the point $(0, b)$, with $0 < b < 1$. Using the explicit formula of the Wu metric for $E_m, 1/2 \leq m < 1$, given in Theorem 6 in Section 3.3, one can easily check that:

$$\begin{aligned} h_{1\bar{1}}(0, b) &= \frac{1}{1 - b^{2m}}, \\ h_{1\bar{2}}(0, b) &= h_{2\bar{1}}(0, b) = 0, \\ h_{2\bar{2}}(0, b) &= \frac{1}{(1 - b^2)^2}, \end{aligned}$$

and

$$\begin{aligned} R_{1\bar{1}1\bar{1}}(0, b) &= \frac{-1}{(1 - b^{2m})^2} + \frac{-b^2}{(1 - b^2)^2 m^2} - \frac{1}{1 - b^{2m}}, \\ R_{1\bar{1}2\bar{2}}(0, b) &= \frac{-b^{2m} m^2}{(1 - b^{2m})^3 b^2}, \\ R_{2\bar{2}1\bar{1}}(0, b) &= \frac{-1 - b^2}{(1 - b^2)^3 m} + \frac{(1 - b^{2m}) b^2}{(1 - b^2)^4 m^2}, \\ R_{1\bar{2}2\bar{1}}(0, b) &= R_{2\bar{1}1\bar{2}}(0, b) = \frac{-b^2 - 1}{(1 - b^2)^3 m} + \frac{b^{2m}}{(1 - b^{2m})(1 - b^2)^2}, \\ R_{2\bar{2}2\bar{2}}(0, b) &= \frac{-2}{(1 - b^2)^4}, \end{aligned}$$

All the other terms vanish. So we have

$$\begin{aligned} \sum R_{i\bar{j}k\bar{\ell}}(0, b) \zeta^i \bar{\zeta}^j \zeta^k \bar{\zeta}^\ell &= \left(\frac{-1}{(1 - b^{2m})^2} + \frac{-b^2}{(1 - b^2)^2 m^2} - \frac{1}{1 - b^{2m}} \right) |\zeta^1|^4 \\ &+ \left(\frac{-b^{2m} m^2}{(1 - b^{2m})^3 b^2} + \frac{-3b^2 - 3}{(1 - b^2)^3 m} + \frac{2b^{2m}}{(1 - b^{2m})(1 - b^2)^2} + \frac{(1 - b^{2m}) b^2}{(1 - b^2)^4 m^2} \right) |\zeta^1|^2 |\zeta^2|^2 \\ &+ \frac{-2}{(1 - b^2)^4} |\zeta^2|^4. \end{aligned}$$

At this stage, we note that

$$mb^m(1 - b^2) < 1 - b^{2m} < m(1 - b^2)b^{m-1}$$

whenever $0 < b < 1$ and $0 < m < 1$. This can be easily verified by checking the

first and second derivatives in the variable b . Using these inequalities, we have

$$\begin{aligned} & \frac{-b^{2m}m^2}{(1-b^{2m})^3b^2} + \frac{2b^{2m}}{(1-b^{2m})(1-b^2)^2} + \frac{-3b^2-3}{(1-b^2)^3m} + \frac{(1-b^{2m})b^2}{(1-b^2)^4m^2} \\ & < \frac{-b^{2m}m^2}{(1-b^{2m})^3b^2} + \frac{2b^{2m}}{(1-b^{2m})(1-b^2)^2} + \frac{-3b^2-3}{(1-b^2)^3m} + \frac{b^{m+1}}{(1-b^2)^3m} \\ & < \frac{-m^2b^{2m}}{(1-b^{2m})^3b^2} + \frac{2}{(1-b^{2m})(1-b^2)^3m} [-(1-b^{2m}) + mb^{2m}(1-b^2)] < 0. \end{aligned}$$

This means that

$$\begin{aligned} & \sum R_{i\bar{j}k\bar{\ell}}(0, b)\zeta^i\bar{\zeta}^j\zeta^k\bar{\zeta}^\ell \\ & \leq \left(\frac{-1}{(1-b^{2m})^2} + \frac{-b^2}{(1-b^2)^2m^2} - \frac{1}{1-b^{2m}} \right) |\zeta^1|^4 + \frac{-2}{(1-b^2)^4} |\zeta^2|^4 \\ & \leq -\frac{1}{2} \sum h_{i\bar{j}}(0, b)h_{k\bar{\ell}}(0, b)\zeta^i\bar{\zeta}^j\zeta^k\bar{\zeta}^\ell. \end{aligned}$$

Therefore, the curvature property in the conclusion of Theorem 3 follows.

Since the Wu metric is at least in the Hölder class $C^{1, [2m]-1}$ at all the points $(z, 0) \in E_m$ in the domain, together with the arguments of Appendix B, the result of Theorem 3 implies

Corollary 1. *The holomorphic curvature of the Wu metric on a Thullen domain E_m with $1/2 \leq m < 1$ is bounded from above by $-1/2$ everywhere in the sense of currents.*

Moreover, from the explicit computation of the components of the curvature tensor above, one immediately obtains

Corollary 2. *The Wu metric on E_m with $1/2 \leq m < 1$ is not Kähler anywhere.*

4.3. Holomorphic curvature on E_m with $m > 1$. Based upon the explicit description of the Wu metric of E_m given in Section 3.4, we will present two separate sections of analysis of the holomorphic curvature tensor of the Wu metric in the following: first at points of M^+ , and then at points in M^- . For the notation, see Section 1.2.

4.3.1. Holomorphic curvature at points of M^+ . We only need to consider the points $(0, b)$ in E_m , for $m > 1$ and $b > 2^{-1/2m}$. Since an explicit formula for the Wu metric is given in Theorem 7, one can easily check that

$$\begin{aligned} R_{1\bar{1}1\bar{1}}(0, b) &= -2g_{1\bar{1}}(0, b)g_{1\bar{1}}(0, b), \\ R_{2\bar{2}2\bar{2}}(0, b) &= -2g_{2\bar{2}}(0, b)g_{2\bar{2}}(0, b), \\ R_{1\bar{2}2\bar{1}}(0, b) &= R_{2\bar{1}1\bar{2}}(0, b) = R_{1\bar{1}2\bar{2}}(0, b) = R_{2\bar{2}1\bar{1}}(0, b) = -g_{1\bar{1}}(0, b)g_{2\bar{2}}(0, b), \\ \text{All the remaining terms } R_{i\bar{j}k\bar{\ell}}(0, b) &= -g_{i\bar{j}}(0, b)g_{k\bar{\ell}}(0, b) = 0. \end{aligned}$$

It follows immediately that the Wu metric has constant holomorphic curvature -2 along every holomorphic direction at $(0, b)$, whenever $b > 2^{-1/2m}$. Due to its invariance under the action of the automorphisms, the holomorphic curvature is identically equal to the constant -2 at every point in $G_m(b)$. This completes the proof of Theorem 4.

4.3.2. *Holomorphic curvature at points of M^- .* The arguments of this section and the preceding section will complete the proof of Theorem 4. We will show the negativity of the holomorphic curvature at every point of M^- by a direct computation. Since the computation is very tedious, we will only outline the main steps here. Again, notice that it is enough to show the negativity of the holomorphic curvature at a typical point $(0, b)$ with $0 < b < 2^{-1/2m}$.

Recall that the Wu metric tensor $h_{i\bar{j}}$ at $(0, b)$ is a diagonal matrix with respect to the Euclidean coordinates as explicitly computed in Theorem 9. Thus we write the Wu metric tensor at $(0, b)$ as

$$(30) \quad h = h_1(0, b)dz_1 \otimes \overline{dz_1} + h_2(0, b)dz_2 \otimes \overline{dz_2}.$$

By Theorem 9,

$$h_1 = \frac{1}{2}t^{2-2m}b^{-2m}B^{-1}, \quad h_2 = \frac{1}{2}m^2t^2b^{-2}B^{-2},$$

where

$$B = 1 - m + mt^2 - t^{2m}b^{2m}$$

and where t is the unique solution in the open interval $(1, 1/b)$ of the algebraic equation (equation (27) in Theorem 9)

$$t^2 + (m - 2)t^{2m}b^{2m} - (m + 1)t^2t^{2m}b^{2m} + 2t^{4m}b^{4m} = 0.$$

Notice that

$$B = (1 - (tb)^{2m}) + m(t^2 - 1) > 0$$

for any t and b with $1 < t < 1/b$. Before we carry out any further detailed estimates, we first point out that throughout this section we have

$$(31) \quad 0 < b < 2^{-1/2m}.$$

A direct computation yields, at the point $(0, b)$,

$$\begin{aligned} R_{1\bar{1}1\bar{1}} &= -\frac{bh'_1}{2m} - 2h_1 - \frac{b^2h_2}{m^2}, \\ R_{1\bar{1}2\bar{2}} &= -\frac{h''_1}{4} - \frac{h'_1}{4b} + \frac{(h'_1)^2}{4h_1} = -\frac{h_1}{4}(\log h_1)'' - \frac{h'_1}{4b}, \\ R_{1\bar{2}2\bar{1}} &= R_{2\bar{1}1\bar{2}} = -\frac{bh'_2}{2m} - \frac{h_2}{m} + \frac{bh'_1h_2}{2mh_1} = -\frac{h_1}{2mb} \left(\frac{b^2h_2}{h_1} \right)', \\ R_{2\bar{2}1\bar{1}} &= -\frac{bh'_2}{2m} - \frac{h_2}{m} + \frac{b^2h_2^2}{m^2h_1}, \\ R_{2\bar{2}2\bar{2}} &= -\frac{h''_2}{4} + \frac{(h'_2)^2}{4h_2} - \frac{h'_2}{4b} = \frac{-h_2(\log h_2)''}{4} - \frac{h'_2}{4b}, \end{aligned}$$

and all the other components vanish, where the “primes” represent the derivative in the real variable b . To show that each component term listed above is real and negative, we need the following lemma.

Lemma 6. *With the notation as above, we have:*

- (i) $h'_1 > 0$,
- (ii) $(\log h_1)'' > 0$,
- (iii) $((b^2 h_2)/h_1)' > 0$,
- (iv) $R_{2\bar{2}1\bar{1}} < R_{1\bar{1}2\bar{2}}$,
- (v) $h'_2 > 0$,
- (vi) $(\log h_2)'' \geq 2(\log h_1)''$,

Lemma 6 will immediately imply that the holomorphic curvature is negative. Note that part (iv) of Lemma 6 also implies the following:

Corollary 3. *The Wu metric of E_m for $m > 1$ is non-Kähler at every point of M^- .*

To prove Lemma 6, we need the following two lemmas.

Lemma 7. *For $(0, b) \in E_m$ with $0 < b < 2^{-1/2m}$ and for t in Theorem 9, we have*

$$(32) \quad \frac{1}{m+1} < t^{2m} b^{2m} < 1/2.$$

Proof. Since $1 < t < 1/b$, we have $(tb)^{2m} < 1$ obviously. Regarding the algebraic equation (27) as a quadratic equation in $(tb)^{2m}$ with coefficients involving t and m and using the properties of convex parabola, we obtain the inequalities

$$0 < (tb)^{2m} < 1/2.$$

Now, the equation (27) can be rewritten as

$$(2t^{2m} b^{2m} - 1)(t^{2m} b^{2m} - 1) = (t^2 - 1)((m+1)t^{2m} b^{2m} - 1).$$

Since the left hand side is positive, so is the right hand side. Since $t > 1$, we now get

$$t^{2m} b^{2m} > \frac{1}{m+1}$$

which completes the proof. \square

We also have

Lemma 8. *Consider t as the function of b defined by the algebraic equation (27), for $0 < b < 2^{-1/2m}$. Then,*

$$\frac{dt}{db} < 0.$$

Proof. Dividing both sides of the equation (27) by t^2 , and differentiating in the variable b , we obtain

$$(33) \quad \frac{dt}{db} = \frac{M_1}{M_2},$$

where

$$(34) \quad \begin{aligned} M_1 &= mt(-m+2 + (m+1)t^2 - 4t^{2m}b^{2m}), \\ M_2 &= -b(-m^2 + 3m - 2 + m(m+1)t^2 - 2(2m-1)t^{2m}b^{2m}). \end{aligned}$$

By Lemma 7, one obtains

$$(35) \quad \begin{aligned} M_1 &\geq mt(m(t^2 - 1) + t^2) \geq mt^3 > 1, \\ M_2 &\leq -b(m^2(t^2 - 1) + mt^2 + (m - 1)) \leq -bmt^2 < 0. \end{aligned}$$

This proves the lemma. □

Proof of Lemma 6. The inequalities in (i), (ii), (iii) and (v) can be obtained by direct computation using Lemma 7 and Lemma 8. From the expression of $R_{a\bar{b}c\bar{d}}$ given at the beginning of Section 4.3.2, we have ,

$$\begin{aligned} R_{1\bar{2}2\bar{1}} - R_{2\bar{2}1\bar{1}} &= \frac{bh_2}{mh_1} \left(\frac{h'_1}{2} - \frac{bh_2}{m} \right) \\ &= \frac{m^3t^{2m}b^{2m}(m - 1)(t^2 - 1)}{2bB^3|M_2|} > 0, \end{aligned}$$

where the last equality is obtained from a direct computation together with exploiting (27) in the statement of Theorem 8. This proves part (iv).

To prove part (vi), we need to regroup terms appropriately. From the computation of h_1 , we obtain

$$(\log h_1)'' = \frac{2m}{B^2M_2^2} \cdot (Y_1 + Y_2 + Y_3),$$

where

$$\begin{aligned} Y_1 &= QB \frac{\partial M_2}{\partial b} + QB \frac{\partial M_2}{\partial t} \cdot \frac{M_1}{M_2}, \\ Y_2 &= -BM_1 \frac{\partial Q}{\partial t} + QM_1 \frac{\partial B}{\partial t}, \\ Y_3 &= -BM_2 \frac{\partial Q}{\partial b} + QM_2 \frac{\partial B}{\partial b}, \\ Q &= (m - 1)t^2(1 - (tb)^{2m}) + m(t^2 - (tb)^{2m}). \end{aligned}$$

Lemmas 7 and 8 can again be used to prove that $Y_1 > 0$ and $Y_2 > 0$. Actually, we have

$$(36) \quad Y_2 > \frac{2(m - 1)M_1(tb)^{2m}}{t} \{m^2(t^4 - 1) + (2m - 1)t^2 + t^2(1 - (tb)^{2m})\} > 0.$$

The term Y_3 is unfortunately negative. A calculation yields

$$Y_3 = \frac{2m^2M_2}{b}(t^4 - 1)(m - 1)t^{2m}b^{2m}.$$

To prove that $Y_2 + Y_3 > 0$ we will extract two positive terms from the expression (36) for Y_2 , and add to Y_3 . Namely we consider

$$Y_4 = \frac{2(m - 1)M_1(tb)^{2m}}{t}(m^2(t^4 - 1) + (2m - 1)t^2) + Y_3.$$

Using inequality (35) and $(tb)^{2m} < 1/2$, after a direct computation, we have

$$Y_4 > 2(m-1)(tb)^{2m}(2m-1)mt^4 > 0.$$

This proves the lemma. \square

Remarks on strict negativity of holomorphic curvature. It turns out that the holomorphic curvature of the Wu metric at points of M^- is bounded from above by a negative constant. This can be obtained by showing that the limits of the holomorphic curvature as $b \rightarrow 0$ and $b \rightarrow 2^{-1/2m}$, respectively, are bounded from above by a negative constant. Then of course by continuity and negativity of the holomorphic curvature mentioned above, one obtains the strict negativity of the holomorphic curvature of the Wu metric at points of M^- , thus proving the final assertion of Theorem 4. We choose not to include any further computation involving this to avoid repeating ourselves with tedious estimates.

APPENDIX A: THE WU METRIC IS C^1 BUT NOT C^2 ACROSS M^0

Earlier, we have shown that the Wu metric of E_m for $m > 1$ is C^1 smooth at points of M^0 . Here, we demonstrate that *the Wu metric for E_2 is not C^2 smooth* through a numerical calculation.

To see this, we first solve for t in terms of b explicitly in the equation in Theorem 9:

$$t = \frac{1}{b\sqrt{2\cos((\cos^{-1}(-b^2))/3)}}.$$

Using this in the expression of $R_{1\bar{1}2\bar{2}}$ at $(0, b) \in M^-$ and $(0, b) \in M^+$, and taking limits as $b \nearrow 2^{-1/4}$ and $b \searrow 2^{-1/4}$ respectively, one sees that the Wu metric is of class C^1 but not C^2 at the point $(0, 2^{-1/4})$, which belongs to M^0 . While we choose not to include any further detailed computations here, we simply mention that a numerical computation on the limit gives the following:

$$\lim_{b \nearrow (2^{-1/4})} R_{1\bar{1}2\bar{2}}(0, b) = -20.113\dots$$

and

$$\lim_{b \searrow (2^{-1/4})} R_{1\bar{1}2\bar{2}}(0, b) = -16\sqrt{2}.$$

This discrepancy implies that the Wu metric at points of M^0 is not C^2 smooth.

APPENDIX B: PROOF OF THEOREM 5

We will begin with the following general situation with a function of two real variables.

Let $u(x, y)$ be a real-valued C^1 smooth function on a neighborhood of the closure of the disk $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$, and let γ be a piecewise C^1 smooth curve without self-intersection passing through the origin such that $D \setminus \gamma$ has exactly two connected components, say D_1 and D_2 . Also assume that γ meets ∂D transversally. Then we have

Lemma. *If $u(x, y)$ is C^1 smooth on D , C^2 smooth on $D_1 \cup D_2$, and satisfies*

$$\Delta u(x, y) \geq ch(x, y) \quad \text{for all } (x, y) \in D_1 \cup D_2$$

for some constant $c > 0$ and a C^1 function $h(x, y)$ defined on D , then for any smooth nonnegative real valued function φ on D with compact support we have

$$\int_D u \Delta \varphi \, dA \geq c \int_D h \varphi \, dA.$$

In particular, we have

$$\partial \bar{\partial} u \geq c \cdot h dz \wedge d\bar{z}$$

on D , in the sense of current.

Proof. First set up the following notation for a C^1 smooth function $f(x, y)$:

$$\delta f = f_x dy - f_y dx.$$

Notice that, in case f is C^2 smooth,

$$d\delta f = (\Delta f) dx \wedge dy.$$

Therefore, for u on D_j ($j = 1, 2$), we now have

$$\begin{aligned} d(u \delta \varphi) &= (u_x \varphi_y + u_y \varphi_x) dx \wedge dy + u \Delta \varphi dx \wedge dy, \\ d(\varphi \delta u) &= (u_x \varphi_y + u_y \varphi_x) dx \wedge dy + \varphi \Delta u dx \wedge dy. \end{aligned}$$

Thus,

$$\varphi \Delta u dx \wedge dy - u \Delta \varphi dx \wedge dy = d(\varphi \delta u - u \delta \varphi)$$

on D_1 and also on D_2 .

Now, apply Stokes' Theorem to this. Let $\varphi(x, y)$ be a C^∞ smooth function with compact support in D . Then we have

$$\begin{aligned} \int_D u \Delta \varphi dx \wedge dy &= \int_{D_1} u \Delta \varphi dx \wedge dy + \int_{D_2} u \Delta \varphi dx \wedge dy \\ &= \int_{D_1} \varphi \Delta u dx \wedge dy + \int_{D_2} \varphi \Delta u dx \wedge dy \\ &\quad + \int_{D_1} d(\varphi \delta u - u \delta \varphi) + \int_{D_2} d(\varphi \delta u - u \delta \varphi). \end{aligned}$$

Since φ does not vanish at every boundary point of D_1 or D_2 , we need to handle the final two integrals with care.

By Stokes' Theorem, we have

$$\begin{aligned} &\int_{D_1} d(\varphi \delta u - u \delta \varphi) + \int_{D_2} d(\varphi \delta u - u \delta \varphi) \\ &= \int_{\partial D_1} \varphi \delta u - u \delta \varphi + \int_{\partial D_2} \varphi \delta u - u \delta \varphi \\ &= \int_{\partial D} \varphi \delta u - u \delta \varphi + \int_{\gamma} \varphi \delta u - u \delta \varphi + \int_{-\gamma} \varphi \delta u - u \delta \varphi \\ &= 0. \end{aligned}$$

Notice that an approximation of γ followed by an application of a convergence theorem for line integrals is required to make the above arguments valid. But this is routine, as the integrand is continuous everywhere on the closure of D . Now,

$$\begin{aligned} \int_D u \Delta \varphi dx \wedge dy &= \int_{D_1} \varphi \Delta u dx \wedge dy + \int_{D_2} \varphi \Delta u dx \wedge dy \\ &\geq c \int_{D_1} \varphi h dx \wedge dy + \int_{D_2} \varphi h dx \wedge dy \\ &= c \int_D \varphi h dx \wedge dy, \end{aligned}$$

as desired. \square

To establish the proof of negativity of the holomorphic curvature current, first observe that the set M^0 at which the Wu metric is only C^1 smooth is explicitly described by the equation

$$|z|^2 + b^{-2m}|w|^{2m} = 1,$$

which is strongly pseudoconvex at $(0, b)$, where $b = 2^{-1/2m}$. Thus any embedded analytic set through $(0, b)$ is either transversal to M^0 or tangential to M^0 with order of contact 2, when restricted to a smaller variety through $(0, b)$ if necessary. One concludes therefore that the intersection between the analytic set and M^0 is either a single point or a piecewise smooth curve. Now, the lemma above together with Theorems 1 through 4 implies the negativity of the holomorphic curvature current at points on M^0 . To complete the proof, we finally consider the points on the set $Z = \{(z, 0) \in \mathbb{C}^2 \mid |z| < 1\}$. It is obvious that any embedded closed analytic disk (which admits a holomorphic extension across the boundary) passing through $(0, 0)$ can meet the set Z along a discrete set of points only, unless it coincides with a subset of Z . When the embedded disk is transversal to Z , the problem reduces to the previous arguments using the lemma above. When the embedded disk is in fact coincides with an open subset of Z containing the origin, the holomorphic curvature of the Wu metric in the direction of Z is realized by the Gauss curvature of the Wu metric restricted to Z , since Z is clearly a totally geodesic complex submanifold with respect to the Wu metric. Then the negativity of the holomorphic curvature in the direction of Z follows immediately. This concludes the proof. \square

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