

LINEAR ADDITIVE FUNCTIONALS OF SUPERDIFFUSIONS AND RELATED NONLINEAR P.D.E.

E. B. DYNKIN AND S. E. KUZNETSOV

ABSTRACT. Let L be a second order elliptic differential operator in a bounded smooth domain D in \mathbb{R}^d and let $1 < \alpha \leq 2$. We get necessary and sufficient conditions on measures η, ν under which there exists a positive solution of the boundary value problem

$$(*) \quad \begin{aligned} -Lv + v^\alpha &= \eta && \text{in } D, \\ v &= \nu && \text{on } \partial D. \end{aligned}$$

The conditions are stated both analytically (in terms of capacities related to the Green's and Poisson kernels) and probabilistically (in terms of branching measure-valued processes called (L, α) -superdiffusions).

We also investigate a closely related subject — linear additive functionals of superdiffusions. For a superdiffusion in an arbitrary domain E in \mathbb{R}^d , we establish a 1-1 correspondence between a class of such functionals and a class of L -excessive functions h (which we describe in terms of their Martin integral representation). The Laplace transform of A satisfies an integral equation which can be considered as a substitute for (*).

1. INTRODUCTION

1.1. Boundary value problem with measures. We start from a differential operator

$$(1.1) \quad Lu = \sum_{i,j} a_{ij} \nabla_i \nabla_j u + \sum_i b_i \nabla_i u$$

(∇_i stands for the partial derivative with respect to x_i) in a bounded smooth domain D of \mathbb{R}^d with coefficients subject to conditions:

1.1.A. (Uniform ellipticity) There exists a constant $\varkappa > 0$ such that

$$\sum_{i,j} a_{ij} \lambda_i \lambda_j \geq \varkappa \sum_i \lambda_i^2 \quad \text{for all } x \in D, \lambda_1, \dots, \lambda_d \in \mathbb{R},$$

1.1.B. $a_{ij} \in C^{2,\lambda}(\bar{D})$, $b_i \in C^{1,\lambda}(\bar{D})$.¹

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¹We follow standard notation in P.D.E. (see, e.g., [21]). [Smooth domain means a domain of class $C^{2,\lambda}$.]

The classical boundary value problem

$$(1.2) \quad \begin{aligned} -Lv + v^\alpha &= \rho && \text{in } D, \\ v &= \sigma && \text{on } \partial D \end{aligned}$$

(with Hölder continuous ρ and continuous σ) is equivalent to an integral equation

$$(1.3) \quad v(x) + \int_D g(x, y)v(y)^\alpha dy = h(x)$$

where

$$(1.4) \quad h(x) = \int_D g(x, y)\rho(y)dy + \int_{\partial D} k(x, y)\sigma(y)a(dy),$$

$g(x, y)$ is Green's function, $k(x, y)$ is the Poisson kernel of L in D and $a(dy)$ is the surface area on ∂D . We interpret v as a (generalized) solution of the problem

$$(1.5) \quad \begin{aligned} -Lv + v^\alpha &= \eta && \text{on } D, \\ v &= \nu && \text{on } \partial D \end{aligned}$$

involving two measures η and ν if the equation (1.3) holds with

$$(1.6) \quad h(x) = \int_D g(x, y)\eta(dy) + \int_{\partial D} k(x, y)\nu(dy).$$

In Theorem 1.1, we establish sufficient conditions on η and ν under which problem (1.5) has a solution. Necessary conditions are established in Theorem 1.2. The equivalence of both sets of conditions follows from results in [17]. [Theorems 1.1 and 1.2 are still valid if D is not smooth. However, in general, the equivalence of conditions imposed on ν in the two theorems is not proved.]

Particular cases of problem (1.5) have been studied before. The case $\nu = 0$ was treated in [2] and the case $\eta = 0$ was considered in [18]. Even earlier, Gmira and Véron [22] have investigated a class of functions ψ such that the problem

$$\begin{aligned} \Delta v &= \psi(v) && \text{on } D, \\ v &= \nu && \text{on } \partial D \end{aligned}$$

has a solution for every finite measure ν . This class contains $\psi(v) = v^\alpha$ with $(\alpha + 1)/(\alpha - 1) > d$.

1.2. L -diffusions. Suppose D is a bounded smooth domain and that L satisfies conditions 1.1.A,B. Then there exists² a strictly positive function $p_t(x, y), t > 0, x, y \in D$ such that:

1.2.A. If f is a continuous function on D with compact support and if

$$(1.7) \quad u_t(x) = \int_D p_t(x, y)f(y)dy,$$

²This is proved (under weaker conditions on L) in Chapter 1 of [19].

then

$$(1.8) \quad \frac{\partial u_t(x)}{\partial t} = Lu_t(x),$$

$$(1.9) \quad u_t(x) \rightarrow f(x) \quad \text{as } t \rightarrow 0$$

and

$$(1.10) \quad u_t(x) \rightarrow 0 \quad \text{as } x \rightarrow z \in \partial D.$$

(All partial derivatives of p which appear in (1.8) are continuous in (t, x, y) .)

Function $p_t(x, y)$ has the following properties:

1.2.B. For all $s, t > 0, x, z \in D$,

$$\int_D p_s(x, y) dy p_t(y, z) = p_{s+t}(x, z).$$

1.2.C. For all $t > 0, x \in D$,

$$\int_D p_t(x, y) dy \leq 1.$$

Therefore $p_t(x, dy) = p_t(x, y)dy$ is a Markov transition function. It is well-known (see, e.g., [6]) that there exists a continuous Markov process $\xi = (\xi_t, \Pi_x)$ in D with this transition function. We call it an *L-diffusion*. If ζ is the life time of ξ , then $\xi_{\zeta-}$ belongs to ∂D . By setting $\xi_t = \xi_{\zeta-}$ for $t \geq \zeta$, we define an *L-diffusion stopped at the exit from D*. Note that ζ can be interpreted as the first exit time of this process from D ; often we use the notation τ for it.

Now suppose that E is an arbitrary domain in \mathbb{R}^d and that L is a differential operator in E which satisfies conditions 1.1.A, B in each bounded domain D with $\bar{D} \subset E$. Consider a sequence of bounded smooth domains D_n such that $\bar{D}_n \subset D_{n+1}$ and $\bigcup D_n = E$. The corresponding functions $p_t^n(x, y)$ increase monotonically and they tend to a limit $p_t(x, y)$ which does not depend on the choice of D_n (this follows from [19, Ch. 1]). There exists a continuous Markov process ξ in E with the transition function $p_t(x, dy) = p_t(x, y)dy$ (see, e.g., [6]). We call it an *L-diffusion in E*.

1.3. G-equation. Markov semigroup, Green's function g and Green's operator G for an *L-diffusion* ξ are defined by the formulae

$$(1.11) \quad T_t f(x) = \int_E p_t(x, dy) f(y),$$

$$(1.12) \quad g(x, y) = \int_0^\infty p_t(x, y) dt$$

and

$$(1.13) \quad Gf(x) = \int_0^\infty T_t f(x) dt = \int_E g(x, y) f(y) dy.$$

A positive Borel function h is called *excessive* if, for all $x \in E$, $T_t h(x) \leq h(x)$ and $T_t h(x) \rightarrow h(x)$ as $t \rightarrow 0$. The case $h(x) = \infty$ for all x is excluded. Since $p_t(x, y) > 0$, the set $\{x : h(x) = \infty\}$ has the Lebesgue measure 0. There exist only two possibilities: either $g(x, y) = \infty$ for all $x, y \in E$ or $g(x, y) < \infty$ for $x \neq y$. In the first case, constants are the only excessive functions and all problems treated in this paper are trivial. Therefore we concentrate on the second case.

Let $1 < \alpha \leq 2$. One of our goals is to find for which excessive functions h the equation

$$(1.14) \quad v + G(v^\alpha) = h$$

(we call it *G-equation*) has a solution.³ Note that if (1.14) holds almost everywhere, then

$$(1.15) \quad \tilde{v} = \begin{cases} h - G(v^\alpha) & \text{on } \{h < \infty\}, \\ \infty & \text{on } \{h = \infty\} \end{cases}$$

satisfies (1.14) everywhere.

Fix an arbitrary point $c \in E$ and put

$$(1.16) \quad k(x, y) = \begin{cases} \frac{g(x, y)}{g(c, y)} & \text{if } y \neq c, \\ 0 & \text{otherwise.} \end{cases}$$

There exist [see, e.g., [7]] a continuous injective mapping from E to a compact metrizable space \hat{E} and an extension of $k(x, y)$ to $E \times \hat{E}$ such that:

1.3.A. For every $x \in E$, $k(x, y) \rightarrow k(x, z)$ as $y \rightarrow z \in \hat{E} \setminus E$.

1.3.B. If $k(\cdot, y_1) = k(\cdot, y_2)$, then $y_1 = y_2$.

We call \hat{E} the *Martin space*. The set $\partial E = \hat{E} \setminus E$ is called the *Martin boundary*. For every $y \in E$, $h(x) = g(x, y)$ is an extremal excessive function.⁴ We denote by E^* the set of all $y \in \partial E$ such that $h(x) = k(x, y)$ is an extremal excessive function. (E^* is a Borel subset of ∂E .) Every excessive function h has a unique representation

$$(1.17) \quad h = G\eta + K\nu$$

where η is a σ -finite measure on E , ν is a finite measure on E^* and

$$(1.18) \quad G\eta(x) = \int_E g(x, y)\eta(dy), \quad K\nu(x) = \int_{E^*} k(x, y)\nu(dy)$$

(cf. (1.6)). Note that $\eta(\Gamma) < \infty$ for every compact $\Gamma \subset E$. Indeed, if $h(x_0) < \infty$, then $a\eta(\Gamma) \leq G\eta(x_0) < \infty$ where $a = \inf_{y \in \Gamma} g(x_0, y) > 0$.

Function $f = K\nu$ is L -harmonic, that is it satisfies equation $Lf = 0$. L -harmonic functions can be also characterized by the following mean value property: for every bounded open set D such that $\bar{D} \subset E$,

$$(1.19) \quad \Pi_x f(\xi_\tau) = f(x) \quad \text{for all } x \in E$$

where τ is the first exit time from D .

³(Cf. (1.3).) When speaking about solutions of G -equation, we always mean positive solutions.

⁴This means if $h = h_1 + h_2$ and if h_1, h_2 are excessive, then h_1, h_2 are proportional to h .

We fix $\alpha \in (1, 2]$. *Green's capacity* CG is defined on compact subsets of E by the formula

$$(1.20) \quad CG(\Gamma) = \sup\{\eta(\Gamma) : \int_E g(c, x)dx \left[\int_\Gamma g(x, y)\eta(dy) \right]^\alpha \leq 1\}.$$

Analogously, the *Martin capacity* CK is defined on compact subsets of ∂E by the formula

$$(1.21) \quad CK(\Gamma) = \sup\{\nu(\Gamma) : \int_E g(c, x)dx \left[\int_\Gamma k(x, y)\nu(dy) \right]^\alpha \leq 1\}.$$

[By a Choquet theorem [3], CG and CK can be extended to all analytic subsets of E and E^* .] If η is a measure on E , then writing $\eta \prec CG$ means that $\eta(\Gamma) = 0$ if $CG(\Gamma) = 0$. Writing $\nu \prec CK$ has an analogous meaning.

It follows from the results in Sections 2 and 3 that:

Theorem 1.1. *If $h = G\eta + K\nu$ and if*

$$(1.22) \quad \eta \prec CG, \quad \nu \prec CK,$$

then G -equation (1.14) has a solution v which is defined uniquely on the set $E(h) = \{h < \infty\}$.

1.4. Operators \mathcal{G} and \mathcal{K} . Let ξ be an L -diffusion in a bounded smooth domain D stopped at the first exit time τ from D . We introduce operators \mathcal{G} and \mathcal{K} acting on functions with the domain $S = \mathbb{R}_+ \times E$ by the formulae

$$(1.23) \quad \mathcal{G}f(t, x) = \int_0^t ds \int_D p_s(x, dy) f(t - s, y) = \Pi_x \int_0^{\tau \wedge t} f(t - s, \xi_s) ds,$$

and⁵

$$(1.24) \quad \mathcal{K}f(t, x) = \Pi_x f(t - \tau, \xi_\tau).$$

If $f(t, x) = f(x)$ does not depend on t , then

$$(1.25) \quad \begin{aligned} \mathcal{G}f(t, x) &= \Pi_x \int_0^t f(\xi_s) ds \rightarrow Gf(x), \\ \mathcal{K}f(t, x) &= \Pi_x f(\xi_\tau) 1_{\tau \leq t} \rightarrow Kf(x) \end{aligned}$$

as $t \rightarrow \infty$. Here G is defined by (1.13) and⁶

$$(1.26) \quad Kf(x) = \Pi_x f(\xi_\tau).$$

⁵We extend each function to $\mathbb{R} \times E$ by setting it equal to zero for negative t .

⁶Operator (1.26) is a particular case of the operator K defined by (1.18): if $E = D$ is a bounded smooth domain, then $E^* = \partial D$ and $\Pi_x f(\xi_\tau) = \int_{\partial D} k(x, y)\nu(dy)$ for $\nu(dy) = f(y)a(dy)$ where k is the Poisson kernel and a is the surface area on ∂D . Writing the same letter for both operators should cause no confusion since one operator is applied only in the context of a smooth domain D and the second one only in the context of the Martin boundary of E .

The boundary of a cylinder $Q = \mathbb{R}_+ \times D$ consists of the side surface $A = (0, \infty) \times \partial D$ and the bottom $B = \{0\} \times \bar{D}$. Besides the boundary value problem (1.2), we consider also a boundary value problem for a parabolic equation

$$(1.27) \quad \begin{aligned} \frac{\partial u}{\partial t} - Lu + u^\alpha &= \rho && \text{in } Q, \\ u &= \sigma && \text{on } A, \\ u &= 0 && \text{on } B. \end{aligned}$$

If ρ and σ are Hölder continuous, then (1.27) is equivalent to the integral equation

$$(1.28) \quad u + \mathcal{G}(u^\alpha) = \mathcal{G}\rho + \mathcal{K}\sigma.$$

1.5. Superdiffusions. Let $\xi = (\xi_t, \Pi_x)$ be a Markov process in a measurable space (E, \mathcal{B}) and let $\mathcal{M} = \mathcal{M}(E)$ be the space of all finite measures on \mathcal{B} . A (ξ, α) -superprocess is a Markov process $X = (X_t, P_\mu)$ in \mathcal{M} which satisfies the condition: for every $\mu \in \mathcal{M}$ and every positive \mathcal{B} -measurable function f ,

$$(1.29) \quad \begin{aligned} P_\mu \exp\langle -f, X_t \rangle &= \exp\langle -u_t, \mu \rangle, \\ u_t(x) + \Pi_x \int_0^t u_{t-s}(\xi_s)^\alpha ds &= \Pi_x f(\xi_t). \end{aligned}$$

We say that X is an (L, α) -superdiffusion if X is a right process and ξ is an L -diffusion. The existence of such processes for $1 < \alpha \leq 2$ is proved, for instance, in [13] (we refer to [8] and [9] for the history of this subject starting from the pioneering work of Watanabe and Dawson).

In the theory of diffusion, a fundamental role is played by random points ξ_τ corresponding to the first exit times from open sets D . An analogous role in the theory of superdiffusion is played by exit measures X_D . In contrast to ξ_τ which can be defined through ξ_t , it is impossible, in general, to define X_D in terms of X_t . The probability distribution of X_D is defined by formulae similar to (1.29):

$$(1.30) \quad \begin{aligned} P_\mu \exp\langle -f, X_D \rangle &= \exp\langle -u, \mu \rangle, \\ u(x) + \Pi_x \int_0^\tau u(\xi_s)^\alpha ds &= \Pi_x f(\xi_\tau). \end{aligned}$$

The joint probability distribution of X_{t_1}, \dots, X_{t_n} is determined by (1.29) and the Markov property of X . Analogously, the joint probability distribution of X_{D_1}, \dots, X_{D_n} can be evaluated by using (1.30) and the following Markov property: for every positive $\mathcal{F}_{\supset D}$ -measurable Y ,

$$(1.31) \quad P_\mu \{Y | \mathcal{F}_{\supset D}\} = P_{X_D} Y$$

where $\mathcal{F}_{\supset D}$ is the σ -algebra generated by $X_{D'}$ with $D' \subset D$ and $\mathcal{F}_{\supset D}$ the σ -algebra generated by $X_{D''}$ with $D'' \supset D$.

We need even a wider class of exit measures [for instance, measures corresponding to the exit from D before time t]. We introduce a random measure (X_Q, P_μ) for

an arbitrary open set Q in $S = \mathbb{R}_+ \times E$ and an arbitrary finite measure μ on the Borel σ -algebra in S . Its probability distribution is defined by the formulae

$$(1.32) \quad \begin{aligned} P_\mu \exp\langle -f, X_Q \rangle &= \exp\langle -u, \mu \rangle, \\ u(r, x) + \Pi_{r,x} \int_r^{\tau^r} u(s, \xi_s)^\alpha ds &= \Pi_{r,x} f(\tau^r, \xi_{\tau^r}) \end{aligned}$$

where

$$(1.33) \quad \tau^r = \inf\{t : t \geq r, (t, \xi_t) \notin Q\}$$

is the first, after r , exit time of ξ from Q and $\Pi_{r,x} Y = \Pi_x \theta_{-r} Y$ describes a Markov process with transition function $p_t(x, dy)$ which starts at time r from point x . The joint probability distribution of X_{Q_1}, \dots, X_{Q_n} is determined by (1.33) and by the property: for every positive $\mathcal{F}_{\supset Q}$ -measurable Y ,

$$(1.34) \quad P_\mu\{Y | \mathcal{F}_{\subset Q}\} = P_{X_Q} Y$$

where $\mathcal{F}_{\subset Q}$ is the σ -algebra generated by $X_{Q'}$ with $Q' \subset Q$ and $\mathcal{F}_{\supset Q}$ the σ -algebra generated by $X_{Q''}$ with $Q'' \supset Q$.

The existence of a family (X_Q, P_μ) subject to conditions (1.32) and (1.34) is proved in [8].

Formula $j_r(x) = (r, x)$ defines a mapping from E to S . If μ is a measure on E , then $j_r(\mu)$ is a measure on S concentrated on $\{r\} \times E$. We set $P_{j_r(\mu)} = P_{r,\mu}$. It follows from (1.32) that

$$(1.35) \quad \begin{aligned} P_{r,\mu} \exp\langle -f, X_Q \rangle &= \exp\left\{ - \int_E u(r, x) \mu(dx) \right\}, \\ u(r, x) + \Pi_x \int_0^\tau u(s+r, \xi_s)^\alpha ds &= \Pi_x f(\tau+r, \xi_\tau). \end{aligned}$$

Formulae (1.29) and (1.30) can be considered as special cases of (1.35) if we identify X_t and X_D with the exit measures from $S_{<t} = [0, t) \times E$ and from $\mathbb{R}_+ \times D$, projected on E .

If τ is the first exit time from D , then $\tau(t) = \tau \wedge t$ is the first exit time from $Q_t = [0, t) \times D$. We call the process $\tilde{X}_t = X_{Q_t}$ an (L, α) -superdiffusion stopped at the exit from D . If $f(t, x) = f(x)$ vanishes outside D and if $v_t(x) = -\log P_x \exp\langle -f, \tilde{X}_t \rangle$, then $v_{t-r}(x) = -\log P_{r,x} \exp\langle -f, X_{Q_t} \rangle$ and (1.35) implies

$$(1.36) \quad \begin{aligned} P_\mu \exp\langle -f, \tilde{X}_t \rangle &= \exp\langle -v_t, \mu \rangle, \\ v_t(x) + \Pi_x \int_0^{\tau(t)} v_{t-s}(\xi_s)^\alpha ds &= \Pi_x f(\xi_{\tau(t)}). \end{aligned}$$

Formula (1.36) can be obtained from (1.29) by replacing ξ with an L -diffusion stopped at the exit from D .

The shift operators θ_t of a time-homogeneous process ξ induce analogous operators for X (see [14, Section 1.12]). We have $X_s(\theta_t \omega) = X_{s+t}(\omega)$ and, if $Q = \mathbb{R}_+ \times D$, then $X_Q(\theta_t \omega, \Gamma) = X_{Q_t}(\omega, \Gamma+t)$ where $Q_t = S_{<t} \cup \{\gamma_t(Q)\}$ with $\gamma_t(r, x) = (r+t, x)$.

It follows from (1.35) that

$$(1.37) \quad P_\mu \int_Q f(s, x) X_Q(ds, dx) = \int \mu(dx) \Pi_x f(\tau, \xi_\tau)$$

[it is sufficient to apply (1.35) to λf and to take the derivative with respect to λ at $\lambda = 0$].

The following result (see Theorem I.1.8 in [8]) provides a link between superprocesses and the G -equation.

Theorem A. *Suppose that \tilde{X} is an (L, α) -superdiffusion stopped at the exit from D , ρ is a positive Borel function on \bar{D} vanishing on ∂D and σ is a positive Borel function on ∂D . Then*

$$(1.38) \quad v(x) = -\log P_x \exp \left\{ - \left[\int_0^\infty \langle \rho, \tilde{X}_t \rangle dt + \langle \sigma, X_D \rangle \right] \right\}$$

is a solution of the G -equation (1.14) where G is Green's operator for L -diffusion in D , K is given by (1.26) and⁷

$$(1.39) \quad h = G\rho + K\sigma.$$

Moreover, for every $\mu \in \mathcal{M}(D)$,

$$(1.40) \quad P_\mu \exp \left\{ - \left[\int_0^\infty \langle \rho, \tilde{X}_t \rangle dt + \langle \sigma, X_D \rangle \right] \right\} = e^{-\langle v, \mu \rangle}.$$

We also need another implication of Theorem I.1.8 in [8] [cf. Theorem 1.1 in [16]].

Theorem B. *Let \tilde{X} , D and ρ be the same as in Theorem A and let σ be a positive Borel function on \bar{D} vanishing on D . Then*

$$(1.41) \quad u(t, x) = -\log P_x \exp \left\{ - \left[\int_0^t \langle \rho, \tilde{X}_s \rangle ds + \langle \sigma, \tilde{X}_t \rangle \right] \right\}$$

is a solution of the equation (1.28). Moreover, for every $\mu \in \mathcal{M}(D)$,⁸

$$(1.42) \quad P_\mu \exp \left\{ - \left[\int_0^t \langle \rho, \tilde{X}_s \rangle ds + \langle \sigma, \tilde{X}_t \rangle \right] \right\} = \exp\langle -u^t, \mu \rangle.$$

The range \mathcal{R} of a superprocess X is the smallest closed subset of E which supports all measures X_t (it supports, a.s., every exit measure X_D). We denote by \mathcal{R}^* the minimal closed subset of the Martin space \hat{E} which supports all measures X_t . A set $\Gamma \subset E$ is called \mathcal{R} -polar if $P_x\{\mathcal{R} \cap \Gamma \neq \emptyset\} = 0$ for all $x \notin \Gamma$. A subset Γ of the Martin boundary ∂E is called \mathcal{R}^* -polar if $P_x\{\mathcal{R}^* \cap \Gamma \neq \emptyset\} = 0$ for all $x \in E$.⁹

We prove in Section 4:

⁷For $x \in \partial D$, $u(x) = h(x) = \sigma(x)$.

⁸We set $u^t(x) = u(t, x)$.

⁹ \mathcal{R}^* -polarity is introduced only on ∂E because $\mathcal{R}^* \cap \Gamma = \mathcal{R} \cap \Gamma$ for every compact $\Gamma \subset E$.

Theorem 1.2. *If $h = G\eta + K\nu$ and if the G -equation (1.14) has a solution, then η does not charge \mathcal{R} -polar sets and ν does not charge \mathcal{R}^* -polar sets.*

We say that $\Gamma \subset E$ is G -polar if $CG(\Gamma) = 0$ and that $\Gamma \subset \partial E$ is K -polar if $CK(\Gamma) = 0$. By Theorem 1.1 in [17] [cf. Theorem 1.6 in [12]], the classes \mathcal{R} -polar and G -polar sets coincide. Theorems 1.1 and 1.2 imply:

1.5.A. All \mathcal{R}^* -polar sets are K -polar.

Indeed, if Γ is compact and if $CK(\Gamma) > 0$, then by (1.21), there exists a measure $\nu \neq 0$ concentrated on Γ such that

$$\int_E g(c, x) dx \left[\int k(x, y) \nu(dy) \right]^\alpha < \infty$$

which implies

$$\int_E g(c, x) dx \left[\int_B k(x, y) \nu(dy) \right]^\alpha < \infty$$

for every B . Hence $\nu(B) = 0$ for all K -polar sets B . By Theorems 1.1 and 1.2, $\nu(B) = 0$ for all \mathcal{R}^* -polar sets B . Therefore Γ is not \mathcal{R}^* -polar.

If ξ is an L -diffusion in a bounded smooth domain of \mathbb{R}^d , then a stronger result than 1.5.A follows from Theorem 1.2 in [17]:

1.5.B. The classes of \mathcal{R}^* -polar and K -polar sets coincide.

It remains an open problem if 1.5.B holds in the general case. If it holds for a diffusion ξ and if X is the corresponding superdiffusion, then each of Theorems 1.1–1.2 gives necessary and sufficient conditions on h for the existence of a solution of (1.14).

1.6. Additive functionals. Let X be a superdiffusion. We denote by \mathcal{F}_t the σ -algebra in Ω generated by the exit measures X_Q for all $Q \subset S_{<t}$. A function $A_t(\omega)$ from $[0, \infty] \times \Omega$ to $[0, \infty]$ is called an *additive functional* of X if:

1.6.A. For every ω , A_t is monotone increasing in t .

1.6.B. A_t is measurable with respect to the completion of \mathcal{F}_t with respect to all measures P_μ , $\mu \in \mathcal{M}(E)$.

1.6.C. For every ω , A_t is left continuous in t .

1.6.D. $A_{s+t} = A_s + \theta_s A_t$ for all pairs s, t and all ω .¹⁰

All these conditions hold for

$$(1.43) \quad A_t = \int_0^t \langle \rho, X_s \rangle ds$$

where ρ is an arbitrary positive Borel function. By a limit procedure, we construct, starting from (1.43), a class of functionals for which a weaker form of condition 1.6.D holds.

We say that a set Λ is ξ -polar if $\Pi_x \{ \xi_t \notin \Lambda \text{ for all } t > 0 \} = 1$ for all x . All ξ -polar sets have the Lebesgue measure 0. A subset \mathcal{N} of $\mathcal{M}(E)$ is called *exceptional* if the set $\{ x : \delta_x \in \mathcal{N} \}$ is ξ -polar and if, for all stopped superdiffusions \tilde{X} and for every $\mu \notin \mathcal{N}$, $P_\mu \{ \tilde{X}_t \notin \mathcal{N} \text{ for all } t \} = 1$.

¹⁰Let $\beta(\omega) = \sup \{ t : A_t(\omega) < \infty \}$. Then there exists a unique measure $A(\omega, dt)$ on $[0, \beta(\omega))$ such that $A[0, t) = A_t$ for all $t < \beta$.

If h is an arbitrary excessive function, then the set $\Lambda(h) = \{x : h(x) = \infty\}$ is ξ -polar and the set $\mathcal{N}(h) = \{\mu : \langle h, \mu \rangle = \infty\}$ is exceptional.

We say that A is an *additive functional with an exceptional set \mathcal{N}* if A satisfies 1.6.A, B, C and:

$$1.6.D^*. A_{s+t} = A_s + \theta_s A_t \text{ for all } s, t, \omega \in \Omega_0 \text{ and } P_\mu(\Omega_0) = 1 \text{ for all } \mu \notin \mathcal{N}.$$

Two additive functionals A and \tilde{A} are called *equivalent* if there exists an exceptional set \mathcal{N} such that $P_\mu\{A_t = \tilde{A}_t \text{ for all } t\} = 1$ for all $\mu \notin \mathcal{N}$.

Let h be an excessive function. An additive functional A with an exceptional set \mathcal{N} is called a *linear additive functional with potential h* if, for every $\mu \notin \mathcal{N}$,

$$(1.44) \quad P_\mu A_\infty = \langle h, \mu \rangle.$$

If $G\rho(x) < \infty$ for some x , then the additive functional (1.43) is linear with potential $G\rho$ (condition (1.44) holds for every μ).

Theorem 1.3. *If $h = G\eta + K\nu$ and if $\eta \prec CG, \nu \prec CK$, then h is the potential of a linear additive functional A of X with an exceptional set \mathcal{N} . For every $\mu \notin \mathcal{N}$,*

$$(1.45) \quad P_\mu e^{-A_\infty} = e^{-\langle v, \mu \rangle}$$

where v is a solution of the G -equation (1.14).

Theorem 1.3 is proved in Section 3. Theorem 1.1 follows immediately from Theorem 1.3 and a uniqueness Theorem 2.1.

Remark. The construction of A in Section 3 implies that A depends linearly on h . More precisely, if A^i corresponds to h^i , then, for every $c_1, c_2 \geq 0$, the functional A corresponding to $c_1 h^1 + c_2 h^2$ is equivalent to $c_1 A^1 + c_2 A^2$. Therefore, if h, \tilde{h} and $h - \tilde{h}$ are excessive functions and if v, \tilde{v} are the solutions of (1.14) corresponding to h and \tilde{h} , then $\tilde{v} \leq v$ outside a ξ -polar set.

In Section 4 we establish:

Theorem 1.4. *If h is the potential of a linear additive functional with an exceptional set \mathcal{N} , then $h = G\eta + K\nu$ with η vanishing on all \mathcal{R} -polar sets and ν vanishing on all \mathcal{R}^* -polar sets.*

Linear additive functionals of superprocesses have been introduced in [11] (in a time-inhomogeneous setting). There a linear additive functional corresponding to a *bounded* excessive function h was constructed for a $(\xi, 2)$ -superprocess where ξ is an arbitrary right Markov process. (No exceptional set is needed in this case.)

The case of an (L, α) -superdiffusion with an arbitrary $\alpha \in (1, 2]$ was investigated in [15]. For $h = G\eta$ with $\eta \prec CG$, a functional A was constructed, subject to conditions 1.6.A, B with the property, for every $\mu \in \mathcal{M}^0$,

$$1.6.D^{**}. A_{s+t} = A_s + \theta_s A_t \text{ } P_\mu\text{-a.s. for all } s, t.$$

Here \mathcal{M}^0 is the set of measures of the form $\mu(dx) = \rho(x)dx$ with $\int \rho(x)^{\alpha'} dx < \infty$ where $\alpha' = \alpha/(\alpha - 1)$. Condition (1.44) was proved also only for $\mu \in \mathcal{M}^0$. (Note that \mathcal{M}^0 is not the complement of an exceptional set!)

Recent results of Le Gall [23] on additive functionals of the Brownian snake can be translated into our language as follows: if $h = K\nu$ with $\nu \prec CK$, then there exists a functional of an $(\Delta, 2)$ -superdiffusion which satisfies conditions 1.6.A, B, C, (1.44) and 1.6.D^{**} for P_x for almost all x .

Additive functionals with an exceptional set have been introduced, in a different context, by Fukushima [20]. In his setting, X is a symmetric Markov process associated with a Dirichlet form and an exceptional set is a polar subset of the state space (in the sense of theory of Dirichlet spaces).

1.7. We have the following logical implications: $\mathcal{A} \implies \mathcal{B} \implies \mathcal{C} \implies \mathcal{D}$ where:

\mathcal{A} : $h = G\eta + K\nu$ with $\eta \prec CG, \nu \prec CK$;

\mathcal{B} : h is the potential of a linear additive functional A with an exceptional set \mathcal{N} .
 Moreover for every $\mu \notin \mathcal{N}$,

$$P_\mu e^{-A_\infty} = e^{-\langle v, \mu \rangle}$$

where v is a solution of the G -equation (1.14).

\mathcal{C} : h is the potential of a linear additive functional A .

\mathcal{D} : $h = G\eta + K\nu$ with η vanishing on all \mathcal{R} -polar sets and ν vanishing on all \mathcal{R}^* -polar sets.

We get $\mathcal{A} \implies \mathcal{B}$ by Theorem 1.3 and $\mathcal{C} \implies \mathcal{D}$ by Theorem 1.4. The implication $\mathcal{B} \implies \mathcal{C}$ is trivial.

If 1.5.B holds for a diffusion ξ and if X is the corresponding superdiffusion, then $\mathcal{D} \implies \mathcal{A}$ and all four statements $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and \mathcal{D} are equivalent. In particular, this is true if ξ is an L -diffusion in a bounded smooth domain D . This also is true for an arbitrary domain E if we consider only excessive functions $h = G\eta$ (in other words if we set $\nu = 0$).

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2. G-EQUATION

2.1. Monotonicity and uniqueness.

Theorem 2.1. *Let ξ be an L -diffusion stopped at the first exit time τ from an open set D , and let G, K be given by (1.13),(1.26). Suppose that η is a measure on D , $u, \hat{u}, \sigma \geq 0$ and, for almost all x ,*

$$(2.1) \quad \hat{u} + G(\hat{u}^\alpha) = u + G(u^\alpha) + G\eta + K\sigma < \infty.$$

Then $\hat{u} \geq u$ at every point of the set (2.1). If $\eta = 0$ and $\sigma = 0$, then $\hat{u} = u$ on the same set.

An analogous result holds for the \mathcal{G} -equation. For every measure η on S we put

$$(2.2) \quad \mathcal{G}\eta(t, x) = \int_0^t \int_D p_{t-s}(x, y)\eta(ds, dy)$$

(cf. (1.23)). If $\eta(ds, dy) = ds\eta(dy)$, then

$$(2.3) \quad \mathcal{G}\eta(t, x) = \int_0^t ds \int_D p_s(x, y)\eta(dy) \rightarrow G\eta(x)$$

as $t \rightarrow \infty$.

Theorem 2.1*. *Let ξ be the same as in Theorem 2.1 and let \mathcal{G}, \mathcal{K} be given by (2.2), (1.24). Suppose that η is a measure on S , $u, \hat{u}, \sigma \geq 0$ and, for almost all t, x ,*

$$(2.4) \quad \hat{u} + \mathcal{G}(\hat{u}^\alpha) = u + \mathcal{G}(u^\alpha) + \mathcal{G}\eta + \mathcal{K}\sigma < \infty.$$

Then $\hat{u} \geq u$ at every point of the set (2.4). If $\eta = 0$ and $\sigma = 0$, then $\hat{u} = u$ on the same set.

We use as a tool a process $(\xi_s, \Pi_x^{t,y})$ with $x, y \in D$. Its finite-dimensional distributions are given by the formula

$$(2.5) \quad \begin{aligned} & \Pi_x^{t,y} \{ \xi_{t_1} \in dy_1, \dots, \xi_{t_n} \in dy_n, t_n < t < \tau \} \\ & = p_{t_1}(x, dy_1) p_{t_2-t_1}(y_1, dy_2) \cdots p_{t_n-t_{n-1}}(y_{n-1}, dy_n) p_{t-t_{n-1}}(y_n, y) \end{aligned}$$

for all $0 < t_1 < \dots < t_n < t$. (Here $p_t(x, dy)$ is the transition function and $p_t(x, y)$ is the transition density of the part of ξ in D .)¹¹

Let f be a positive Borel function. Formula

$$(2.6) \quad p_t^\varphi(x, y) = \Pi_x^{t,y} \left\{ \exp \left\{ - \int_0^t \varphi(\xi_s) ds \right\} \right\}$$

defines the transition density of a Markov process obtained from ξ by killing with rate $f(x)$ at point x .

Operator G_φ corresponding to p^φ by (2.2) can be expressed by formula

$$(2.7) \quad \mathcal{G}_\varphi \rho(t, x) = \Pi_x \int_0^{\tau \wedge t} ds \rho(t-s, \xi_s) \exp \left\{ - \int_0^s \varphi(t-r, \xi_r) dr \right\}.$$

We prove Theorem 2.1*. (Proof of Theorem 2.1 is similar but simpler.) We need two lemmas.

Lemma 2.1. ¹² (i) *Let ρ be a Borel function on S . Equation*

$$(2.8) \quad \mathcal{G}\rho - \mathcal{G}_\varphi \rho = \mathcal{G}_\varphi(\varphi \mathcal{G}\rho)$$

holds on the set $\{\mathcal{G}|\rho| < \infty\}$.

(ii) *If η is a measure on S , then*

$$(2.9) \quad \mathcal{G}\eta - \mathcal{G}_\varphi \eta = \mathcal{G}_\varphi(\varphi \mathcal{G}\eta)$$

on the set $\{\mathcal{G}\eta < \infty\}$.

(iii) *For every positive Borel σ , equation*

$$(2.10) \quad \mathcal{K}\sigma - \mathcal{K}_\varphi \sigma = \mathcal{G}_\varphi(\varphi \mathcal{K}\sigma)$$

holds on the set $\{\mathcal{K}\sigma < \infty\}$.

¹¹Normalized measure $\Pi_x^{t,y}$ can be obtained by conditioning the diffusion ξ started from point x to come at point y at time t .

¹²Cf. [5]. This result can be interpreted as the resolvent form of the Feynman-Kac formula.

Proof. 1°. It is sufficient to check (2.8) for $\rho \geq 0$. We use (1.23) and (2.7), the Markov property of ξ , Fubini's theorem and relation

$$\int_0^s da Y_a \exp \left\{ - \int_0^a Y_r dr \right\} = 1 - \exp \left\{ - \int_0^s Y_r dr \right\}$$

which we apply to $Y_s = \varphi(t - s, \xi_s)$.

2°. Put

$$(2.11) \quad \rho_\varepsilon(x) = \int_D p_\varepsilon(x, y) \eta(dy).$$

Note that

$$(2.12) \quad G\rho_\varepsilon(x) = \int_\varepsilon^\infty dt \int_D p_t(x, y) \eta(dy).$$

We get (2.9) by applying (2.8) to ρ_ε and by passing to the limit as $\varepsilon \rightarrow 0$.

3°. Formula (2.10) can be proved in the same way as (2.8).

Lemma 2.2. *Suppose that $\varphi, \sigma \geq 0$ and that, for almost all t, x ,*

$$(2.13) \quad \mathcal{G}\eta + \mathcal{K}\sigma + \mathcal{G}|\varphi w| < \infty$$

and

$$(2.14) \quad w + \mathcal{G}(\varphi w) = \mathcal{G}\eta + \mathcal{K}\sigma.$$

Then

$$(2.15) \quad w = \mathcal{G}_\varphi\eta + \mathcal{K}_\varphi\sigma$$

at every point (t, x) where (2.13) and (2.14) hold.

Proof. We have

$$(2.16) \quad \mathcal{G}_\varphi(\varphi w) + \mathcal{G}_\varphi[\varphi\mathcal{G}(\varphi w)] = \mathcal{G}_\varphi(\varphi\mathcal{G}\eta) + \mathcal{G}_\varphi(\varphi\mathcal{K}\sigma).$$

On the set defined by (2.13) and (2.14), the left side in (2.16) is equal to $\mathcal{G}(\varphi w)$ by (2.8) and, the right side is equal to $\mathcal{G}\eta + \mathcal{K}\sigma - \mathcal{G}_\varphi\eta - \mathcal{K}_\varphi\sigma$ by (2.8) and (2.10). Therefore $\mathcal{G}(\varphi w) = \mathcal{G}\eta + \mathcal{K}\sigma - \mathcal{G}_\varphi\eta - \mathcal{K}_\varphi\sigma$ and (2.15) follows from (2.14).

Proof of Theorem 2.1.* Denote by \hat{S} the set defined by (2.4). Put $w = \hat{u} - u$ on \hat{S} and $w = 0$ on $E \setminus \hat{S}$. There exists a function $\varphi \geq 0$ such that $\hat{u}^\alpha - u^\alpha = \varphi w$ a.e. Equation (2.4) implies (2.14). Since $G|\varphi w| \leq G(u^\alpha) + G(\hat{u}^\alpha) < \infty$ on \hat{S} , Theorem 2.1* follows from Lemma 2.2.

2.2. Properties of G and \mathcal{G} . In this subsection we deal with operators corresponding to an L -diffusion ξ in a bounded smooth domain D . We denote by $\|u\|$ the norm of u in $L^1(D)$. For a function f on $S = \mathbb{R}_+ \times D$ and for $b \in \mathbb{R}_+$, we set

$$\ell_b(f) = \int_0^b \int_D |f(r, x)| dr dx.$$

We need the following results.

2.2.A. There is a constant C such that

$$\int_D g(x, y) dx \leq C \quad \text{for all } y \in D.$$

2.2.A*. For every $b > 0$, there exists a constant C such that

$$\int_D p_t(x, y) dx \leq C \quad \text{for all } y \in D, 0 < t \leq b.$$

2.2.B. If f_n is a sequence of functions such that $\ell_b(f_n)$ are bounded for every b , then the sequence $\mathcal{G}f_n$ contains a subsequence which converges a.e. (relative to $dr dx$).

2.2.C. Let

$$\theta = \sup_{x \in D} c^*(x)$$

where

$$(2.17) \quad c^* = \sum_{i,j=1}^d \nabla_i \nabla_j a_{ij} - \sum_{i=1}^d \nabla_i b_i.$$

Then

$$(2.18) \quad \int_D f \operatorname{sign} Gf dx \geq -\theta \|Gf\|$$

for all $f \in L^1(D)$.

Properties 2.2.A and 2.2.A* follow from well-known bounds for $g(x, y)$ ([24, Chapter 3]) and $p_t(x, y)$ ([19, Chapter 1]).

Proof of 2.2.B. Denote by φ_δ a function equal to 0 for $|t| < \delta/2$, equal to 1 for $|t| > \delta$ and linear on $[-\delta, -\delta/2]$ and on $[\delta/2, \delta]$. Formula

$$\mathcal{G}_\delta f(t, x; s, y) = \varphi_\delta(t - s) p_{t-s}(x, y)$$

defines a continuous kernel on $S_b = [0, b] \times \bar{D}$. The corresponding operator \mathcal{G}_δ is compact in $L^1(S_b)$ because functions $\mathcal{G}_\delta f_n$ are equicontinuous for every sequence f_n bounded in $L^1(S_b)$.

By 2.2.A* and Fubini's theorem,

$$\begin{aligned} \ell_b(\mathcal{G}f - \mathcal{G}_\delta f) &= \int_{S_b} dt dx \int_{S_b} [1 - \varphi_\delta(t - s)] p_{t-s}(x, y) |f(s, y)| ds dy \\ &\leq \int_{S_b} ds dy |f(s, y)| \int_s^{(s+\delta) \wedge b} dt dx p_{t-s}(x, y) \leq C \delta \ell_b(f). \end{aligned}$$

Therefore \mathcal{G} is a compact operator in $L^1(S_b)$. We get 2.2.B by the diagonal procedure.

Proof of 2.2.C. 1°. Suppose that φ is a bounded increasing continuously differentiable function on \mathbb{R} such that $\varphi(0) = 0$. Suppose that

$$(2.19) \quad u \in C^2(\bar{D}), \quad u = 0 \quad \text{on } \partial D.$$

Put $\Phi(t) = \int_0^t \varphi(s) ds$. By integration by parts, we get

$$(2.20) \quad \begin{aligned} - \int_D \varphi(u) Lu dx &= \int_D \left[\sum_{i,j} a_{ij} \varphi'(u) \nabla_i u \nabla_j u + \sum_i \left(\sum_j \nabla_j a_{ij} + b_i \right) \varphi(u) \nabla_i u \right] dx \\ &= \int_D \left[\sum a_{ij} \varphi'(u) \nabla_i u \nabla_j u - c^* \Phi(u) \right] dx \end{aligned}$$

and therefore

$$(2.21) \quad - \int_D dx \varphi(u) Lu \geq -\theta \int_D \Phi(u) dx.$$

2°. Suppose $u = Gf$ with $f \in C^2$. Then u satisfies (2.19) and $Lu = -f$. By (2.21),

$$(2.22) \quad \int_D \varphi(u) f dx \geq -\theta \int_D \Phi(u) dx.$$

An arbitrary $f \in L^1(D)$ is the strong limit of a sequence $f_n \in L^1(D) \cap C^2$. Let $u_n = Gf_n, u = Gf$. We have

$$(2.23) \quad \int \varphi(u) f dx - \int \varphi(u_n) f_n dx = \int \varphi(u_n) (f - f_n) dx + \int (\varphi(u) - \varphi(u_n)) f dx.$$

By 2.2.A, $u_n \rightarrow u$ in $L^1(D)$. Therefore a subsequence u_{n_k} converges to u a.e. and the second term in the right side of (2.23) converges to 0 along this subsequence. The first term also converges to 0. Since (2.22) holds for f_n , it holds also for f .

3°. By applying (2.22) to a sequence of functions φ_n which converge boundedly to sign u and by passing to the limit, we get

$$\int_D f \text{sign } u \, dx \geq -\theta \int_D |u| dx$$

which is equivalent to (2.18).

2.3. Existence. Suppose that ξ is an L -diffusion stopped at the first exit time τ from a bounded smooth domain D , L satisfies conditions 1.1.A–B, p is defined by condition 1.2.A and g is the corresponding Green’s function defined by (1.12). We consider a function in D defined by the formula

$$(2.24) \quad h = G\eta + K\sigma$$

where η is a finite measure on D and σ is a positive bounded Borel function on ∂D . Put $D(h) = \{h < \infty\}$, $D(h, \alpha) = \{h + G(h^\alpha) < \infty\}$ and $\mathcal{N}(h, \alpha) = \{\mu : \langle h + G(h^\alpha), \mu \rangle = \infty\}$. Note that $D(h, \alpha)$ is either empty or is the complement of a ξ -polar set. Let $Q(h) = \mathbb{R}_+ \times D(h)$ and $Q(h, \alpha) = \mathbb{R}_+ \times D(h, \alpha)$.

Theorem 2.2. *Suppose that $D(h, \alpha)$ is nonempty. Then there exists $v \geq 0$ such that*

$$(2.25) \quad v + G(v^\alpha) = h \quad \text{on } D(h, \alpha).$$

Equation (2.25) determines v uniquely on $D(h, \alpha)$. We have

$$(2.26) \quad \|v^\alpha\| \leq 2C\eta(D) + C_1(\sigma)$$

where C is defined in 2.2.A and $C_1(\sigma)$ does not depend on η . Let \tilde{X} be an (L, α) -superdiffusion stopped at exit from D and let

$$(2.27) \quad v_\varepsilon(x) = -\log P_x \exp \left\{ - \left[\int_0^\infty \langle \rho_\varepsilon, \tilde{X}_t \rangle dt + \langle \sigma, X_D \rangle \right] \right\},$$

$$(2.28) \quad u_\varepsilon(t, x) = -\log P_x \exp \left\{ - \left[\int_0^t \langle \rho_\varepsilon, \tilde{X}_s \rangle ds + \langle \sigma, \tilde{X}_t \rangle \right] \right\}$$

where $\sigma = 0$ in D , ρ_ε is given by (2.11) in D and it is equal to 0 on ∂D .

We have

$$(2.29) \quad \lim_{\varepsilon \rightarrow 0} v_\varepsilon(x) = v(x) \quad \text{on } D(h, \alpha),$$

$$(2.30) \quad \lim_{\varepsilon \rightarrow 0} u_\varepsilon(t, x) = u(t, x) \quad \text{on } Q(h, \alpha)$$

where v is the solution of (2.25) and u is the solution of the equation

$$(2.31) \quad u + \mathcal{G}(u^\alpha) = \mathcal{G}\eta + \mathcal{K}\sigma \quad \text{on } Q(h, \alpha).$$

Moreover, if $\mu \notin \mathcal{N}(h, \alpha)$, then

$$(2.32) \quad \langle v_\varepsilon, \mu \rangle \rightarrow \langle v, \mu \rangle$$

and

$$(2.33) \quad \langle u_\varepsilon^t, \mu \rangle \rightarrow \langle u^t, \mu \rangle$$

for all t .

Finally,

$$(2.34) \quad u(t, x) \uparrow v(x) \quad \text{as } t \rightarrow \infty \quad \text{on } Q(h, \alpha).$$

Proof. By Theorems 2.1 and 2.1*, each of the equations (2.25) and (2.31) has no more than one solution. We split the proof of Theorem 2.2 into three steps. First, we establish a bound for $\|v_\varepsilon^\alpha\|$. Then we use this bound to prove formulae (2.30) and (2.33). Finally, we establish (2.34), (2.29), (2.32), (2.25), (2.31) and (2.26).

1°. It follows from (2.12) that

$$(2.35) \quad h_\varepsilon \leq h \quad \text{and} \quad h_\varepsilon \uparrow h \quad \text{as} \quad \varepsilon \rightarrow 0$$

where $h_\varepsilon = G\rho_\varepsilon + K\sigma$. By Theorem A and (1.19), v_ε given by (2.27) satisfies equation

$$(2.36) \quad v_\varepsilon + G([v_\varepsilon]^\alpha) = h_\varepsilon$$

and

$$(2.37) \quad w(x) = -\log P_x \exp\{-\langle \sigma, X_D \rangle\}$$

satisfies equation

$$(2.38) \quad w + G(w^\alpha) = K\sigma.$$

Note that functions ρ_ε, w and $K\sigma$ are bounded and

$$(2.39) \quad v_\varepsilon - w = G(F_\varepsilon)$$

where

$$(2.40) \quad F_\varepsilon = \rho_\varepsilon - v_\varepsilon^\alpha + w^\alpha.$$

By 2.2.C,

$$\int F_\varepsilon \operatorname{sign}(v_\varepsilon - w) dx = \int F_\varepsilon \operatorname{sign} G F_\varepsilon dx \geq -\theta \|v_\varepsilon - w\|$$

and, since $\operatorname{sign}(v_\varepsilon^\alpha - w^\alpha) = \operatorname{sign}(v_\varepsilon - w)$, we have

$$(2.41) \quad \|v_\varepsilon^\alpha - w^\alpha\| = \int (v_\varepsilon^\alpha - w^\alpha) \operatorname{sign}(v_\varepsilon^\alpha - w^\alpha) dx \leq \|\rho_\varepsilon\| + \theta \|v_\varepsilon - w\|.$$

By 2.2.A* and (2.11),

$$(2.42) \quad \|\rho_\varepsilon\| \leq C\eta(D).$$

Note that, if $\alpha > 1$, then for every $\delta > 0$, there exists a constant C_δ such that

$$(2.43) \quad |b - a| \leq \delta |b^\alpha - a^\alpha| + C_\delta$$

for all reals a, b . It follows from (2.41), (2.42) and (2.43) that

$$(2.44) \quad \|v_\varepsilon^\alpha - w^\alpha\| \leq \theta \delta \|v_\varepsilon^\alpha - w^\alpha\| + C\eta(D) + \theta C_\delta.$$

If $\delta \theta \leq 1/2$, then

$$(2.45) \quad \|v_\varepsilon^\alpha - w^\alpha\| \leq 2C\eta(D) + 2\theta C_\delta.$$

Since $w \leq K\sigma$ and σ is bounded, (2.45) implies

$$(2.46) \quad \|v_\varepsilon^\alpha\| \leq 2C\eta(D) + C_1(\sigma)$$

where $C_1(\sigma) = 2\theta C_\delta + \|(K\sigma)^\alpha\|$.

2°. By (1.23), (2.12) and (2.24),

$$(2.47) \quad \mathcal{G}\rho_\varepsilon + \mathcal{K}\sigma \leq G\rho_\varepsilon + K\sigma \leq h.$$

By Theorem B,

$$(2.48) \quad u_\varepsilon + \mathcal{G}(u_\varepsilon^\alpha) = \mathcal{G}\rho_\varepsilon + \mathcal{K}\sigma$$

and

$$W(t, x) = -\log P_x \exp\{-\langle \sigma, \tilde{X}_t \rangle\}$$

is a solution of the equation

$$W + \mathcal{G}[W^\alpha] = \mathcal{K}\sigma.$$

We have

$$(2.49) \quad u_\varepsilon - W = \mathcal{G}(F_\varepsilon)$$

where

$$(2.50) \quad F_\varepsilon = \rho_\varepsilon - u_\varepsilon^\alpha + W^\alpha.$$

By (2.28) and (2.27),

$$(2.51) \quad u_\varepsilon(t, x) \leq v_\varepsilon(x) \quad \text{for all } t, x.$$

For every b , by (2.46) and (2.51),

$$(2.52) \quad b^{-1}\ell_b[(u_\varepsilon)^\alpha] \leq \|v_\varepsilon^\alpha\| \leq 2C\eta(D) + C_1(\sigma).$$

It follows from (2.11) and 1.2.B that

$$(2.53) \quad \mathcal{G}\rho_\varepsilon(t, x) = \int_\varepsilon^{t+\varepsilon} ds \int_D p_s(x, y)\eta(dy) \leq h(x)$$

and therefore

$$(2.54) \quad \mathcal{G}\rho_\varepsilon \rightarrow \mathcal{G}\eta \quad \text{as } \varepsilon \rightarrow 0 \quad \text{on } D(h).$$

For every b , $\ell_b(F_\varepsilon)$ are bounded by (2.50), (2.42) and (2.52). By (2.49) and 2.2.B, every sequence u_{ε_n} contains a subsequence which converges, a.e., to a limit u . Suppose $u_{\varepsilon_n} \rightarrow u$ a.e. By (2.48) and (2.47), $u_\varepsilon \leq h$. It follows from (1.23) and the dominated convergence theorem that

$$(2.55) \quad \mathcal{G}[(u_{\varepsilon_n})^\alpha] \rightarrow \mathcal{G}[(u)^\alpha] \quad \text{on } Q(h, \alpha).$$

By (2.55) and (2.48), u satisfies (2.31) a.s. Formula (2.30) holds because, otherwise, $|u_{\varepsilon_n} - u| > \delta$ for some $(r, x) \in Q(h, \alpha), \delta > 0$ and for some sequence $\varepsilon_n \rightarrow 0$. By applying once more the dominated convergence theorem, we get (2.33).

3°. It is clear from (2.28) that $u_\varepsilon(t, x)$ is monotone increasing in t . Therefore for every $x \in D(h, \alpha)$, $u(t, x)$ is also monotone increasing in t . By the monotone convergence theorem, $v(x) = \lim_{t \rightarrow \infty} u(t, x)$ satisfies (2.25). Formula (2.26) follows from (2.52) and (2.34).

Note that $u_\varepsilon \leq v_\varepsilon$ and, by (2.30) and (2.34), $\liminf_{\varepsilon \rightarrow 0} v_\varepsilon \geq v$ on $D(h, \alpha)$. On the other hand, it follows from (1.25) and (1.13) that $\mathcal{G}(u_\varepsilon^\alpha) \leq \mathcal{G}(v_\varepsilon^\alpha) \leq G(v_\varepsilon^\alpha)$ and, by (2.48), (2.36), (2.12), (2.53), (1.25) and (1.26),

$$0 \leq v_\varepsilon - u_\varepsilon^t \leq \int_{t+\varepsilon}^\infty ds \int_D p_s(x, y) \eta(dy) + \Pi_x \sigma(\xi_\tau) 1_{\tau > t}$$

and therefore $\limsup_{\varepsilon \rightarrow 0} v_\varepsilon \leq v$ on $D(h, \alpha)$. Clearly, this implies (2.29). Formula (2.32) can be deduced from (2.33) in an analogous way.

3. PROOF OF THEOREM 1.3

3.1. We use several times a property of exit measures which will be established in Lemma 3.1. We start from a functional

$$(3.1) \quad B_t(\varepsilon) = \int_0^t \langle \rho_\varepsilon, \tilde{X}_s \rangle ds + C_t$$

where ρ_ε is given by (2.11) and C_t is a left continuous modification of $\langle \sigma, \tilde{X}_t \rangle$ which we define in Lemma 3.2. Put

$$(3.2) \quad B_t = \lim_{k \rightarrow \infty} \text{med } B_t(1/k) \quad \text{for all } t > 0$$

where $\lim \text{med}$ is Mokobodzki's medial limit. It is defined for every sequence $a_n \in [0, \infty]$ and it takes values in $[0, \infty]$. We need the following properties of this limit (see, e.g., [4, X.56, X.57]):

- 3.1.A. $\liminf a_n \leq \lim \text{med } a_n \leq \limsup a_n$;
- 3.1.B. $\lim \text{med}(a_n + b_n) = \lim \text{med } a_n + \lim \text{med } b_n$;
- 3.1.C. If $a_n \leq b_n$ for all n , then $\lim \text{med } a_n \leq \lim \text{med } b_n$;
- 3.1.D. Let Z_n be measurable mappings from a measurable space (Ω, \mathcal{F}) to $[0, \infty]$. Then $Z(\omega) = \lim \text{med } Z_n(\omega)$ is measurable with respect to the universal completion of \mathcal{F} . If P is a probability measure on (Ω, \mathcal{F}) and if $Z_n \rightarrow Y$ in P -probability, then $Y = Z$ P -a.s.

In Theorem 3.1, we construct a functional B of an (L, α) -superdiffusion \tilde{X} stopped at the exit from a bounded smooth domain D which satisfies conditions 1.6.A, B and the following condition:

$$(3.3) \quad B_{s+t} \leq B_s + \theta_s B_t \quad \text{a.s. for every } s, t.$$

Moreover, for every $\mu \notin \mathcal{N}(h, \alpha)$:

$$(3.4) \quad B_t = \lim_{\varepsilon \rightarrow 0} B_t(\varepsilon) \quad \text{in } P_\mu\text{-probability for all } t \in \mathbb{R}_+$$

and

$$(3.5) \quad -\log P_\mu e^{-Bt} = \langle u^t, \mu \rangle$$

where u satisfies (2.31).

The next step is a passage to the limit from bounded smooth domains to an arbitrary domain E . We assume that h is given by (1.17) and that $E(h, \alpha) = \{h < \infty, G(h^\alpha) < \infty\}$ is nonempty. We consider a sequence of bounded smooth domains D_n which approximate E and we denote by G^n, K^n the Green and Poisson operators corresponding to the L -diffusion stopped at the exit from D_n . Put $\sigma_n = 1_{E \setminus D_n} K\nu$ and denote by B^n the function corresponding to

$$(3.6) \quad h_n(x) = \int_{D_n} g^n(x, y) \eta(dy) + K^n \sigma_n(x)$$

by Theorem 3.1. By 3.1.C, D, function

$$(3.7) \quad B_t = \lim_{n \rightarrow \infty} \text{med } B_t^n$$

satisfies 1.6.A, B. We show that, for every $\mu \notin \mathcal{N}(h, \alpha)$ and every t ,

$$(3.8) \quad B_t = \lim_{n \rightarrow \infty} B_t^n \quad P_\mu - \text{a.s.}$$

Function $A_t = B_{t-}$ satisfies 1.6.A-C and 1.6.D** with $\mathcal{N} = \mathcal{N}(h, \alpha)$.

At the final stage, we use Lemma 3.3 to decompose measures η, ν , subject to condition (1.22), into series of measures η_n, ν_n for which $E(h_n, \alpha) \neq \emptyset$ (here $h_n = G\eta_n + K\nu_n$). The functional corresponding to η, ν is defined as the sum of functionals corresponding to η_n, ν_n .

This way we obtain a functional of X , subject to conditions 1.6.A, B, C, for which 1.6.D** and (1.45) hold for all μ outside of an exceptional set \mathcal{N} . Then we refer to a result in [4] to prove the existence of an equivalent functional which satisfies 1.6.D*.

3.2. A property of exit measures.

Lemma 3.1. *Suppose that $Q_1 \supset Q_2$ are open subsets of S and $\Gamma \cap Q_1 = \emptyset$. Then $X_{Q_1}(\Gamma) \geq X_{Q_2}(\Gamma)$ a.s.*

Proof. For every $\nu \in \mathcal{M}(E)$, $P_\nu\{X_{Q_1}(\Gamma) \geq \nu(\Gamma)\} = 1$. Indeed, $\Pi_{r,x}\{\tau^r = r\} = 1$ for every $(r, x) \notin Q_1$ and, by (1.32), for every $\lambda > 0$,

$$P_\nu e^{-\lambda X_{Q_1}(\Gamma)} = e^{-\langle v_\lambda, \nu \rangle}$$

with $v_\lambda = \lambda 1_\Gamma$ on Γ . Hence,

$$(3.9) \quad P_\nu e^{-\lambda X_{Q_1}(\Gamma)} \leq e^{-\lambda \nu(\Gamma)}.$$

Put $Y = X_{Q_1}(\Gamma) - \nu(\Gamma)$. By (3.9), $P_\nu e^{-\lambda Y} \leq 1$ for all $\lambda > 0$ and therefore $Y \geq 0$ P_ν -a.e.

It follows from (1.34) that, for every positive measurable f , $P_\mu f(X_{Q_2}, X_{Q_1}) = P_\mu F(X_{Q_2})$ where $F(\nu) = P_\nu f(\nu, X_{Q_1})$. If $f(\nu_1, \nu_2) = 1_{\nu_1(\Gamma) \leq \nu_2(\Gamma)}$, then $F(\nu) = P_\nu\{\nu(\Gamma) \leq X_{Q_1}(\Gamma)\} = 1$.

3.3. Regularization of $\langle \sigma, \tilde{X}_t \rangle$.

Lemma 3.2. *Let \tilde{X} be an (L, α) -superdiffusion stopped at the exit from a bounded smooth domain D and let σ be a positive Borel function on \bar{D} which vanishes on D . There exists a function C_t subject to conditions 1.6.A–C such that, for every t ,*

$$(3.10) \quad C_t = \langle \sigma, \tilde{X}_t \rangle \quad \text{a.s.}$$

Proof. Put $Y_t = \langle \sigma, \tilde{X}_t \rangle$. Recall (see Section 1.5) that $\tilde{X}_t = X_{Q_t}$ where $Q_t = [0, t) \times D$. It follows from Lemma 3.1 that $\tilde{X}_r(\Gamma) \leq \tilde{X}_s(\Gamma)$ a.s. if $r < s$ and $\Gamma \cap Q_s = \emptyset$. Since $\sigma = 0$ in Q_s , $Y_r \leq Y_s$ a.s. Denote by \mathbb{Q}_+ the set of positive rationals. The set

$$\Omega_t = \{Y_r \leq Y_s \text{ for all } r < s \in \mathbb{Q}_+ \cap [0, t)\}$$

belongs to \mathcal{F}_t and $P_\mu\{\Omega_t\} = 1$ for all $\mu \in \mathcal{M}(D)$. Function

$$C_t = \begin{cases} \lim_{s \uparrow t, s \in \mathbb{Q}_+} Y_s & \text{on } \Omega_t, \\ \infty & \text{otherwise} \end{cases}$$

satisfies conditions 1.6.A–C. It remains to prove that $Y_t = C_t$ a.s. By Theorem B,

$$(3.11) \quad -\log P_\mu e^{-Y_t} = \langle u^t, \mu \rangle$$

where u is a solution of the equation

$$(3.12) \quad u + \mathcal{G}[u^\alpha] = \mathcal{K}\sigma.$$

By (3.11), $u(t, x)$ is monotone increasing in t . Put $u_-(t, x) = u(t-, x)$. Since $\Pi_x\{\tau = t\} = 0$ for all t , function $\mathcal{K}\sigma$ is continuous in t . By passing to the limit in (3.12), we get

$$u_- + \mathcal{G}[u_-^\alpha] = \mathcal{K}\sigma.$$

By (1.25), functions $\mathcal{K}\sigma \leq K\sigma$ are bounded and, by Theorem 2.1*, $u_- = u$. By (3.11), $P_\mu e^{-Y_t} = P_\mu e^{-C_t}$. Since $Y_t \leq C_t$, this implies (3.10).

3.4.

Lemma 3.3. *Let η and ν satisfy condition (1.22). Then there exist measures η_n, ν_n such that*

$$\eta = \eta_1 + \dots + \eta_n + \dots, \quad \nu = \nu_1 + \dots + \nu_n + \dots$$

and

$$(3.13) \quad G(h_n^\alpha)(c) < \infty$$

where $h_n = G\eta_n + \mathcal{K}\nu_n$ and c is the same as in formula (1.16).

Proof. Since $(\frac{a+b}{2})^\alpha \leq \frac{1}{2}(a^\alpha + b^\alpha)$ for all $\alpha, a, b \geq 0$, we can assume that $\eta = 0$ or $\nu = 0$. We refer to [18, Theorem 2.2] in the first case and [2, Lemma 4.2] in the second case.

3.5.

Theorem 3.1. *Let \tilde{X} be an (L, α) -superdiffusion stopped at the exit from a bounded smooth domain D and let $h, \eta, \rho_\varepsilon$ and σ be as in Theorem 2.2. If $B_t(\varepsilon)$ is defined by (3.1), then function B_t given by (3.2) satisfies conditions 1.6.A, B, (3.3), (3.4) and (3.5).*

Proof. Properties 1.6.A, B follow from 3.1.C, D. By Theorem B,

$$(3.14) \quad u_{\delta\varepsilon}(t, x) = -\log P_x \exp\left\{-\frac{1}{2}(B_t(\delta) + B_t(\varepsilon))\right\}$$

satisfies the equation

$$u_{\delta\varepsilon} + \mathcal{G}[u_{\delta\varepsilon}^\alpha] = \frac{1}{2}(\mathcal{G}\rho_\delta + \mathcal{G}\rho_\varepsilon) + \mathcal{K}\sigma.$$

The same arguments as in the proof of Theorem 2.2 show that, for all $\mu \notin \mathcal{N}(h, \alpha)$ and all t ,

$$(3.15) \quad \langle u_{\delta\varepsilon}^t, \mu \rangle \rightarrow \langle u^t, \mu \rangle \quad \text{as } \delta, \varepsilon \rightarrow 0$$

where u is the unique solution of (2.31).

By Theorem B,

$$(3.16) \quad P_\mu \left[e^{-B_t(\varepsilon)/2} - e^{-B_t(\delta)/2} \right]^2 = e^{-\langle u_{\varepsilon\varepsilon}^t, \mu \rangle} + e^{-\langle u_{\delta\delta}^t, \mu \rangle} - 2e^{-\langle u_{\delta\varepsilon}^t, \mu \rangle}$$

for every $\mu \in \mathcal{M}(D)$. If $\mu \notin \mathcal{N}(h, \alpha)$, then, by (3.15), the right side in (3.16) tends to 0 as $\delta, \varepsilon \rightarrow 0$. Hence $e^{-B_t(\varepsilon)}$ converges in $L^2(P_\mu)$ as $\varepsilon \rightarrow 0$ which implies that $B_t(\varepsilon)$ converges in P_μ -probability to a limit B_t^μ . It follows from 3.1.D that $B_t = B_t^\mu$ P_μ -a.s. which implies (3.4). To prove (3.3), we note that $\tilde{X}_t = X_{Q(t)}$ where $Q(t) = [0, t] \times D$. Therefore (see Section 1.5)) $\theta_s \tilde{X}_t = X_{Q(s,t)}$ where $Q(s, t) = S_{<s} \cup Q(s+t)$ and, by Lemma 3.1, $\langle \rho_\varepsilon, \theta_s \tilde{X}_t \rangle \geq \langle \rho_\varepsilon, \tilde{X}_{s+t} \rangle$ and $\langle \sigma, \theta_s \tilde{X}_t \rangle \geq \langle \sigma, \tilde{X}_{s+t} \rangle$ a.s. Clearly, restrictions of measures \tilde{X}_s and \tilde{X}_{s+t} to $[0, s] \times \partial D$ coincide, and, by (3.1),

$$(3.17) \quad B_s(\varepsilon) + \theta_s B_t(\varepsilon) \geq B_{s+t}(\varepsilon) \quad \text{a.s.}$$

and (3.3) follows from (3.2).

Let $\mu \notin \mathcal{N}(h, \alpha)$. By (1.42),

$$-\log P_\mu e^{-B_t(\varepsilon)} = \langle u_\varepsilon^t, \mu \rangle$$

where u_ε is given by (2.28). By (3.4) and (2.33), this implies (3.5).

3.6. The next step in the program outlined in Section 3.1 is a passage to the limit from bounded smooth domains to an arbitrary domain E . Recall that, according to Section 1.2, L -diffusion ξ in E can be constructed by using a sequence of bounded smooth domains D_n such that $\bar{D}_n \subset D_{n+1}$ and $E = \bigcup D_n$: the transition density $p_t(x, y)$ of ξ is the limit of monotone increasing sequence $p_t^n(x, y)$ defined in 1.2.A (it is convenient to set $p_t^n(x, y) = 0$ if $x \notin D_n$ or $y \notin D_n$). Green's functions $g^n(x, y)$,

$g(x, y)$ and Green's operators G^n, G corresponding to p^n, p are determined by (1.12) and (1.13). Operators K^n correspond by (1.26) to the first exit times τ_n from D_n .

Let X be an (L, α) -superdiffusion in E and let X^n be an (L, α) -superdiffusion stopped at the exit from D_n . Denote by Y_t^n the restriction of X_t^n to D_n . By Lemma 3.1, for every t and every n ,

$$(3.18) \quad Y_t^n \leq Y_t^{n+1} \quad \text{a.s.}$$

By (1.37),

$$P_\mu Y_t^n(B) = \int_E \mu(dx) \int_B p_t^n(x, y) dy \uparrow \int_E \mu(dx) \int_B p_t(x, y) dy = P_\mu X_t(B)$$

and therefore

$$(3.19) \quad Y_t^n \uparrow X_t \quad \text{a.s.}$$

3.7. Let h be given by (1.17) with finite measures η and ν . Suppose that $E(h, \alpha) \neq \emptyset$. Put $f = K\nu$. By (1.19) and (1.26),

$$f(x) = \Pi_x \sigma_n(\xi_{\tau_n}) = K^n \sigma_n(x)$$

where $\sigma_n = 1_{E \setminus D_n} f$. We define B^n and B as in Section 3.1. By 3.1.D, to prove formula (3.8), we need only show that B_t^n converges P_μ -a.s. as $n \rightarrow \infty$. Put

$$(3.20) \quad Z_t^n(\varepsilon) = \int_0^t \langle \rho_\varepsilon^n, Y_r^n \rangle dr.$$

By 3.1.B,

$$B_t^n = Z_t^n + C_t^n$$

where

$$Z_t^n = \lim_{k \rightarrow \infty} \text{med } Z_t^n(1/k).$$

For every n , $\rho_\varepsilon^{n+1} \geq \rho_\varepsilon^n$ and, by (3.20) and (3.18), $Z_t^n(\varepsilon)$ is, a.s., monotone increasing in n . By 3.1.C, sequence Z_t^n has the same property and therefore it converges P_μ -a.s.

On the other hand, since f is L -harmonic, it follows from the Markov property (1.34) that the sequence $W_n = \langle f, X_t^n \rangle$ is a martingale with respect to P_μ . Therefore C_t^n also converges a.s.

3.8. Put $S(h, \alpha) = \mathbb{R}_+ \times E(h, \alpha)$. By Theorem 2.2, for every $\mu \notin \mathcal{N}(h, \alpha)$,

$$-\log P_\mu e^{-B_t^n} = \int_{D_n} u_n(t, x) \mu(dx)$$

where u_n satisfies the equation

$$(3.21) \quad u_n(t, x) + \int_0^t ds \int_{D_n} p_{t-s}^n(x, y) u_n(s, y)^\alpha dy = H_n(t, x) \quad \text{on } S(h, \alpha)$$

with

$$(3.22) \quad H_n(t, x) = \int_0^t ds \int_{D_n} p_{t-s}^n(x, y) \eta(dy) + \Pi_x f(\xi_{\tau_n}) 1_{\tau_n < t}.$$

Moreover, by (2.28), (2.30) and (3.20),

$$(3.23) \quad u_n(t, x) = - \lim_{\varepsilon \rightarrow 0} \log P_x \exp\{-B_t^n(\varepsilon)\} \quad \text{on } S(h, \alpha).$$

By (3.4),

$$u_n(t, x) = - \log P_x e^{-B_t^n} \quad \text{on } S(h, \alpha).$$

By (3.8),

$$(3.24) \quad u_n(t, x) \rightarrow u(t, x) = - \log P_x e^{-B_t} \quad \text{on } S(h, \alpha).$$

Note that

$$\Pi_x f(\xi_{\tau_n}) 1_{\tau_n < t} = f(x) - \Pi_x f(\xi_t) 1_{\tau_n \geq t}$$

and therefore H_n converges to

$$(3.25) \quad H(t, x) = \int_0^t ds \int_E p_{t-s}(x, y) \eta(dy) + F(t, x)$$

where

$$(3.26) \quad F(t, x) = f(x) - \Pi_x f(\xi_t).$$

By (3.21), (3.22) and (1.17), $u_n \leq h$. The second term in (3.21) converges to $\mathcal{G}[u^\alpha]$ by (2.2) and the dominated convergence theorem. Hence, (3.21) implies

$$(3.27) \quad u + \mathcal{G}[u^\alpha] = H \quad \text{on } S(h, \alpha).$$

3.9. Note that $u(t, x)$ increases in t by (3.24) and 1.6.A. Put $u_-(t, x) = u(t-, x)$. An L -excessive function f has a representation

$$f = f_0 + \int_0^\infty \varphi_s ds$$

where $T_t f_0 = f_0$ and $T_t \varphi_s = \varphi_{s+t}$ for all t, s (see [10, Section 2.8]). Therefore

$$H(t, x) = \int_0^t ds \left[\int_E p_s(x, y) \eta(dy) + \varphi_s \right]$$

is increasing and continuous in t . By passing to the limit in (3.24) and (3.27), we get

$$(3.28) \quad \begin{aligned} u_-(t, x) &= - \log P_x e^{-B_{t-}}, \\ u_- + \mathcal{G}[u_-^\alpha] &= H \quad \text{on } S(h, \alpha). \end{aligned}$$

By Theorem 2.1*, this implies $u_- = u$. Since $B_{t-} \leq B_t$, (3.24) and (3.28) yield $B_{t-} = B_t$ a.s. Function $A_t = B_{t-}$ satisfies conditions 1.6.A–C.

We claim that 1.6.D** holds for $\mathcal{N} = \mathcal{N}(h, \alpha)$. Indeed, if $\mu \notin \mathcal{N}(h, \alpha)$, then

$$B_t^n(\varepsilon) = Z_t^n(\varepsilon) + C_t^n$$

converges in P_μ -probability to B_t^n by (3.4). By the Markov property (1.34),

$$P_\mu e^{-B_{s+t}^n(\varepsilon)} = P_\mu \left[e^{-B_s^n(\varepsilon)} P_{Y_s^n} e^{-B_t^n(\varepsilon)} \right]$$

for all $s, t > 0$. This implies

$$P_\mu e^{-B_{s+t}^n} = P_\mu \left[e^{-B_s^n} P_{Y_s^n} e^{-B_t^n} \right]$$

and therefore

$$(3.29) \quad |P_\mu e^{-B_{s+t}^n} - P_\mu \left[e^{-B_s^n} P_{X_s} e^{-B_t^n} \right]| \leq P_\mu |P_{Y_s^n} e^{-B_t^n} - P_{X_s} e^{-B_t^n}|$$

By (3.5), the right side is equal to

$$P_\mu |e^{-\langle v_n^t, Y_s^n \rangle} - e^{-\langle v_n^t, X_s \rangle}|$$

where $v_n^t(x) = u_n^t(0, x)$. By (3.7), $v_n^t \leq h$ and therefore (3.29) does not exceed

$$P_\mu |1 - e^{-\langle h, X_s - Y_s^n \rangle}|.$$

By (3.19), this tends to 0 and we conclude from (3.29) and the Markov property of X that

$$(3.30) \quad P_\mu e^{-B_{s+t}} = P_\mu e^{-B_s} P_{X_s} e^{-B_t} = P_\mu e^{-(B_s + \theta_s B_t)}.$$

By (3.1), $B_{s+t} \leq B_s + \theta_s B_t$ and (3.30) implies 1.6.D**.

We get (1.45) by passing to the limit in (3.5) and (3.27) as $t \rightarrow \infty$.

3.10. Let h be an arbitrary function of the form (1.17) with η and ν subject to conditions (1.22). Consider measures η_n and ν_n defined in Lemma 3.3. Denote by A^n the functional of X corresponding to $h_n(x) = G\eta_n + K\nu_n$ by Section 3.9 and put

$$A = A_1 + \dots + A_n + \dots$$

Clearly, conditions 1.6.A,B,C and 1.6.D** hold for A . Formula (1.45) holds if $\mu \notin \mathcal{N} = \bigcup \mathcal{N}(h_n, \alpha)$ and v satisfies (1.14) on $E = \bigcap E(h_n, \alpha)$. Function \tilde{v} defined by (1.15) is a solution of (1.14) everywhere. It also satisfies (1.45).

Formula (1.44) can be obtained from (1.45) in the same way as (1.37) was deduced from (1.35).

By the “perfection” theorem [4, 15.8], there exists a functional equivalent to A which satisfies 1.6.D*. (In [4] functionals without an exceptional set are considered, but the proof is applicable without any change to our case.)

4. PROOF OF THEOREMS 1.2 AND 1.4

4.1.

Lemma 4.1. *Let an excessive function be given by formula (1.17). If there exists u such that*

$$(4.1) \quad u + G(u^\alpha) = h,$$

then there exists v such that

$$(4.2) \quad v + G(v^\alpha) = K\nu.$$

Proof. Let D_n, G^n and K^n have the same meaning as in Section 3.6. By the strong Markov property of ξ , (4.1) implies

$$(4.3) \quad u + G^n(u^\alpha) = G^n\eta + K^n u \quad \text{on } D_n.$$

By Theorem A,

$$v_n(x) = -\log P_x e^{-\langle u, X_{D_n} \rangle}$$

satisfies the equation

$$(4.4) \quad v_n + G^n(v_n^\alpha) = K^n u \quad \text{on } D_n.$$

We use again the strong Markov property of ξ to get from here that, for each $m > n$,

$$(4.5) \quad v_m + G^n(v_m^\alpha) = K^n v_m \quad \text{on } D_n.$$

We conclude from Theorem 2.1, by comparing (4.3) and (4.4), that $v_n \leq u$ in $D_n \cap E(h)$, and, by comparing (4.4) and (4.5), that $v_m \leq v_n$ in $D_n \cap E(h)$. Therefore there exists a limit

$$v = \lim_{n \rightarrow \infty} v_n \quad \text{on } E(h).$$

It follows from (4.3) by monotone convergence theorem that

$$u + G(u^\alpha) = G\eta + \lim K^n u.$$

In combination with (4.1), this yields $\lim K^n u = K\nu$ on $E(h)$. By (4.1), $G(u^\alpha) < \infty$ on $E(h)$ and, by the dominated convergence theorem, $\lim G^n(v_n^\alpha) = G(v^\alpha)$. Therefore (4.4) implies that (4.2) holds on $E(h)$. It holds everywhere for a function v modified by formula (1.15).

4.2. Proof of Theorem 1.2. Suppose that u is a solution of (4.1). By Lemma 4.1, equation (4.2) has a solution and ν does not charge \mathcal{R}^* -polar sets by Theorem 3.1 in [18].

It remains to prove that $\eta(\Gamma) = 0$ for \mathcal{R} -polar sets Γ . We can assume that Γ is compact. Let D be a bounded smooth domain such that $\Gamma \subset D$ and $\bar{D} \subset E$. Equation (4.1) implies

$$(4.6) \quad u + G_D(u^\alpha) = G_D\eta + K_D u \quad \text{in } D$$

(cf. (4.3)). By Theorem E° in [17], $\text{Cap}_{2,\alpha'}(\Gamma) = 0$. We use the following fact (see Lemma 4.1 in [2]): a signed measure γ does not charge sets Γ with $\text{Cap}_{2,\alpha'}(\Gamma) = 0$ if

$$\int_D \varphi(x)\gamma(dx) \leq \text{const} \cdot \|\varphi\|_{2,\alpha'} \quad \text{for all } \varphi \in C_0^\infty(D)$$

(here $\|\varphi\|_{2,\alpha'}$ is the norm of φ in the Sobolev space $W^{2,\alpha'}(D)$).

By Lemma 4.1, there exists $v \geq 0$ such that

$$(4.7) \quad v + G_D(v^\alpha) = K_D u \quad \text{in } D.$$

By Theorem 2.1, $w = u - v \geq 0$. There exists a function $q \geq 0$ such that $u^\alpha - v^\alpha = qw$ a.e. (cf. proof of Theorem 2.1*). It follows from (4.6) and (4.7) that

$$(4.8) \quad w + G_D(qw) = G_D \eta.$$

Let $\gamma(dx) = \eta(dx) - (qw)(x)dx$ and $\varphi \in C_0^\infty(D)$. Put $\psi = -L^* \varphi$. Note that $\|\psi\|_{\alpha'} \leq \|\varphi\|_{2,\alpha'}$ and

$$\varphi(y) = \int_D dx \psi(x) g_D(x, y).$$

By (4.8), $G_D \gamma = w$ and therefore

$$\begin{aligned} \int_D \varphi(x)\gamma(dx) &= \int_{D \times D} dx \psi(x) g_D(x, y) \gamma(dy) = \int_D w(x)\psi(x)dx \\ &\leq \|w\|_\alpha \|\psi\|_{\alpha'} \leq \|w\|_\alpha \|\varphi\|_{2,\alpha'}. \end{aligned}$$

If $h(c) < \infty$, then $G(u^\alpha)(c) < \infty$ by (4.1) and $u \in L^\alpha(D)$ because $\inf_D g(c, y) > 0$.

We have $0 \leq w \leq u$ and therefore $w \in L^\alpha(D)$. Hence $\gamma(\Gamma) = 0$. Since $\text{Cap}_{2,\alpha'}(\Gamma) = 0$ implies that the Lebesgue measure of Γ is equal to 0, we get $\eta(\Gamma) = 0$.

4.3. Localization. To prove Theorem 1.4, we need some preparations. Suppose that h is the potential of a linear additive functional A with exceptional set \mathcal{N} and let η, ν correspond to h by (1.17). For every positive bounded continuous function φ on $E \cup E^*$, we put $h^\varphi(x) = G(\eta^\varphi) + K(\nu^\varphi)$ where $\eta^\varphi(dx) = \varphi(x)\eta(dx)$, $\nu^\varphi(dx) = \varphi(x)\nu(dx)$. It follows from 1.6.D*, the strong Markov property of X and (1.44) that

$$\langle h, X_T \rangle = P_\mu \{A_\infty | \mathcal{F}_T\} - A_T \quad P_\mu\text{-a.s.}$$

for every \mathcal{F}_t -stopping time T and for every $\mu \notin \mathcal{N}$. It is easy to see from here that $\langle h, X_t \rangle$ is a supermartingale of class (D) relative to P_μ (cf. [4, V.15]). Since $\langle h^\varphi, X_t \rangle \leq \text{const} \langle h, X_t \rangle$, the same is true for $\langle h^\varphi, X_t \rangle$. By [4, Th. XV.6] or [25, Th. 38.1]¹³, there exists a *natural* additive functional A^φ ¹⁴ such that:

4.3.A. $P_\mu A_\infty^\varphi = \langle h^\varphi, \mu \rangle$ for all $\mu \notin \mathcal{N}$.

We call it the φ -localization of A . In the same way as in Theorem 3.3 of [17], we establish:

4.3.B. If $\varphi_1 \leq \varphi_2$, then $A^{\varphi_1} \leq A^{\varphi_2}$ P_μ -a.s. for all $\mu \notin \mathcal{N}$.

4.3.C. If $\varphi = 0$ on D , then $\{A^\varphi = 0\} \supset \{\mathcal{R} \subset \bar{D}\}$ P_μ -a.s. and $\{A^\varphi = 0\} \supset \{\mathcal{R}^* \subset \bar{D}\}$ P_μ -a.s. for all $\mu \notin \mathcal{N}$.

¹³As in the case of "perfection", these theorems can be easily extended to functionals with an exceptional set.

¹⁴An additive functional A is natural if the process A_{t+} is predictable (cf. [4, IV.61] or the Appendix to [9]). We believe that functional A constructed in Theorem 1.3 is natural but this is not proved in the present paper.

4.4. Proof of Theorem 1.4. This proof is similar to that of Theorem 3.3 in [18]. Let, for instance, $\Gamma \subset E$ be a compact \mathcal{R} -polar set. Put

$$D_n = \{x \in E : d(x, \Gamma) > \frac{1}{n}\}$$

where d is the distance in the Martin space \hat{E} . Bounded positive continuous functions

$$\varphi_n(x) = (1 - nd(x, \Gamma))_+$$

vanish on D_n . Consider the corresponding localizations A^{φ_n} . For every $\mu \notin \mathcal{N}$,

$$A^1 \geq A^{\varphi_1} \geq \dots \geq A^{\varphi_n} \geq \dots, \quad P_\mu\text{-a.s.}$$

by 4.3.B and

$$\{\mathcal{R} \subset D_n\} \subset \{A_\infty^{\varphi_n} = 0\}, \quad P_\mu\text{-a.s.}$$

by 4.3.C. Let $\mu(\Gamma) = 0$. Since Γ is \mathcal{R} -polar, $1_{\mathcal{R} \subset D_n} \uparrow 1$ P_μ -a.s. and therefore $A_\infty^{\varphi_n} \rightarrow 0$ P_μ -a.s. By the dominated convergence theorem,

$$(4.9) \quad \lim P_\mu A_\infty^{\varphi_n} = 0$$

On the other hand, by 4.3.A,

$$\begin{aligned} P_\mu A_\infty^{\varphi_n} &= \int \mu(dx) \int_E g(x, y) \varphi_n(y) \eta(dy) \\ &\quad + \int \mu(dx) \int_{E^*} \mu(dx) \int_{E^*} k(x, y) \varphi_n(y) \nu(dy) \downarrow \int \mu(dx) \int_\Gamma g(x, y) \eta(dy). \end{aligned}$$

In combination with (4.9), this implies $\eta(\Gamma) = 0$. The case of \mathcal{R}^* -polar set $\Gamma \subset E^*$ can be treated in a similar way.

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DEPARTMENT OF MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NEW YORK 14853-7901
E-mail address: ebd1@cornell.edu

CENTRAL ECONOMICS AND MATHEMATICAL INSTITUTE, RUSSIAN ACADEMY OF SCIENCES, 117418, MOSCOW, RUSSIA

Current address: Department of Mathematics, Cornell University, Ithaca, New York 14853-7901
E-mail address: sk47@cornell.edu