

## TENSOR PRODUCTS OVER ABELIAN $W^*$ -ALGEBRAS

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ABSTRACT. Tensor products of  $C^*$ -algebras over an abelian  $W^*$ -algebra  $Z$  are studied. The minimal  $C^*$ -norm on  $A \odot_Z B$  is shown to be just the quotient of the minimal  $C^*$ -norm on  $A \odot B$  if  $A$  or  $B$  is exact.

### 1. INTRODUCTION

The author's initial motivation for studying tensor products of  $C^*$ -algebras as modules over central abelian subalgebras was the fact that the Haagerup tensor product of modules (see [1], [6] and [17]) and some tensor products of von Neumann algebras (see [23], [24] and [28]) have nice properties. Later (five months after the first version of this paper was submitted) we were informed<sup>1</sup> that  $C^*$ -tensor products over abelian  $C^*$ -algebras have been studied recently also by Blanchard ([3], [4]) and Giordano and Mingo [10] (in the present revised version some of the overlapping results are omitted). Moreover such tensor products are closely related to the recent work of Kirchberg and Wassermann [15] and occurred previously also in [14], [8] and [9].

By a  $C^*$ -algebra over an abelian unital  $C^*$ -algebra  $C$  we mean a  $C^*$ -algebra  $A$  together with a unital  $*$ -homomorphism  $\varphi$  from  $C$  into the center of the multiplier algebra  $M(A)$  of  $A$ . Usually we shall write  $ca$  instead of  $\varphi(c)a$  for  $c \in C$  and  $a \in A$ . If there is a faithful representation  $\pi$  of  $M(A)$  on a Hilbert space such that  $\pi\varphi(C)$  is closed in the weak operator topology, then  $A$  is called a *proper  $C^*$ -algebra over  $C$* . In this case the unique normal extension of  $\pi\varphi$  to the universal enveloping  $W^*$ -algebra  $Z$  of  $C$  maps  $Z$  onto  $\pi\varphi(C)$ , hence we can regard  $A$  as a  $C^*$ -algebra over  $Z$  instead of  $C$ .

Given  $C^*$ -algebras  $A$  and  $B$  over  $C$ , the algebraic tensor product  $A \odot_C B$  is by definition the quotient  $(A \odot B)/N$ , where  $A \odot B$  denotes the algebraic tensor product of  $A$  and  $B$  viewed as complex algebras and  $N$  is the linear span of all elements of the form  $ac \otimes b - a \otimes cb$  ( $a \in A$ ,  $b \in B$ ,  $c \in C$ ). Since by assumption  $C$  commutes with  $A$  and  $B$ , the subspace  $N$  is a self-adjoint two-sided ideal in  $A \odot B$ ; hence  $A \odot_C B$  is an algebra (over  $C$ ) with an involution. It turns out that most of the basic theory of tensor products of  $C^*$ -algebras (see [13], [19], [26], [7]) can be extended to the context of proper  $C^*$ -algebras over  $C$ . In fact, some basic results can be extended to the setting of general  $C^*$ -algebras over  $C$  (see [4]), but proper

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$C^*$ -algebras over  $C$  can be regarded as  $C^*$ -algebras over  $Z = C''$  and represented faithfully on self-dual Hilbert  $Z$ -modules, which can play the role of Hilbert spaces.

In the preliminary Section 2 of this paper we recall some basic theory of Hilbert modules in a concrete form suitable for our application. Then in Section 3 we study algebraic properties of the product  $A \odot_Z B$  and introduce the *spatial* norm on  $A \odot_Z B$  by faithfully representing  $A$  and  $B$  on some Hilbert  $Z$ -modules  $E$  and  $F$  and then regarding  $A \odot_Z B$  as an algebra of operators on  $E \otimes_Z F$ . In the case  $A$  or  $B$  is exact we show that the kernel of the natural map from  $A \otimes B$  to  $A \otimes_Z B$  (= the completion of  $A \odot_Z B$  in the spatial norm) is just the closure of the ideal  $N$  defined above. If  $A$  and  $B$  are not proper, then it can happen that there is no  $C^*$ -norm on  $A \odot_C B$  (see Example 3.9). This is essentially an answer to the question of Elliott [8, p. 48], which has been solved recently and independently also by Blanchard [4]. In Section 4 we derive a formula for the spatial tensor norm which is analogous to the well known classical one (in the case  $Z = \mathbb{C}$ ), but using  $Z$ -valued states instead of usual scalar valued states.

Throughout this paper  $C$  will denote a commutative unital  $C^*$ -algebra,  $Z$  a commutative von Neumann algebra (usually the universal  $W^*$ -envelope of  $C$ ) and  $A, B$  will be general  $C^*$ -algebras over  $C$  or  $Z$ .

## 2. PRELIMINARIES CONCERNING HILBERT $C^*$ -MODULES

A (*right*) *Hilbert module over a  $C^*$ -algebra*  $A$  is a right  $A$ -module  $E$  equipped with an  $A$ -valued inner product  $\langle \cdot, \cdot \rangle$ , which is linear over  $A$  in the second factor, such that, with the norm  $\|x\| \stackrel{\text{def}}{=} \|\langle x, x \rangle\|^{1/2}$  ( $x \in E$ ),  $E$  is a Banach space (see [16], [20] and [21]). A Hilbert  $A$ -module  $E$  is *self-dual* if each bounded  $A$ -module homomorphism  $\rho : E \rightarrow A$  is of the form  $\rho(x) = \langle x_0, x \rangle$  for some  $x_0 \in E$ .

Given a Hilbert  $A$ -module  $E$ , we denote by  $L(E)$  the  $C^*$ -algebra of all operators on  $E$  that have an adjoint (each such operator is automatically a bounded homomorphism of the  $A$ -module  $E$ ). The *linking algebra*,  $\Lambda(E)$ , of  $E$  is the  $C^*$ -subalgebra of  $L(E \oplus A)$  consisting of all matrices of the form

$$\begin{bmatrix} b & x \\ y^* & a \end{bmatrix} \quad (a \in A, x, y \in E, b \in L(E)).$$

The concept of linking algebra was introduced in [5], but with  $L(E)$  replaced by the subalgebra of ‘compact’ operators. We shall regard  $A, E$  and  $L \stackrel{\text{def}}{=} L(E)$  as subsets of  $\Lambda(E)$  in the obvious way:

$$A \cong \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}, \quad E \cong \begin{bmatrix} 0 & E \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad L \cong \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix}.$$

Then the inner product  $\langle x, y \rangle$  in  $E$  becomes simply the product  $x^*y$  in  $\Lambda(E)$  and the module multiplication becomes a part of the internal multiplication in  $\Lambda(E)$ .

A Hilbert module  $E$  over a von Neumann algebra  $R$  is *faithful* if the ideal  $\langle E, E \rangle$  is weak\* dense in  $R$ . Equivalently,  $E$  is faithful iff  $Ea = 0$  implies  $a = 0$  ( $a \in R$ ).

From the results of Paschke [20] and Rieffel [21] one can deduce the following concrete realization of Hilbert modules.

**Theorem 2.1.** *Let  $E$  be a faithful Hilbert module over a von Neumann algebra  $R \subseteq B(\mathcal{H})$  and let  $Z$  be the center of  $R$ . Then there exists a von Neumann algebra  $\bar{\Lambda}$  and two projections  $p, q \in \bar{\Lambda}$  such that  $R = q\bar{\Lambda}q$ ,  $E$  is contained in  $\bar{E} = p\bar{\Lambda}q$  as a weak\* dense Hilbert  $R$ -submodule and  $L(E)$  is weak\* dense in  $L(\bar{E}) = p\bar{\Lambda}p$ ,*

where the  $R$ -valued inner product in  $\overline{E}$  is defined by  $\langle x, y \rangle = x^*y$ . Moreover, each bounded  $R$ -module homomorphism  $\rho : E \rightarrow R$  is of the form  $\rho(y) = \langle y_0, y \rangle$  for some  $y_0 \in \overline{E}$ . In particular,  $E$  is self-dual iff  $\overline{E} = E$ , and in this case  $L(E) = L(\overline{E})$  is a  $W^*$ -algebra and  $\Lambda(E) = \overline{\Lambda}$ .

Moreover, if  $R = Z$  (that is, if  $R$  is abelian) and  $E$  is self-dual, then  $E$  is (isometrically) isomorphic to a Hilbert module of the form  $Le$ , where  $L = L(E)$  is realized as a von Neumann algebra on some Hilbert space such that its commutant  $L'$  is isomorphic to  $Z$  and  $e$  is an abelian projection with central carrier  $c(e) = 1$  in  $L$ . Here the inner product in  $E$  is defined by  $\langle x, y \rangle = \tau^{-1}(x^*y)$ , where  $\tau$  is the isomorphism from  $Z$  onto  $Ze = eLe$  given by  $\tau(z) = ze$ .

The first part of Theorem 2.1 can be proved easily also by extending the identity representation of  $R$  to a representation of  $\Lambda(E)$  on some Hilbert space  $\mathcal{L} \supseteq \mathcal{H}$ , and more details of this proof will appear in the expository article [18]. To prove the last part of the theorem, we may then assume that  $E = p\Lambda q$ , where  $\Lambda$  is a von Neumann algebra and the central carriers of the projections  $p$  and  $q$  in  $\Lambda$  may be assumed to be 1 (otherwise reduce  $\Lambda$ ). Then, since  $q\Lambda q = Z$  is abelian, it follows that  $q$  is equivalent to a projection  $e \leq p$  in  $\Lambda$ . This implies that  $E$  is isomorphic to  $p\Lambda e = Le$ , since  $L = p\Lambda p$ .

From Theorem 2.1 it follows in particular that a self-dual Hilbert module  $E$  over a von Neumann algebra is a dual Banach space (this was first proved in [20]). Note that the weak\* topology on  $E$  can be defined intrinsically (without using any representations): on bounded sets it is given by the family of seminorms  $x \mapsto \rho(\langle y, x \rangle)$ , where  $y \in E$  and  $\rho$  is a normal functional on  $R$ . From Theorem 2.1 we see also that each Hilbert  $R$ -module can be embedded into a self-dual Hilbert  $R$ -module  $\overline{E}$  such that  $E$  is weak\* dense in  $\overline{E}$  and each  $b \in L(E)$  extends uniquely to an operator in  $L(\overline{E})$  (all this was also proved in [20]).  $\overline{E}$  will be called *the self-dual hull of  $E$* .

### 3. THE SPATIAL TENSOR PRODUCT

If  $E$  and  $F$  are Hilbert modules over  $C$ , then (regarding  $F$  as a left  $C$ -module by  $cy \stackrel{\text{def}}{=} yc$  for  $y \in F$  and  $c \in C$ ) the algebraic tensor product  $E \odot_C F$  can be equipped with the  $C$ -valued inner product by

$$\langle x \odot_C y, u \odot_C v \rangle = \langle x, u \rangle \langle y, v \rangle \quad (x, u \in E, y, v \in F).$$

*Remark 3.1.* This construction is just a special case of the internal tensor product of Hilbert modules over not necessarily abelian  $C^*$ -algebras (see Chapter 4 in [16]). In particular, it follows from Proposition 4.5 in [16] that the above inner product is indeed positive definite.

The completion of  $E \odot_C F$  in the norm induced from the inner product is a Hilbert module over  $C$  and will be denoted by  $E \otimes_C F$ . If  $E$  and  $F$  are Hilbert modules over an abelian  $W^*$ -algebra  $Z$ , then the self-dual hull of  $E \otimes_Z F$  will be denoted by  $E \otimes_Z F$ .

Using the techniques from [16, Chapter 4] it is easy to see that for each  $a \in L(E)$  and  $b \in L(F)$  there exists a unique operator  $a \otimes_C b \in L(E \otimes_C F)$  such that  $(a \otimes_C b)(x \otimes_C y) = ax \otimes_C by$  for all  $x \in E$  and  $y \in F$ , and that  $\|a \otimes_C b\| \leq \|a\| \|b\|$ .

We denote by  $M_n(E)$  the set of all  $n \times n$  matrices with entries in a given set  $E$ . The following lemma is formally similar to Theorem 5.5.4 in [13] and is equivalent to Theorem 3.1 of [10]. We shall give a direct proof.

**Lemma 3.2.** *Let  $E$  and  $F$  be Hilbert modules over an abelian  $W^*$ -algebra  $Z$  and  $a_i \in L(E)$ ,  $b_i \in L(F)$  ( $i = 1, \dots, n$ ). Then the identity  $\sum_{i=1}^n a_i \otimes_Z b_i = 0$  holds in  $L(E \otimes_Z F)$  if and only if there exists a projection  $p = [p_{ij}] \in M_n(Z)$  such that*

$$\sum_{j=1}^n a_j p_{ji} = 0 \quad \text{and} \quad \sum_{j=1}^n p_{ij} b_j = b_i$$

for all  $i = 1, \dots, n$ .

*Proof.* For arbitrary  $x \in E$ ,  $y \in F$  put

$$\mathbf{x} = \begin{bmatrix} a_1 x & \dots & a_n x \\ 0 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} b_1 y & \dots & b_n y \\ 0 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & 0 \end{bmatrix}$$

and regard  $M_n(E)$  and  $M_n(F)$  as subspaces in  $M_n(\Lambda(E))$  and  $M_n(\Lambda(F))$  (respectively). Let  $a_x = [\langle a_i x, a_j x \rangle] = \mathbf{x}^* \mathbf{x}$  and let  $b_y = [\langle b_i y, b_j y \rangle]$  be the transpose of the matrix  $\mathbf{y}^* \mathbf{y}$  in  $M_n(Z)$ . Suppose that  $w \stackrel{\text{def}}{=} \sum_{i=1}^n a_i \otimes_Z b_i = 0$ . Then we have

$$(3.1) \quad \sum_{i,j=1}^n \langle a_i x, a_j x \rangle \langle b_i y, b_j y \rangle = \langle w(x \otimes_Z y), w(x \otimes_Z y) \rangle = 0.$$

Since  $a_x$  is positive, there exists a positive  $g = [g_{ij}] \in M_n(Z)$  such that  $a_x = g^2$ , and (3.1) can be rewritten as

$$0 = \sum_{i,j=1}^n \sum_{k=1}^n g_{ik} g_{kj} (b_y)_{ji} = \sum_{k=1}^n (g b_y g)_{kk}.$$

Thus all the diagonal entries of the positive operator matrix  $g b_y g$  are 0 (note that  $b_y$  is positive as the transpose of the positive matrix  $\mathbf{y}^* \mathbf{y}$  over the abelian  $C^*$ -algebra  $Z$ ); hence  $g b_y g = 0$ , which implies that

$$(3.2) \quad a_x b_y = 0.$$

Let  $p_y \in M_n(Z)$  be the range projection of  $b_y$ . Then from (3.2) we have  $a_x p_y = 0$  and  $p_y^\perp b_y = 0$  (where  $p_y^\perp = 1 - p_y$ ), which implies (by a simple computation, writing  $p_y$  as a matrix and using the definiteness of the inner product) that

$$[a_1 x, \dots, a_n x] p_y = 0 \quad \text{and} \quad p_y^\perp \begin{bmatrix} b_1 y \\ \vdots \\ b_n y \end{bmatrix} = 0.$$

Hence, with  $p = \bigvee_{y \in F} p_y$ , we have

$$[a_1, \dots, a_n] p = 0 \quad \text{and} \quad p \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

With  $[p_{ij}] = p$  ( $p_{ij} \in Z$ ), we now see that  $\sum_{i=1}^n a_i \otimes_Z b_i = \sum_{i=1}^n a_i \otimes_Z \sum_{j=1}^n p_{ij} b_j = \sum_{j=1}^n \sum_{i=1}^n a_i p_{ij} \otimes_Z b_j = 0$ . This proves the lemma in one direction. The proof of the converse is trivial.  $\square$

The following result is an immediate consequence of Lemma 3.2.

**Corollary 3.3.** *If  $A \subseteq L(E)$  and  $B \subseteq L(F)$  are  $C^*$ -algebras over  $Z$ , then the natural map  $\Theta : A \odot_Z B \rightarrow L(E \otimes_Z F)$  is one to one. Thus, if  $A_0 \subseteq A$  and  $B_0 \subseteq B$  are  $C^*$ -algebras over  $Z$ , then the natural map  $A_0 \odot_Z B_0 \rightarrow A \odot_Z B$  is also one to one.*

To see that Corollary 3.3 cannot be extended to general abelian  $C^*$ -algebras in place of  $Z$ , let  $C \subseteq l^\infty$  be the  $C^*$ -algebra of all convergent sequences and let  $E$  consists of all sequences  $(c_i)$  with the entries in  $C$  such that  $\sum_i c_i^* c_i$  converges in norm. Then  $E$  is a Hilbert  $C$  module (the direct sum of countably many copies of  $C$ ) and  $l^\infty$  can be regarded as a  $C^*$ -subalgebra of  $L(E)$  by the diagonal action  $(\alpha_i)(c_i) = (\alpha_i c_i)((\alpha_i) \in l^\infty, (c_i) \in E)$ . Let  $a = (1, 0, 1, 0, \dots)$  and  $b = (0, 1, 0, 1, \dots)$  and let  $e_1, e_2, \dots$  be the standard basic elements of  $E$ . Note that  $e_i \otimes_C e_j = 0$  if  $i \neq j$  and  $(a \otimes_C b)(e_i \otimes_C e_j) = 0$  for all  $i, j$ . Thus  $a \otimes_C b = 0$  in  $L(E \otimes_C E)$ . On the other hand,  $a \otimes_C b \neq 0$  in  $l^\infty \otimes_C l^\infty$ . Indeed, if  $\phi$  and  $\psi$  are characters on  $l^\infty$  annihilating the sequences converging to 0 (hence  $\phi|_C = \psi|_C$ ) such that  $\phi(a) = 1 = \psi(b)$ , then the correspondence  $x \otimes_C y \mapsto \phi(x)\psi(y)$  can be extended to a linear map  $\rho : l^\infty \otimes_C l^\infty \rightarrow C$  such that  $\rho(a \otimes_C b) = 1$ .

*Remark 3.4.* It follows from [14, 1.5] that each  $C^*$ -algebra  $A$  over a  $C^*$ -algebra  $C$  can be embedded into  $L(E)$  for some self-dual Hilbert  $Z$ -module  $E$ , where  $Z$  is the universal von Neumann envelope of  $C$ . To see this more directly, we may assume that  $A \subseteq B(\mathcal{H})$  for some Hilbert space  $\mathcal{H}$  such that  $[A\mathcal{H}] = \mathcal{H}$ . Let  $\varphi$  be the  $*$ -homomorphism from  $C$  into the center of  $M(A) \subseteq B(\mathcal{H})$ , which determines the  $C$ -module structure on  $A$ , let  $\bar{\varphi}$  be the normal extension of  $\varphi$  to  $Z$  and let  $p \in Z$  be the projection such that  $\bar{\varphi}|_{Zp}$  is an isomorphism onto  $\bar{\varphi}(Z)$ . Put  $Z_0 = \bar{\varphi}(Z)$  and  $L = Z'_0$ . Let  $e$  be any abelian projection in  $L$  with central carrier 1 and let  $E \stackrel{\text{def}}{=} Le$  have the  $Z_0$ -valued inner product defined as in the last sentence of Theorem 2.1. Then  $L(E) = L$ ; hence  $A \subseteq L(E)$ . Since  $Z_0 \cong Zp$ ,  $E$  can be regarded as a Hilbert  $Z$ -module, and by adding the direct summand  $Zp^\perp$  to  $E$  we obtain a faithful self-dual Hilbert  $Z$ -module.

If  $A$  and  $B$  are proper  $C^*$ -algebras over  $C$  and  $Z$  is the universal  $W^*$ -envelope of  $C$ , then by Remark 3.4 we may assume that  $A \subseteq L(E)$  and  $B \subseteq L(F)$  for some (faithful self-dual) Hilbert  $Z$ -modules  $E$  and  $F$ , and  $A \odot_Z B$  can be regarded as a subalgebra of  $L(E \otimes_Z F)$  by Corollary 3.3 and therefore inherits a  $C^*$ -norm, which will be called *the spatial norm* on  $A \odot_Z B (= A \odot_C B)$ . We shall obtain a formula for this norm which will be independent of the choice of embedding of  $A$  and  $B$  into operators on Hilbert  $Z$ -modules. The completion of  $A \odot_Z B$  in the spatial norm will be denoted by  $A \otimes_Z B$  (or  $A \otimes_C B$ ) and called *the spatial tensor product of  $A$  and  $B$* . More generally, we denote by  $A \otimes_Z^\alpha B$  the completion of  $A \odot_Z B$  in a given  $C^*$ -norm  $\alpha$ .

*Remark 3.5.* The theory of such tensor products can be reduced to the case where  $Z$  is contained in the centers of  $A$  and  $B$ . To see this, we may assume that  $A \subseteq B(\mathcal{H})$  and  $B \subseteq B(\mathcal{K})$  and let  $\varphi, \psi$  be the normal  $*$ -homomorphisms from  $Z$  into the centers of the multiplier algebras  $M(A)$  and  $M(B)$  (resp.) that determine the  $Z$ -module structure on  $A$  and  $B$ . As in the classical case (see [27, T.6.1]) there is a natural  $*$ -homomorphism from  $M(A) \odot_Z M(B)$  into  $M(A \otimes_Z^\alpha B)$ . This homomorphism is injective, since a simple argument (somewhat similar to the one in the proof of Lemma 3.2) shows that an element  $w$  of  $M(A) \odot_Z M(B)$  annihilates all  $x \otimes_Z y \in A \odot_Z B$  only if  $w = 0$ . Consequently, we may regard  $M(A) \odot_Z M(B)$

as a subalgebra of  $M(A \otimes_Z^\alpha B)$ , from which  $M(A) \odot_Z M(B)$  inherits a norm which extends the norm  $\alpha$  on  $A \odot_Z B$ . This shows that in studying  $C^*$ -norms there is no essential loss of generality in restriction to unital algebras. Then  $\varphi$  and  $\psi$  are normal  $*$ -homomorphisms from  $Z$  into the centers of  $A$  and  $B$  (resp.), hence  $\ker \varphi = e^\perp Z$  and  $\ker \psi = f^\perp Z$  for some projections  $e, f \in Z$ . Put  $p = ef$ . Then  $A \odot_Z B = Ap \odot_{Zp} pB$ ; hence, replacing  $A$ ,  $B$  and  $Z$  with  $Ap$ ,  $pB$  and  $Zp$ , we may assume that  $\varphi$  and  $\psi$  are  $*$ -isomorphisms. Moreover, the decomposition of  $\varphi(Z)'$  and  $\psi(Z)'$  into homogeneous parts (see [13] or [26]) induces the decomposition of  $A \odot_Z B$ , since  $\varphi(Z)$  and  $\psi(Z)$  are contained in centers of  $A$  and  $B$ . It suffices to study each summand separately. Finally, replacing  $A$ ,  $B$  and  $Z$  by suitable isomorphic algebras (of the form  $A \otimes 1, \dots$ ), we may assume that  $Z$  is contained in the centers of  $A$  and  $B$  and that  $Z'$  is homogeneous.

For a unital abelian  $C^*$ -algebra  $C$  we shall always denote by  $\Delta$  the spectrum of  $C$  (= the space of pure states) and identify  $C$  with the  $C^*$ -algebra of all continuous functions on  $\Delta$ . For each  $s \in \Delta$  we denote by  $C_s$  the kernel of  $s$ . If  $C$  is contained in the center of a  $C^*$ -algebra  $A$ , then the closed ideal of  $A$  generated by  $C_s$  will be denoted by  $A_s$  and called the *Glimm ideal at  $s$* . It is well known that  $A_s$  is the intersection of left kernels of all pure states  $\rho$  on  $A$  satisfying  $\rho(C_s) = 0$  and that

$$(3.3) \quad \|a\| = \sup_{s \in \Delta} \|a(s)\|$$

for each  $a \in A$ , where  $a(s)$  denotes the coset of  $a$  in  $A(s) \stackrel{\text{def}}{=} A/A_s$ . This implies that the natural  $*$ -homomorphism

$$\Psi_A : A \rightarrow \bigoplus_{s \in \Delta} A(s), \quad \Psi_A(a)(s) = a(s)$$

is one to one. For each  $a \in A$  the function  $s \mapsto \|a(s)\|$  is upper semi-continuous and, if  $C$  is a  $W^*$ -algebra, then this function is continuous. All this was proved by Glimm in [11, Section 4] for the case  $C$  is the center of  $A$ , but the same arguments apply if  $C$  is a  $C^*$ -subalgebra of the center. (To prove the continuity of the function  $s \mapsto \|a(s)\|$  in the case  $C$  is a von Neumann algebra, regard  $A$  as a subalgebra of  $C'$  and use [11, Lemma 10] and Remark 3.6 below.)

*Remark 3.6.* If  $B$  is a  $C^*$ -subalgebra of  $A$  containing  $C$ , then  $B_s = A_s \cap B$  for each  $s \in \Delta$ , since each pure state on  $B$  extends to a pure state on  $A$ . It follows that the natural  $*$ -homomorphism  $B(s) \rightarrow A(s)$  is injective; hence  $B(s)$  will be regarded as a  $C^*$ -subalgebra of  $A(s)$ .

If  $C$  is contained in the centers of two  $C^*$ -algebras  $A, B \subseteq B(\mathcal{H})$ , then the two quotient mappings  $\alpha_s : A \rightarrow A(s)$  and  $\beta_s : B \rightarrow B(s)$  induce a map  $\iota_s : A \odot_C B \rightarrow A(s) \odot B(s)$  (where  $\odot$  means  $\odot_C$ ), and the collection of all maps  $\iota_s$  ( $s \in \Delta$ ) defines a  $*$ -homomorphism

$$(3.4) \quad \iota : A \odot_C B \rightarrow \bigoplus_{s \in \Delta} A(s) \otimes B(s), \quad \iota(x)(s) = \iota_s(x).$$

Given a Hilbert  $C$ -module  $E$  and  $s \in \Delta$ , let

$$E_s = \{x \in E : \langle x, x \rangle(s) = 0\}.$$

Then  $E_s$  is a Hilbert submodule of  $E$  and the quotient  $E(s) = E/E_s$  is a Hilbert space with the inner product

$$\langle x(s), y(s) \rangle = \langle x, y \rangle(s),$$

where  $x(s)$  and  $y(s)$  denote the cosets of  $x$  and  $y$  in  $E(s)$ . (The completeness of  $E(s)$  follows since the norm  $\|x(s)\|$  in  $E(s)$  induced from the inner product coincides with the quotient norm as can be seen by a standard application of an approximate unit of  $C_s$ , but this will not be important here.) There is a representation  $\pi_s$  of  $L(E)$  on  $E(s)$  defined by  $\pi_s(a)x(s) = (ax)(s)$ .

**Proposition 3.7.** *The  $*$ -homomorphism*

$$\pi : L(E) \rightarrow \bigoplus_{s \in \Delta} B(E(s)), \quad \pi(a)(s) = \pi_s(a)$$

is one to one. If  $C = Z$  is an abelian  $W^*$ -algebra and  $E$  is faithful and self-dual, then the kernel of  $\pi_s$  is the Glimm ideal  $L(E)_s$  for each  $s \in \Delta$ .

*Proof.* If  $a \in \ker \pi$ , then  $\langle ax, ax \rangle(s) = 0$  for all  $x \in E$  and  $s \in \Delta$ , hence  $ax = 0$  and  $a = 0$ .

Assume now that  $C = Z$  is a  $W^*$ -algebra and  $E$  is self-dual and faithful. Put  $L = L(E)$ . Then by Theorem 2.1 we may assume that  $E = Le$  for some abelian projection  $e \in L$  with the central carrier  $c(e) = 1$ , and we may identify  $L'$  with  $Z$  (in some representation of  $L$  on a Hilbert space). Since the function  $t \mapsto \|e(t)\|$  (where  $e(t)$  denotes the coset of  $e$  in  $L/L_t$ , with  $L_t$  the Glimm ideal of  $L$  at  $t$ ) is continuous on  $\Delta$  and can take the values 1 or 0 only, while  $c(e) = 1$ , it follows that  $\|e(t)\| = 1$  for all  $t \in \Delta$ . Given  $x \in L$  and  $s \in \Delta$ , we then have by the definition of the inner product in  $E = Le$  that  $\langle xe, xe \rangle(s) = \tau^{-1}(ex^*xe)(s)$ , where  $\tau : Z \rightarrow Ze$  is the  $*$ -isomorphism  $z \mapsto ze$ . Since  $e(s) \neq 0$  and  $z(s) \in \mathbb{C}$  for each  $z \in Z$ , it follows easily that  $\langle xe, xe \rangle(s) = 0$  if and only if  $xe \in L_s$ ; thus  $E_s = Le \cap L_s = L_s e$ . Given  $a \in L$ , it follows that  $a \in \ker \pi_s$  if and only if  $aLe \subseteq L_s$ . Since  $L_s$  is a prime ideal by [11, Lemma 11] and  $e \notin L_s$  (for  $\|e(s)\| = 1$ ), we see that  $aLe \subseteq L_s$  if and only if  $a \in L_s$ . Thus,  $\ker \pi_s = L_s$ .  $\square$

**Corollary 3.8.** *If  $Z$  is contained in the centers of  $C^*$ -algebras  $A, B \subseteq B(\mathcal{H})$ , then the  $*$ -homomorphism  $\iota : A \odot_Z B \rightarrow \bigoplus_{s \in \Delta} A(s) \otimes B(s)$  (defined by (3.4)) is isometric if  $A \odot_Z B$  is equipped with the spatial  $C^*$ -norm.*

*Proof.* We may assume that  $A \subseteq L(E)$  and  $B \subseteq L(F)$  for some faithful self-dual Hilbert  $Z$ -modules  $E$  and  $F$  (see Remark 3.4). It is easy to see that for each  $s \in \Delta$  the natural map  $E \odot_Z F \rightarrow E(s) \otimes F(s)$  induces an (isometric) isomorphism of Hilbert spaces  $(E \otimes_Z F)(s) \rightarrow E(s) \otimes F(s)$ , hence we shall identify these two Hilbert spaces. The representation  $\pi_s$  of  $L(E)$  on  $E(s)$  induces a  $*$ -homomorphism  $\alpha(s) : A(s) \rightarrow B(E(s))$  (which is in fact injective by Proposition 3.7 and Remark 3.6) and in the same way we have a  $*$ -homomorphism  $\beta(s) : B(s) \rightarrow B(F(s))$ . Let  $\psi(s) = \alpha(s) \otimes \beta(s)$  and consider now the commutative diagram

$$\begin{array}{ccc} A \odot_Z B & \xrightarrow{\Theta} L(E \otimes_Z F) \xrightarrow{\pi} & \bigoplus_{s \in \Delta} B(E(s) \otimes F(s)) \\ & \searrow \iota & \uparrow \psi \\ & & \bigoplus_{s \in \Delta} A(s) \otimes B(s) \end{array}$$

where  $\Theta$  is the injection from Corollary 3.3 (and isometric by the definition of the spatial norm),  $\pi$  is the  $*$ -monomorphism from Proposition 3.7 (with  $E \otimes_Z F$

instead of  $E$ ) and  $\psi$  is the direct sum of the  $*$ -homomorphisms  $\psi(s)$ . Since  $\psi$ ,  $\pi$  and  $\Theta$  are isometric, it follows that  $\iota$  is isometric.  $\square$

For a result related to Corollary 3.8 in a more general context, see [3, 3.21] and [4, 3.1].

**Example 3.9.** For a general abelian (unital)  $C^*$ -algebra  $C$  the above  $*$ -homomorphism  $\iota$  is not always injective. (This answers a question of Elliott [8, p. 48].) Namely, let  $A$  be the  $C^*$ -algebra of all bounded functions on the circle  $\Delta = [0, 1]/\{0, 1\}$  and let  $C$  be the subalgebra of all continuous functions. Choose two irrational numbers  $\alpha$  and  $\beta$  such that 1,  $\alpha$  and  $\beta$  are linearly independent over the rationals, define  $a \in A$  by

$$a(s) = \begin{cases} 1/k & \text{if } s - k\alpha \in \mathbb{Z} \text{ for some } k \in \mathbb{Z} \setminus \{0\}, \\ 0 & \text{otherwise,} \end{cases}$$

and define  $b \in A$  similarly to  $a$  (but using  $\beta$  instead of  $\alpha$ ). It is well known that the orbits of  $\alpha$  and  $\beta$  are dense in  $\Delta$ , and it is easy to see that the functions  $a$  and  $b$  are upper semi-continuous. Note that  $a$  is in the Glimm ideal  $A_s$  if  $a(s) = 0$  (where  $a(s)$  denotes here the value of  $a$  at  $s$ , not the coset in  $A(s)$ ). Indeed, given  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $|a(t)| < \varepsilon$  for all  $t$  satisfying  $|t - s| < \delta$ . Choose  $c \in C$  such that  $0 \leq c \leq 1$ ,  $c(s) = 0$  and  $c(t) = 1$  if  $|t - s| \geq \delta$ . Then  $\|a - ca\| < \varepsilon$ , hence (letting  $\varepsilon \rightarrow 0$ )  $a \in A_s$ . Similarly,  $b \in B_s$  if  $b(s) = 0$ .

Since for each  $s \in \Delta$  at least one of  $a(s)$  or  $b(s)$  is 0, it follows that  $(a + A_s) \otimes (b + A_s) = 0$  in  $A(s) \otimes A(s)$  for all  $s$ , thus  $\iota(a \otimes_C b) = 0$ . On the other hand, since  $Ca$  and  $Cb$  are free  $C$ -modules (note that  $ca = 0$  for  $c \in C$  implies that  $c = 0$  and similarly for  $b$ ), the element  $a \otimes_C b$  is a generator of the free  $C$ -module  $Ca \otimes_C Cb \cong C$ . To see that  $a \otimes_C b \neq 0$  in  $A \otimes_C A$ , let  $\hat{C}$  be the injective hull of  $C$  (which coincides as a  $C$ -module with the Johnson ring of quotients of  $C$ ; see [22, 3.3.15]), let  $\varphi_0 : Ca \rightarrow C \subseteq \hat{C}$  and  $\psi_0 : Cb \rightarrow \hat{C}$  be defined by  $\varphi_0(ca) = c$  and  $\psi_0(cb) = c$  (resp.), and let  $\varphi, \psi : A \rightarrow \hat{C}$  be extensions of  $\varphi_0$  and  $\psi_0$  as  $C$ -module homomorphisms. Defining  $\rho : A \otimes_C A \rightarrow \hat{C}$  by  $\rho(x \otimes_C y) = \varphi(x)\psi(y)$ , we see that  $\rho(a \otimes_C b) = 1$ , hence  $a \otimes_C b \neq 0$ .

In general, if  $A$  and  $B$  are abelian  $C^*$ -algebras containing  $C$ , then it is easy to see that for each possible  $C^*$ -norm  $\alpha$  on  $A \otimes_C B$  the spectrum  $\mathcal{P}$  of  $A \otimes_C B$  is homeomorphic with

$$\{(\varphi, \psi) \in P(A) \times P(B) : \varphi|_C = \psi|_C\},$$

where a pair  $(\varphi, \psi)$  corresponds to the character

$$(\varphi \otimes_C \psi)(a \otimes_C b) \stackrel{\text{def}}{=} \varphi(a)\psi(b).$$

(See [13] for the proof in the special case  $C = \mathbb{C}$ ; the proof in general is almost the same.) Hence there is at most one  $C^*$ -norm on  $A \otimes_C B$ . This  $C^*$ -(semi)norm is given of course by

$$(3.5) \quad \alpha(g) = \sup_{\rho \in \mathcal{P}} \rho(g).$$

It is easy to see that the quantity on the right side of (3.5) is equal to the norm of  $\iota(g)$  for each  $g \in A \otimes_C B$ , where  $\iota$  is the  $*$ -homomorphism defined by (3.4). Since we have just seen that  $\iota$  is not always injective, it follows that the quantity in (3.5) is not always a norm.

If  $Z$  is a von Neumann algebra contained in the centers of two  $C^*$ -algebras  $A \subseteq L(E)$  and  $B \subseteq L(F)$  (where  $E$  and  $F$  are faithful self-dual Hilbert  $Z$ -modules),  $q : A \odot B \rightarrow A \odot_Z B$  is the quotient homomorphism and  $\Theta$  and  $\pi$  are as in the diagram in the proof of Corollary 3.8, then  $\pi\Theta q : A \odot B \rightarrow \bigoplus_{s \in \Delta} B(E(s)) \otimes F(s)$  coincides with the restriction to  $A \odot B$  of the direct sum over  $(s \in \Delta)$  of maps  $\pi_s^A \otimes \pi_s^B : A \otimes B \rightarrow B(E(s)) \otimes B(F(s))$  (where  $\pi_s^A : A \rightarrow B(E(s))$  is the restriction of the representation of  $L(E)$  on  $E(s)$  defined immediately before Proposition 3.7, and  $\pi_s^B$  is defined similarly). Since  $\pi$  and  $\Theta$  are isometric, it follows that  $q$  is bounded (when  $A \odot B$  and  $A \odot_Z B$  are equipped with the spatial norms) and extends uniquely to a  $*$ -homomorphism  $\varphi : A \otimes B \rightarrow A \otimes_Z B$ . Obviously  $\varphi$  is onto and the kernel of  $\varphi$  contains the closed linear span  $[N]$  of the set  $\{az \otimes b - a \otimes zb : a \in A, b \in B, z \in Z\}$ . Thus,  $\varphi$  induces a  $*$ -epimorphism  $\Phi : (A \otimes B)/[N] \rightarrow A \otimes_Z B$ .

From now on denote

$$A \check{\otimes}_Z B = (A \otimes_Z B)/[N]$$

and

$$J_s = A_s \otimes B + A \otimes B_s.$$

Observe that  $[N] \subseteq J_s$  for all  $s \in \Delta$ . Indeed, for each pure state  $\rho$  on  $A \otimes B$  satisfying  $\rho((Z \otimes Z)_{(s,s)}) = 0$  we have  $\rho(ac \otimes b - a \otimes cb) = \rho(a \otimes b)\rho(c \otimes 1) - \rho(a \otimes b)\rho(1 \otimes c) = \rho(a \otimes b)(c(s) - c(s)) = 0$  for all  $a \in A, b \in B$  and  $c \in Z$  (see [13, 4.3.14]). Note that  $Z \otimes Z$  is contained in the center of  $A \otimes B$ , the spectrum of  $Z \otimes Z$  is  $\Delta \times \Delta$  and the kernel of the character  $(s, t)$  is  $Z_s \otimes Z + Z \otimes Z_t$  for all  $s, t \in \Delta$  (by an application of the Stone - Weierstrass theorem). This implies that the Glimm ideal  $(A \otimes B)_{(s,t)} = [(A \otimes B)(Z \otimes Z)_{(s,t)}]$  is equal to  $A_s \otimes B + A \otimes B_t$ ; in particular  $(A \otimes B)_{(s,s)} = J_s$ . We need a simple lemma.

**Lemma 3.10.** *The  $*$ -homomorphism*

$$\Psi : A \check{\otimes}_Z B \rightarrow \bigoplus_{s \in \Delta} (A \otimes B)/J_s, \quad \Psi(a \otimes b + [N])(s) = a \otimes b + J_s$$

is one to one.

*Proof.* For each  $g \in A \otimes B$  denote by  $\dot{g}$  the coset of  $g$  in  $A \check{\otimes}_Z B$  and by  $g(s, t)$  the coset of  $g$  in  $(A \otimes B)/(A \otimes B)_{(s,t)}$ . Assume that  $\dot{g} \in \ker \Psi$ . We shall prove that  $\|\dot{g}\| \leq \varepsilon$  for each  $\varepsilon > 0$ , hence  $\dot{g} = 0$ . Given  $\varepsilon > 0$ , let  $W = \{(s, t) \in \Delta \times \Delta : \|g(s, t)\| \geq \varepsilon\}$ . Since the function  $(s, t) \mapsto \|g(s, t)\|$  is upper semicontinuous by [11],  $W$  is closed. Since  $\dot{g} \in \ker \Psi, g(s, s) = 0$  for each  $s \in \Delta$ ; hence there exists an open neighborhood  $\mathcal{U}$  of the diagonal  $\{(s, s) : s \in \Delta\}$  in  $\Delta \times \Delta$  such that  $\overline{\mathcal{U}} \cap W = \emptyset$ . By Urysohn's lemma there exists  $d \in Z \otimes Z$  such that  $0 \leq d \leq 1, d|_{\overline{\mathcal{U}}} = 0$  and  $d|_W = 1$ . Then, with  $g_0 = dg$ , we have

$$(3.6) \quad \|g_0 - g\| \leq \varepsilon$$

and  $g_0(s, t) = 0$  for all  $(s, t) \in \mathcal{U}$ .

Let  $\{\mathcal{V}_i : i = 1, \dots, n\}$  be a partition of the extremely disconnected compact space  $\Delta$  into closed and open subsets such that  $\mathcal{V}_i \times \mathcal{V}_i \subseteq \mathcal{U}$  for all  $i$ , and denote by  $p_i$  the projection in  $Z$  corresponding to  $\mathcal{V}_i$ . Since  $g_0|_{\mathcal{U}} = 0$ , we have

$$g_0(s, t) = \sum_{i,j=1}^n p_i(s)g_0(s, t)p_j(t) = \sum_{i \neq j} p_i(s)g_0(s, t)p_j(t)$$

for all  $s, t \in \Delta$ ; hence

$$(3.7) \quad g_0 = \sum_{i \neq j} (p_i \otimes 1)g_0(1 \otimes p_j) = \sum_{i \neq j} (p_i \otimes 1 - 1 \otimes p_i)g_0(1 \otimes p_j).$$

Approximating each term  $g_0(1 \otimes p_j)$  by a sum of simple tensors, we see from (3.7) that  $g_0 \in [N]$ ; hence we conclude from (3.6) that  $\|\dot{g}\| \leq \varepsilon$ .  $\square$

**Corollary 3.11.** *If  $A_0 \subseteq A$  and  $B_0 \subseteq B$  are concrete  $C^*$ -algebras containing  $Z$  in their centers, then the natural homomorphisms  $A_0 \otimes_Z B_0 \rightarrow A \otimes_Z B$  and  $A_0 \check{\otimes}_Z B_0 \rightarrow A \check{\otimes}_Z B$  are injective.*

*Proof.* This result follows immediately by application of Remark 3.6, Corollary 3.8, Lemma 3.10 and the fact that the usual spatial tensor product preserves inclusions.  $\square$

The injectivity of the second homomorphism in Corollary 3.11 means that the two closed ideals,  $N(A_0, B_0)$  in  $A_0 \otimes B_0$  and  $N(A, B)$  in  $A \otimes B$ , generated by all elements of the form  $w = c \otimes 1 - 1 \otimes c$  ( $c \in Z$ ), satisfy  $N(A_0, B_0) = N(A, B) \cap (A_0 \otimes B_0)$ . The last equality can be proved easily also by using the fact that a left ideal in a  $C^*$ -algebra is equal to the intersection of the left kernels of all (pure) states annihilating it (and that states can be extended to larger  $C^*$ -algebras), which shows that the result is true also if  $Z$  is just a unital abelian  $C^*$ -algebra (not necessarily a  $W^*$ -algebra).

By (3.4) and Corollary 3.8 we have a natural  $*$ -homomorphism

$$\iota : A \otimes_Z B \rightarrow \bigoplus_{s \in \Delta} A(s) \otimes B(s)$$

and obviously  $\Phi : A \check{\otimes}_Z B \rightarrow A \otimes_Z B$  must be one to one if the composition  $\Gamma = \iota\Phi$  is one to one. For each  $g \in A \otimes B$  let  $g[s, t]$  denote the image of  $g$  in  $A(s) \otimes B(t)$  under the tensor product of the two quotient maps. If the function  $(s, t) \mapsto \|g[s, t]\|$  is continuous on  $\Delta \times \Delta$  for each  $g \in A \otimes B$ , then the argument from the proof of Lemma 3.10 shows that  $\Gamma$  is one to one. If  $A$  or  $B$  is exact, then by [15, 4.6, 2.9] the function  $\|g[\cdot, \cdot]\|$  is indeed continuous. (A  $C^*$ -algebra  $A$  is called exact, if for each exact sequence of  $C^*$ -algebras  $0 \rightarrow J \rightarrow B \rightarrow B/J \rightarrow 0$  the sequence  $0 \rightarrow A \otimes J \rightarrow A \otimes B \rightarrow A \otimes B/J \rightarrow 0$  is exact. This class is large, it includes for example all nuclear  $C^*$ -algebras.) This proves the following theorem.

**Theorem 3.12.** *If  $A$  or  $B$  is exact, then the natural  $*$ -epimorphism  $\Phi : A \check{\otimes}_Z B \rightarrow A \otimes_Z B$  is one to one.*

The author was not able to prove Theorem 3.12 without the assumption of exactness. Since both tensor products  $\check{\otimes}_Z$  and  $\otimes_Z$  preserve inclusions by Corollary 3.11, it would suffice to study the case when  $A$  and  $B$  are type I von Neumann algebras containing  $Z$  in their centers (see also [10, Remark 3.5]).

#### 4. THE SPATIAL NORM AND $Z$ -VALUED STATES

Based on the work of Halpern [12], it would be not hard to build a theory of  $Z$ -valued functionals and then to extend the classical results of Takesaki [25] (for example, the minimality of the spatial norm) to our present context. But since such an extension has been recently achieved in [4] and [10] (in a different way) in the setting when the role of  $Z$  is played by a general abelian  $C^*$ -algebra, we shall

limit our discussion here to results which are more specific for the setting where  $Z$  is an abelian  $W^*$ -algebra.

A  $Z$ -state on  $A$  is a positive  $Z$ -module homomorphism  $\omega$  from  $A$  to  $Z$  such that  $\omega(1) = 1$  (hence  $\|\omega\| = 1$ ). The set of all  $Z$ -states on  $A$  will be denoted by  $S_Z(A)$  (while  $S(A)$  denotes the usual state space of  $A$ ). If  $A \subseteq L(E)$  for some Hilbert  $Z$ -module  $E$ , then each  $x \in E$  with  $\langle x, x \rangle = 1$  defines a vector  $Z$ -state  $\omega_x$  on  $A$  by  $\omega_x(a) = \langle x, ax \rangle$ . The space of all bounded  $Z$ -module homomorphisms from  $A$  to  $Z$ , denoted by  $A^\natural$ , is a  $Z$ -bimodule by  $(z\omega)(a) = (\omega z)(a) \stackrel{\text{def}}{=} \omega(za)$  ( $z \in Z$ ). The weak\* topology on bounded sets of  $A^\natural$  is defined by the family of seminorms  $\omega \mapsto |\nu(\omega(a))|$  ( $a \in A, \nu \in Z_\sharp$ ), where  $Z_\sharp$  is the predual of  $Z$ . ( $A^\natural$  is the dual of the projective tensor product  $A \hat{\otimes}_Z Z_\sharp$ , but we shall not need this fact.) The unit ball of  $A^\natural$  and its weak\* closed subset  $S_Z(A)$  are weak\* compact.

Given a  $Z$ -module  $M$ , a subset  $S \subseteq M$  is  $Z$ -convex if  $cx + (1 - c)y \in S$  for all  $x, y \in S$  and all  $c \in Z$  with  $0 \leq c \leq 1$ . A point  $v \in S$  is  $Z$ -extreme if the condition  $v = cx + (1 - c)y$ , where  $x, y \in S$  and  $c \in Z$  with  $0 \leq c \leq 1$  and  $c, 1 - c$  invertible, implies that  $x = y = v$ . The set of all  $Z$ -extreme points of  $S_Z(A)$  (pure  $Z$ -states on  $A$ ) is denoted by  $P_Z(A)$ . Here is a variant of the Krein-Milman theorem.

**Theorem 4.1.** *If  $K$  is a  $Z$ -convex weak\* compact set of positive contractions in  $A^\natural$ , then  $K$  is the weak\* closure of the smallest  $Z$ -convex set containing the  $Z$ -extreme points of  $K$ . In particular,  $S_Z(A)$  is the weak\* closure of the  $Z$ -convex hull of  $P_Z(A)$ .*

In the case  $A$  is a von Neumann algebra and  $Z$  the center of  $A$ , Theorem 4.1 was proved by Halpern in [12, 2.2]; since essentially the same arguments apply to the more general case where  $Z$  is contained in the center of a  $C^*$ -algebra  $A$ , we shall omit the proof (the result will not be important for what follows in this paper).

As it may be expected, in the case  $A \subseteq L(E)$ ,  $S_Z(A)$  is equal to the weak\* closure of the  $Z$ -convex hull of the set of all vector  $Z$ -states. This is an immediate consequence of the following result.

**Theorem 4.2.** *If  $S$  is a subset of  $S_Z(A)$  such that  $\sup_{\omega \in S} \|\omega(a)\| = \|a\|$  for each self-adjoint  $a \in A$ , then  $S_Z(A) = \overline{\text{co}_Z S}$ , where  $\overline{\text{co}_Z S}$  is the weak\* closure of the  $Z$ -convex hull of  $S$ .*

*Proof.* Suppose the contrary, that  $\rho \notin \overline{\text{co}_Z S}$  for some  $\rho \in S_Z(A)$ , and put  $S_0 = \overline{\text{co}_Z S}$ . Then by the Hahn - Banach theorem there exists a linear functional  $\theta$  on  $A^\natural$  of the form

$$(4.1) \quad \theta(\omega) = \sum_{i=1}^m \nu_i(\omega(a_i)) \quad (\omega \in A^\natural),$$

where  $a_i \in A$  and  $\nu_i \in Z_\sharp$  are fixed, such that

$$(4.2) \quad \sup_{\omega \in S_0} \text{Re}(\theta(\omega)) < \text{Re}(\theta(\rho)).$$

Put  $\nu = \sum_{i=1}^m |\nu_i|$  ( $\in Z_\sharp$ ). Note that for each  $i$  there exists  $c_i \in Z$  such that  $\nu_i = \nu c_i$ , and put  $a = \sum_{i=1}^m c_i a_i$ . Since each  $\omega \in A^\natural$  is a homomorphism of  $Z$ -modules, we have from (4.1) that  $\theta(\omega) = \nu(\omega(a))$ ; hence (4.2) can be rewritten as

$$(4.3) \quad \sup_{\omega \in S_0} \text{Re}(\nu(\omega(a))) < \text{Re}(\nu(\rho(a))).$$

Replacing  $a$  by  $(1/2)(a + a^*)$  in (4.3), we may assume that  $a^* = a$ . Observe that the subset  $Z_0 = \{\omega(a) : \omega \in S_0\}$  of  $Z$  is an increasing net, that is, the maximum of two (and hence of finitely many) elements  $f$  and  $g$  of  $Z_0$  is in  $Z_0$ . (Indeed, let  $\mathcal{U} = \{t \in \Delta : f(t) > g(t)\}$  and let  $p$  be the characteristic function of the closure of the open set  $\mathcal{U}$ . Then  $p$  is a projection in  $Z$  (since  $\Delta$  is extremely disconnected) and  $\max\{f, g\} = pf + p^\perp g$ . If  $f = \omega_1(a)$  and  $g = \omega_2(a)$ , with  $\omega_1, \omega_2 \in S_0$ , then  $\max\{f, g\} = \omega(a)$ , where  $\omega = p\omega_1 + p^\perp\omega_2 \in S_0$ .) Denoting

$$c = \sup_{\omega \in S_0} \omega(a),$$

it follows now from (4.3), since  $\nu$  is normal, that

$$(4.4) \quad \nu(c) < \nu(\rho(a)).$$

Replacing in (4.4)  $a$  by  $a + t1$  (and consequently  $c$  by  $c + t1$ ) for a suitable real  $t$ , we may further assume that  $a$  is positive.

We claim that there exists a projection  $p_1 \in Z$  dominated by the support projection  $p_0 \in Z$  of  $\nu$  such that

$$(4.5) \quad \|cp_1\| < \|\rho(a)p_1\|.$$

Otherwise  $\|\rho(a)p\| \leq \|cp\|$  for each projection  $p \leq p_0$  in  $Z$ ; hence

$$\rho(a)(s)p_0(s) \leq c(s)p_0(s)$$

for each  $s$  in the spectrum  $\Delta$  of  $Z$ , and consequently  $\rho(a)p_0 \leq cp_0$ . But, since  $\nu$  is positive, this would imply that  $\nu(\rho(a)) \leq \nu(c)$ , which would contradict (4.4).

By the hypothesis of the lemma and the definition of  $c$  we have that

$$\|ap\| = \sup_{\omega \in S} \|\omega(ap)\| \leq \sup_{\omega \in S_0} \|\omega(a)p\| = \|cp\|$$

for each projection  $p \in Z$ . Therefore (4.5) implies that  $\|ap_1\| \leq \|cp_1\| < \|\rho(ap_1)\|$ , which is a contradiction since  $\|\rho\| = 1$  for each  $\rho \in S_Z(A)$ .  $\square$

Given vector  $Z$ -states  $\omega_x$  on  $A$  and  $\omega_y$  on  $B$ , the vector  $Z$ -state  $\omega_{x \otimes_Z y}$  is clearly the unique  $Z$ -state  $\tau$  on  $A \otimes_Z B$  satisfying  $\tau(a \otimes_Z b) = \omega_x(a)\omega_y(b)$  for all  $a \in A$  and  $b \in B$ . Since Theorem 4.2 implies that the  $Z$ -convex hull of vector  $Z$ -states is weak\* dense in the space of all  $Z$ -states, it follows as in the classical case  $Z = \mathbb{C}$  (see [13, 11.1.1]) that for arbitrary  $\omega \in S_Z(A)$  and  $\rho \in S_Z(B)$  there is a unique  $\tau \in S_Z(A \otimes_Z B)$  satisfying  $\tau(a \otimes_Z b) = \omega(a)\rho(b)$  for all  $a \in A$  and  $b \in B$ . This state  $\tau$ , called *the product  $Z$ -state*, will be denoted by  $\omega \otimes_Z \rho$ .

The *s-topology* on a self-dual Hilbert module  $E$  over a von Neumann algebra  $R$  is defined by the family of seminorms

$$x \mapsto (\rho(\langle x, x \rangle))^{1/2},$$

where  $\rho$  is a normal state on  $R$ . In a concrete representation of  $E \subseteq \Lambda(E)$  (Theorem 2.1) the *s-topology* coincides on bounded sets with the usual strong operator topology. Hence, *s-continuous* linear functionals are weak\* continuous and the same subspaces of  $E$  are closed in both topologies. (This was proved in a different way in [2].) Note also that the operators in  $L(E)$  are weak\* (and *s-*) continuous on  $E$ .

*Remark 4.3.* (i) Let  $F$  be an  $R$ -submodule of a self-dual Hilbert  $R$ -module  $E$  (where  $R$  is a  $W^*$ -algebra) and let  $\overline{F}$  be the weak\* closure of  $F$ . Then an application of the Kaplansky density theorem in  $\Lambda(E)$  shows that the unit ball of  $F$  is dense in the unit ball of  $\overline{F}$  in the weak\* and in the *s-topology*.

(ii) The weak\* closed submodules of a self-dual Hilbert  $R$ -module  $E$  are precisely the submodules of the form  $pE$ , where  $p$  is a projection in  $L(E)$ . This was observed in [2, p. 205] and can be proved also by considering the weak\* closed right ideal in  $\Lambda(E)$  generated by a submodule of  $E$ , which is well known to be generated by a projection in  $\Lambda(E)$ .

**Theorem 4.4.** *Let  $E$  and  $F$  be faithful self-dual Hilbert  $Z$ -modules and  $A \subseteq L(E)$ ,  $B \subseteq L(F)$  two  $C^*$ -algebras containing (an isomorphic copy of)  $Z$ . Then the spatial norm of each  $g \in A \odot_Z B$  can be expressed as*

$$(4.6) \quad \|g\|^2 = \sup_{\tau, h} \frac{\|\tau(h^*g^*gh)\|}{\|\tau(h^*h)\|},$$

where the supremum is over all elements  $h \in A \odot_Z B$  and all product  $Z$ -states  $\tau$  on  $A \otimes_Z B$  such that  $\tau(h^*h) \neq 0$ .

*Proof.* Suppose first that  $E$  and  $F$  are cyclic in the sense that  $E = [Ax]$  and  $F = [By]$  for some unit vectors  $x \in E$  and  $y \in F$ . Here  $[S]$  denotes the s-closed linear span of a subset  $S$  of  $E$  or  $F$ . Then the unit balls of  $E_0 \stackrel{\text{def}}{=} Ax$  and  $F_0 \stackrel{\text{def}}{=} By$  are s-dense in the unit balls of  $E$  and  $F$  (respectively) by Remark 4.3(i); hence, given  $u \in E$  and  $v \in F$ , there exist bounded nets  $(u_k)$  in  $E_0$  and  $(v_k)$  in  $F_0$  converging to  $u$  and  $v$  (resp.) in the s-topology. Then the net  $(u_k \otimes_Z v_k)$  converges to  $u \otimes_Z v$  in the s-topology of  $E \otimes_Z F$ . To see this, write

$$u_k \otimes_Z v_k - u \otimes_Z v = (u_k - u) \otimes_Z v_k + u \otimes_Z (v_k - v)$$

and note that a net  $(x_k \otimes_Z y_k)$  converges to 0 in the s-topology whenever the nets  $(x_k)$  and  $(y_k)$  are bounded and one of them, say  $(x_k)$ , converges to 0, since for each normal state  $\nu$  on  $Z$  we have

$$(\nu(\langle x_k \otimes_Z y_k, x_k \otimes_Z y_k \rangle))^{1/2} \leq \|y_k\|(\nu(\langle x_k, x_k \rangle))^{1/2} \rightarrow 0.$$

This shows that  $[E_0 \otimes_Z F_0]$  contains all vectors of the form  $u \otimes_Z v$ , hence  $[E_0 \otimes_Z F_0] = E \otimes_Z F$ , so by Remark 4.3(i) the unit ball of  $E_0 \otimes_Z F_0$  is dense in the unit ball of  $E \otimes_Z F$  in the s-topology. Therefore for each  $g \in A \odot_Z B$  we have

$$\begin{aligned} \|g\|^2 &= \sup\left\{ \frac{\|g(w)\|^2}{\|w\|^2} : w \in E_0 \otimes_Z F_0, w \neq 0 \right\} \\ &= \sup\left\{ \frac{\|gh(x \otimes_Z y)\|^2}{\|h(x \otimes_Z y)\|^2} : h \in A \odot_Z B, h(x \otimes_Z y) \neq 0 \right\} \\ &= \sup\left\{ \frac{\|\tau_0(h^*g^*gh)\|}{\|\tau_0(h^*h)\|} : h \in A \odot_Z B, \tau_0(h^*h) \neq 0 \right\}, \end{aligned}$$

where  $\tau_0 = \omega_x \otimes_Z \omega_y$ . This implies that  $\|g\|^2$  is dominated by the right side of (4.6), but the reverse inequality is obvious.

In general, a standard maximality argument (using Remark 4.3(ii)) shows that  $E$  and  $F$  are orthogonal sums of s-closed cyclic submodules, and the theorem follows by the same reasoning as in the special case  $Z = \mathbb{C}$  (see [13, 11.1.2]).  $\square$

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