EXTREMAL FUNCTIONS FOR MOSER’S INEQUALITY

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Abstract. Let Ω be a bounded smooth domain in \( \mathbb{R}^n \), and \( u(x) \) a \( C^1 \) function with compact support in \( \Omega \). Moser’s inequality states that there is a constant \( c_0 \), depending only on the dimension \( n \), such that

\[
\frac{1}{|\Omega|} \int_{\Omega} e^{\frac{n}{n-1} \frac{1}{\|\nabla u\|_n}} \, dx \leq c_0,
\]

where \( |\Omega| \) is the Lebesgue measure of \( \Omega \), and \( \omega_{n-1} \) the surface area of the unit ball in \( \mathbb{R}^n \). We prove in this paper that there are extremal functions for this inequality. In other words, we show that the

\[
\sup \left\{ \frac{1}{|\Omega|} \int_{\Omega} e^{\frac{n}{n-1} \frac{1}{\|\nabla u\|_n}} \, dx : u \in W^{1,n}, \|\nabla u\|_n \leq 1 \right\}
\]

is attained. Earlier results include Carleson-Chang (1986, \( \Omega \) is a ball in any dimension) and Flucher (1992, \( \Omega \) is any domain in 2-dimensions).

1. Introduction

Let \( \Omega \) be a bounded smooth domain in \( \mathbb{R}^n \), and \( u(x) \) a \( C^1 \) function supported in \( \Omega \) with \( \|\nabla u\|_q < n \). Sobolev’s Imbedding Theorem says that if \( 1 \leq q < n \), then

\[
\|u\|_p \leq C(n,q),
\]

where \( \frac{1}{p} = \frac{1}{q} - \frac{1}{n} \), and \( C(n,q) \) is a constant independent of the function \( u \), as well as the domain \( \Omega \). The imbedding is no longer valid when \( q = n \). Indeed, there are unbounded functions whose gradients are in \( L^n \). However, Trudinger [14] in 1967 proved that if \( \|\nabla u\|_n \leq 1 \), then \( u \) is in an exponential class. More precisely, the integral

\[
\int_{\Omega} e^{\beta_0 \|\nabla u\|_n^p} \, dx,
\]

is uniformly bounded, for some positive \( \beta_0 \) depending only on dimension. Moser [12] in 1971 then found the best exponent \( \beta_0 \). He showed if \( \|\nabla u\|_n \leq 1 \), then

\[
\frac{1}{|\Omega|} \int_{\Omega} e^{\frac{n}{n-1} \frac{1}{\|\nabla u\|_n}} \, dx \leq c_0,
\]

where \( c_0 \) is a constant depending only on \( n \). (\( \omega_{n-1} \) is the surface area of the unit ball in \( \mathbb{R}^n \).)

The aim of this paper is to prove the following:

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Theorem 1. There are extremal functions for Moser’s inequality (2). In other words, the
\[
\sup\{ \frac{1}{|\Omega|} \int_{\Omega} e^{\int_{0}^{u} \frac{1}{n-1} u^{\frac{n}{n-1}}} \, dx : u \in W^{1,n}_{\partial \Omega}, \| \nabla u \|_{n} \leq 1 \}
\]
is attained.

The first result in this direction is due to Carleson-Chang [3], who proved in 1986 that there are extremals when \( \Omega \) is a ball in any dimension. Their result came as a surprise, since it was known at that time that no extremals exist for Sobolev’s inequality (1) when \( \Omega \) is a ball. (See an account of this in the more expository article [10].) In 1992, M. Flucher [5] proved the same existence for any bounded smooth domain in 2-dimensions. Though our result is an improvement, the method of the proof relies on both heavily. The key ingredient is the use of \( n \)-Green’s functions, the singular solutions to the \( n \)-Laplacian. As to the solvability of the corresponding Euler equation, see Adimurthi [1] and Struwe [6].

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2. Outline of Proof

By \( W^{1,n}_{\partial \Omega} \) we mean the Sobolev space of functions vanishing on the boundary \( \partial \Omega \) with \( \| \nabla u \|_{n} < \infty \), and we denote by \( F_{\Omega}(u) \) the Moser functional
\[
\int_{\Omega} \left( e^{\int_{0}^{u} \frac{1}{n-1} u^{\frac{n}{n-1}}} - 1 \right) \, dx.
\]
(the term \(-1\) in the integrand is introduced for convenience). We now describe the outline of the proof. Let \( \{ u_{j} \} \) be a maximizing sequence for (2), that is, \( \{ u_{j} \} \subset W^{1,n}_{\partial \Omega}, \| \nabla u_{j} \|_{n} \leq 1 \), and \( F_{\Omega}(u_{j}) \) tends to the supremum. We get for free from funtional analysis that we can extract a subsequence, still denoted by \( \{ u_{j} \} \), which satisfies
\[
\text{\( \| \nabla u_{j} \|_{n} \leq 1, \ u_{j} \rightarrow u \) weakly, and \( \| \nabla u(x) \|_{n} dx \rightarrow d \mu \) weakly},
\]
where \( u \) is a function in \( W^{1,n}_{\partial \Omega} \), and \( d \mu \) a finite measure on \( \Omega \). Our goal is to prove that \( F_{\Omega}(u_{j}) \rightarrow F_{\Omega}(u) \) (\( u \) will then be an extremal).

The main difficulty in this type of problem is that the Moser funtional \( F_{\Omega}(u) \) is not compact. In other words, there exists a sequence of functions \( \{ u_{j} \} \) which satisfies all the conditions in (3), but \( F_{\Omega}(u_{j}) \) fails to converge to \( F_{\Omega}(u) \). Here is an example. Take \( \Omega \) to be the unit ball in \( \mathbb{R}^{n} \), and define \( u_{a} \) to be \( c_{a} \log \frac{1}{|x|} \) for \( a \leq |x| \leq 1 \), and a constant \( d_{a} \) for \( 0 \leq |x| \leq a \), where \( d_{a} \) is chosen so that the functions are continuous, and \( d_{a} \) chosen so that \( \| \nabla u_{a} \|_{n} = 1 \). It is easy to see that as \( a \rightarrow 0, u_{a} \rightarrow u = 0 \) weakly, \( \| \nabla u_{a}(x) \|_{n} dx \rightarrow \delta_{0} \) the Dirac measure at 0 weakly, and that \( \lim \sup F_{\Omega}(u_{a}) > F_{\Omega}(0) \).

All is not lost, however. P. L. Lions [11] was able to show that this is the only thing that can go wrong.

Theorem 2 (P. L. Lions). Suppose \( \{ u_{j} \} \) is a sequence satisfying (3). Then, either
(a) the compactness holds, i.e., \( F_{\Omega}(u_{j}) \rightarrow F_{\Omega}(u) \); or (b) \( \{ u_{j} \} \) concentrates at some point \( x_{0} \), i.e., \( u_{j} \rightarrow u = 0 \) weakly, and \( \| \nabla u(x) \|_{n} dx \rightarrow \delta_{x_{0}} \).
This is the so-called the concentration-compactness principle for the Moser functional. See Flucher [3] for another proof. So far, we haven’t used the condition that $u_j$ is maximizing. In the following, we’ll show that maximizing sequences never concentrate. To do this, we first quantify the concentration phenomenon. The following notion was introduced in Flucher [5].

**Definition 1.** Let $x_0$ be a point in $\tilde{\Omega}$. The concentration function at $x_0$ is defined to be

$$C_{\Omega}(x_0) = \sup \{ \limsup_{j \to \infty} F_{\Omega}(u_j) : \|\nabla u\|_n \leq 1, \{u_j\} \text{ concentrates at } x_0 \}.$$ 

Obviously, we have $\sup_u F_{\Omega}(u) \geq \sup_x C_{\Omega}(x)$. And, in view of Lions’s concentration-compactness principle, it now suffices to prove $\sup_u F_{\Omega}(u) > \sup_x C_{\Omega}(x)$.

In fact, this was how Carleson-Chang [3] proved their theorem on a ball:

**Theorem 3** (Carleson-Chang). Let $B$ be the unit ball in $\mathbb{R}^n$. Then

(a) $\sup_x C_B(x) = C_B(0) = e^{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}} |B|$, 

(b) $\sup_u F_B(u) > e^{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}} |B|$.

Now we switch from balls to a general domain $\Omega$. When we do so, both the sup of Moser functional and that of the concentration function will change. The key observation is that the ratio of functional over concentration will only increase.

**Theorem 4.**

$$\frac{\sup_u F_{\Omega}(u)}{\sup_x C_{\Omega}(x)} \geq \frac{\sup_u F_B(u)}{\sup_x C_B(x)}.$$ 

Thus our Theorem 1 is a consequence of Lions’s Theorem 2, Carleson-Chang’s Theorem 3 and Theorem 4. The appearance of Theorem 4 bears some resemblance to the classical isoperimetric inequality. Indeed, somewhere in the proof, we do use the classical isoperimetric inequality.

The proof of Theorem 4 consists of two parts: one is the comparison of the concentration function on $\Omega$ and on the ball $B$; the other is the comparison of the Moser functional on these two domains. More, precisely, we’ll prove

**Theorem 5.** (a) For every $x$ in $\Omega$, $C_{\Omega}(x) = r^n_{\Omega}(x) C_B(0)$,

(b) $\sup_u F_{\Omega}(u) \geq (\sup_x r^n_{\Omega}(x)) \sup_u F_B(u)$.

The factor (without the $n$-th power), $r_{\Omega}(x)$, that appears in both formulas is what we call the $n$-harmonic radius, which will depend only on the point $x$ and the domain $\Omega$. It is obvious that Theorem 5 implies Theorem 4. We now digress to define the $n$-Green’s functions and the $n$-harmonic radius.

**Definition 2.** Let $x_0$ be a point in $\Omega$. The $n$-Green’s function $G = G_{x_0} = G_{\Omega,x_0}$ on $\Omega$ with pole at $x_0$ is the singular solution to the $n$-Laplacian:

$$\Delta_n G = \text{Div}(|\nabla G|^{n-2} \nabla G) = \delta_{x_0} \text{ in } \Omega,$$

$$G = 0 \text{ on } \partial \Omega.$$

In terms of distributions, equation (4) means

$$\int_{\Omega} |\nabla G|^{n-2} \nabla G \cdot \nabla \phi \, dx = \phi(x_0),$$

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for every compactly supported smooth function $\phi(x)$ on $\Omega$. Of course, when $n = 2$, the 2-Laplacian is the usual Laplacian, and 2-Green’s function is the usual Green’s function. In higher dimensions, the existence and uniqueness of this $G$ is also well-known, see [6] and [7], for example. The $n$-Green’s function on the ball $B$ with pole at the origin is $G_0 = -\omega_{n-1}^{\frac{1}{n-1}} \log |x|$, and for general domain we have the following asymptotic expansion:

$$G_{x_0}(x) = -\omega_{n-1}^{\frac{1}{n-1}} \log |x - x_0| - H_{x_0}(x),$$

where $H_{x_0}(x)$ is a continuous function on $\Omega$ and is $C^{1,\alpha}$ in $\Omega \setminus \{x_0\}$.

**Definition 3.** The $n$-harmonic radius at $x_0$ is defined to be

$$r_{\Omega}(x_0) = e^{-\omega_{n-1}^{\frac{1}{n-1}} H_{x_0}(x_0)}.$$

**Remark 1.** When $n = 2$, and $\Omega$ simply-connected, one can use the invariance of Green’s functions under conformal mappings to see that the $n$-harmonic radius is nothing but $|f'(0)|$, where $f(z)$ is a conformal mapping from the unit disc to $\Omega$ with $f(0) = x_0$. See [2].

**Remark 2.** In higher dimensions, $n$-Green’s functions are invariant under M"obius transformations. (The usual Green’s functions are not.) As a consequence, one can compute the conformal radius at $x_0 \in B$ as $1 - |x_0|^2$.

We return to the discussion of the proof of our Theorem 1, which now reduces to that of Theorem 5. To prove Theorem 5, we’ll need to transplant functions, either from a general domain $\Omega$ to a ball, or from a ball to $\Omega$. In doing so, we have to keep the functions in the same class, i.e., $\|\nabla u\|_n \leq 1$, and, at the same time, to obtain a relation between the functional on the two domains. For the direction from $\Omega$ to $\Omega$ (the symmetrized domain of $\Omega$, which is a ball), the classical rearrangement $u^*$ is the main tool. (See [7].) Recall that $|\{u^* > t\}| = |\{u > t\}|$, and

**Theorem 6.** For $u \in W_0^{1,n}(\Omega)$, we have

(a) $\|\nabla u^*\|_{L^\infty(\Omega^*)} \leq \|\nabla u\|_{L^\infty(\Omega)}$,

(b) $F_{\Omega^*}(u^*) = F_{\Omega}(u)$.

For the other direction, from $B$ to the unit ball $\Omega$, we use the $n$-harmonic transplantation, which is defined via the level sets of $n$-Green’s functions.

**Definition 4.** Let $x_0$ be a point in $\Omega$, and $v_0$ a decreasing, radial function on $B$. The $n$-harmonic transplantation of $v_0$ on $\Omega$ at $x_0$ is defined to be $v_{x_0} = v_{\Omega,x_0} = v_0 \circ G_{B,\Omega,x_0}^{-1} \circ G_{\Omega,x_0}$.

So, $v_{x_0}$ has the same level sets as $G_{x_0}$ does. Furthermore, $v_{x_0}$ and $v_0$ agree on the corresponding level sets of $G_{x_0}$ and $G_0$. The analogous result to Theorem 6, when we move functions from $B$ to $\Omega$, is:

**Theorem 7.** For a radial, decreasing function $v_0$ in $W_0^{1,n}(B)$, we have

(a) $\|\nabla v_{x_0}\|_{L^\infty(\Omega)} = \|\nabla v_0\|_{L^\infty(B)}$,

(b) $F_{\Omega}(v_{x_0}) \geq r_{\Omega}(x_0) F_{B}(v_0)$.

We will prove some properties about $n$-Green’s functions in the next section. The proofs of Theorems 7 and 5 are presented in the last section.
3. n-Green’s Functions

We develop in this section some important properties about the n-Green’s function.

Lemma 1. Let $G = G_{x_0}(x)$.

(a) \[
\int_{\{G < t\}} |\nabla G|^n \, dx = t \quad \text{for every } t,
\]

(b) \[
\int_{\partial\{G > t\}} |\nabla G|^{n-1} \, dx = 1 \quad \text{for every } t.
\]

(c) The sets \(\{G > t\}\) form a sequence of approximately small balls of radii \(r_t = r_\Omega(x_0) e^{-\omega_{n-1}^{-1} t}\). In other words, \(B(x_0, \rho_t - r_t) \subset \{G > t\} \subset B(x_0, \rho_t + r_t)\), with \(r_t/\rho_t \to 0\) as \(t \to \infty\). In particular,

\[
\lim_{t \to \infty} \frac{|\{G > t\}|}{\alpha_n e^{-\omega_{n-1}^{-1} t}} = r_\Omega^n(x_0),
\]

where \(\alpha_n\) is the volume of the unit ball in \(\mathbb{R}^n\).

(d) On the set \(\{G = t\}\), we have

\[
|\nabla G(x)| = \omega_{n-1}^{-1} \frac{1}{\rho_t} + O(1) \quad \text{uniformly, as } t \to \infty.
\]

Proof. (a) Choose a smooth approximation of the function \(\phi(x) = \inf \{G(y) + t\}\) in equation (5).

(b) follows from equation (4) via an integration by parts.

(c) Solving for \(|x - x_0|\) in (6), we get

\[
|x - x_0| = e^{-\frac{1}{\omega_{n-1}}} t e^{-\frac{1}{\omega_{n-1}}} H_{x_0}(x) = e^{-\frac{1}{\omega_{n-1}}} t e^{-\frac{1}{\omega_{n-1}}} H_{x_0}(x_0) + (e^{-\frac{1}{\omega_{n-1}}} H_{x_0}(x) - e^{-\frac{1}{\omega_{n-1}}} H_{x_0}(x_0)) e^{-\frac{1}{\omega_{n-1}}} t = \rho_t + r_t.
\]

It is easy to see that \(r_t/\rho_t \to 0\) as \(t \to \infty\), by the continuity of \(H_{x_0}(x)\) at \(x_0\).

(d) On \(\{G = t\}\), we have

\[
|\nabla G(x)| = \left| -\omega_{n-1}^{-1} \frac{x - x_0}{|x - x_0|^2} - \nabla H_{x_0}(x) \right| = \omega_{n-1}^{-1} \frac{1}{\rho_t} + O(1),
\]

by the \(C^{1,\alpha}\) property of \(H_{x_0}(x)\) in \(\Omega \setminus \{x_0\}\) and (c).

\[
\square
\]

Lemma 2. For domains \(\Omega\) in \(\mathbb{R}^n\),

\[
\sup_x r_\Omega(x) \leq \sup_x r_{\Omega^*}(x) = r_{\Omega^*}(0).
\]

Proof. We have from (c) of Lemma 1:

\[
r_\Omega^n(x) = \lim_{t \to \infty} \frac{|\{G_{\Omega,x} > t\}|}{\alpha_n e^{-\omega_{n-1}^{-1} t}},
\]

and

\[
r_{\Omega^*}^n(0) = \lim_{t \to \infty} \frac{|\{G_{\Omega^*,0} > t\}|}{\alpha_n e^{-\omega_{n-1}^{-1} t}}.
\]

\((G_{\Omega^*,0}\) is the n-Green’s function on \(\Omega^*\) with pole at 0.)
Now we compare the two sets, \( \{G_{\Omega,x} > t\} \) and \( \{G_{\Omega^*,0} > t\} \). Part (a) of Lemma 1 and Theorem 6 implies
\[
t = \int_{\{G_{\Omega,x} < t\}} |\nabla G_{\Omega,x}|^n \, dx \geq \int_{\{G_{\Omega,x} < t\}} |\nabla G_{\Omega,x}^*|^n \, dx \geq \int_{\{v_t < t\}} |\nabla v_t|^n \, dx,
\]
where \( v_t \) is the \( n \)-harmonic function sharing the same boundary values as \( G_{\Omega,x}^* \) on \( \{G_{\Omega,x}^* < t\} \), and the last inequality is Dirichlet’s principle. This \( v_t \) must be a constant multiple of \( G_{\Omega^*,0} \), say, \( v_t = \lambda_t G_{\Omega^*,0} \). So, we have
\[
t \geq \int_{\{G_{\Omega^*,0} < t/\lambda_t\}} \lambda_t^n |\nabla v_t|^n \, dx = t \lambda_t^{n-1}.
\]
Hence \( \lambda_t \leq 1 \). Therefore,
\[
r^n_\Omega(x) = \lim_{t \to \infty} \frac{|\{G_{\Omega,x} > t\}|}{\alpha_n e^{-\omega_{n-1} \frac{t}{n}}} = \lim_{t \to \infty} \frac{|\{G_{\Omega,x}^* > t\}|}{\alpha_n e^{-\omega_{n-1} \frac{t}{n}}} = \lim_{t \to \infty} \frac{|\{G_{\Omega^*,0} > t/\lambda_t\}|}{\alpha_n e^{-\omega_{n-1} \frac{t}{n}}} = r^n_\Omega(0). \]

Lemma 3. For every \( 0 < r \leq 1 \), we have
\[
\frac{1}{(\omega_{n-1}^{-1} r)^n} \int_{\partial\{G > -\frac{1}{\omega_{n-1}} \log r\}} \frac{1}{|\nabla G|} \, ds \geq r_n^\Omega(x_0),
\]
and the inequality tends to be an equality, as \( r \to 0 \).

Proof. The isoperimetric inequality for domains \( A \) in \( \mathbb{R}^n \) says that
\[
|A| \leq \alpha_n \omega_{n-1}^{-\frac{n}{n-1}} \left( \int_{\partial A} ds \right)^{\frac{n}{n-1}}.
\]
If we take \( A \) to be \( \{G > -\frac{1}{\omega_{n-1}} \log r\} \), then we have
\[
|A| \leq \alpha_n \omega_{n-1}^{-\frac{n}{n-1}} \left( \int_{\partial A} |\nabla G|^{-\frac{n}{n-1}} \frac{1}{|\nabla G|^{-\frac{1}{n}}} \, ds \right)^{\frac{n}{n-1}}
\]
\[
\leq \alpha_n \omega_{n-1}^{-\frac{n}{n-1}} \left( \int_{\partial A} |\nabla G|^{-1} \, ds \right)^{\frac{n}{n-1}} \left( \int_{\partial A} \frac{1}{|\nabla G|} \, ds \right)^{\frac{n-1}{n-1}}
\]
\[
= \alpha_n \omega_{n-1}^{-\frac{n}{n-1}} \int_{\partial A} \frac{1}{|\nabla G|} \, ds.
\]
On the other hand, we can estimate \( |A| \) from below in terms of \( r_\Omega(x_0) \). Since \( G_{A,x_0}(x) = G_{\Omega,x_0}(x) + \omega_{n-1}^{-1} \log r \), we have \( H_{A,x_0}(x) = H_{\Omega,x_0}(x) - \omega_{n-1}^{-\frac{1}{n}} \log r \). Thus \( r_A(x_0) = r \cdot r_\Omega(x_0) \). And Lemma 2 gives
\[
|A| \geq \alpha_n r^n_A(x_0) = \alpha_n r^n \cdot r^n_\Omega(x_0).
\]
Combining these two inequalities gives the one in the lemma. Furthermore, we have from Lemma 1,
\[
\frac{1}{|\nabla G|} \sim \omega_{n-1}^{-\frac{1}{n}} r \cdot r_\Omega(x_0), \quad \text{and}
\]
\[
|\{G = -\omega_{n-1}^{-\frac{1}{n}} \log r\}| \sim \omega_{n-1}^{-1} r^{n-1} r^n_\Omega(x_0),
\]
as \( r \to 0 \). The asymptotic equality then follows. \( \square \)
4. Proofs of Theorems

Proof of Theorem 7. (a) By the co-area formula (see [2]), the definition of $v_{x_0}$ (which yields $\nabla v_{x_0} = \frac{\nabla v_0}{|\nabla G_{B,0}|}\nabla G_{x_0}$), and part (b) of Lemma 1, we have

$$\|\nabla v_{x_0}\|_L^n(\Omega) = \int_\Omega |\nabla v_{x_0}|^n dx = \int_0^\infty \int_{\partial(v_{x_0} > t)} |\nabla v_{x_0}|^{n-1} dt ds$$

$$= \int_0^\infty \frac{|\nabla v_0|^{n-1}}{|\nabla G_{B,0}|^{n-1}} \int_{\partial(v_{x_0} > t)} |\nabla G_{x_0}|^{n-1} ds dt = \int_0^\infty \frac{|\nabla v_0|^{n-1}}{|\nabla G_{B,0}|^{n-1}} dt.$$

The last integral is independent of domains, so it is equal to $\|\nabla v_0\|_{L^n(B)}$.

(b) We let $f(t) = e^{\frac{-|x|^2}{4}} - 1.$ Again, by the co-area formula,

$$F_\Omega(v_{x_0}) = \int_0^\infty \int_{\partial(v_{x_0} > t)} \frac{f(t)}{|\nabla v_{x_0}|} \frac{|\nabla G_{B,0}(v(t))|}{|\nabla v_0(z(t))|} \frac{1}{|\nabla G_{x_0}|} ds dt$$

$$= \int_0^\infty f(t) \int_{\partial(v_{x_0} > t)} \frac{|\nabla G_{B,0}(v(t))|}{|\nabla v_0(z(t))|} \frac{1}{|\nabla G_{x_0}|} ds dt,$$

$$= \int_0^\infty f(t) \left( \frac{1}{\omega_{n-1} |z(t)|} \right) \left( \frac{1}{\omega_{n-1} |z(t)|} \right)^{n-1} \times \left( \int_{\partial(v_{x_0} > t)} \frac{1}{|\nabla v_0|} ds \right) \left( \int_{\partial(G>B,0(z(t)))} \frac{1}{|\nabla G|} ds \right) dt$$

$$= \int_0^\infty f(t) \left( \int_{\partial(v_{x_0} > t)} \frac{1}{|\nabla v_0|} ds \right) \left( \frac{1}{\omega_{n-1} |z(t)|} \right)^{n} \int_{\partial(G>B,0(z(t)))} \frac{1}{|\nabla G|} ds dt$$

$$\geq r_\Omega^n(x_0) \int_0^\infty f(t) \int_{\partial(v_{x_0} > t)} \frac{1}{|\nabla v_0|} ds dt = r_\Omega^n(x_0) \int_B f(v_0) dx = r_\Omega^n(x_0) F_B(v_0).$$

In the above formulas, $z(t)$ is a point in $B$ such that $v_0(z(t)) = t.$

Proof of Theorem 5. We prove part (b) first. Let $v_0(x)$ be an extremal function which realizes $\sup_u F_B(u)$, as assured by Carleson-Chang’s Theorem 3. We may assume this $v_0(x)$ is radial and decreasing on $B$, by Theorem 6. Now, Theorem 7 says every conformal rearrangement $v_{x_0}$ satisfies $F_\Omega(v_{x_0}) \geq r_\Omega^n(x_0) F_B(v_0)$. Taking the supremum over $x_0$ in $B$ gives us (b).

To prove (a), we first take a concentrating sequence $\{v_j\}$ on $B$ which realizes $C_B(0)$. Theorem 7 gives us a sequence $\{v_j, x_0\}$ on $\Omega$. The same argument for proving (a) of Theorem 7 yields

$$\int_{\{G_{1,x_0} < t\}} |\nabla v_{j,x_0}|^n dx = \int_{\{G_{B,0} < t\}} |\nabla v_j|^n dx,$$
which tends to 0, as \( j \to \infty \), for every \( t \). So \( \{v_j,x_0\} \) concentrates at \( x_0 \). Futhermore, as in the proof of (b) of Theorem 7, we have

\[
F_{\Omega}(v_j,x_0) = \int_0^\infty f(t) \left( \int_{\partial(\{v_j>0\})} \frac{1}{|\nabla v_j|} ds \right) \times \left( \frac{1}{(\omega_{n-1}|z_j(t)|^n} \int_{\partial(G_{x_0}>G(z_j(t)))} \frac{1}{|\nabla G_{x_0}|} ds \right) dt.
\]

The second inner integral converges to \( r_{\Omega}^n(x_0) \) uniformly in \( t \), as \( j \to \infty \), by Lemma 3. And the rest of the integral is nothing but \( F_B(v_j) \). So, \( F_{\Omega}(v_j,x_0) \to r_{\Omega}^n(x_0)C_B(0) \). This gives \( C_\Omega(x_0) \geq r_{\Omega}^n(x_0)C_B(0) \).

For the other direction of (a), take a sequence \( \{u_j\} \) on \( \Omega \) realizing \( C_\Omega(x_0) \). We first argue that that \( u_j \) must behave like \( \lambda_jG_{x_0} \) off \( \{x_0\} \), where \( \lambda_j \to 0 \). To see this, note the sets \( \{u_j > 1\} \) are contained in balls \( B(x_0,r_j) \), with \( r_j \to 0 \). We then replace \( u_j \) on \( A_j = \{u_j \leq 1\} \) by an \( n \)-harmonic function which agrees with \( u_j \) on \( \partial A_j \). (We still call the new sequence \( \{u_j\} \).) This will not increase the norm of the gradient, by Dirichlet’s principle. Futhermore, if we fix a point \( y \neq x_0 \), and set \( \lambda_j = u_j(y)/G_{x_0}(y) \), then \( \lambda_j \to 0 \), and \( u_j/\lambda_j \to G_{x_0} \) locally uniformly off \( x_0 \). To see the last statement, take a compact set \( K \), containing \( y \), but not \( x_0 \). Harnack’s inequality (see [4]) says that the sequence \( \{u_j/\lambda_j\} \) is uniformly bounded on \( K \), so it is equicontinuous on \( K \). Hence it converges uniformly on \( K \). The limit must be \( n \)-harmonic, and equal to \( G_{x_0} \).

Next, we obtain from Theorem 6 the sequence of symmetrized functions \( u_j^* \) on \( \Omega^* \), which satisfies \( \|
abla u_j^*\|_{L^\infty(\Omega^*)} \leq 1 \), and \( F_{\Omega^*}(u_j^*) = F_{\Omega}(u_j) \). It is easy to see that \( \{u_j^*\} \) concentrates at \( 0 \). To get the conformal factor \( r_{\Omega}^n(x_0) \), we would like to dilate \( u_j(x) \) to \( u_j^*(\frac{x}{r_{\Omega}(x_0)}) \). This will not change the norm of the gradient, and the functional will have the desired conformal factor. However, the new function \( u_j^*(\frac{x}{r_{\Omega}(x_0)}) \) is supported on the set \( 1/r_{\Omega}(x_0) : \Omega^* \), which is larger than the unit ball \( B \).

To remedy the situation, we take the part of \( u_j^* \), where \( u_j^*>1 \), over the unit ball \( B \), and dilate it so that it matches with \( \lambda_jG_{B,0} \). (The latter is defined on the rest of \( B \).) In other words, we are defining a function \( v_j \) on \( B \), so that \( v_j(z) = \lambda_jG_{B,0}(z) \) for values \( \leq 1 \); and \( v_j(z) = u_j^*(\eta_j z) \) for values \( > 1 \), where \( \eta_j \) is chosen so that the two pieces fit together. Notice, by part (c) of Lemma 1, that the radii of the sets \( \{u_j^*>1\} \) and \( \{\lambda_jG_{B,0}>1\} \) are asymptotically equal to \( r_{\Omega}(x_0) \exp(-\omega_{n-1}/1) \) and \( \exp(-\omega_{n-1}/1) \), respectively. So, \( \eta_j \to r_{\Omega}(x_0) \), as \( j \to \infty \).

The sequence \( \{v_j\} \) concentrates at \( 0 \), and \( \|
abla v_j\|_{L^\infty(B)} \leq \|
abla u_j\|_{L^\infty(\Omega)} \leq 1 \). Moreover, we have

\[
\lim_{j \to \infty} F_{\Omega}(u_j) = \lim_{j \to \infty} \int_{\{u_j>1\}} f(u_j) \, dx = \lim_{j \to \infty} \int_{\{u_j^*>1\}} f(u_j^*) \, dx
\]

\[
\lim_{j \to \infty} \int_{\{v_j(x)>1\}} f(v_j(x)) \, dx = \lim_{j \to \infty} \eta_j^n \int_{\{v_j^*(x)>1\}} f(v_j^*(x)) \, dx
\]

\[
= r_{\Omega}^n(x_0) \lim_{j \to \infty} F_B(v_j) \leq r_{\Omega}^n(x_0)C_B(0).
\]

This proves the other half of (a). The proof of Theorem 5 is now complete. \( \square \)
REFERENCES


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