SPECTRAL CONVERGENCE FOR DEGENERATING SEQUENCES OF THREE DIMENSIONAL HYPERBOLIC MANIFOLDS

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Abstract. For degenerating sequences of three dimensional hyperbolic manifolds of finite volume, we prove convergence of their eigenfunctions, heat kernel and spectral measure.

1. Introduction

A natural and important question for a degenerating sequence of Riemannian manifolds concerns their spectral behavior, i.e., spectral degeneration. The works of Wang [23] and Thurston [10] show that among locally symmetric spaces of finite volume, only hyperbolic manifolds of dimension 2 and 3 admit degenerating sequences. The spectral degeneration for hyperbolic surfaces has been intensively studied in [8] [9] [11] [12] [13] [16] [17] [24] [25] [26]. One of the results says roughly that eigenfunctions of compact surfaces converge to linear combinations of Eisenstein series of the limit non-compact surfaces [25, Theorem 3.4] [11, Theorem 1.2]. Besides its interest from the point of view of spectral convergence, this result plays an important role in [27] to verify the Phillips & Sarnak conjecture on finiteness of cuspidal spectrum for generic non-compact surfaces in special families, and is used in [15] to show that the Phillips & Sarnak conjecture implies that for a generic compact surface, almost all eigenvalues are simple. In this paper, we prove that the same result holds for degenerating sequences of three dimensional hyperbolic manifolds. We also prove the convergence of their heat kernel, Green function and spectral measure.

Let $M_i, i \geq 1,$ be a degenerating sequence of three dimensional hyperbolic manifolds of finite volume converging to $M_0$. That is, $M_i, i \geq 1,$ has finite volume and is not necessarily compact, and the lengths of several geodesics in $M_i$ converge to zero as $i \to \infty$. Then $M_0$ has continuous spectrum $[1, +\infty)$, and the generalized eigenfunctions are Eisenstein series $E_{\xi_j}(u; s), u \in M_0, s \in \mathbb{C}$, one for each end $\xi_j$ of $M_0$. Our first result is the following:

Theorem 1.1. Let $\varphi_i$ be a $L^2$-eigenfunction of $M_i$ with eigenvalue $\lambda_i$. Assume that $\lambda_0 = \lim_{i \to \infty} \lambda_i$ exists and $\lambda_0 \geq 1$. Then for any subsequence $i'$, there is a further subsequence $i''$ such that suitable multiples of $\varphi_{i''}$ converge uniformly over compact...
subsets to a non-zero function $\psi_0$ on $M_0$, which satisfies $\Delta \psi_0 = \lambda_0 \psi_0$. Furthermore, there exist constants $a_j$ and a $L^2$-eigenfunction $\varphi_0$ on $M_0$ of eigenvalue $\lambda_0$, which could be zero, such that

$$ \psi_0(u) = \sum_j a_j E_{\xi_j}(u; s_0) + \varphi_0(u), $$

where $s_0(2 - s_0) = \lambda_0$, $\text{Re}(s_0) = 1$.

Remarks. (1) If $\lambda_0 = 1$ and $E_{\xi_j}(u; 1)$ are linearly dependent, then linear combinations of derivatives $\frac{d^k}{ds^k} E_{\xi_j}(u; 1)$ should be used. (2) The same result holds if $M_i$ is noncompact and has finite volume, and $\varphi_i$ is of moderate growth instead of being square integrable (see 5.1 for the definition of moderate growth).

Our second result is the following:

**Theorem 1.2.** Let $H_i(u, v, t)$ be the heat kernel of $M_i$, and $G_{i,s}(u, v)$, $\text{Re}(s) > 2$, be the resolvent kernel of the Beltrami-Laplace operator $\Delta$ of $M_i$. Then $H_i(u, v, t)$ converges to $H_0(u, v, t)$ uniformly over compact subsets, and $G_{i,s}(u, v)$ converges to $G_{0,s}(u, v)$ uniformly over compact subsets away from the diagonal.

**Corollary 1.3.** The spectral measure of $M_i$ converges to the spectral measure of $M_0$.

Since there are no canonical maps between $M_i$ and $M_0$, the precise definition of the convergence in the above theorems is given in 3.6. The precise statement of Corollary 1.3 is given in Proposition 6.5.

If $M_i$, $i \geq 1$, are assumed to be compact, Chavel and Dodziuk [3] showed that the eigenvalues of the Beltrami-Laplace operator of $M_i$ become dense in the continuous spectrum $[1, \infty)$ of $M_0$ as $i \to \infty$ and they obtained the asymptotics of the rate of the spectral accumulation in terms of the lengths of the pinching geodesics. Dodziuk and McGowan [4] obtained similar results for the Hodge-Laplacian acting on differential forms. Colbois and Courtois [1] [2] showed that the eigenvalues of $M_i$ which are less than 1 and their eigenfunctions converge to those of $M_0$. Theorem 1.1 generalizes the result of Colbois and Courtois and gives refined spectral convergence for the continuous spectrum. All the results in this paper hold for the Hodge-Laplacian acting on differential forms, and this will be treated elsewhere.

The organization of the rest of this paper is as follows. In §2, we recall the thick-thin decomposition of three dimensional hyperbolic manifolds. In §3, we recall deformation theory for non-compact hyperbolic manifolds and give the precise definition of the convergence of functions on $M_i$ mentioned earlier. In §4, we derive the Maass-Selberg relation for $M_i$; and in §5, we prove Theorem 1.1. In §6, we prove Theorem 1.2.

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2. The Thick-Thin Decomposition

Let $M$ be a three dimensional complete hyperbolic manifold of finite volume. For any point $u \in M$, let $\iota(u)$ be the injectivity radius at $u$. For any $\varepsilon > 0$, let $M_{(\varepsilon, \infty)} = \{ u \in M \mid \iota(u) \geq \varepsilon \}$ be the thick part of $M$, and $M_{(0, \varepsilon)} = \{ u \in M \mid \iota(u) < \varepsilon \}$ be the thin part of $M$. Then the decomposition $M = M_{(\varepsilon, \infty)} \sqcup M_{(0, \varepsilon)}$ is called the
thick-thin decomposition of $M$. According to a theorem of Kazhdan and Margulis [7, §2], there exists a universal constant $\varepsilon_0 > 0$, called Margulis constant, such that $M(0,\varepsilon_0)$ consists of finitely many components, which can be described explicitly as follows.

Since $M$ has constant curvature $-1$, there exists a Kleinian group $\Gamma \subset \text{PSL}(2, \mathbb{C})$ such that $M = \Gamma \setminus \mathbb{H}^3$, where $\mathbb{H}^3 = \{(z, t) \mid z = x + \sqrt{-1}y \in \mathbb{C}, t > 0\}$ with the hyperbolic metric $ds^2 = t^{-2}(dx^2 + dy^2 + dt^2)$.

Assume that $\Gamma \setminus \mathbb{H}^3$ is non-compact. Then $\Gamma$ contains parabolic elements. Suppose that $\infty$ is a parabolic fixed point of $\Gamma$, then its stabilizer $\Gamma_{\infty}$ in $\Gamma$ consists of parabolic elements of the form $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, and hence can be identified with a lattice in $\mathbb{R}^2 \cong \mathbb{C}$.

For any $\tau > 0$, $\Gamma_{\infty}$ preserves the horoball $H^3_\tau = \{(z, t) \mid t > \tau\}$ at $\infty$.

Lemma 2.1. [7, §2] There exists a positive number $\tau_{\infty}$ that depends only on the parabolic fixed point $\infty$ and $\Gamma$ such that for any $\tau \geq \tau_{\infty}$, two points in $\mathbb{H}^3_{\tau}$ are $\Gamma$ equivalent if and only if they are $\Gamma_{\infty}$ equivalent. In particular, there is an embedding: $\Gamma_{\infty} \setminus \mathbb{H}^3_{\tau} \to \Gamma \setminus \mathbb{H}^3$ given by $\Gamma_{\infty}(z, t) \to \Gamma(z, t)$.

The open subset $\Gamma_{\infty} \setminus \mathbb{H}^3_{\tau_{\infty}} \subset M$ is called the cusp associated with the parabolic fixed point and is denoted by $C_{\infty}$. It is isometric to the space $[\tau_{\infty}, +\infty) \times \Gamma_{\infty} \setminus \mathbb{R}^2$ with the metric $t^{-2}(dt^2 + d\omega^2)$, where $d\omega^2$ is the metric on the torus $\Gamma_{\infty} \setminus \mathbb{R}^2$ induced from the metric $dx^2 + dy^2$ on $\mathbb{R}^2$. The constant $\tau_{\infty}$ satisfies the following property: For any $u \in \partial C_{\infty}$, the injectivity radius $\iota(u) = \varepsilon_0$. For any $a > 0$, denote the subcusp $\Gamma_{\infty} \setminus \mathbb{H}^3_{\tau_{\infty} + \exp a}$ of $C_{\infty}$ by $C_{\infty}(a)$. Clearly, $d(\partial C_{\infty}, C_{\infty}(a)) = a$.

For any other parabolic fixed point $\xi$ of $\Gamma$, we can also define its cusp $C_{\xi} \subset M$ by conjugating $\xi$ to $\infty$ so that for any $u \in \partial C_{\xi}$, $\iota(u) = \varepsilon_0$. Similarly, for any $a > 0$, we can define its subcusp $C_{\xi}(a) \subset C_{\xi}$ so that $d(\partial C_{\xi}, C_{\xi}(a)) = a$.

Any hyperbolic element $\gamma \in \Gamma$ leaves invariant a unique geodesic in $\mathbb{H}^3$ and acts on it by translation. This geodesic is called the axis of $\gamma$. For any simple closed geodesic $c(s)$ in $M$, its lift $\tilde{c}(s)$ in $\mathbb{H}^3$ is the axis of the deck transformation associated to $\gamma_c$. Let $(r, s, \theta)$ be the Fermi coordinates of $\mathbb{H}^3$ based on the geodesic $\tilde{c}$, where $r$ is the distance to $\tilde{c}$, $s$ is the arc length of $\tilde{c}$, and $\theta$ is the angular coordinate in the plane perpendicular to $\tilde{c}$ at $\tilde{c}(s)$. Then the deck transformation $\gamma_c$ is given by $\gamma_c(r, s, \theta) = (r, s + |c|, \theta + \alpha)$, where $|c|$ is the length of the geodesic $c$ in $M$, and $\alpha \in \mathbb{R}/2\pi$. In these coordinates, the hyperbolic metric on $\mathbb{H}^3$ is given by $ds^2 = dr^2 + \cosh^2 r ds^2 + \sinh^2 r d\theta^2$. Then for any $R > 0$, the tubular neighborhood $B(\tilde{c}, R) = \{(z, t) \in \mathbb{H}^3 \mid d((z, t), \tilde{c}) < R\}$ of $\tilde{c}$ is invariant under $\gamma_c$.

Lemma 2.2. [7, §2] Let $\varepsilon_0$ be the Margulis constant. For any simple closed geodesic $c(s)$ with $|c| < 2\varepsilon_0$, there exists a constant $R_c$ which depends only on $c$ and $M$, and $R_c \sim 2\log 1/|c|$ as $|c| \to 0$ such that for any $0 < R \leq R_c$, the quotient $B(\tilde{c}, R)/\gamma_c$ embeds into $M$ and hence is a $R$ tubular neighborhood of $c$, denoted by $B(c, R)$; and for any $u \in \partial B(c, R_c)$, the injectivity radius $\iota(u) = \varepsilon_0$.

Let $\xi_1, \ldots, \xi_p$ be the maximal set of $\Gamma$ inequivalent parabolic fixed points, and $C_{\xi_1}, \ldots, C_{\xi_p}$ their cusps. Let $c_1, \ldots, c_m$ be all the simple closed geodesics in $M$ with length less than $2\varepsilon_0$, and $R_1, \ldots, R_m$ the corresponding constants in Lemma 2.2. Then we get the following decomposition.

Lemma 2.3. [7, p. 44] With the above notation, all the cusps $C_{\xi_1}, \ldots, C_{\xi_p}$ and the tubular neighborhoods $B(c_1, R_1), \ldots, B(c_m, R_m)$ of the simple closed geodesics
c_1, \ldots, c_m \text{ are disjoint, and}
\[ M_{(c_0, \infty)} = M \setminus \left( \bigcup_{i=1}^{p} C_i \bigcup_{j=1}^{m} B(c_j, R_j) \right), \quad M_{(0, c_0)} = \bigcup_{i=1}^{p} C_i \bigcup_{j=1}^{m} B(c_j, R_j). \]

In particular, there is a bijective correspondence between the ends of M and the \( \Gamma \)-equivalence classes of parabolic fixed points.

3. Deformation of Hyperbolic Structure

Let \( M_0 = \Gamma_0 \setminus \mathbb{H}^3 \) be a non-compact complete hyperbolic three dimensional manifold of finite volume. Then the deformation space of \( \Gamma_0 \) or \( M_0 \) in \( \text{PSL}(2, \mathbb{C}) \) is defined by
\[ \text{Def}(\Gamma_0) = \text{Hom}(\Gamma_0, \text{PSL}(2, \mathbb{C}))/\text{PSL}(2, \mathbb{C}), \]
where \( \text{PSL}(2, \mathbb{C}) \) acts by conjugation.

**Lemma 3.1.** [10, Theorem 5.6] [21, Proposition 2.3] If \( \Gamma_0 \) has \( p \) inequivalent parabolic fixed points, then \( \dim_{\mathbb{R}} \text{Def}(\Gamma_0) = 2p. \)

Let \( \mathbb{R}^2 = \mathbb{R}^2 \coprod \{ \infty \} \) be the one point compactification of \( \mathbb{R}^2. \) Then we have the following parametrization of the deformation space \( \text{Def}(\Gamma_0). \)

**Lemma 3.2.** [22, p. 235] For any choice of generators of the stabilizers \( \Gamma_0, \xi_1, \cdots, \Gamma_0, \xi_p, \) of all the inequivalent parabolic fixed points \( \xi_1, \cdots, \xi_p \) of \( \Gamma_0, \) there exist a neighborhood \( D \) in \( \text{Def}(\Gamma_0) \) of the inclusion \( i_0 : \Gamma_0 \hookrightarrow \text{PSL}(2, \mathbb{C}) \) and a homeomorphism from \( D \) onto a neighborhood of \( (\infty, \cdots, \infty) \in (\mathbb{R}^2)^p \) such that \( i_0 \) is mapped to \( (\infty, \cdots, \infty). \)

For any deformation \( \rho \in D \subset \text{Def}(\Gamma_0), \) let \( ((r_1(\rho), u_1(\rho)); \cdots; (r_p(\rho), u_p(\rho))) \in (\mathbb{R}^2)^p \) be the corresponding coordinates, where some of \( (r_i(\rho), u_i(\rho)) \) could be \( \infty. \)

**Lemma 3.3.** [21, Theorem 5.8.2, Lemma 5.8.1] [21, Proposition 2.3] [22, Theorem 1.7, p. 235] For any \( \rho \in D \subset \text{Def}(\Gamma_0), \) the subgroup \( \rho(\Gamma_0) \subset \text{PSL}(2, \mathbb{C}) \) is a discrete torsion free cofinite subgroup if and only if for any \( i = 1, \cdots, p \) either \( r_i(\rho), u_i(\rho) \) are coprime or \( (r_i(\rho), u_i(\rho)) = \infty. \) For such a deformation \( \rho, \) the total number of inequivalent parabolic fixed points of \( \rho(\Gamma_0) \) is equal to the number of \( i \) with \( (r_i(\rho), u_i(\rho)) = \infty. \)

**Definition 3.4.** Assume that \( M_i, i \geq 0, \) is a sequence of connected complete hyperbolic manifolds of three dimensions. Then \( M_i \) is defined to converge to \( M_0 \) when \( i \to \infty \) if for any \( \delta > 0 \) and \( \varepsilon > 0, \) there exist \( n_0 > 0 \) and a homeomorphism \( f_i : M_{0,[\varepsilon, \infty)} \to M_{i,[\varepsilon, +\infty)} \) for all \( i \geq n_0 \) such that
\[ L(f_i) = \sup_{u,v \in M_{0,[\varepsilon, \infty)}} \left| \log \frac{d(u,v)}{d(f_i(u), f_i(v))} \right| \leq \delta. \]

For any \( \rho_0 \in D \) with at least one \( (r_i(\rho_0), u_i(\rho_0)) = \infty, \) there are only countably infinitely many \( \rho \) in a neighborhood of \( \rho_0 \) such that \( \rho(\Gamma_0) \) is a torsion free cofinite subgroup, according to Lemma 3.3. Denote them by \( \rho_1, \cdots, \rho_i, \cdots. \) Then \( \rho_i \to \rho_0 \) in \( D \) as \( i \to \infty. \)
Proposition 3.5. [10, Theorem 5.12] The sequence of manifolds $\rho_i(\Gamma_0)\backslash H^3$ converges to $\rho_0(\Gamma_0)\backslash H^3$. Conversely, if $M_i$ is a sequence of complete hyperbolic three dimensional manifolds converging to $\rho_0(\Gamma_0)\backslash H^3$, then $M_i$ is eventually of the form $\rho_i(\Gamma_0)\backslash H^3$ for some $\rho_i \in D$, and $\rho_i \to \rho_0$ in $D$ as $i \to \infty$.

From this proposition and the Mostow rigidity theorem, the sequence $\rho_i \in D$ with $\rho_i \neq \rho_0$ and $\rho_i \to \rho_0$ gives rise to a degenerating sequence with limit $\rho_0(\Gamma_0)\backslash H^3$, since $\rho_0(\Gamma_0)\backslash H^3$ is not homeomorphic to $\rho_i(\Gamma_0)\backslash H^3$. By the Mostow rigidity theorem again, any convergent sequence $M_i$ is degenerate unless it is eventually constant.

From now on, for any degenerate sequence of hyperbolic three dimensional manifolds $M_i$ with limit $M_0$, we fix a uniformization group $\Gamma_i$, i.e., $M_i \cong \Gamma_i \backslash H^3$, such that for $i \gg 1$, $\Gamma_i = \rho_i(\Gamma_0)$ and $\rho_i(\Gamma_0) \to \Gamma_0$ in $\text{PSL}(2, \mathbb{C})$, i.e., for any $\gamma \in \Gamma_0$, $\rho_i(\gamma) \to \gamma$ as $i \to \infty$. Since there is no natural map to compare functions on $M_i$ and $M_0$, we introduce the following convention.

Definition 3.6. [Convergence of Functions]. Assume that $M_i$ converges to $M_0$ as $i \to \infty$. (1) Let $\varphi_i$ be a continuous function on $M_i$ for $i \geq 0$ and $\tilde{\varphi}_i$ its lift to $H^3$. Then $\varphi_i$ is defined to converge to $\varphi_0$ uniformly over compact subsets if $\tilde{\varphi}_i$ converges uniformly over all compact subsets of $H^3$. (2) Assume that $\psi_i$ is a function on $M_i \times M_i$ which is continuous off the diagonal, and $\psi_i$ is its lift on $H^3 \times H^3$. Then $\psi_i$ is defined to converge to $\psi_0$ uniformly over compact subsets away from the diagonal if for any compact subsets $K_1, K_2 \subset H^3$ with $K_1 \cap \gamma K_2 = \emptyset$ for any $\gamma \in \Gamma_0$, $\tilde{\psi}_i$ converges to $\tilde{\psi}_0$ uniformly over $K_1 \times K_2$. Similarly, we can define the uniform convergence of $\psi_i$ to $\psi_0$ over all compact subsets of $H^3 \times H^3$.

This definition of convergence is justified by the following.

Lemma 3.7. Assume that $\varphi_i$ converges to $\varphi_0$ uniformly over compact subsets according to Definition 3.6. Then for any fixed $\varepsilon > 0$ and $f_i : M_{0,|\varepsilon,\infty}) \to M_{i,|\varepsilon,\infty})$ as in Definition 3.4 with $L(f_i) \to 0$ as $i \to \infty$, $\varphi_i \circ f_i$ converges to $\varphi_0$ uniformly over $M_{0,|\varepsilon,\infty})$.

Proof. Let $\tilde{M}_{i,|\varepsilon,\infty})$ be the inverse image in $H^3$ of the thick part $M_{i,|\varepsilon,\infty})$, and $\tilde{f}_i : \tilde{M}_{0,|\varepsilon,\infty}) \to \tilde{M}_{i,|\varepsilon,\infty}) \subset H^3$ the lift of $f_i$. Note that $M_{0,|\varepsilon,\infty})$ is homeomorphic to $M_{i,|\varepsilon,\infty})$ and hence such a lift exists. Since $L(f_i) \to 0$, by the normalization of the uniformization groups $\Gamma_i$ and hence the lifting, $\tilde{f}_i$ converges uniformly over compact subsets of $\tilde{M}_{i,|\varepsilon,\infty})$ to the inclusion map $\tilde{M}_{0,|\varepsilon,\infty}) \hookrightarrow H^3$. By assumption, $\varphi_i$ converges to $\varphi_0$ uniformly over compact subsets, and hence $\tilde{\varphi}_i \circ \tilde{f}_i$ converges to $\tilde{\varphi}_0$ uniformly over compact subsets. Therefore $\varphi_i \circ f_i$ converges to $\varphi_0$.

Deformation of $\Gamma_0$ or $M_0$ can also be studied via deformation of fundamental domains in $H^3$. For a detailed account, see [22, pp. 232-237]. Hence we recall only a few statements needed for our proof of Theorem 1.1.

Let $D_0 = D(\Gamma_0, u_0)$ be the Dirichlet domain of $\Gamma_0$ with center $u_0 \in H^3$, i.e.,

$$D_0 = D(\Gamma_0, u_0) = \{u \in H^3 \mid d(u, u_0) \leq d(u, \gamma u_0), \gamma \in \Gamma_0\}.$$  

Then $D_0$ is a closed convex domain in $H^3$ with finitely many faces. Points in $D_0 \cap H^3(\infty)$ are parabolic fixed points of $\Gamma_0$, and any parabolic fixed point of $\Gamma_0$ is equivalent to one of them. If $u_0$ is generic, i.e., outside countably many totally geodesic hyperplanes in $H^3$ determined by $\Gamma_0$, then any two points in $D_0 \cap H^3(\infty)$
are not \( \Gamma_0 \) equivalent, and hence there is a one-to-one correspondence between the points in \( D_0 \cap H^3(\infty) \) and the cuspidal ends of \( M_0 \) [14, Lemmas 3.4 and 4.5].

For simplicity, we assume that \( D_0 \cap H^3(\infty) = \infty \), in particular, that \( M_0 \) has only one cusp. Then there are exactly four faces of \( D_0 \) passing through \( \infty \), and they are paired by two generators \( \alpha \) and \( \beta \) of \( \Gamma_{0,\infty} \). Recall from \( \S 2 \) that for any \( \tau > 0 \), \( H^3_\tau = \{ (z,t) \in H^3 \mid t > \tau \} \). Then \( D_0 \cap H^3_\tau = D(\Gamma_{0,\infty}) \times (\tau, \infty) \) (see [22, Figure 20, p. 234]), where \( D(\Gamma_{0,\infty}) \) is the Dirichlet fundamental domain of \( \Gamma_{0,\infty} \) acting on \( \mathbb{C} = \mathbb{R}^2 \) with center \( z_0 \), where \( u_0 = (z_0, t_0) \). The truncated domain \( D_0 \setminus (D_0 \cap H^3_\tau) \) is compact.

Any deformation of \( \Gamma_0 \) corresponds to a deformation of \( D_0 \) in the category of convex polyhedra. For the deformation \( \Gamma_i = \rho_i(\Gamma_0) \), the compact part \( D_0 \setminus (D_0 \cap H^3_\tau) \) is deformed slightly, while \( D_0 \cap H^3_\tau \) is replaced by a wedge along the common axis of helical motions \( \rho_i(\alpha) \) and \( \rho_i(\beta) \) (see [22, Figures 21 and 22, pp. 236-237]). Since \( \rho_i(\alpha) \to \alpha \) and \( \rho_i(\beta) \to \beta \), this wedge converges to \( D_0 \cap H^3_\tau \).

In particular, the distance from the common axis to \( u_0 \) goes to infinity, and the dihedral angle along the wedge goes to zero as \( i \to \infty \). Denote the deformed polyhedra by \( D_i \). Then \( D_i \) is a fundamental polyhedron for \( \Gamma_i \) [22, Theorem 1.7, p. 235].

Since \( M_0 \) has only one cusp, \( M_i \) has exactly one pinching geodesic, denoted by \( c_i \). Let \( B(c_i, R_i) \) be the tubular neighborhood of \( c_i \) defined in Lemma 2.2. Denote the projection from \( H^3 \) to \( \Gamma_i \setminus \Gamma_0 \) by \( \pi_i \). Then we have the following:

**Lemma 3.8.** For any \( a > 0 \), \( D_i \cap \pi_i^{-1}(B(c_i, R_i) \setminus B(c_i, R_i - a)) \) converges to \( D_0 \cap \pi_0^{-1}(C_\infty - C_\infty(a)) \) as \( i \to \infty \).

**Proof.** For any \( r \geq 0 \), as \( i \to \infty \), \( D_i \setminus (D_i \cap H^3_{\tau_{\infty} + r}) \to D_0 \setminus (D_0 \cap H^3_{\tau_{\infty} + r}) \), and hence \( i(\pi_i(u)) \to i(\pi_0(u)) \) for any \( u \in H^3 \), where \( i(\pi_i(u)) \) is the injectivity radius of \( \pi_i(u) \) in \( M_i \). Since \( \pi_0(D_0 \cap (H^3_{\tau_{\infty} - H^3_{\tau_{\infty} + r})) = C_\infty \setminus C_\infty(a) \) and \( C_\infty \) is defined in terms of the injectivity radius, it follows that
\[
d(\pi_i(D_i \cap (H^3_{\tau_{\infty} - H^3_{\tau_{\infty} + r}))), B(c_i, R_i) \setminus B(c_i, R_i - a)) \to 0,
\]
and hence \( d(D_i \cap \pi_i^{-1}(B(c_i, R_i) \setminus B(c_i, R_i - a)), D_i \cap \pi_i^{-1}(H^3_{\tau_{\infty} - H^3_{\tau_{\infty} + r}))) \to 0 \) as \( i \to \infty \). Since \( D_i \cap \pi_i^{-1}(B(c_i, R_i) \setminus B(c_i, R_i - a)) \), \( D_i \cap \pi_i^{-1}(H^3_{\tau_{\infty} - H^3_{\tau_{\infty} + r}}) \), it then follows that \( D_i \cap \pi_i^{-1}(B(c_i, R_i) \setminus B(c_i, R_i - a)) \to D_0 \cap \pi_0^{-1}(C_\infty \setminus C_\infty(a)) \) as \( i \to \infty \).

**Lemma 3.9.** For any compact subset \( K \subset H^3 \), there exist finitely many elements \( \gamma_1, \ldots, \gamma_n \in \Gamma_0 \) and a positive constant \( a \) such that \( K \) is contained in \( (\rho_i(\gamma_1)D_i \cup \cdots \cup \rho_i(\gamma_n)D_i) \cap \pi_i^{-1}(M_i \setminus B(c_i, R_i - a)) \) for \( i \gg 1 \).

**Proof.** Since \( K \) is compact, there exist \( a > 0 \) and \( \gamma_1, \ldots, \gamma_n \in \Gamma_0 \) such that \( K \subset \text{Interior}(\gamma_1 D_0 \cup \cdots \cup \gamma_n D_0 \setminus H^3_{\infty + a}) \). Since \( (\rho_i(\gamma_1)D_i \cup \cdots \cup \rho_i(\gamma_n)D_i) \setminus H^3_{\infty + a} \) goes to \( -\infty \), \( \infty \), it is clear that when \( i \gg 1 \), \( K \subset (\rho_i(\gamma_1)D_i \cup \cdots \cup \rho_i(\gamma_n)D_i \setminus \pi_0^{-1}(C_\infty(a)) \), and hence \( K \subset (\rho_i(\gamma_1)D_i \cup \cdots \cup \rho_i(\gamma_n)D_i) \cap \pi_i^{-1}(M_i \setminus B(c_i, R_i - a)) \).

4. MAASS-SELBERG RELATION

The Maass-Selberg relation is essential to spectral analysis for non-compact hyperbolic surfaces of finite area [18, Theorems 2.3.1 and 2.3.2]. In this section, we establish a Maass-Selberg relation for compact hyperbolic three dimensional manifolds with short geodesics. Such a formula for hyperbolic surfaces was given by

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Proposition 4.2. [Maass-Selberg Relation]. Let \( \varphi \) be a smooth function on \( M_i \) satisfying \( \Delta \varphi = \lambda \varphi \), \( \lambda \geq 0 \). Then

\[
\int_{M_i} |\nabla \varphi|^2 = \lambda \int_{M_i} |\varphi|^2 + \frac{d \varphi (R_i - a)}{dr} \varphi (T_i(R_i - a)).
\]

Proof. By Green’s formula,

\[
\int_{M_i \setminus B(c_i,R_i - a)} -\Delta \varphi \varphi + |\nabla \varphi|^2 = \int_{T_i(R_i - a)} \frac{\partial \varphi (R_i - a, w)}{\partial r} \varphi (R_i - a, w) d\mu(w),
\]

where \( w \in T_i(R_i - a) = \partial B(c_i, R_i - a) \), and \( d\mu(w) \) is the area form on \( T_i(R_i - a) \) induced from \( M_i \). Then by Lemma 4.1,

\[
\int_{M_i \setminus B(c_i,R_i - a)} -\Delta \varphi \varphi + |\nabla \varphi|^2 = \int_{T_i(R_i - a)} \frac{\partial \varphi (R_i - a, w)}{\partial r} \varphi (R_i - a, w)
\]

\[
+ \frac{d \varphi (R_i - a)}{dr} \varphi (R_i - a) d\mu(w).
\]

Similarly,

\[
\int_{B(c_i,R_i - a)} -\Delta \varphi \varphi + |\nabla \varphi|^2 = \int_{T_i(R_i - a)} \frac{\partial \varphi (R_i - a, w)}{\partial r} \varphi (R_i - a, w) d\mu(w).
\]
Adding these two equations yields that
\[
\int_{M_i} -\Delta \varphi^a \varphi^a + |\nabla \varphi^a|^2 = \frac{d \varphi (R_t - a)}{dr} \varphi (R_t - a) \text{area}(T_i(R_t - a)).
\]
From the assumption on \( \varphi \), it follows that \( \Delta \varphi^a = \lambda \varphi^a \) outside \( T_i(R_t - a) \), and hence
\[
-\lambda \int_{M_i} |\varphi^a|^2 + \int_{M_i} |\nabla \varphi^a|^2 = \frac{d \varphi (R_t - a)}{dr} \varphi (R_t - a) \text{area}(T_i(R_t - a)).
\]

5. **Proof of Theorem 1.1**

First, we need a few lemmas.

**Definition 5.1.** A function \( f \) on \( M_0 \) is said to have moderate growth if there exist constants \( \alpha, \beta > 0 \) such that
\[
|f(u)| \leq e^{\alpha d(u, u_0)}
\]
for some \( u_0 \in M_0 \), where \( d(\cdot, \cdot) \) is the distance function.

For each cusp \( \xi_j \) of \( M_0 \), an Eisenstein series \( E_{\xi_j}(u; s) \), \( u \in M_0, \ s \in \mathbb{C} \), is defined and has moderate growth [5]. If \( \infty \) is a parabolic fixed point, then its Eisenstein series \( E_\infty \) is defined for \( \text{Re}(s) > 2 \) as follows:
\[
E_\infty((z, \ell); s) = \sum_{\gamma \in \Gamma_0, s \not\in \Gamma_0} (t_{\gamma(z, \ell)})^s,
\]
where \( t_{\gamma(z, \ell)} \) is the \( t \) component of the point \( \gamma(z, \ell) \in \mathbb{H}^3 \).

By a similar argument to that of [19, Satz 10 and Satz 11], we get the following result.

**Lemma 5.2.** If \( f \) is a solution of \( \Delta f = s(2 - s)f \), \( \text{Re}(s) = 1 \), of moderate growth, then there exist constants \( a_i \) and a \( L^2 \) solution \( \varphi_0 \) of \( \Delta \varphi_0 = s(2 - s)\varphi_0 \), which is zero if \( s(2 - s) \) is not an eigenvalue of \( M_0 \), such that
\[
f = \sum_j a_j E_{\xi_j}(\cdot; s_0) + \varphi_0.
\]

We also need the following fact.

**Lemma 5.3.** [3, Lemma 3.9] For each pinching geodesic \( c_{i,j} \) in \( M_i \), let \( \nu_{i,j}(r) \) be the first positive eigenvalue of the torus \( T_{i,j}(r) = \partial B(c_{i,j}, r) \), then \( \nu_{i,j}(r) \) is a decreasing function of \( r \). Moreover, there exists a positive constant \( c \) independent of \( i \) such that \( \nu_{i,j}(r) \geq ce^{2(R_{i,j} - r)} \) when \( i \gg 1 \), where \( R_{i,j} \) is the width of the collar of \( c_{i,j} \) in Lemma 2.2.

We are ready to prove Theorem 1.1. Identify the function \( \varphi_i \) with its lift to \( \mathbb{H}^3 \). Then \( \varphi_i \) is a \( \Gamma_i \) invariant function on \( \mathbb{H}^3 \). The proof consists of three steps:

1. Show that after suitable scaling, for any subsequence \( i' \) of \( i \), there is a further subsequence \( i'' \) such that \( \varphi_i'' \) converges uniformly over compact subsets to a function \( \psi_0 \) on \( \mathbb{H}^3 \) which is invariant under \( \Gamma_0 \).
2. Show that \( \psi_0 \) is not identically zero.
3. Show that \( \psi_0 \) has moderate growth.
STEP 1

For simplicity, we assume that $M_0$ has only one cuspidal end, and hence $M_i$ is compact with only one pinching geodesic $c_i$ (see the remark at the end for the general case). We normalize $\varphi_i$ by

$$\int_{M_i} |\varphi_i^a|^2 = 1,$$

where $\varphi_i^a$ is the truncated function from §4. It follows from Theorem 1.1 that this normalization is essentially equivalent to two other more common normalizations:

$$\int_{M_i, [\varepsilon_0, \infty)} |\varphi_i|^2 = 1, \quad \text{or} \quad \sup_{M_i, [\varepsilon_0, \infty)} |\varphi_i| = 1.$$

However, the normalization we use here is the most convenient for the proof. From

$$\int_{M_i} |\varphi_i^a|^2 = \int_{M_i \setminus B(c_i, R_i - a)} |\varphi_i|^2 + \int_{B(c_i, R_i - a)} |\varphi_i| \overline{\varphi_i}|^2$$

it follows that

$$\int_{B(c_i, R_i) \setminus B(c_i, R_i - a)} |\varphi_i| \overline{\varphi_i}|^2 \leq 1.$$

Since $\varphi_i$ satisfies the following ordinary differential equation [3, Equations 2.8 and 2.9]

$$(-\frac{d^2}{dr^2} - 2 \coth(2r) \frac{d}{dr}) \varphi_i = \lambda_i \varphi_i$$

and $\coth(r) \to 1$ as $i \to \infty$ uniformly for $r \geq R_i - a$, by the stability of solutions of ordinary differential equations, we show that $\varphi_i(R_i - a)$ and $\frac{d}{dr} \varphi_i(R_i - a)$ are bounded uniformly for $i \geq 1$. By Lemma 3.8, area($T_i(R_i - a))$ is bounded for $i \geq 1$. Then from Proposition 4.2 and the assumption $\lim_{i \to \infty} \lambda_i < +\infty$, it follows that

$$(5.1) \int_{M_i} |\nabla \varphi_i^a|^2 \leq c_1,$$

where $c_1$ is a constant independent of $i$. By the stability of ordinary differential equations again, we get that for any $b > a$,

$$\int_{B(c_i, R_i - a) \setminus B(c_i, R_i - b)} |\varphi_i|^2 + |\frac{d}{dr} \varphi_i|^2 \leq c_2$$

for some constant $c_2$ independent of $i$. These inequalities imply that

$$\int_{M_i \setminus B(c_i, R_i - b)} |\varphi_i|^2 + |\nabla \varphi_i|^2 \leq \int_{M_i} |\varphi_i^a|^2 + |\nabla \varphi_i^a|^2$$

$$+ \int_{B(c_i, R_i - a) \setminus B(c_i, R_i - b)} |\varphi_i|^2 + |\frac{d}{dr} \varphi_i|^2$$

$$\leq 1 + c_1 + c_2.$$

Let $K \subset \mathbb{H}^3$ be any compact subset. Using Lemma 3.9, we get that

$$\int_K |\varphi_i|^2 + |\nabla \varphi_i|^2 \leq c_3,$$
where \( c_3 \) is a constant independent of \( i \). Since \( \varphi_i \) satisfies \( \Delta \varphi_i = \lambda_i \varphi_i \) and \( \lim \lambda_i \) exists, by elliptic regularity theory [6, Theorems 8.8 and 8.10], for any \( k \geq 1 \), there exists a constant independent of \( i \) such that

\[
\| \varphi_i \|_{W^{k,2}(K)} \leq c_4,
\]

where \( W^{k,2}(K) \) is the Sobolev norm. Take an increasing sequence of compact subsets \( K_n \subset \mathbb{H}^3 \) with \( \bigcup_{n=1}^{\infty} K_n = \mathbb{H}^3 \). Then by the Sobolev embedding theorem [6, §7.7] and a diagonal argument, for any subsequence \( i' \), there is a further subsequence \( i'' \) such that \( \varphi_{i''} \) converges \( C^k \)-uniformly over compact subsets to function \( \psi_0 \) on \( \mathbb{H}^3 \) for all \( k \geq 1 \). This function \( \psi_0 \) is clearly smooth and satisfies the equation \( \Delta \psi_0 = \lambda_0 \psi_0 \), where \( \lambda_0 = \lim \lambda_i \).

Since \( \Gamma_i \to \Gamma_0 \) as \( i \to \infty \) (see the convention before Definition 3.6), and \( \varphi_{i''} \) is \( \Gamma_{i''} \) invariant, it follows from the uniform convergence that \( \psi_0 \) is \( \Gamma_0 \) invariant, and hence descends to a function on \( M_0 \).

**Step 2**

We show that \( \psi_0 \) is not identically zero by contradiction. If \( \psi_0 \equiv 0 \), then for any \( b > a \), \( \varphi_{i''} \) converges uniformly to zero on \( M_{i''} \setminus B(c_{i''}, R_{i''} - b) \) as \( i'' \to +\infty \). Let \( \chi_{i''} \) be a cut-off function on \( M_{i''} \): \( \chi_{i''} = 1 \) on \( M_{i''} \setminus B(c_{i''}, R_{i''} - b - 1) \), \( \chi_{i''} = 0 \) on \( B(c_{i''}, R_{i''}) \setminus B(c_{i''}, R_{i''} - b) \), \( \chi_{i''}(u) \) only depends on \( r(u) \), \( 0 \leq \chi_{i''} \leq 1 \), and \( |\frac{d}{dr} \chi_{i''}| \leq 2 \).

Then

\[
\int_{\mathcal{B}(c_{i''}, R_{i''} - b)} |\nabla (\chi_{i''} \varphi_{i''})|^2 = \int_{\mathcal{B}(c_{i''}, R_{i''} - b)} |\nabla \varphi_{i''}|^2 + \varepsilon_{i''}(b),
\]

where

\[
\varepsilon_{i''}(b) = \int_{\mathcal{B}(c_{i''}, R_{i''} - b) \setminus \mathcal{B}(c_{i''}, R_{i''} - b - 1)} |\varphi_{i''} \nabla \chi_{i''}|^2 \to 0
\]

as \( i'' \to \infty \) for any fixed \( b > a \).

By Equation 5.1,

\[
\int_{\mathcal{B}(c_{i''}, R_{i''} - b)} |\nabla \varphi_{i''}|^2 \leq \int_{M_{i''}} |\nabla \varphi_{i''}|^2 \leq c_1,
\]

where \( c_1 \) is a constant independent of \( i \), and hence

\[
(5.2) \quad \int_{\mathcal{B}(c_{i''}, R_{i''} - b)} |\nabla (\chi_{i''} \varphi_{i''})|^2 \leq c_1 + |\varepsilon_{i''}(b)|.
\]

On the other hand, by Lemma 5.3, there exists \( b_0 > 0 \) such that for any \( b > b_0 \), \( \nu_{i''}(R_{i''} - b) \geq c_1 + 1 \), and hence
\[
\int_{B(c_{\iota},R_{\iota \iota} - b)} |\nabla (\chi_{\iota''} \bar{\varphi}_{\iota''})|^2 \geq \int_0^{R_i - b} \int_{T_i(r)} |\nabla' (\chi_{\iota''} \bar{\varphi}_{\iota''})|^2 \\
\geq \int_0^{R_i - b} \nu_{\iota''}(r) \int_{T_i(r)} |\chi_{\iota''} \bar{\varphi}_{\iota''}|^2 \\
\geq \nu_{\iota''}(R_{\iota''} - b) \int_0^{R_i - b} \int_{T_i(r)} |\chi_{\iota''} \bar{\varphi}_{\iota''}|^2 \\
\geq (c_1 + 1) \int_{B(c_{\iota''},R_{\iota''} - b)} |\chi_{\iota''} \bar{\varphi}_{\iota''}|^2 \\
\geq c_1 + \frac{1}{2},
\]
for \( i'' \gg 1 \), since \( \int_{M_{\iota''}} |\varphi_{\iota''}|^2 = 1 \) and \( \varphi_{\iota''} \) converges uniformly to zero on \( M_{\iota''} \setminus B(c_{\iota''},R_{\iota''} - b) \). In the first inequality, \( \nabla' \) stands for the gradient on the torus \( T_i(r) \). This contradicts Inequality 5.2 when \( b \to \infty \).

**Step 3**

We next show that \( \psi_0 \) has moderate growth on \( M_0 \). Assume that \( \infty \) is a parabolic fixed point of \( \Gamma_\iota \). By Lemma 3.8, for any fixed \( b > a \), as \( i \to +\infty \),

\[
D_i \cap \pi^{-1}_i(B(c_i,R_i) \setminus B(c_i,R_i - b)) \to D_0 \cap \pi_0^{-1}(C_\infty \setminus C_\infty(b)),
\]
where \( D_i \subset \mathbb{H}^3 \) is a polyhedral fundamental domain for \( \Gamma_\iota \), and \( \pi_i : \mathbb{H}^3 \to \Gamma_\iota \setminus \mathbb{H}^3 \) as in §3. From the uniform convergence of \( \varphi_{\iota''} \to \psi_0 \) over compact subsets, it then follows that the non-constant Fourier term \( \bar{\varphi}_{\iota''} \) on \( D_{\iota''} \cap \pi_{\iota''}^{-1}(B(c_{\iota''},R_{\iota''}) \setminus B(c_{\iota''},R_{\iota''} - b)) \) converges uniformly to \( \bar{\psi}_0 \) on \( D_0 \cap \pi_0^{-1}(C_\infty \setminus C_\infty(b)) \), and hence

\[
\int_{C_\infty \setminus C_\infty(b)} |\bar{\psi}_0|^2 = \int_{D_0 \cap \pi_0^{-1}(C_\infty \setminus C_\infty(b))} |\bar{\psi}_0|^2 \\
= \lim_{i'' \to \infty} \int_{D_{\iota''} \cap \pi_{\iota''}^{-1}(B(c_{\iota''},R_{\iota''}) \setminus B(c_{\iota''},R_{\iota''} - b))} |\bar{\varphi}_{\iota''}|^2 \\
\leq \lim_{i'' \to \infty} \int_{B(c_{\iota''},R_{\iota''})} |\varphi_{\iota''}|^2 \\
= \lim_{i'' \to \infty} (1 - \int_{M_{\iota''} \setminus B(c_{\iota''},R_{\iota''})} |\varphi_{\iota''}|^2) \\
= 1 - \int_{M_0 \setminus C_\infty} |\psi_0|^2.
\]

Letting \( b \to +\infty \), we get

\[
(5.3) \quad \int_{C_\infty} |\bar{\psi}_0|^2 \leq 1.
\]

Write \( \lambda_0 = s_0(2 - s_0) \) for some \( s_0 \in \mathbb{C} \) with \( \text{Re}(s_0) = 1 \). From the equation \( \Delta \bar{\psi}_0 = \lambda_0 \bar{\psi}_0 = s_0(2 - s_0) \psi_0 \), \( \bar{\psi}_0 \) has the following full Fourier expansion on the cusp \( C_\infty \):

\[
\bar{\psi}_0(z,t) = \sum_{\tau \in \Gamma_\infty, \tau \neq 0} \alpha_{\tau} t K_{s_0 - 1}(|\tau| t) e^{2\pi \sqrt{-1}(z,\tau)} + b_\tau t I_{s_0 - 1}(|\tau| t) e^{2\pi \sqrt{-1}(z,\tau)},
\]
where $\Gamma_{0,\infty}$ is the dual lattice of $\Gamma_{0,\infty}$ in $\mathbb{R}^2$. From the asymptotics $K_{\gamma_{0}^{-1}}(t) \sim \sqrt{\pi/2t} \exp(-t)$, $I_{\gamma_{0}^{-1}}(t) \sim 1/\sqrt{2t\pi} \exp t$ as $t \to +\infty$, and inequality (5.3) above, it is clear that $b_{\gamma} = 0$, and hence $\psi_{0}$ has moderate growth. Then by Lemma 5.2, there exist a constant $a_{\infty}$ and a $L^2$ solution $\varphi_{0}$ of $\Delta \varphi_{0} = s(2-s)\varphi_{0}$ such that

$$\psi_{0}(u) = a_{\infty} E_{\infty}(u; s_{0}) + \varphi_{0}(u).$$

This completes the proof of Theorem 1.1.

Remarks. (1). In the above proof, we assume that $M_{0}$ has only one cusp and $M_{i}$ are compact. If $M_{0}$ has more than one cusp and $M_{i}$ are still compact, we define the truncated function $\varphi_{i}^{n}$ by cutting off the constant Fourier terms at all the pinching geodesics. If $M_{i}$ are non-compact, then the function $\varphi_{i}^{n}$ is defined by further cutting off the constant Fourier term at every cusp of $M_{i}$; and in Step 2 we use the following counterpart of Lemma 5.3 for a cusp $C_{k}(b)$: The first positive eigenvalue of the surface $\partial C_{k}(b)$ is strictly increasing to $\infty$ as $b \to +\infty$.

(2). If $M_{i}$ is noncompact, $\varphi_{i}$ in Theorem 1.1 can be taken to be a linear combination of a square integrable eigenfunction and Eisenstein series of $M_{i}$, i.e., a generalized eigenfunction; and the same result holds. This condition on $\varphi_{i}$ is equivalent to being of moderate growth, according to Lemma 5.2.

6. Proof of Theorem 1.2

We prove Theorem 1.2 in several propositions (Propositions 6.1, 6.3, 6.6). In the following, we use Definition 3.6 for the convergence of functions on $M_{i}$.

Proposition 6.1. The heat kernel $H_{i}(u,v,t)$ converges to $H_{0}(u,v,t)$ uniformly over compact subsets in $H^{3} \times H^{3}$ as $i \to \infty$.

Proof. By assumption, $M_{i}$ is a degenerate sequence of hyperbolic manifolds with limit $M_{0}$. By Proposition 3.5, $M_{i} = \Gamma_{i} \setminus H^{3}$, $M_{0} = \Gamma_{0} \setminus H^{3}$, and $\Gamma_{i} = \rho_{i}(\Gamma_{0})$, $\rho_{i}(\gamma) \to \gamma$ in $\mathrm{PSL}(2, \mathbb{C})$ as $i \to \infty$ for any $\gamma \in \Gamma_{0}$. For any $u_{0} \in H^{3}$, there exists a small ball $B_{0}$ around $u_{0}$ of radius less than 1 such that for any $\gamma \in \Gamma_{i}$, $\gamma \neq 1$, $\gamma B_{0} \cap B_{0} = \emptyset$. This follows either from the convergence of manifolds $M_{i} \to M_{0}$ or from the convergence of the fundamental domain $D_{i}$ for $\Gamma_{i}$ to the fundamental domain $D_{0}$ for $\Gamma_{0}$ in §3. Fix another $v_{0} \in H^{3}$. For any $R > 0$, let

$$B(\Gamma_{i}, R) = \{ \gamma \in \Gamma_{i} \mid d(u,0, v_{0}) < R \}.$$ 

Then

$$\bigcup_{\gamma \in B(\Gamma_{i}, R)} \gamma B_{0} \subset \{ v \in H^{3} \mid d(v, u_{0}) \leq R + d(u_{0}, v_{0}) + 1 \} = B(u_{0}, R + d(u_{0}, v_{0}) + 1) \subset H^{3},$$

and hence

$$|B(\Gamma_{i}, R)| \mathrm{vol}(B_{0}) \leq \mathrm{vol}(B(u_{0}, R + d(u_{0}, v_{0}) + 1)) \leq c_{1} e^{2R},$$

where $c_{1}$ is a constant depending only on $d(u_{0}, v_{0})$. This implies

$$|B(\Gamma_{i}, R)| \leq c_{2} e^{2R}, \quad |B(\Gamma_{i}, R) - B(\Gamma_{i}, R - 1)| \leq c_{2} e^{2R},$$

where $c_{2}$ is a constant independent of $i$. By [20, p. 219], the heat kernel $P(u, v, t)$ of $H^{3}$ is as follows:

$$P(u, v, t) = \frac{d(u, v)}{\sinh d(u, v)} (4\pi t)^{-3/2} e^{-t} e^{-d^{2}(u, v)/4t}.$$
Therefore we have
\[ H_i(u_0, v_0, t) = \sum_{\gamma \in \Gamma_i} \frac{d(u_0, \gamma v_0)}{\sinh d(u_0, \gamma v_0)} \frac{(4\pi t)^{-3/2} e^{-t} e^{-d^2(u_0, \gamma v_0)/4t}}{4}, \]
and for any \( R > 0 \),
\[ |H_i(u_0, v_0, t) - \sum_{\gamma \in \Gamma_i, d(u_0, \gamma v_0) < R} P(u_0, \gamma v_0, t)| \leq \sum_{n \geq R} e^{2(n+1)} \frac{n}{\sinh n} \frac{d(u_0, \gamma v_0)}{\sinh d(u_0, \gamma v_0)} \frac{(4\pi t)^{-3/2} e^{-t} e^{-d^2(u_0, \gamma v_0)/4t}}{4}, \]
and hence
\[ |H_i(u_0, v_0, t) - \sum_{\gamma \in \Gamma_i, d(u_0, \gamma v_0) < R} P(u_0, \gamma v_0, t)| \to 0 \]
uniformly for \( t \) in compact subsets of \((0, +\infty)\).

On the other hand, from the convergence of \( \Gamma_i \to \Gamma_0 \), it follows that for any \( R > 0 \) with \( d(u_0, \gamma v_0) \neq R \) for all \( \gamma \in \Gamma_0 \),
\[ \sum_{\gamma \in \Gamma_i, d(u_0, \gamma v_0) < R} P(u_0, \gamma v_0, t) \to \sum_{\gamma \in \Gamma_0, d(u_0, \gamma v_0) < R} P(u_0, \gamma v_0, t) \]
uniformly for \( t \) in compact subsets of \((0, +\infty)\). These results immediately imply the uniform convergence of \( H_i(u, v, t) \) to \( H_0(u, v, t) \) over compact subsets of \( H^3 \times H^3 \times (0, +\infty) \).

**Proposition 6.2.** For any \( t_0 > 0 \) and any compact subsets \( K_1, K_2 \subset H^3 \) with \( K_1 \cap K_2 = \emptyset \) for all \( \gamma \in \Gamma_0 \), \( H_i(u, v, t) \) converges to \( H_0(u, v, t) \) uniformly over \( K_1 \times K_2 \times [0, t_0] \).

**Proof.** By the proof of the previous proposition, it suffices to show that for any \( \gamma \in B(\Gamma_i, \delta) \), \( P(u, \gamma v, t) \to 0 \) uniformly for \( (u, v) \in K_1 \times K_2 \) as \( t \to 0 \). By assumption, \( K_1 \cap \gamma K_2 = \emptyset \), and hence there exists \( \delta > 0 \) such that for any \( \gamma \in B(\Gamma_i, \delta) \), \( d(K_1, \gamma K_2) \geq \delta \), i.e., \( d(u, \gamma v) \geq \delta \) for all \( (u, v) \in K_1 \times K_2 \). Then the convergence is clear.

**Proposition 6.3.** For any \( s \in \mathbb{C} \) with \( \text{Re}(s) > 2 \), the resolvent kernel \( G_{i,s}(u, v) \) converges to \( G_{0,s}(u, v) \) uniformly over compact subsets away from the diagonal.

**Proof.** For simplicity, we assume that \( s \in \mathbb{R}, s > 2 \). For any \( u \neq v \),
\[ G_{i,s}(u, v) = \int_0^{+\infty} e^{s(2-t)} H_i(u, v, t) dt. \]
Since
\[ H_i(u, v, t) \leq H_i(u, u, t)^{\frac{1}{2}} H_i(v, v, t)^{\frac{1}{2}}, \]
and $H_i(u, u, t)$ is decreasing in $t$, $H_i(u, v, t) \leq H_i(u, u, 1)^{\frac{1}{2}} H_i(v, v, 1)^{\frac{1}{2}}$ when $t \geq 1$. Therefore, for any $T > 1$,
\[
|G_{i,s}(u, v) - G_{0,s}(u, v)|
\leq \int_0^T e^{s(2-s)t} |H_i(u, v, t) - H_0(u, v, t)| dt
\]
\[+ \int_T^{+\infty} e^{s(2-s)t} (H_i(u, v, t) + H_0(u, v, t)) dt \]
\[= \int_0^T e^{s(2-s)t} |H_i(u, v, t) - H_0(u, v, t)| dt \]
\[+ \left( \sqrt{H_i(u, u, 1) H_i(v, v, 1)} + \sqrt{H_0(u, u, 1) H_0(v, v, 1)} \right) \frac{1}{s(s - 2)} e^{s(2-s)T}. \]

Since $e^{s(2-s)T} \to 0$ as $T \to +\infty$, the conclusion follows immediately from the previous proposition.

If we use the formula for the resolvent kernel of $\Delta$ acting on $L^2(\mathbb{H}^3)$ in [20, Proposition 3.2], we can prove as in Proposition 6.1 a slightly stronger result.

**Proposition 6.4.** For any $s_1, s_2 \in \mathbb{C}$ with $\text{Re}(s_j) > 2$, $j = 1, 2$, then $G_{i,s_1}(u, v) - G_{i,s_2}(u, v)$ converges to $G_{0,s_1}(u, v) - G_{0,s_2}(u, v)$ uniformly over all compact subsets.

For any $\lambda > 0$, let $F_i(\lambda; u, v)$ be the spectral measure for $M_i$. If $M_i$ is compact and $\{\varphi_j\}$ is an orthonormal basis of eigenfunctions with eigenvalues $\{\lambda_j\}$, then the spectral measure $F_i(\lambda; u, v)$ of $M_i$ is given by
\[F_i(\lambda; u, v) = \sum_{\lambda_j \leq \lambda} \varphi_j(u) \varphi_j(v). \]

If $M_i$ is noncompact, the Eisenstein series has to be taken into account. See [8, Equation 7.2] for definition of $F_i$ in the hyperbolic surfaces case. The same definition works for finite volume hyperbolic three dimensional manifolds.

Using $F_i(\lambda; u, v)$, Corollary 1.3 can be stated more precisely as follows:

**Proposition 6.5.** For any $\lambda > 0$ which is not an eigenvalue of $M_0$, $F_i(\lambda; u, v)$ converges to $F_0(\lambda; u, v)$ uniformly over compact subsets as $i \to \infty$.

The proof is the same as that of Theorem 4 in [8]. As in [8, Corollary C], this proof gives the following improvement of Proposition 6.3:

**Proposition 6.6.** For $s \in \mathbb{C}$, $\text{Re}(s) > 1$ and $s(2 - s) \notin \text{Spec}(M_0)$, $G_{i,s}(u, v)$ converges to $G_{0,s}(u, v)$ uniformly over compact subsets away from the diagonal.

Since $M_i$ has finite volume, $0 \in \text{Spec}(M_i)$ for all $i \geq 0$, and hence $s = 2$ is a pole of $G_{i,s}(u; v)$. Let $L^2(M_i)'$ be the orthogonal complement of constants in $L^2(M_i)$. Then the Beltrami-Laplace operator $\Delta$ of $M_i$ acts on $L^2(M_i)'$ and is invertible on this subspace. Let $G_i(u, v)$ be the Schwartz kernel of this inverse, called the Green function of $M_i$.

**Proposition 6.7.** The Green function $G_i(u, v)$ converges to $G_0(u, v)$ uniformly over compact subsets away from the diagonal.

**Proof.** Let $\sigma$ be a small circle in the $\lambda$ plane around $\lambda = 0$, disjoint from $\text{Spec}(M_0)$. Let $s = s(\lambda)$ be the analytic branch of $s(2 - s) = \lambda$ around $\lambda = 0$ with $s(0) = 2$. 

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Then for any $i \geq 0$,
\[ G_i(u,v) = \frac{1}{2\pi \sqrt{-1}} \int_{\sigma} (G_{i,s}(\lambda)(u,v) - \frac{1}{\operatorname{vol}(M_i)} - \lambda) d\lambda. \]

By a theorem of Jørgensen [10, Theorem 5.12] [7, p. 46], $\operatorname{vol}(M_i) \to \operatorname{vol}(M_0)$ as $i \to \infty$. Then the conclusion follows from the previous proposition. \qed

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