

PRESENTATION AND CENTRAL EXTENSIONS OF MAPPING CLASS GROUPS

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ABSTRACT. We give a presentation of the mapping class group \mathcal{M} of a (possibly bounded) surface, considering either all twists or just non-separating twists as generators. We also study certain central extensions of \mathcal{M} . One of them plays a key role in studying TQFT functors, namely the mapping class group of a p_1 -structure surface. We give a presentation of this extension.

INTRODUCTION

All surfaces that we shall consider are connected and oriented. We shall denote by $\Sigma_{g,r}$ a surface with genus g and r boundary components, or simply Σ when there is no ambiguity. We shall denote its mapping class group by $\mathcal{M}_{g,r}$ (or \mathcal{M}_Σ). Recall that \mathcal{M}_Σ is the group of diffeomorphisms of Σ which leave fixed its boundary, modulo those which are isotopic to the identity. The aim of this article is to find a presentation of $\mathcal{M}_{g,r}$ with very simple geometrical relations and to describe more precisely this presentation as central extensions.

These groups are generated by Dehn twists ([D]). Lickorish ([L1, L2]) proved that $3g-1$ of these twists generate $\mathcal{M}_{g,0}$. Humphries ([Hu]) reduced this number to $2g+1$ and showed that it was minimal.

Using the presentation of $\mathcal{M}_{g,0}$ found by Hatcher and Thurston in [H-T], Wajnryb gave in [W] an explicit presentation of $\mathcal{M}_{g,1}$ and $\mathcal{M}_{g,0}$ with Humphries generators. We shall also use their result to obtain a presentation of $\mathcal{M}_{g,r}$ with the set of all Dehn twists for generators.

Four types of relations will appear in this presentation: “braids”, “lanterns”, “stars” and “chains”.

The braids. It is well known that if α is a simple closed curve (s.c.c.) in Σ and h is a diffeomorphism of Σ , then $\tau_{h(\alpha)} = h\tau_\alpha h^{-1}$ (where τ_α represents the twist along α). More specifically, if h is a twist τ_β , one has the relation (called a braid relation)

$$(T) \quad \tau_\gamma = \tau_\beta \tau_\alpha \tau_\beta^{-1},$$

where $\gamma = \tau_\beta(\alpha)$. We will denote this relation by T_n when $|\alpha \cap \beta| = n$ and by T_{20} when $|\alpha \cap \beta| = 2$ and the algebraic intersection is zero (denoted $|\alpha \cap \beta| = 2_0$).

Remark. If G is a multiplicative group, \bar{x} will denote x^{-1} and $y(x)$ will denote the conjugate of x by y (i.e. $y x \bar{y}$).

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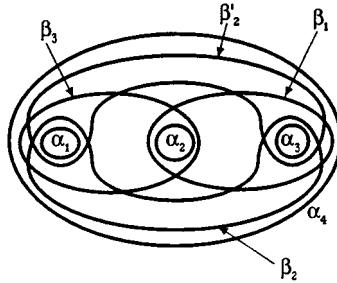


FIGURE 1

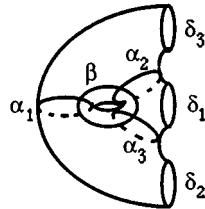


FIGURE 2

The lanterns. These relations were discovered by Johnson ([J]). Let us consider a subsurface of $\Sigma_{g,r}$ which is homeomorphic to a sphere with four holes. Let $\alpha_1, \alpha_2, \alpha_3$ and α_4 be the boundary components and β_1, β_2 and β_3 curves as shown in Figure 1. The relation (called a lantern relation) is:

$$(L) \quad \tau_{\alpha_1} \tau_{\alpha_2} \tau_{\alpha_3} \tau_{\alpha_4} = \tau_{\beta_3} \tau_{\beta_2} \tau_{\beta_1}.$$

Remark. By a homology calculation, one can check that if two curves α and β in a surface Σ are such that $|\alpha \cap \beta| = 2_0$, then a neighborhood of $\alpha \cup \beta$ in Σ is homeomorphic to a sphere with four holes. So, a lantern (L) is completely determined by the curves β_1 and β_3 . The curves α_i are the four boundary components of a neighborhood of $\beta_1 \cup \beta_3$. There are two choices for β_2 , but only one gives the equality (L) (the other choice, the curve β'_2 shown in Figure 1, gives the equality $\tau_{\alpha_1} \tau_{\alpha_2} \tau_{\alpha_3} \tau_{\alpha_4} = \tau_{\beta_1} \tau_{\beta'_2} \tau_{\beta_3}$).

The stars. We consider a subsurface of $\Sigma_{g,r}$ which is homeomorphic to a surface of genus one with three boundary components. The relation is

$$(E) \quad (\tau_{\alpha_1} \tau_{\alpha_2} \tau_{\alpha_3} \tau_{\beta})^3 = \tau_{\delta_1} \tau_{\delta_2} \tau_{\delta_3},$$

where the curves are described in Figure 2.

Remark. The star (E) is uniquely determined by four curves $\alpha_1, \alpha_2, \alpha_3$ and β such that $|\alpha_i \cap \alpha_j| = 0$ if $i \neq j$ and $|\beta \cap \alpha_i| = 1$ for all $i \in \{1, 2, 3\}$ (a neighborhood of these curves in Σ is homeomorphic to $\Sigma_{1,3}$).

The chains. These are just a particular case of the stars: we suppose that the curve δ_1 bounds a disc in Σ . Then, we have $\tau_{\delta_1} = 1$, $\tau_{\alpha_2} = \tau_{\alpha_3}$, and the relation (E) becomes:

$$(C) \quad (\tau_{\alpha_1} \tau_{\beta} \tau_{\alpha_2})^4 = \tau_{\delta_2} \tau_{\delta_3}.$$

Remark. Using relations T_0 and T_1 , it is easy to show that $(\tau_{\alpha_1}\tau_{\alpha_2}\tau_{\alpha_2}\tau_{\beta})^3 = (\tau_{\alpha_1}\tau_{\beta}\tau_{\alpha_2})^4$. This kind of calculation will be often done in this article, and the reader can refer to them. Note that a chain is determined by the curves α_1 , β and α_2 , with $|\alpha_i \cap \beta| = 1$ and $\alpha_1 \cap \alpha_2 = \emptyset$.

Theorem A. *For any oriented surface Σ of genus $g \geq 0$, with r boundary components ($r \geq 0$), \mathcal{M}_Σ has the following presentation:*

generators: $\{\tau_\alpha / \alpha \text{ s.c.c. in } \Sigma\}$ (if α bounds a disc in Σ , then $\tau_\alpha = 1$).

relations: (I) All the braids T_0 , T_1 and T_{2_0} .

(II) All the chains.

(III) All the lanterns.

If $g \geq 1$, one can replace (II) and (III) by (IV):

(IV) All the stars.

Remark. If $g = 0$, only the relations T_0 , T_{2_0} and (III) remain.

Harer proved in [H] that when $g \geq 3$, $\mathcal{M}_{g,r}$ is generated by the twists along the non-separating curves for any r . We shall see that this result is still true when $g = 2$, but not when $g = 1$. We shall also give a presentation of $\mathcal{M}_{g,r}$ ($g \geq 2$) with these generators.

Notation. If the curves involved in a lantern L (resp. a star E or a chain C) are all non-separating curves, we will denote L by L_0 (resp. E by E_0 and C by C_0).

Theorem B. *For any orientable surface Σ of genus $g \geq 2$, with r boundary components ($r \geq 0$), \mathcal{M}_Σ has the following presentation:*

generators: $\{\tau_\alpha / \alpha \text{ non-separating s.c.c. in } \Sigma\}$.

relations: (I)' All the braids T_0 and T_1 (with non-separating curves).

(II)' One chain C_0 .

(III)' One lantern L_0 .

When $g \geq 3$, one can replace (II)' or (III)' by a star E_0 .

Remark. In the case of genus two, there are only the relations (I)' and (II)' (the lanterns L_0 and the stars E_0 exist only if $g \geq 3$).

Let $S_{g,r}$ be the group generated by $\{a_\alpha / \alpha \text{ non-separating s.c.c. in } \Sigma\}$ and defined by the relations T_0 and T_1 between these curves. Theorem B shows that $\mathcal{M}_{g,r}$ is a quotient of $S_{g,r}$ when $g \geq 2$. We shall prove the following more precise result.

Theorem C. *For any orientable surface Σ , there exists a central extension*

$$(*) \quad 0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{T}_\Sigma \longrightarrow \mathcal{M}_\Sigma \longrightarrow 1$$

satisfying the following properties:

- 1) $\Sigma \mapsto \mathcal{M}_\Sigma$, $\Sigma \mapsto S_\Sigma$ and $\Sigma \mapsto \mathcal{T}_\Sigma$ are three functors from the category of surfaces (where morphisms are embeddings) to the category of groups, and the extension $(*)$ is natural. When $g \geq 3$ (g denotes the genus of Σ), this extension is the universal central extension of \mathcal{M}_Σ .

Furthermore, there exist two natural morphisms $\varphi: S_\Sigma \rightarrow \mathcal{T}_\Sigma$ and $\psi: S_\Sigma \rightarrow \mathbb{Z}$ such that:

- 2) if $g \geq 3$, φ and ψ induce an isomorphism from S_Σ to $\mathcal{T}_\Sigma \times \mathbb{Z}$.
- 3) if $g = 2$, φ is an isomorphism from S_Σ to \mathcal{T}_Σ .

1. PRELIMINARY RESULTS

In this section, Σ is a surface of genus $g \geq 0$ with r boundary components ($r \geq 0$).

Let R_Σ (resp. S_Σ) be the group generated by $\{a_\alpha / \alpha \text{ a s.c.c. in } \Sigma\}$ (resp. $\{a_\alpha / \alpha \text{ a non-separating s.c.c. in } \Sigma\}$) and defined by the relations T_0 , T_1 and T_{2_0} (resp. the relations T_0 and T_1 between non-separating curves). The goal of this section will be to prove that all the braid relations are satisfied in R and S and to find an equality between the lanterns, the stars and the chains.

We start by proving two lemmas that will be used often.

Lemma 1.1. (i) Let α and β be two disjoint s.c.c. (resp. non-separating s.c.c.) in Σ . Then, a_α commutes with a_β in R (resp. S).

(ii) If α and β are two s.c.c. in Σ such that $|\alpha \cap \beta| = 1$, then $a_\alpha a_\beta a_\alpha = a_\beta a_\alpha a_\beta$ in R and S .

Notation. In the following, we will refer to these relations respectively by (0) and (1).

Proof. (i) is exactly the relation T_0 because $\tau_\beta(\alpha) = \alpha$. (ii) is a consequence of T_1 and the fact that $\tau_\beta(\alpha) = \tau_\alpha^{-1}(\beta)$. \square

Lemma 1.2. Let $(\alpha_1, \alpha_2, \alpha_3, \beta)$ be a star in Σ and set

$$X_1 = a_{\alpha_1} a_{\alpha_2}, \quad X_2 = a_\beta X_1 a_\beta, \quad X_3 = a_{\alpha_3} X_2 a_{\alpha_3}.$$

Then, one has in R and S :

- (i) $X_i X_j = X_j X_i$,
- (ii) $(a_{\alpha_1} a_{\alpha_2} a_{\alpha_3} a_\beta)^3 = X_1 X_2 X_3$,
- (iii) $(a_{\alpha_1} a_\beta a_{\alpha_2})^4 = X_1^2 X_2^2$.

Proof. (i) Using (1), we obtain

$$(2) \quad a_{\alpha_1} X_2 = a_{\alpha_1} a_\beta a_{\alpha_1} a_{\alpha_2} a_\beta = a_\beta a_{\alpha_1} a_\beta a_{\alpha_2} a_\beta = a_\beta a_{\alpha_1} a_{\alpha_2} a_\beta a_{\alpha_2} = X_2 a_{\alpha_2}$$

and in the same way, $a_{\alpha_2} X_2 = X_2 a_{\alpha_1}$. From this, we conclude that $X_1 X_2 = X_2 X_1$ and, since a_{α_3} commutes with a_{α_1} and a_{α_2} (by (0)), $X_1 X_3 = X_3 X_1$.

One has by (0) and (1)

$$\begin{aligned} a_\beta X_3 &= a_\beta a_{\alpha_3} a_\beta a_{\alpha_1} a_{\alpha_2} a_\beta a_{\alpha_3} = a_{\alpha_3} a_\beta a_{\alpha_3} a_{\alpha_1} a_{\alpha_2} a_\beta a_{\alpha_3} \\ &= a_{\alpha_3} a_\beta a_{\alpha_1} a_{\alpha_2} a_\beta a_{\alpha_3} a_\beta = X_3 a_\beta \end{aligned}$$

so X_2 commutes with X_3 .

$$\begin{aligned} (ii) \quad X_1 X_2 X_3 &= a_{\alpha_1} a_{\alpha_2} a_\beta a_{\alpha_1} a_{\alpha_2} a_\beta a_{\alpha_3} a_\beta a_{\alpha_1} a_{\alpha_2} a_\beta a_{\alpha_3} \\ &= a_{\alpha_1} a_{\alpha_2} a_\beta a_{\alpha_1} a_{\alpha_2} a_{\alpha_3} a_\beta a_{\alpha_3} a_{\alpha_1} a_{\alpha_2} a_\beta a_{\alpha_3} \quad \text{by (1)} \\ &= a_{\alpha_1} a_{\alpha_2} a_\beta a_{\alpha_3} a_{\alpha_1} a_{\alpha_2} a_\beta a_{\alpha_2} a_{\alpha_1} a_{\alpha_3} a_\beta a_{\alpha_3} \quad \text{by (0)} \\ &= a_{\alpha_1} a_{\alpha_2} a_\beta a_{\alpha_3} a_{\alpha_1} a_\beta a_{\alpha_2} a_\beta a_{\alpha_1} a_\beta a_{\alpha_3} a_\beta \quad \text{by (1)} \\ &= a_{\alpha_1} a_{\alpha_2} a_\beta a_{\alpha_3} a_{\alpha_1} a_\beta a_{\alpha_2} a_{\alpha_1} a_\beta a_{\alpha_1} a_{\alpha_3} a_\beta \quad \text{by (1)} \\ &= a_{\alpha_1} a_{\alpha_2} a_\beta a_{\alpha_3} a_{\alpha_1} a_\beta a_{\alpha_2} a_{\alpha_1} a_\beta a_{\alpha_2} a_{\alpha_1} a_{\alpha_3} a_\beta \quad \text{by (2)} \\ &= (a_{\alpha_1} a_{\alpha_2} a_{\alpha_3} a_\beta)^3 \quad \text{by (0) and (1)}. \end{aligned}$$

$$\begin{aligned} (iii) \quad X_1^2 X_2^2 &= a_{\alpha_1} a_{\alpha_2} a_\beta a_{\alpha_1} a_{\alpha_2} a_\beta a_{\alpha_1} a_{\alpha_2} a_\beta a_{\alpha_1} a_{\alpha_2} a_\beta \quad \text{by (i)} \\ &= a_{\alpha_1} a_\beta a_{\alpha_2} a_\beta a_{\alpha_1} a_\beta a_{\alpha_2} a_\beta a_{\alpha_1} a_\beta a_{\alpha_2} a_\beta \quad \text{by (0) and (1)} \\ &= a_{\alpha_1} a_\beta a_{\alpha_2} a_{\alpha_1} a_\beta a_{\alpha_2} a_{\alpha_1} a_\beta a_{\alpha_1} a_{\alpha_2} a_\beta a_{\alpha_2} \quad \text{by (0) and (1)} \\ &= (a_{\alpha_1} a_\beta a_{\alpha_2})^4 \quad \text{by (0)}. \end{aligned}$$

\square

1.1. The braids.

Theorem 1.3. *For any simple closed curves α and β in Σ , one has in R_Σ $a_\gamma = a_\beta a_\alpha \overline{a_\beta}$ where $\gamma = \tau_\beta(\alpha)$.*

If α and β do not separate, this relation is also true in S_Σ .

Corollary 1.4. *Let h be a diffeomorphism of Σ and suppose that it is decomposed into a product of Dehn twists; denote by m the corresponding word in R_Σ . Let α be a s.c.c. in Σ and $\gamma = h(\alpha)$. Then, in R_Σ one has the relation $a_\gamma = m a_\alpha \overline{m}$.*

Furthermore, if the twists involved in the decomposition of h are all twists along non-separating curves and if α does not separate, then the relation is also true in S_Σ .

Proof. Write $h = \tau_{\beta_n}^{\varepsilon_n} \dots \tau_{\beta_1}^{\varepsilon_1}$ ($\varepsilon_i = \pm 1$); then, $m = a_{\beta_n}^{\varepsilon_n} \dots a_{\beta_1}^{\varepsilon_1}$ in R (or S).

Denote, for $k \in \{0, \dots, n\}$, $\gamma_k = \tau_{\beta_k}^{\varepsilon_k} \dots \tau_{\beta_1}^{\varepsilon_1}(\alpha)$ ($\gamma_0 = \alpha$ and $\gamma_n = \gamma$). Then, Theorem 1.3 implies

$$a_{\gamma_k} = a_{\beta_k}^{\varepsilon_k} a_{\gamma_{k-1}} a_{\beta_k}^{-\varepsilon_k},$$

thus

$$a_\gamma = a_{\gamma_n} = a_{\beta_n}^{\varepsilon_n} a_{\gamma_{n-1}} a_{\beta_n}^{-\varepsilon_n} = \dots = m a_\alpha \overline{m}.$$

□

Before we prove Theorem 1.3, let us show that the relations T_{2_0} are true in S :

Lemma 1.5. *Let α and β be two non-separating curves in Σ such that $|\alpha \cap \beta| = 2_0$. Then, if $\gamma = \tau_\beta(\alpha)$, one has in S*

$$a_\gamma = a_\beta a_\alpha \overline{a_\beta}.$$

Proof. A neighborhood of $\alpha \cup \beta$ in Σ is a lantern where α and β are respectively identified with the curves β_1 and β_2 of the Figure 1. Since these curves do not separate, there exists a curve δ such that

$$|\delta \cap \alpha| = |\delta \cap \beta| = |\delta \cap \alpha_1| = |\delta \cap \alpha_2| = 1.$$

(We might have to replace $\{\alpha_1, \alpha_2\}$ by $\{\alpha_3, \alpha_4\}$.)

It can be seen that

$$\tau_\delta \tau_{\alpha_1} \tau_{\alpha_2} \tau_\delta(\beta) = \alpha \quad \text{and} \quad \tau_\delta^{-1} \tau_{\alpha_1}^{-1} \tau_{\alpha_2}^{-1} \tau_\delta^{-1}(\beta) = \gamma.$$

And since $|\delta \cap \beta| = |\alpha_2 \cap \tau_\delta(\beta)| = \dots = |\delta \cap \tau_{\alpha_1}^{-1} \tau_{\alpha_2}^{-1} \tau_\delta^{-1}(\beta)| = 1$, the relation T_1 implies

$$X a_\beta \overline{X} = a_\alpha \quad \text{and} \quad \overline{X} a_\beta X = a_\gamma, \quad \text{where} \quad X = a_\delta a_{\alpha_1} a_{\alpha_2} a_\delta$$

and so $a_\beta a_\alpha \overline{a_\beta} = a_\beta X a_\beta \overline{X} \overline{a_\beta}$.

Since $(\alpha_1, \alpha_2, \beta, \delta)$ defines a star in Σ , Lemma 1.2 shows that X commutes with $a_\beta X a_\beta$. Thus, we obtain:

$$a_\beta a_\alpha \overline{a_\beta} = \overline{X} a_\beta X a_\beta \overline{a_\beta} = a_\gamma. \quad \square$$

Proof of Theorem 1.3. Suppose inductively that the braid relations T_k are satisfied in R and S for $k \leq n-1$, let α and β be two s.c.c. in Σ such that $|\alpha \cap \beta| = n$, and set $\gamma = \tau_\beta(\alpha)$ (for S , we suppose also that α and β do not separate). Let us fix an orientation on α and β .

First case. There exist two points A and B in $\alpha \cap \beta$ which are consecutive on α and such that the signs of the intersection of α and β at these points are the same.

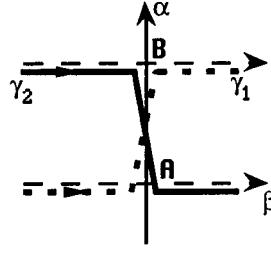


FIGURE 3

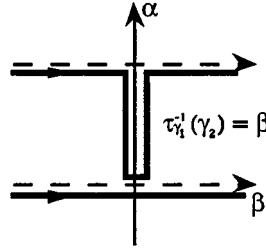


FIGURE 4

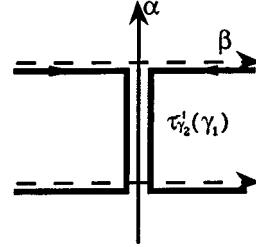


FIGURE 5

Let us consider the curves γ_1 and γ_2 shown on Figure 3. As shown on Figure 4, one has $\tau_{\gamma_1}^{-1}(\gamma_2) = \beta$ and so, using T_1 and (1),

$$(3) \quad a_\beta = \overline{a_{\gamma_1}} a_{\gamma_2} a_{\gamma_1} = \overline{a_{\gamma_1}} \overline{a_{\gamma_1}} a_{\gamma_1} a_{\gamma_2} a_{\gamma_1} = \overline{a_{\gamma_1}} \overline{a_{\gamma_1}} a_{\gamma_2} a_{\gamma_1} a_{\gamma_2}.$$

Let us denote by δ_1 , δ_2 , δ_3 and δ_4 the curves

$$\delta_1 = \tau_{\gamma_2}(\alpha), \quad \delta_2 = \tau_{\gamma_1}(\delta_1), \quad \delta_3 = \tau_{\gamma_2}(\delta_2), \quad \delta_4 = \tau_{\gamma_1}^{-1}(\delta_3).$$

Then, we have $\gamma = \tau_{\gamma_1}^{-1}(\delta_4)$. We will successively show that $|\gamma_2 \cap \alpha|$, $|\gamma_1 \cap \delta_1|$, $|\gamma_2 \cap \delta_2|$, $|\gamma_1 \cap \delta_3|$ and $|\gamma_1 \cap \delta_4|$ are smaller than $n-1$. Then, by the induction hypothesis we obtain:

$$a_\gamma = \overline{a_{\gamma_1}}(a_{\delta_4}) = \overline{a_{\gamma_1}} \overline{a_{\gamma_1}}(a_{\delta_3}) = \dots = \overline{a_{\gamma_1}} \overline{a_{\gamma_1}} a_{\gamma_2} a_{\gamma_1} a_{\gamma_2}(a_\alpha),$$

that is to say, $a_\gamma = a_\beta a_\alpha \overline{a_\beta}$ using (3).

- By definition, $|\gamma_2 \cap \alpha| < n$.
- $|\gamma_1 \cap \delta_1| = |\gamma_1 \cap \tau_{\gamma_2}(\alpha)| = |\tau_{\gamma_2}^{-1}(\gamma_1) \cap \alpha| = n-2$ as shown on Figure 5.
- $|\gamma_2 \cap \delta_2| = |\gamma_2 \cap \tau_{\gamma_1} \tau_{\gamma_2}(\alpha)| = |\tau_{\gamma_2}^{-1} \tau_{\gamma_1}^{-1}(\gamma_2) \cap \alpha| = |\tau_{\gamma_2}^{-1} \tau_{\gamma_2}(\gamma_1) \cap \alpha| = |\gamma_1 \cap \alpha| < n$.

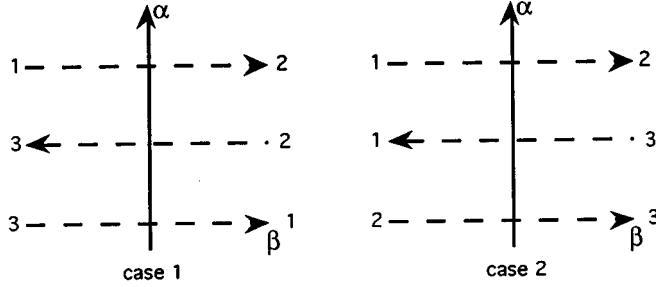


FIGURE 6

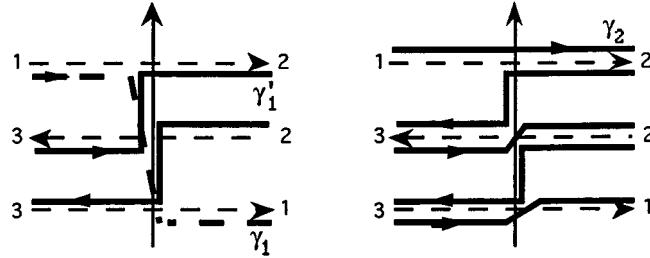


FIGURE 7

- $|\gamma_1 \cap \delta_3| = |\gamma_1 \cap \tau_{\gamma_2} \tau_{\gamma_1} \tau_{\gamma_2}(\alpha)| = |\tau_{\gamma_2}^{-1} \tau_{\gamma_1}^{-1} \tau_{\gamma_2}^{-1}(\gamma_1) \cap \alpha| = |\tau_{\gamma_2}^{-1} \tau_{\gamma_1}^{-1} \tau_{\gamma_2}(\gamma_2) \cap \alpha| = |\gamma_2 \cap \alpha| < n.$
- $|\gamma_1 \cap \delta_4| = |\tau_{\gamma_1}(\gamma_1) \cap \tau_{\gamma_1}(\delta_4)| = |\gamma_1 \cap \delta_3| < n.$

Remark. Note that the curves γ_1 and γ_2 do not separate. So, the proof above remains valid for S .

Second case. There are not two points in $\alpha \cap \beta$ consecutive on α and with the same intersection sign.

If n is equal to two, then T_n is the relation T_{2_0} (wich is true in S by Lemma 1.5); so, suppose that $n \geq 3$. As we can see in Figure 6 there are only two different ways to join the strands in order to obtain the simple closed curve β . Denote by n_k the number of points in $\alpha \cap \beta$ which are on the arc $k-k$ of β ($k=1,2$ or 3).

Let us consider the first case. If γ_1 and γ_2 are the curves shown in Figure 7, one has $|\gamma_1 \cap \beta|=2_0$ and $\tau_\beta(\gamma_1)=\gamma_2$. So, the relation T_{2_0} yields

$$a_{\gamma_2} = a_\beta a_{\gamma_1} \overline{a_\beta}, \quad \text{i.e.,} \quad a_\beta = \overline{a_{\gamma_2}} a_\beta a_{\gamma_1}.$$

Now, if $\delta_1=\tau_{\gamma_1}(\alpha)$ and $\delta_2=\tau_\beta(\delta_1)$, then $\gamma=\tau_{\gamma_2}^{-1}(\delta_2)$.

Since $|\gamma_1 \cap \alpha|=n_1+1 \leq n-2 < n$, one has by the induction hypothesis that $a_{\delta_1}=a_{\gamma_1}(a_\alpha)$. We have (Figure 8 shows $\tau_{\gamma_1}^{-1}(\beta)$)

$$|\beta \cap \delta_1|=|\beta \cap \tau_{\gamma_1}(\alpha)|=|\tau_{\gamma_1}^{-1}(\beta) \cap \alpha|=n-2 < n$$

so $a_{\delta_2}=a_\beta(a_{\delta_1})=a_\beta a_{\gamma_1}(a_\alpha)$. Finally,

$$|\gamma_2 \cap \delta_2|=|\gamma_2 \cap \tau_\beta \tau_{\gamma_1}(\alpha)|=|\tau_\beta^{-1}(\gamma_2) \cap \tau_{\gamma_1}(\alpha)|=|\gamma_1 \cap \tau_{\gamma_1}(\alpha)|=|\gamma_1 \cap \alpha|=n_1+1 < n$$

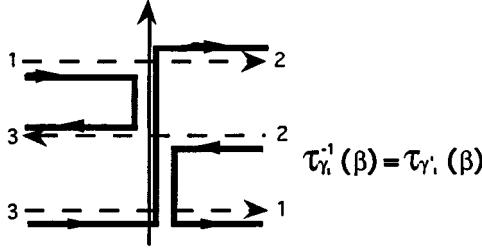


FIGURE 8

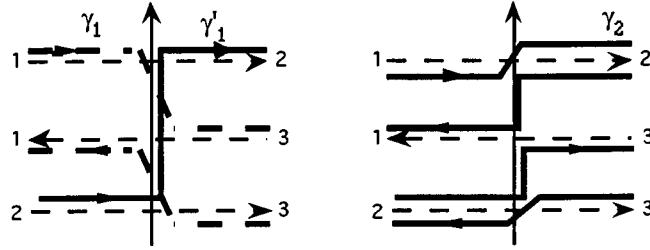


FIGURE 9

and we obtain

$$a_\gamma = \overline{a_{\gamma_2}}(a_{\delta_2}) = \overline{a_{\gamma_2}} a_\beta a_{\gamma_1}(a_\alpha) = a_\beta(a_\alpha).$$

Note that the curve γ_1 (and also γ_2) may separate Σ , and in this case the proof above is incorrect for S . But, if γ'_1 is the curve shown in Figure 7, then $[\gamma_1] + [\gamma'_1] = [\beta]$ (where $[\lambda]$ is the homology class modulo two of the curve λ) and so γ'_1 is a non-separating curve if β does not separate and γ_1 does.

Let us set $\gamma'_2 = \tau_\beta(\gamma'_1)$; then, since $|\beta \cap \gamma'_1| = 2_0$, one has by Lemma 1.5

$$a_{\gamma'_2} = a_\beta a_{\gamma'_1} \overline{a_\beta}, \quad \text{and thus} \quad a_\beta = a_{\gamma'_2} a_\beta \overline{a_{\gamma'_1}}.$$

If δ'_1 and δ'_2 are respectively the curves $\tau_{\gamma'_1}^{-1}(\alpha)$ and $\tau_\beta(\delta'_1)$, then $\gamma = \tau_{\gamma'_2}(\delta'_2)$.

- $|\gamma'_1 \cap \alpha| = 2 + n_2 + n_3 < n$.
- $|\beta \cap \delta'_1| = |\beta \cap \tau_{\gamma'_1}^{-1}(\alpha)| = |\tau_{\gamma'_1}(\beta) \cap \alpha|$.

Since $\tau_{\gamma'_1}(\beta) = \tau_{\gamma_1}^{-1}(\beta)$ (see Figure 8), one has $|\beta \cap \delta'_1| = n - 2$.

- $|\gamma'_2 \cap \delta'_2| = |\gamma'_2 \cap \tau_\beta \tau_{\gamma'_1}^{-1}(\alpha)| = |\tau_{\gamma'_1} \tau_\beta^{-1}(\gamma'_2) \cap \alpha| = |\gamma'_1 \cap \alpha| < n$.

So, by the induction hypothesis,

$$a_\gamma = a_{\gamma'_2}(a_{\delta'_2}) = a_{\gamma'_2} a_\beta(a_{\delta'_1}) = a_{\gamma'_2} a_\beta \overline{a_{\gamma'_1}}(a_\alpha) = a_\beta(a_\alpha).$$

For the second case of the Figure 6, the proof is the same using the curves γ_1 , γ'_1 , and γ_2 of Figure 9 \square

1.2. A relation between the lanterns, the stars and the chains. For any lantern determined by the curves β_1 and β_3 , we denote by $L(\beta_1, \beta_3)$ the element $a_{\alpha_1} a_{\alpha_2} a_{\alpha_3} a_{\alpha_4} \overline{a_{\beta_1}} \overline{a_{\beta_2}} \overline{a_{\beta_3}}$ of R , and by \mathcal{L} the subgroup of R generated by all these elements. Similarly, we define elements $E(\alpha_1, \alpha_2, \alpha_3, \beta)$ and $C(\alpha_1, \beta, \alpha_2)$ associated

to the stars and the chains and the subgroups \mathcal{E} and \mathcal{C} . When the curves involved do not separate, we will also consider these elements in S , but we will denote them with a subscript “0”.

- Theorem 1.6. a)**
- (i) *The subgroups \mathcal{L} , \mathcal{E} and \mathcal{C} are in the center of R .*
 - (ii) *If $g \geq 1$, then $\mathcal{E} = \mathcal{CL}$, where \mathcal{CL} is the subgroup generated by \mathcal{C} and \mathcal{L} .*
 - b)**
 - (i) *The elements $L_0(\cdot, \cdot)$, $E_0(\cdot, \cdot, \cdot)$ and $C_0(\cdot, \cdot, \cdot)$ are central in S .*
 - (ii) *For any lanterns (β_1, β_3) and (β'_1, β'_3) with non-separating curves, one has in S*

$$L_0(\beta_1, \beta_3) = L_0(\beta'_1, \beta'_3).$$

The same result holds for stars and chains. Let us denote by L_0 , E_0 and C_0 the three elements of S obtained this way.

- (iii) $C_0 = L_0 E_0$ in S .

Remark. Part b) makes sense only if the elements L_0 , E_0 and C_0 exists, that is to say, when $g \geq 2$ for C_0 and when $g \geq 3$ for L_0 and E_0 .

Proof. The assertion (i) of a) and b) is an easy consequence of Corollary 1.4, and (ii) of b) is a consequence of (i) and the classification of surfaces. Thus, let us look at the relation $\mathcal{E} = \mathcal{CL}$. To do this, we first consider a lantern $L = L(\beta_1, \beta_3)$ where the curves are denoted as shown in Figure 1.

First case: At least one of the curves α_i does not separate. Then, there is a second curve α_j which does not separate. Without loss of generality, one can suppose that these two curves are α_1 and α_2 . Then, if β and δ are the curves shown in Figure 10, one has by Lemma 1.2

$$C = C(\alpha_1, \beta, \alpha_2) = X_1^2 X_2^2 \overline{a_\delta} \overline{a_{\beta_3}}$$

and

$$E = E(\alpha_1, \beta_1, \alpha_2, \beta) = X_1 X_2 X_3 \overline{a_\delta} \overline{a_{\alpha_4}} \overline{a_{\alpha_3}}$$

where $X_1 = a_{\alpha_1} a_{\alpha_2}$, $X_2 = a_\beta X_1 a_\beta$ and $X_3 = a_{\beta_1} X_2 a_{\beta_1}$. Thus, we obtain by (0), Lemma 1.2 and (i)

$$C \overline{E} = a_{\alpha_1} a_{\alpha_2} a_{\alpha_3} a_{\alpha_4} \overline{a_{\beta_3}} \overline{a_{\beta_1}} \overline{X_2} \overline{a_{\beta_1}} X_2.$$

But $(\tau_\beta \tau_{\alpha_1} \tau_{\alpha_2} \tau_\beta)^{-1}(\beta_1) = \beta_2$, so, by Corollary 1.4, $a_{\beta_2} = \overline{X_2}(a_{\beta_1})$ and we obtain $C \overline{E} = L$. Thus, since $\mathcal{C} \subset \mathcal{E}$, we have $L \in \mathcal{E}$ (by (i), $L = a_{\alpha_1} a_{\alpha_2} a_{\alpha_3} a_{\alpha_4} \overline{a_{\beta_1}} \overline{a_{\beta_2}} \overline{a_{\beta_3}} = a_{\alpha_1} a_{\alpha_2} a_{\alpha_3} a_{\alpha_4} \overline{a_{\beta_2}} \overline{a_{\beta_3}} \overline{a_{\beta_1}} = a_{\alpha_1} a_{\alpha_2} a_{\alpha_3} a_{\alpha_4} \overline{a_{\beta_3}} \overline{a_{\beta_1}} \overline{a_{\beta_2}}$).

Note that this shows the relation (iii) of b) because if β_3 does not separate, then δ does not separate either.

Second case: All the curves α_i separate in Σ . Since g is greater than or equal to one, there exists i such that the component of $\Sigma \setminus \alpha_i$ which does not contain α_j (for all $j \neq i$) is of genus greater than or equal to one. We can suppose, without loss of generality, that $i=3$ and consider the curves $\gamma_1, \dots, \gamma_6$ shown in Figure 11.

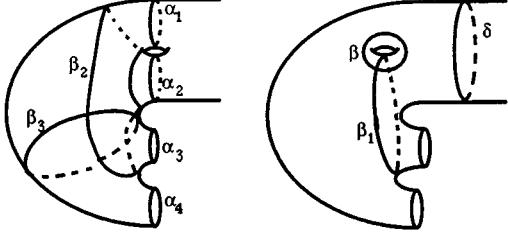


FIGURE 10

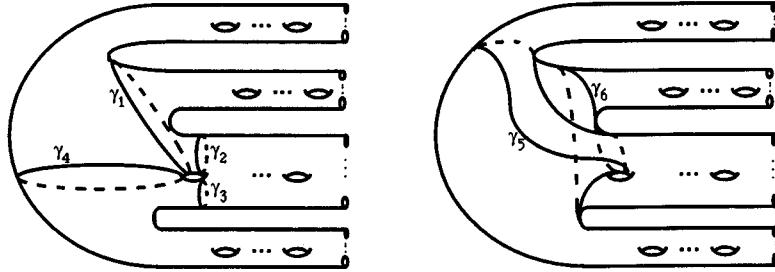


FIGURE 11

Denote respectively by L_1 , L_2 and L_3 the lanterns $L(\gamma_6, \gamma_1)$, $L(\gamma_6, \gamma_4)$ and $L(\gamma_5, \gamma_1)$. Then, using (0) and (i), one has

$$\begin{aligned} \overline{L_1} L_2 L_3 &= (a_{\alpha_2} a_{\beta_1} a_{\gamma_2} a_{\gamma_3} \overline{a_{\gamma_6}} \overline{a_{\alpha_3}} \overline{a_{\gamma_1}})^{-1} (a_{\alpha_2} a_{\alpha_4} a_{\gamma_3} a_{\gamma_5} \overline{a_{\gamma_6}} \overline{a_{\beta_2}} \overline{a_{\gamma_4}}) \\ &\quad \times (a_{\alpha_1} a_{\alpha_2} a_{\gamma_2} a_{\gamma_4} \overline{a_{\gamma_5}} \overline{a_{\beta_3}} \overline{a_{\gamma_1}}) \\ &= a_{\alpha_3} \overline{a_{\gamma_3}} \overline{a_{\gamma_2}} \overline{a_{\beta_1}} a_{\alpha_4} a_{\gamma_3} \overline{a_{\beta_2}} a_{\alpha_1} a_{\alpha_2} a_{\gamma_2} \overline{a_{\beta_3}} \\ &= a_{\alpha_3} \overline{a_{\beta_1}} a_{\alpha_4} \overline{a_{\beta_2}} a_{\alpha_1} a_{\alpha_2} \overline{a_{\beta_3}} \\ &= L. \end{aligned}$$

The first case proves that $L_k \in \mathcal{E}$, so we have $L \in \mathcal{E}$.

Thus, we have proved that $\mathcal{L} \subset \mathcal{E}$ and so $\mathcal{LC} \subset \mathcal{E}$. Now, let $(\alpha_1, \alpha_2, \alpha_3, \beta)$ be a star E . We shall find a lantern L and a chain C such that $E = CL$ in R . Considering the curves shown in Figure 12, one has

$$\begin{aligned} E &= X_1 X_2 X_3 \overline{a_{\delta_1}} \overline{a_{\delta_2}} \overline{a_{\delta_3}}, \\ C(\alpha_1, \beta, \alpha_2) &= X_1^2 X_2^2 \overline{a_{\delta_3}} \overline{a_{\gamma_1}} = C, \\ L(\gamma_2, \gamma_1) &= a_{\alpha_1} a_{\alpha_2} a_{\delta_1} a_{\delta_2} \overline{a_{\alpha_3}} \overline{a_{\gamma_2}} \overline{a_{\gamma_1}} = L, \end{aligned}$$

where $X_1 = a_{\alpha_1} a_{\alpha_2}$, $X_2 = a_\beta X_1 a_\beta$ and $X_3 = a_{\alpha_3} X_2 a_{\alpha_3}$.

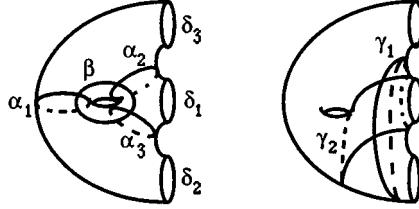


FIGURE 12

Then, using (0) and (i), we obtain

$$\begin{aligned} C\bar{L} &= X_1 X_2 \overline{a_{\delta_3}} X_1 X_2 \overline{a_{\gamma_1}} a_{\gamma_1} a_{\gamma_2} a_{\alpha_3} \overline{a_{\alpha_1}} \overline{a_{\alpha_2}} \overline{a_{\delta_1}} \overline{a_{\delta_2}} \\ &= X_1 X_2 \overline{a_{\delta_3}} X_2 a_{\gamma_2} a_{\alpha_3} \overline{a_{\delta_1}} \overline{a_{\delta_2}} \\ &= X_1 X_2 X_2 a_{\gamma_2} a_{\alpha_3} \overline{a_{\delta_1}} \overline{a_{\delta_2}} \overline{a_{\delta_3}}. \end{aligned}$$

But one can see that $\tau_\beta \tau_{\alpha_2} \tau_{\alpha_1} \tau_\beta(\gamma_2) = \alpha_3$, so $a_{\alpha_3} = X_2 a_{\gamma_2} \overline{X_2}$ by Corollary 1.4. Thus,

$$\begin{aligned} C\bar{L} &= X_1 X_2 a_{\alpha_3} X_2 a_{\alpha_3} \overline{a_{\delta_1}} \overline{a_{\delta_2}} \overline{a_{\delta_3}} \\ &= X_1 X_2 X_3 \overline{a_{\delta_1}} \overline{a_{\delta_2}} \overline{a_{\delta_3}} \\ &= E. \end{aligned}$$

This shows that $E \in \mathcal{CL}$ and concludes the proof of Theorem 1.6. \square

We conclude this section by a lemma which gives us a new relation in \mathcal{M} .

Lemma 1.7. *Let Σ' be a subsurface of Σ which is diffeomorphic to a surface of genus one with four boundary components. Consider the curves shown on Figure 13 and set (in R and S)*

$$X_1 = a_{\alpha_1} a_{\alpha_3}, \quad X_2 = a_\beta X_1 a_\beta, \quad X_3 = a_{\alpha_2} X_2 a_{\alpha_2}, \quad X'_3 = a_{\alpha_4} X_2 a_{\alpha_4}.$$

Then, one can find two stars E_1, E_2 , and a chain C such that

$$X_3 X'_3 = a_{\delta_1} a_{\delta_2} a_{\delta_3} a_{\delta_4} E_1 E_2 C^{-1} \text{ in } R.$$

The same result holds in S if $\delta_1, \dots, \delta_4$ do not separate in Σ .

Proof. By Lemma 1.2, one has

$$\begin{aligned} E_1 &= E(\alpha_1, \alpha_3, \alpha_2, \beta) = X_1 X_2 X_3 \overline{a_{\delta_2}} \overline{a_{\delta_3}} \overline{a_\delta}, \\ E_2 &= E(\alpha_1, \alpha_3, \alpha_4, \beta) = X_1 X_2 X'_3 \overline{a_{\delta_1}} \overline{a_{\delta_4}} \overline{a_{\delta'}}, \\ C &= C(\alpha_1, \beta, \alpha_3) = X_1^2 X_2^2 \overline{a_\delta} \overline{a_{\delta'}}, \end{aligned}$$

where δ and δ' are the curves shown on Figure 14. Thus, we obtain

$$\begin{aligned} E_1 E_2 C^{-1} &= \overline{a_{\delta_1}} \overline{a_{\delta_2}} \overline{a_{\delta_3}} \overline{a_{\delta_4}} X_1 X_2 X_3 \overline{a_\delta} X_1 X_2 X'_3 \overline{a_{\delta'}} a_\delta a_{\delta'} \overline{X_1^2} \overline{X_2^2} \text{ by (0)} \\ &= \overline{a_{\delta_1}} \overline{a_{\delta_2}} \overline{a_{\delta_3}} \overline{a_{\delta_4}} X_3 \overline{a_\delta} X'_3 a_\delta \text{ by Lemma 1.2 and (0)} \\ &= \overline{a_{\delta_1}} \overline{a_{\delta_2}} \overline{a_{\delta_3}} \overline{a_{\delta_4}} X_3 X'_3 \text{ by (0) and (i) of Theorem 1.6.} \end{aligned}$$

Suppose now that the four curves $\delta_1, \dots, \delta_4$ do not separate. The proof above remains valid for S only if both δ and δ' do not separate Σ . But if they separate, the curves γ and γ' shown on Figure 14 do not separate. Then, one can conclude in the

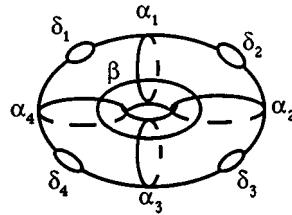


FIGURE 13

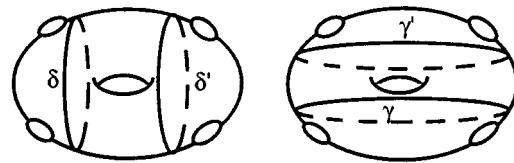


FIGURE 14

same way using the stars $(\alpha_2, \alpha_4, \alpha_1, \beta)$, $(\alpha_2, \alpha_4, \alpha_3, \beta)$ and the chain $(\alpha_2, \beta, \alpha_4)$.

□

2. PROOF OF THEOREMS A AND B

Let Σ be a surface of genus $g \geq 0$ with r boundary components ($r \geq 1$), $\mathcal{M} = \mathcal{M}_{g,r}$ its mapping class group (we consider Σ to be a disc with g handles attached and $r-1$ holes).

In order to simplify notation, we will denote by β the twist along a curve β . First, we shall recall the results obtained by Hatcher-Thurston ([H-T]) and Harer ([H]) about presentations of \mathcal{M} . To do this, we will consider the curves shown on Figures 15, 16 and 17. If $D_1, D_2, \dots, D_{2g+r-1}$ are the holes obtained when we cut Σ along the curves α_i or corresponding to $\partial\Sigma$ (except δ , the boundary of the disc on which the handles were attached), then $\delta_{ij}, 1 \leq i < j \leq 2g+r-1$ is a curve in $\Sigma \setminus (\bigcup \alpha_i)$ which encircles the holes D_i and D_j and $\varphi_{k, 3 \leq k \leq 2g+r-1}$ a curve which encircles D_1, D_2 and D_k (see Figure 17).

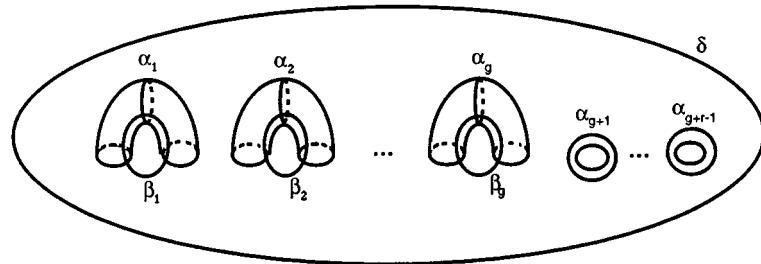


FIGURE 15

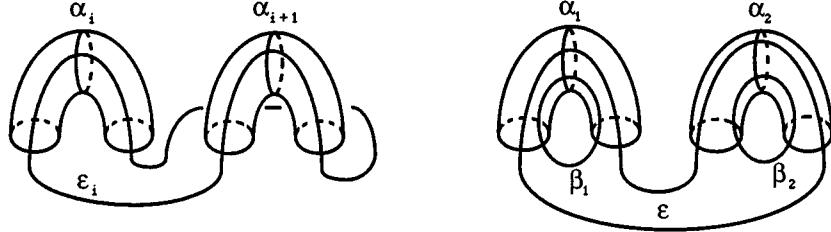


FIGURE 16

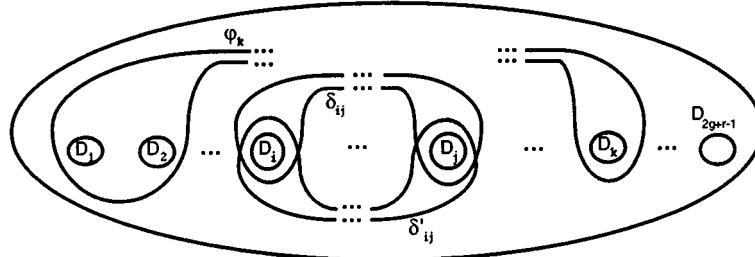


FIGURE 17

Denote by H_0 the subgroup of elements of \mathcal{M} which leave the curves α_i fixed. Hatcher and Thurston have proved in [H-T] that H_0 is isomorphic to $P_{2g+r-1} \times \mathbb{Z}^{g+r-1}$, where P_n is the pure braid group on n strands. It is generated by α_i 's and $\delta_{i,j}$'s, and admits a presentation with relations (see [B] for a presentation of P_n)

- (i)_a $\alpha_i \alpha_j = \alpha_j \alpha_i \quad \forall i, j \in \langle 1, g+r-1 \rangle,$
- (i)_b $\alpha_k \delta_{ij} = \delta_{ij} \alpha_k \quad \forall i < j \in \langle 1, 2g+r-1 \rangle \text{ and } k \in \langle 1, g+r-1 \rangle,$
- (i)_c $\delta_{ij} \lambda_{ijkl} = \lambda_{ijkl} \delta_{ij},$ with

$$\lambda_{ijkl} = \begin{cases} \delta_{kl} & \text{if } 1 \leq k < l < i < j \leq 2g+r-1 \\ & \text{or } 1 \leq i < k < l < j \leq 2g+r-1, \\ \delta_{kj} \delta_{kl} & \text{if } 1 \leq k < l = i < j \leq 2g+r-1, \\ \delta_{lj} \delta_{ij} \delta_{kl} & \text{if } 1 \leq i = k < l < j \leq 2g+r-1, \\ [\delta_{lj}^{-1}, \delta_{kj}^{-1}] \delta_{kl} & \text{if } 1 \leq k < i < l < j \leq 2g+r-1. \end{cases}$$

H is the subgroup of elements of \mathcal{M} which leave the set $\{\alpha_1, \dots, \alpha_g\}$ fixed (an element of H can permute the α_i 's and reverse their orientations). So H is defined by the following exact sequences (see [W]):

$$1 \rightarrow H_0 \rightarrow H \xrightarrow{\theta} \pm \mathcal{S}_n \rightarrow 1, \quad 1 \rightarrow (\mathbb{Z}/2\mathbb{Z})^n \rightarrow \pm \mathcal{S}_n \rightarrow \mathcal{S}_n \rightarrow 1,$$

where \mathcal{S}_n is the symmetric group of rank n and θ the morphism which assigns to each element of H the induced signed permutation on the α_i 's. Therefore, H is generated by H_0 , $\xi = \beta_1 \alpha_1 \alpha_1 \beta_1$ and $(\theta_i = \varepsilon_i \alpha_i \alpha_{i+1} \varepsilon_i)_{1 \leq i \leq g-1}$, and is defined by the relations (i) and

- (ii)_a $\theta_{i+1} \theta_i \theta_{i+1} = \theta_i \theta_{i+1} \theta_i$ for $1 \leq i \leq g-2$, and $\theta_i \theta_j = \theta_j \theta_i$ for $|i-j| \geq 2$,
- (ii)_b $\theta_i^2 \in H_0$ for $1 \leq i \leq g-1$,
- (ii)_c $\xi^2 \in H_0$,
- (ii)_d $[\xi_i, \xi_j] \in H_0$ for $1 \leq i < j \leq g$ where $\xi_i = \theta_{i-1} \theta_{i-2} \dots \theta_1 \xi \overline{\theta_1} \dots \overline{\theta_{i-2}} \overline{\theta_{i-1}}$,
- (ii)_e $[\theta_i, \xi_j] \in H_0$ for $i \neq j, j-1$,
- (iii)_a $\xi(\alpha_i) \in H_0$ for $1 \leq i \leq g+r-1$,
- (iii)_b $\xi(\delta_{ij}) \in H_0$ for $1 \leq i < j \leq 2g+r-1$,
- (iii)_c $\theta_k(\alpha_i) \in H_0$ for $1 \leq i \leq g+r-1$ and $1 \leq k \leq g-1$,
- (iii)_d $\theta_k(\delta_{ij}) \in H_0$ for $1 \leq i < j \leq 2g+r-1$ and $1 \leq k \leq g-1$.

Remark. θ_i permutes α_i and α_{i+1} , and ξ_i reverses the orientation of α_i ($\xi_1 = \xi$).

Theorem 2.1 ([H-T][H]). \mathcal{M} is generated by H and $\sigma = \alpha_1 \beta_1 \alpha_1$. All relations involving σ come from (A)–(E) below and, together with (i)–(iii), they suffice for the presentation of \mathcal{M} .

- (A) σ commutes with $H(\alpha_1, \beta_1)$, where $H(\alpha_1, \beta_1)$ is the subgroup of elements of H which leave α_1 and β_1 fixed.
- (B) $\sigma^2 \in H$.
- (C) Consider the curves $\gamma_1, \dots, \gamma_N$ shown on Figure 18. For any γ among these curves and any $h \in H$ such that $h\sigma(\alpha_1) = \gamma$, $h\sigma(\gamma) = \beta_1$ and $h\sigma(\beta_1) = \alpha_1$, one has $\sigma h \sigma h \sigma \in H$. In the presence of (A), the choice of one h suffices.
- (D) For any $h \in H$ such that $h(\beta_2) = \beta_1$, σ commutes with $\bar{h}(\sigma)$. In the presence of (A), the choice of one h suffices.
- (E) If ε is the curve shown on Figure 16, then $\sigma h_5 \sigma h_4 \sigma h_3 \sigma h_2 \sigma h_1 \sigma \in H$ whenever there are $h_1, h_2, h_3, h_4, h_5 \in H$ such that $h_1 \sigma(\delta_{2,3}) = \beta_1$, $h_2 \sigma h_1 \sigma(\varepsilon) = \beta_1$, $h_3 \sigma h_2 \sigma h_1 \sigma(\alpha_1) = \beta_1$, $h_4 \sigma h_3 \sigma h_2 \sigma h_1 \sigma(\beta_2) = \beta_1$ and

$$h_5 \sigma h_4 \sigma h_3 \sigma h_2 \sigma h_1 \sigma(\alpha_2) = \beta_1.$$

The choice of one 5-tuple $(h_1, \dots, h_5) \in H^5$ suffices.

Remark. Relations (C) and (E) are slightly different from [H-T]. Moreover, Hatcher and Thurston worked only with closed surfaces. The result above is due to Harer ([H]).

Corollary 2.2. Let Σ be a surface with at least one boundary component. Then \mathcal{M}_Σ admits a presentation with generators α_i , $\delta_{i,j}$, ξ , θ_i , σ and relations (i)–(iii) above added to the following:

- (iv) σ commutes with $(\alpha_i)_{2 \leq i \leq g+r-1}$, $(\theta_i)_{2 \leq i \leq g-1}$, $\xi' = \overline{\theta_1} \delta_{2,4} \overline{\alpha_1}(\xi)$, $\delta_{1,2}$, $(\delta_{ij})_{3 \leq i < j \leq 2g+r-1}$ and $(\varphi_i)_{3 \leq i \leq 2g+r-1}$, where $\varphi_i = \alpha_1^2 (\xi \delta_{2,i})^2 \overline{\alpha_{k_i}}$ with $k_i = [(i+1)/2]$ if $i \leq 2g$ and $k_i = i-g$ if $i \geq 2g+1$.
- (v) $\sigma^2 \in H$.
- (vi) $\sigma h \sigma h \sigma \in H$ for all $h \in \mathcal{H} = \{\alpha_1 ; \delta_{1,3} ; \delta_{1,4} \delta_{1,3} \overline{\alpha_1} ; \delta_{1,5} \delta_{1,3} \overline{\alpha_1} ; \delta_{1,2g+1}, \dots, \delta_{1,2g+r-1} ; \delta_{1,2g+1} \delta_{1,3} \overline{\alpha_1}, \dots, \delta_{1,2g+r-1} \delta_{1,3} \overline{\alpha_1} ; \delta_{1,k} \delta_{1,j} \overline{\alpha_1}$ for $2g+1 \leq j < k \leq 2g+r-1\}$.

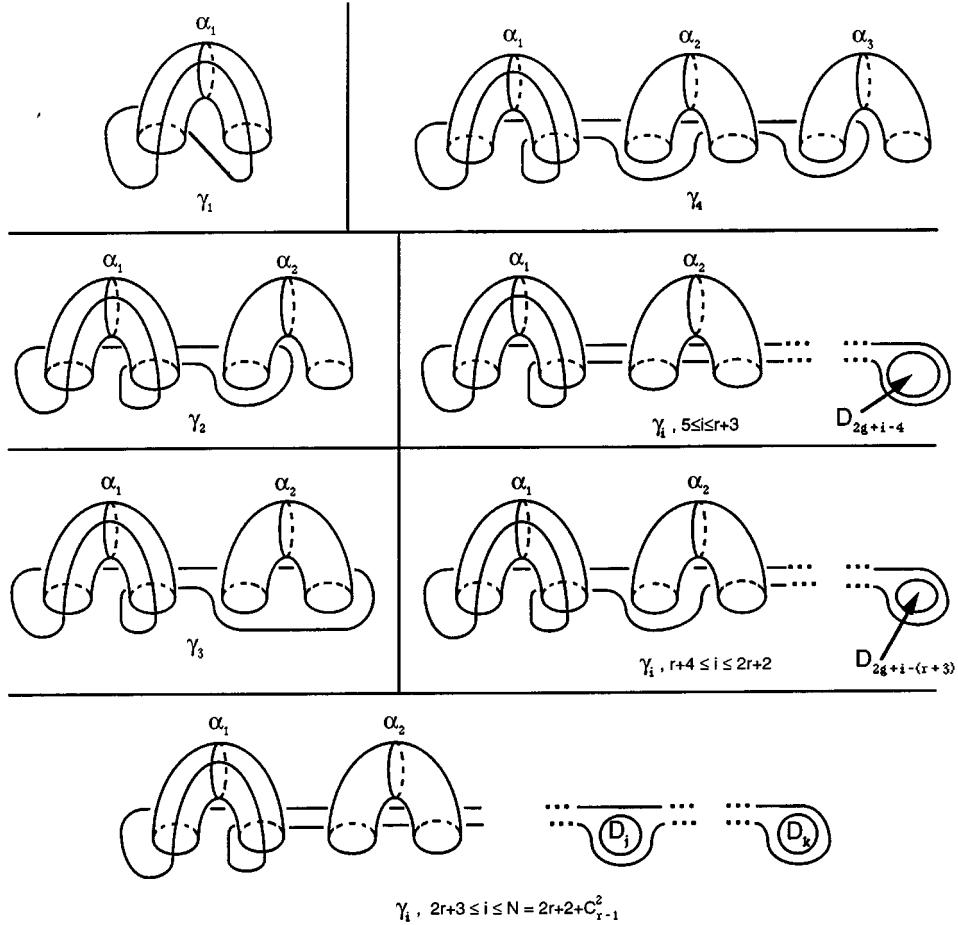


FIGURE 18

(vii) σ commutes with $\bar{h}\sigma h$, where $h = \alpha_1 \overline{\delta_{2,4}} \theta_1$.

(viii) $\sigma h_5 \sigma h_4 \sigma h_3 \sigma h_2 \sigma h_1 \sigma \in H$, where $h_1 = h_3 = h_5 = \overline{\alpha_1} \delta_{2,3}$, $h_2 = \overline{\delta_{2,3}} \theta$ and $h_4 = \delta_{2,3} \overline{\alpha_1} \bar{\theta}$; here $\theta = \overline{\varphi_3} \xi \alpha_1^2 \alpha_2(\theta_1)$.

Remarks. (i) φ_i is the curve shown in Figure 17. The chain $(\alpha_1, \beta_1, \delta_{2,i})$ and the computations of Lemma 1.2 prove that

$$\varphi_i = (\alpha_1 \beta_1 \delta_{2,i})^4 \overline{\alpha_{k_i}} = \alpha_1^2 (\xi \delta_{2,i})^2 \overline{\alpha_{k_i}}.$$

(ii) As diffeomorphisms of Σ , ξ and θ can be seen to be equal to $\beta_2 \alpha_2 \alpha_2 \beta_2$ and $\varepsilon \alpha_1 \alpha_2 \varepsilon$ respectively, where the curves are whose which are shown on Figures 15 and 16.

Proof. We have to prove that the relations (iv)–(viii) are equivalent to the relations (A)–(E).

(iv) Let $H_0(\alpha_1, \beta_1)$ be the subgroup of elements of $H(\alpha_1, \beta_1)$ which leave each curve α_i fixed. Then, $H(\alpha_1, \beta_1)$ is generated by $H_0(\alpha_1, \beta_1)$ and elements permuting the α_i 's for $2 \leq i \leq g$ or reversing their orientation. We have seen that θ_i permutes α_i and α_{i+1} . The diffeomorphism ξ' preserves α_1 , β_1 and α_i for

$i \geq 3$ and reverses the orientation of α_2 (using braid relations, one can see that $\xi' = \beta_2 \alpha_2 \alpha_2 \beta_2$). So, $H(\alpha_1, \beta_1)$ is generated by $H_0(\alpha_1, \beta_1)$, ξ' and $(\theta_i)_{2 \leq i \leq g-1}$. Furthermore, following the proof of Hatcher-Thurston for H_0 , one can prove that $H_0(\alpha_1, \beta_1)$ is isomorphic to $P_{2g+r-2} \times \mathbb{Z}^{g+r-2}$ and deduce that it is generated by $\delta_{1,2}$, $(\alpha_i)_{2 \leq i \leq g+r-1}$, $(\delta_{ij})_{3 \leq i < j \leq 2g+r-1}$ and $(\varphi_i)_{3 \leq i \leq 2g+r-1}$. So, we have found the required generators for $H(\alpha_1, \beta_1)$, and this proves that (iv) is equivalent to (A).

(vi) For any s.c.c. γ in $\{\gamma_1, \dots, \gamma_N\}$, we have to find an element h of H such that $h\sigma(\alpha_1) = \gamma$, $h\sigma(\gamma) = \beta_1$ and $h\sigma(\beta_1) = \alpha_1$. \mathcal{H} is exactly the set of the elements that we have found:

$$\begin{aligned} * \quad \gamma = \gamma_1 &\Rightarrow h = \alpha_1, \\ * \quad \gamma = \gamma_2 &\Rightarrow h = \delta_{1,3}, \\ * \quad \gamma = \gamma_3 &\Rightarrow h = \delta_{1,4} \delta_{1,3} \overline{\alpha_1}, \\ * \quad \gamma = \gamma_4 &\Rightarrow h = \delta_{1,5} \delta_{1,3} \overline{\alpha_1}, \\ * \quad \gamma = \gamma_i, \quad i=5, \dots, r+3 &\Rightarrow h = \delta_{1,2g+i-4}, \\ * \quad \gamma = \gamma_i, \quad i=r+4, \dots, 2r+2 &\Rightarrow h = \delta_{1,2g+i-r-3} \delta_{1,3} \overline{\alpha_1}, \\ * \quad \gamma = \gamma_i, \quad i=2r+3, \dots, N &\Rightarrow h = \delta_{1,k} \delta_{1,j} \overline{\alpha_1} \text{ if } \gamma_i \text{ encircles } D_j \text{ and } D_k. \end{aligned}$$

(vii) $h = \alpha_1 \overline{\delta_{2,4}} \theta_1$ is an element of H satisfying $h(\beta_2) = \beta_1$, so (vii) is equivalent to (D).

(viii) If we choose $h_1 = h_3 = h_5 = \overline{\alpha_1} \delta_{2,3}$, $h_2 = \overline{\delta_{2,3}} \theta$ and $h_4 = \delta_{2,3} \overline{\alpha_1} \bar{\theta}$, then $h_i \in H$ for all i and the 5-tuple $(h_1, h_2, h_3, h_4, h_5)$ satisfies the condition of the relation (E). \square

2.1. Proof of Theorem A. Let G be a group with generators $\{a \equiv a_\alpha / \alpha \text{ s.c.c. in } \Sigma\}$ and relations (I), (II) and (III) (Theorem A). We shall denote by H'_0 the subgroup of G generated by the a_i 's and the d_{ij} 's, and by H' the subgroup generated by H'_0 , $x = b_1 a_1 a_1 b_1$, and the t_i 's, where $t_i = e_i a_i a_{i+1} e_i$ ($i=1, \dots, g-1$).

Remark. Since G is a quotient of R , the relations which have been shown in Section 1 are also true in G .

Let $\varphi: G \rightarrow \mathcal{M}$ be the epimorphism defined by $\varphi(a) = \alpha$. In order to prove that φ is an isomorphism, we shall construct an inverse morphism $\psi: \mathcal{M} \rightarrow G$. So, let us define ψ on the generators of \mathcal{M} as follows:

$$\psi(\alpha) = a \quad \forall \alpha \in \left\{ (\alpha_i)_{1 \leq i \leq g+r-1}; (\delta_{ij})_{1 \leq i < j \leq 2g+r-1} \right\},$$

$$\psi(\xi) = x, \quad \psi(\theta_i) = t_i, \quad \psi(\sigma) = a_1 b_1 a_1 = s.$$

First of all, let us prove that ψ is a homomorphism defined on \mathcal{M} ; to do this, we will show that the relations (i)–(viii) are mapped by ψ into true relations in G .

Relations (i). (i)_a and (i)_b are consequences of (0), and since $\lambda_{ijkl}(\delta_{ij}) = \delta_{ij}$, (i)_c is a consequence of Corollary 1.4.

Relation (ii)a. (0) implies $t_i t_j = t_j t_i$ when $|i-j| \geq 2$. For $1 \leq i \leq g-1$, one has

$$\begin{aligned}
t_{i+1}(t_i) &= (e_{i+1} a_{i+1} a_{i+2} e_{i+1})(e_i a_i a_{i+1} e_i)(\overline{e_{i+1}} \overline{a_{i+1}} \overline{a_{i+2}} \overline{e_{i+1}}) \\
&= e_{i+1} a_{i+1} a_{i+2} e_i a_i e_{i+1} a_{i+1} \overline{e_{i+1}} e_i \overline{a_{i+1}} \overline{a_{i+2}} \overline{e_{i+1}} \quad \text{by (0)} \\
&= e_{i+1} a_{i+1} a_{i+2} e_i a_i \overline{a_{i+1}} e_{i+1} a_{i+1} e_i \overline{a_{i+1}} \overline{a_{i+2}} \overline{e_{i+1}} \quad \text{by (1)} \\
&= e_{i+1} a_{i+2} a_{i+1} e_i \overline{a_{i+1}} a_i e_{i+1} a_{i+1} e_i \overline{a_{i+1}} \overline{a_{i+2}} \overline{e_{i+1}} \quad \text{by (0)} \\
&= e_{i+1} a_{i+2} \overline{e_i} a_{i+1} e_i a_i e_{i+1} \overline{e_i} a_{i+1} e_i \overline{a_{i+2}} \overline{e_{i+1}} \quad \text{by (1)} \\
&= \overline{e_i} e_{i+1} a_{i+2} a_{i+1} e_i a_i \overline{e_i} e_{i+1} a_{i+1} \overline{a_{i+2}} \overline{e_{i+1}} e_i \quad \text{by (0)} \\
&= \overline{e_i} e_{i+1} a_{i+2} a_{i+1} \overline{a_i} e_i a_i e_{i+1} a_{i+1} \overline{a_{i+2}} \overline{e_{i+1}} e_i \quad \text{by (1)} \\
&= \overline{e_i} \overline{a_i} e_{i+1} a_{i+1} e_i a_{i+2} e_{i+1} \overline{a_{i+2}} a_{i+1} \overline{e_{i+1}} a_i e_i \quad \text{by (0)} \\
&= \overline{e_i} \overline{a_i} e_{i+1} a_{i+1} e_i \overline{e_{i+1}} a_{i+2} e_{i+1} a_{i+1} \overline{e_{i+1}} a_i e_i \quad \text{by (1)} \\
&= \overline{e_i} \overline{a_i} e_{i+1} a_{i+1} \overline{e_{i+1}} e_i a_{i+2} e_{i+1} a_{i+1} \overline{e_{i+1}} a_i e_i \quad \text{by (0)} \\
&= \overline{e_i} \overline{a_i} \overline{a_{i+1}} e_{i+1} a_{i+1} e_i a_{i+2} \overline{a_{i+1}} e_{i+1} a_{i+1} a_i e_i \quad \text{by (1)} \\
&= \overline{e_i} \overline{a_i} \overline{a_{i+1}} e_{i+1} a_{i+1} e_i \overline{a_{i+1}} a_{i+2} e_{i+1} a_{i+1} a_i e_i \quad \text{by (0)} \\
&= \overline{e_i} \overline{a_i} \overline{a_{i+1}} e_{i+1} a_{i+1} \overline{e_i} a_{i+1} e_i a_{i+2} e_{i+1} a_{i+1} a_i e_i \quad \text{by (1)} \\
&= \overline{e_i} \overline{a_i} \overline{a_{i+1}} \overline{e_i} e_{i+1} a_{i+1} a_{i+2} e_{i+1} e_i a_{i+1} a_i e_i \quad \text{by (0)} \\
&= \overline{t_i}(t_{i+1}),
\end{aligned}$$

which implies $t_{i+1} t_i t_{1+1} = t_i t_{i+1} t_i$.

Relation (ii)_b. Let δ'_{ij} be the curves shown in Figure 17.

Lemma 2.3. $d'_{ij} \in H'_0$ for all $1 \leq i < j \leq 2g+r-1$, and H'_0 is generated by a_i 's and d'_{ij} 's.

Proof. Since $\delta_{i,j} = \tau_{\delta_{i,j-1}} \dots \tau_{\delta_{i,i+1}}(\delta'_{i,j})$, one has

$$d'_{i,j} = \overline{d_{i,i+1}} \dots \overline{d_{i,j-1}}(d_{i,j}) \in H'_0$$

(use Corollary 1.4). Symmetrically, we obtain $d_{i,j} = d'_{i,i+1} \dots d'_{i,j-1}(d'_{i,j})$, which proves the lemma. \square

By Lemma 1.2, one has $(a_i e_i a_{i+1})^4 = a_i^2 a_{i+1}^2 t_i^2$. But the chain $(\alpha_i, \varepsilon_i, \alpha_{i+1})$ gives:

$$(a_i e_i a_{i+1})^4 = d'_{2i-1, 2i+1} d_{2i, 2i+2}.$$

Thus, by Lemma 2.3, t_i^2 is an element of H'_0 .

Relation (ii)_c. The chain $(\alpha_1, \beta_1, \alpha_1)$ yields the relation $a_1^4(b_1 a_1 a_1 b_1)^2 = d_{1,2}$ (using Lemma 1.2). Thus we have $x^2 = d_{1,2} a_1^{-4} \in H'_0$.

Relation (iii)_a. For $i \geq 2$, $x(a_i) = a_i \in H'_0$ by (0). One has

$$x(a_1) = b_1 a_1 a_1 b_1(a_1) = b_1 a_1 a_1 \overline{a_1}(b_1) = b_1 \overline{b_1}(a_1) = a_1 \in H'_0 \quad \text{by (1).}$$

Relation (iii)_b. When $3 \leq i < j \leq 2g+r-1$ or $(i, j) = (1, 2)$, one has $x(d_{i,j}) = d_{i,j} \in H'_0$ by (0).

When $3 \leq j \leq 2g+r-1$, one has $\tau_{\delta_{1,2}}^{-1} \xi(\delta_{1,j}) = \delta_{2,j}$ so, according to Corollary 1.4, $x(d_{1,j}) = d_{1,2}(d_{2,j}) \in H'_0$.

Similarly, when $3 \leq j \leq 2g+r-1$, we have $x(d_{2,j}) = d_{1,j} \in H'_0$.

Relation (iii)_c. We have seen that θ_k permutes α_k and α_{k+1} and leaves α_i fixed when $i \neq k, k+1$. So, by Corollary 1.4, one has $t_k(a_i) \in H'_0$.

Relation (iii)_d. For all i, j, k , we shall find $h \in H_0$ such that $\theta_k(\delta_{ij}) = h(\delta_{i',j'})$ for some i', j' . Then, if h is given as a word on the generators a_i 's and $\delta_{i,j}$'s, and m is the associated word in H'_0 , one has by Corollary 1.4

$$t_k(d_{ij}) = m(d_{i',j'}) \in H'_0.$$

The cases of $\delta_{2k-1,2k+1}, \delta_{2k+1,j}$ and $\delta_{i,2k+1}$ will be proved in a slightly different way:

- * $i, j \neq 2k-1, 2k, 2k+1, 2k+2 \Rightarrow \theta_k(\delta_{ij}) = \delta_{ij}.$
- * $j \geq 2k+3 \Rightarrow \theta_k(\delta_{2k-1,j}) = \delta'_{2k-1,2k+1} \delta'_{2k,2k+1}(\delta_{2k+1,j}).$
- * $i \leq 2k-2 \Rightarrow \theta_k(\delta_{i,2k-1}) = \delta'_{2k-1,2k+1} \delta'_{2k,2k+1}(\delta_{i,2k+1}).$
- * $j \geq 2k+3 \Rightarrow \theta_k(\delta_{2k,j}) = \delta_{2k,2k+2}(\delta_{2k+2,j}).$
- * $i \leq 2k-2 \Rightarrow \theta_k(\delta_{i,2k}) = \delta_{2k,2k+2}(\delta_{i,2k+2}).$
- * $j \geq 2k+3 \Rightarrow \theta_k(\delta_{2k+2,j}) = \delta_{2k+1,j}.$
- * $i \leq 2k-2 \Rightarrow \theta_k(\delta_{i,2k+2}) = \delta_{i,2k+1}.$
- * $\theta_k(\delta_{2k-1,2k}) = \delta'_{2k-1,2k+1}(\delta_{2k+1,2k+2}).$
- * $\theta_k(\delta_{2k-1,2k+2}) = \delta_{2k-1,2k}^{-1}(\delta_{2k,2k+1}).$
- * $\theta_k(\delta_{2k,2k+1}) = \delta_{2k+1,2k+2}^{-1}(\delta'_{2k-1,2k+2}).$
- * $\theta_k(\delta_{2k,2k+2}) = \delta_{2k,2k+2}.$
- * $\theta_k(\delta_{2k+1,2k+2}) = \delta_{2k,2k+2}(\delta_{2k-1,2k}).$
- * Since $\delta_{2k-1,2k+1} = \delta_{2k-1,2k}(\delta'_{2k-1,2k+1})$, one has

$$\theta_k(\delta_{2k-1,2k+1}) = \theta_k \delta_{2k-1,2k}(\delta'_{2k-1,2k+1}) = \theta_k \delta_{2k-1,2k} \overline{\theta_k} \theta_k(\delta'_{2k-1,2k+1}).$$

But since

$$\theta_k(\delta'_{2k-1,2k+1}) = \delta'_{2k-1,2k+1},$$

one has using Corollary 1.4

$$t_k(d_{2k-1,2k+1}) = t_k d_{2k-1,2k} \overline{t_k}(d'_{2k-1,2k+1}).$$

We have seen that $t_k(d_{2k-1,2k}) \in H'_0$; therefore $t_k(d_{2k-1,2k+1}) \in H'_0$ by Lemma 2.3.

- * We saw in Lemma 2.3 that

$$d_{2k+1,j} = d_{2k+1,j-1} \dots d_{2k+1,2k+2}(d'_{2k+1,j}).$$

Since, for $j \geq 2k+3$, $\theta_k(\delta'_{2k+1,j}) = \delta'_{2k-1,2k+1}(\delta'_{2k-1,j})$, one has $t_k(d'_{2k+1,j}) \in H'_0$ by Corollary 1.4. Furthermore, we already have $t_k(d_{2k+1,2k+2}) \in H'_0$, thus, $t_k(d_{2k+1,j}) \in H'_0$ by induction on j .

- * Similarly, one has $d_{i,2k+1} = d_{i,2k} \dots d_{i,i+1}(d'_{i,2k+1})$ and, for i less than $2k-1$, $t_k(d'_{i,2k+1}) = d'_{2k-1,2k+1}(d'_{i,2k-1}) \in H'_0$. Since $t_k(d_{i,l}) \in H'_0$ for $l = i+1, \dots, 2k$, we have $t_k(d_{i,2k+1}) \in H'_0$.

Relation (ii)_d.

Lemma 2.4. t_1 commutes with $x t_1 x$ in G .

Proof. We have

$$\begin{aligned}
x t_1 x t_1 &= (b_1 a_1 a_1 b_1)(e_1 a_1 a_2 e_1)(b_1 a_1 a_1 b_1)(e_1 a_1 a_2 e_1) \\
&= b_1 a_1 a_1 e_1 b_1 a_1 a_2 e_1 a_1 a_1 b_1 e_1 a_1 a_2 e_1 \quad \text{by (0)} \\
&= b_1 a_1 a_1 e_1 a_1 b_1 a_1 a_2 e_1 a_1 a_1 b_1 e_1 a_1 a_2 e_1 \quad \text{by (1)} \\
&= b_1 a_1 e_1 a_1 e_1 b_1 a_2 a_1 e_1 a_1 a_1 e_1 b_1 a_1 a_2 e_1 \quad \text{by (0) and (1)} \\
&= b_1 e_1 a_1 e_1 b_1 e_1 a_2 e_1 a_1 e_1 a_1 e_1 b_1 a_1 a_2 e_1 \quad \text{by (0) and (1)} \\
&= e_1 b_1 a_1 b_1 e_1 a_2 e_1 a_1 a_1 e_1 b_1 a_1 a_2 e_1 \quad \text{by (0) and (1)} \\
&= e_1 a_1 b_1 a_1 e_1 a_2 e_1 a_1 a_1 e_1 b_1 a_1 b_1 a_2 e_1 \quad \text{by (1)} \\
&= e_1 a_1 b_1 a_1 a_2 e_1 a_2 a_1 a_1 a_2 e_1 b_1 a_1 e_1 b_1 \quad \text{by (0) and (1)} \\
&= e_1 a_1 a_2 b_1 a_1 e_1 a_1 a_2 a_1 b_1 e_1 a_2 e_1 a_1 e_1 b_1 \quad \text{by (0) and (1)} \\
&= e_1 a_1 a_2 b_1 e_1 a_1 e_1 a_1 b_1 a_2 e_1 a_2 a_1 e_1 a_1 b_1 \quad \text{by (0) and (1)} \\
&= e_1 a_1 a_2 e_1 b_1 a_1 e_1 a_1 b_1 e_1 a_2 e_1 a_1 e_1 a_1 b_1 \quad \text{by (0) and (1)} \\
&= e_1 a_1 a_2 e_1 b_1 a_1 e_1 a_1 e_1 b_1 a_2 a_1 e_1 a_1 a_1 b_1 \quad \text{by (0) and (1)} \\
&= e_1 a_1 a_2 e_1 b_1 a_1 a_1 e_1 b_1 a_1 a_2 e_1 a_1 a_1 b_1 \quad \text{by (1)} \\
&= e_1 a_1 a_2 e_1 b_1 a_1 a_1 e_1 b_1 a_1 a_2 e_1 b_1 a_1 a_1 b_1 \quad \text{by (0)} \\
&= t_1 x t_1 x. \quad \square
\end{aligned}$$

Denote $\psi(\xi_i) = t_{i-1} t_{i-2} \dots t_1(x)$ by x_i ; we have to prove that $[x_i, x_j] \in H'_0$ ($i < j$). Relations (iii)_{c-d} show that H'_0 is stable under conjugation by t_i . So, one can conjugate by $g = \overline{t_2} \dots \overline{t_i} \overline{t_1} \dots \overline{t_{j-1}}$:

$$g t_{j-1} \dots t_1(x) = \overline{t_2} \dots \overline{t_i} \overline{t_1}(x) = x \quad \text{by (0)},$$

and

$$\begin{aligned}
g t_{i-1} \dots t_1(x) &= \overline{t_2} \dots \overline{t_i} \overline{t_1} \dots \overline{t_{i-1}} \overline{t_i} \dots \overline{t_{j-1}} t_{i-1} \dots t_1(x) \\
&= \overline{t_2} \dots \overline{t_i} \overline{t_1} \dots \overline{t_{i-1}} \overline{t_i} t_{i-1} \dots t_1(x) \quad \text{by (ii)_a and (0)} \\
&= \overline{t_2} \dots \overline{t_i} \overline{t_1} \dots \overline{t_{i-2}} \overline{t_i} \overline{t_{i-1}} \overline{t_i} t_{i-2} \dots t_1(x) \quad \text{by (ii)_a} \\
&= \overline{t_2} \dots \overline{t_i} \overline{t_i} \overline{t_1} \dots \overline{t_{i-2}} \overline{t_{i-1}} t_{i-2} \dots t_1(x) \quad \text{by (ii)_a and (0)} \\
&= \overline{t_2} \dots \overline{t_{i-1}} \overline{t_1} \dots \overline{t_{i-3}} \overline{t_{i-1}} \overline{t_{i-2}} \overline{t_{i-1}} t_{i-3} \dots t_1(x) \quad \text{by (ii)_a} \\
&\vdots \\
&= \overline{t_2} \overline{t_1} \overline{t_2} t_1(x) \\
&= \overline{t_2} t_2 \overline{t_1} \overline{t_2}(x) \quad \text{by (ii)_a} \\
&= \overline{t_1}(x) \quad \text{by (0).}
\end{aligned}$$

Thus, (ii)_d is equivalent to $[\overline{t_1}(x), x] \in H'_0$. But, by Lemma 2.4, we have

$$[\overline{t_1}(x), x] = \overline{t_1} x t_1 x \overline{t_1} \overline{x} t_1 \overline{x} = \overline{t_1}^2 x t_1^2 \overline{x}.$$

Then, relations (ii)_b and (iii)_{a-b} yield (ii)_d.

Relation (ii)_e. For $i \geq j+1$, relations (ii)_a and (0) give

$$t_i(x_j) = t_i t_{j-1} \dots t_1(x) = t_{j-1} \dots t_1 t_i(x) = x_j$$

i.e.

$$t_i, x_j] = 1 \in H'_0.$$

For $i \leq j-2$, the same relations give

$$\begin{aligned} t_i(x_j) &= t_i t_{j-1} \dots t_1(x) \\ &= t_{j-1} \dots t_i t_{i+1} t_i \dots t_1(x) \\ &= t_{j-1} \dots t_{i+1} t_i t_{i+1} \dots t_1(x) \\ &= t_{j-1} \dots t_{i+1} t_i \dots t_1 t_{i+1}(x) \\ &= t_{j-1} \dots t_{i+1} t_i \dots t_1(x) = x_j. \end{aligned}$$

Thus $[t_i, x_j] = 1 \in H'_0$.

Relation (iv). The relation (0) shows that s commutes with $(a_i)_{2 \leq i \leq g+r-1}$, $(t_i)_{2 \leq i \leq g-1}$, $d_{1,2}$, $(d_{ij})_{3 \leq i < j \leq 2g+r-1}$.

By Corollary 1.4, one has $\psi(\xi') = \overline{t_1} d_{2,4} \overline{a_1}(x) = b_2 a_2 a_2 b_2$ and so (0) implies that s commutes with $\psi(\xi')$.

Chain relations and Lemma 1.2 show that $\psi(\varphi_i) = a_1^2 (x d_{2,i})^2 \overline{a_{k_i}}$ is equal to f_i . Thus, s commutes with $\psi(\varphi_i)$ by (0).

Relation (v). $s^2 = a_1 x a_1 \in H'$.

Relation (vi). Suppose first that $m = \psi(h) \in \{a_1; d_{1,3}; d_{1,2g+1}, \dots, d_{1,2g+r-1}\}$. Then

$$m a_1 = a_1 m \text{ by (0)} \quad \text{and} \quad b_1 m b_1 = m b_1 b_1 \text{ by (1).}$$

So,

$$\begin{aligned} s m s m s &= a_1 b_1 a_1 m a_1 b_1 a_1 m a_1 b_1 a_1 = a_1 b_1 a_1 a_1 m b_1 m a_1 a_1 b_1 a_1 \\ &= a_1 x m x a_1 \in H'. \end{aligned}$$

Suppose next that

$$\begin{aligned} m \in \{ &d_{1,4} d_{1,3} \overline{a_1}; d_{1,5} d_{1,3} \overline{a_1}; d_{1,2g+1} d_{1,3} \overline{a_1}, \dots, d_{1,2g+r-1} d_{1,3} \overline{a_1}; d_{1,k} d_{1,j} \overline{a_1} \\ &\text{for } 2g+1 \leq j < k \leq 2g+r-1 \}; \end{aligned}$$

m can be seen as a product $d_{1,k} d_{1,j} \overline{a_1}$ with $j = 3$ and $k \in \{4, 5, 2g+1, \dots, 2g+r-1\}$ or $2g+1 \leq j < k \leq 2g+r-1$.

If μ_{jk} is a curve in $\Sigma \setminus \{\alpha_i\} \subset \Sigma$ which encircles D_1 , D_j and D_k , the lantern $L(\delta_{j,k}, \delta_{1,k})$ yields

$$(4) \quad a_1 a_{i_j} a_{i_k} m_{jk} = d_{1,k} d_{1,j} d_{jk}.$$

Thus, $m = a_{i_j} a_{i_k} m_{jk} \overline{d_{jk}}$ and

$$\begin{aligned} s m s m s &= (a_1 b_1 a_1)(\overline{d_{jk}} a_{i_j} a_{i_k} m_{jk})(a_1 b_1 a_1)(\overline{d_{jk}} a_{i_j} a_{i_k} m_{jk})(a_1 b_1 a_1) \\ &= \overline{d_{jk}}^2 a_{i_j}^2 a_{i_k}^2 a_1 b_1 a_1 a_1 m_{jk} b_1 m_{jk} a_1 a_1 b_1 a_1 \text{ by (0)} \\ &= \overline{d_{jk}}^2 a_{i_j}^2 a_{i_k}^2 a_1 x m_{jk} x a_1 \text{ by (1)}. \end{aligned}$$

It is already known that a_1 , a_{i_j} , a_{i_k} , x and d_{jk} are elements of H' ; (4) shows that $m_{jk} \in H'$. Therefore, $s m s m s \in H'$.

Relation (vii). If $h = \tau_{\alpha_1} \tau_{\delta_{2,4}}^{-1} \tau_{\varepsilon_1} \tau_{\alpha_1} \tau_{\alpha_2} \tau_{\varepsilon_1}$, then $h^{-1}(\alpha_1) = \alpha_2$ and $h^{-1}(\beta_1) = \beta_2$. Thus, by Corollary 1.4, one has $\overline{m}(a_1) = a_2$ and $\overline{m}(b_1) = b_2$, where $m = \psi(h) = a_1 d_{2,4} t_1$, so $\overline{m}(s) = a_2 b_2 a_2$. Then, s commutes with $\overline{m}(s)$ by (0).

Relation (viii). Set $m_i = \psi(h_i)$ and $t = \psi(\theta) = \psi(\varphi_3)^{-1} x a_1^2 a_2 (t_1)$. We have seen that $\psi(\varphi_3) = f_3$ and so, using Corollary 1.4, we can see that $t = e a_1 a_2 e$ where e is the curve shown in Figure 16.

Recall that by relation (2) (see the proof of Lemma 1.2), one has $a_1 t_1 = t_1 a_2$ and $a_2 t_1 = t_1 a_1$. Then one can compute:

$$\begin{aligned}
& s m_5 s m_4 s m_3 s m_2 s m_1 s \\
&= (a_1 b_1 a_1)(\overline{a_1} d_{2,3})(a_1 b_1 a_1)(d_{2,3} \overline{a_1} \bar{t})(a_1 b_1 a_1)(\overline{a_1} d_{2,3}) \\
&\quad \times (a_1 b_1 a_1)(\overline{d_{2,3}} t)(a_1 b_1 a_1)(\overline{a_1} d_{2,3})(a_1 b_1 a_1) \\
&= a_1 b_1 a_1 d_{2,3} b_1 d_{2,3} \bar{t} a_1 b_1 d_{2,3} a_1 b_1 a_1 \overline{d_{2,3}} t a_1 b_1 d_{2,3} a_1 b_1 a_1 \text{ by (0)} \\
&= (a_1 b_1 d_{2,3})^2 \bar{t} b_1 a_1 b_1 d_{2,3} b_1 \overline{d_{2,3}} a_1 t a_1 b_1 d_{2,3} a_1 b_1 a_1 \text{ by (0) and (1)} \\
&= (a_1 b_1 d_{2,3})^2 \bar{t} b_1 a_1 d_{2,3} b_1 d_{2,3} \overline{d_{2,3}} a_2 t a_2 b_1 d_{2,3} a_1 b_1 a_1 \text{ by (1) and (2)} \\
&= (a_1 b_1 d_{2,3})^2 \bar{t} a_2 b_1 d_{2,3} a_1 b_1 e a_1 a_2 e b_1 d_{2,3} a_1 b_1 a_1 a_2 \text{ by (0)} \\
&= (a_1 b_1 d_{2,3})^2 a_1 \bar{t} b_1 d_{2,3} a_1 e a_1 b_1 a_1 a_2 e d_{2,3} a_1 b_1 a_1 a_2 \text{ by (0) and (2)} \\
&= (a_1 b_1 d_{2,3})^2 a_1 \bar{t} b_1 d_{2,3} a_1 e a_1 b_1 a_1 a_2 e d_{2,3} a_1 b_1 a_1 a_2 \text{ by (1)} \\
&= (a_1 b_1 d_{2,3})^2 a_1 \bar{t} b_1 d_{2,3} e a_1 e b_1 a_1 a_2 e d_{2,3} a_1 b_1 a_1 a_2 \text{ by (1)} \\
&= (a_1 b_1 d_{2,3})^2 a_1 \bar{e} \overline{a_1} \overline{a_2} \bar{e} b_1 d_{2,3} e a_1 b_1 t d_{2,3} a_1 b_1 a_1 a_2 \text{ by (0)} \\
&= (a_1 b_1 d_{2,3})^2 a_1 \bar{e} \overline{a_1} \overline{a_2} b_1 d_{2,3} a_1 b_1 t d_{2,3} a_1 b_1 a_1 a_2 \text{ by (0)} \\
&= (a_1 b_1 d_{2,3})^2 a_1 \bar{e} b_1 a_1 \overline{b_1} d_{2,3} b_1 t \overline{a_1} d_{2,3} a_1 b_1 a_1 a_2 \text{ by (1) and (2)} \\
&= (a_1 b_1 d_{2,3})^2 a_1 b_1 \bar{e} a_1 d_{2,3} b_1 \overline{d_{2,3}} t d_{2,3} b_1 a_1 a_2 \text{ by (0) and (1)} \\
&= (a_1 b_1 d_{2,3})^2 a_1 b_1 d_{2,3} \bar{e} a_1 b_1 t b_1 a_1 a_2 \text{ by (0)} \\
&= (a_1 b_1 d_{2,3})^3 \bar{e} a_1 b_1 e a_1 a_2 e b_1 a_1 a_2 \\
&= (a_1 b_1 d_{2,3})^3 \bar{e} a_1 e b_1 a_1 b_1 a_2 e a_1 a_2 \text{ by (0)} \\
&= (a_1 b_1 d_{2,3})^3 a_1 e \overline{a_1} a_1 b_1 a_1 a_2 e a_1 a_2 \text{ by (1)} \\
&= (a_1 b_1 d_{2,3})^3 a_1 b_1 e a_1 a_2 e a_1 a_2 \text{ by (0)} \\
&= (a_1 b_1 d_{2,3})^4 \overline{d_{2,3}} t a_1 a_2 \\
&= a_1^2 (x d_{2,3})^2 \overline{d_{2,3}} t a_1 a_2 \text{ by Lemma 1.2.}
\end{aligned}$$

It is known that a_i , x and $d_{2,3}$ are elements of H' . Since $\psi(\varphi_3) = a_1^2 (x d_{2,3})^2 \overline{a_{k_i}} \in H'$, one has $t = \psi(\theta) = \psi(\varphi_3) x a_1^2 a_2 (t_1) \in H'$. This concludes the proof of (viii).

This ends the proof that $\psi : \mathcal{M} \rightarrow G$ is a homomorphism. One has $\varphi\psi = Id_{\mathcal{M}}$. So, to prove Theorem A, it is enough to show that ψ is onto. Let α be a s.c.c. in Σ . We have to show that a is in K , the subgroup of G generated by $\{a_i\text{'s}, d_{ij}\text{'s}, x, t_i\text{'s}, s\}$ (one has $\psi(\mathcal{M}) = K$).

First case: α does not separate Σ . There exists a diffeomorphism h of Σ such that $h(\alpha_1) = \alpha$. If we decompose h into a product of elements of $\{\alpha_i\text{'s}, \delta_{ij}\text{'s}, \xi, \theta_i\text{'s}, \sigma\}$ (these elements generate \mathcal{M} by Corollary 2.2) and set $m = \psi(h)$, then Corollary 1.4 implies $a = m(a_1) \in K$.

Second case: α separates in Σ . Denote by g_α the genus of the component of $\Sigma \setminus \alpha$ which does not contain δ , and by r_α its number of boundary components. Up to diffeomorphism, α is the curve shown in Figure 19.

If, in $\Sigma \setminus (\bigcup \alpha_i)$, α encircles only one hole, then $\alpha = \alpha_{g+1}$ and $a \in K$. If α encircles two holes, then α is one of the curves $\delta_{2g-1,2g}$ or $\delta_{2g+1,2g+2}$ and $a \in$

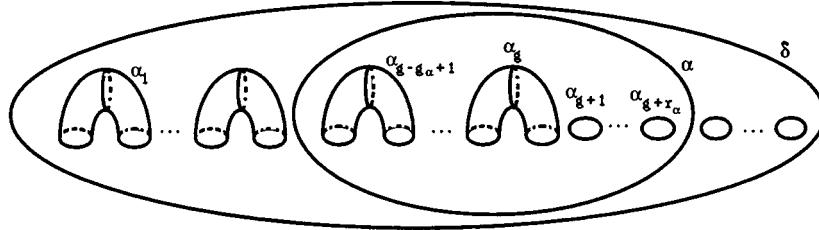


FIGURE 19

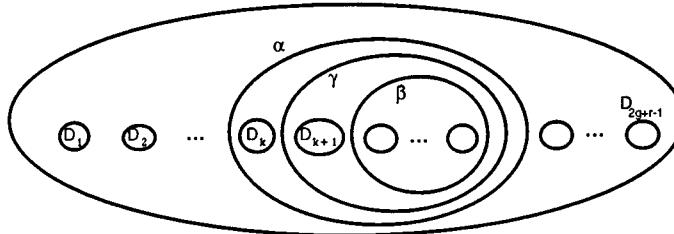


FIGURE 20

K . Otherwise, one can find three curves β, γ, λ verifying the lantern relation $a a_{i_k} a_{i_{k+1}} b = d_{k,k+1} c l$ in G and such that if one of them separates, it encircles fewer holes than α does (see Figure 20). Then we conclude, using the first case and an inductive argument.

We have proved the first part of Theorem A when $r \geq 1$. If $r = 0$, it is clear that the presentation given by Wajnryb in [W] together with Theorem 1.3 implies Theorem A. The last statement of the theorem was proven in Theorem 1.6.

2.2. Proof of Theorem B. Harer proved in [H] that, when $g \geq 3$, $\mathcal{M}_{g,r}$ is generated by twists along non-separating curves. The following two lemmas tell us what happens in genus one and two.

Lemma 2.5. *For any $r \geq 0$, the twists on non-separating curves generate $\mathcal{M}_{2,r}$.*

Proof. We only need to prove that a twist on a separating curve can be decomposed into a product of twists on non-separating curves. So, let α be a separating curve in $\Sigma_{2,r}$.

First case. One of the two components of $\Sigma \setminus \alpha$ has genus zero. Then the star $(\alpha_1, \alpha_2, \alpha_3, \beta)$ (see Figure 21) gives $(\tau_{\alpha_1} \tau_{\alpha_2} \tau_{\alpha_3} \tau_\beta)^3 = \tau_\alpha \tau_\delta \tau_{\delta_2}$.

Second case. The two components of $\Sigma \setminus \alpha$ have genus one. The chain $(\alpha_1, \beta, \alpha_2)$ (see Figure 22) gives $(\tau_{\alpha_1} \tau_\beta \tau_{\alpha_2})^4 = \tau_\alpha \tau_\delta$ and we conclude using the first case. \square

Lemma 2.6. *For any $r \geq 1$, $\mathcal{M}_{1,r}$ is generated by twists on $r-1$ boundary curves together with those on non-separating curves.*

This lemma will be proved in Section 5.

We shall now prove Theorem B. From now on, Σ will denote a surface of genus $g \geq 2$ with r boundary components ($r \geq 0$). Let G' be a group with generators

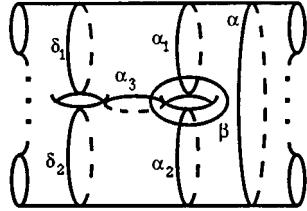


FIGURE 21

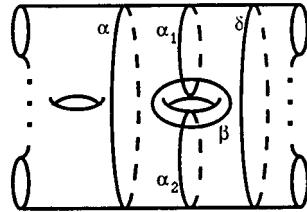


FIGURE 22

$\{a \equiv a_\alpha / \alpha \text{ non-separating s.c.c. in } \Sigma\}$ and relations (I)', (II)' and (III)'. Since G' is a quotient of S , the relations which have been shown in Section 1 will be true in G' . In particular, all lanterns, stars and chains are equal to one if the curves involved do not separate (Theorem 1.6).

Define $\varphi: G' \rightarrow \mathcal{M}$ by $\varphi(a) = \alpha$. φ is clearly a homomorphism. Harer's result and Lemma 2.5 show that φ is onto. We shall construct an inverse morphism $\psi: \mathcal{M} \rightarrow G'$. To do this, we consider \mathcal{M} given by the presentation of Theorem A. So, let α be a s.c.c. in Σ . If α does not separate, we set $\psi(\alpha) = a$. Otherwise, there are the two following cases.

First case. One of the two components of $\Sigma \setminus \alpha$ has genus greater than or equal to two. Consider the curves shown in Figure 21 and define $\psi(\alpha)$ by $\psi(\alpha) = (a_1 a_2 a_3 b)^3 \overline{d_1} \overline{d_2}$. We have to show that $\psi(\alpha)$ is independent of the choice of the star. So, let $(\alpha'_1, \alpha'_2, \alpha'_3, \beta')$ be a second star and denote $(a_1 a_2 a_3 b)^3 \overline{d_1} \overline{d_2}$ by a and $(a'_1 a'_2 a'_3 b')^3 \overline{d'_1} \overline{d'_2}$ by a' . We want to show that $a = a'$.

If all the curves involved lie in the same component Σ_1 of $\Sigma \setminus \alpha$, there is a diffeomorphism h of Σ_1 which sends the first star onto the second one fixing α (this is a consequence of the classification of surfaces). Decompose h into a product of twists on non-separating curves in Σ_1 (it can be done since Σ_1 has genus greater than or equal to two) and extend it to Σ by the identity. Then, if m denotes the word of G' associated to h , one has $a' = m(a)$ using Corollary 1.4. But if γ is a curve in the decomposition of h , then γ and α are disjoint. So, one has

$$\gamma = \tau_\alpha(\gamma) = (\tau_{\alpha_1} \tau_{\alpha_2} \tau_{\alpha_3} \tau_\beta)^4 \tau_{\delta_1}^{-1} \tau_{\delta_2}^{-2}(\gamma)$$

and by Corollary 1.4, $a(c) = a$. This proves that m commutes with a and so $a' = a$.

We suppose now that the two stars lie in distinct components of $\Sigma \setminus \alpha$ (see Figure 23).

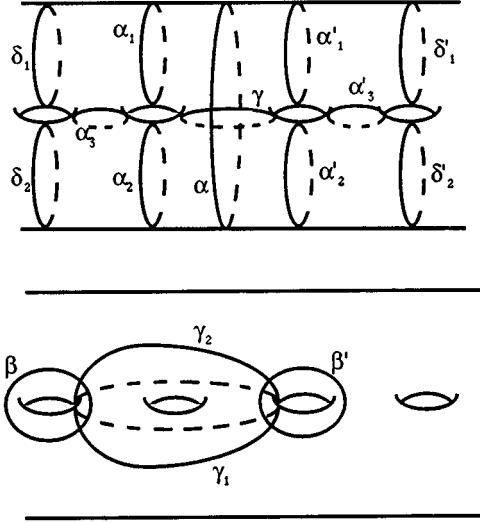


FIGURE 23

By Lemma 1.7 and stars and chains relations E_0 and C_0 , one has

$$(c b a_1 a_2 b c)(a_3 b a_1 a_2 b a_3) = a'_1 a'_2 d_1 d_2,$$

$$(c b' a'_1 a'_2 b' c)(a'_3 b' a'_1 a'_2 b' a'_3) = a_1 a_2 d'_1 d'_2.$$

Thus, we obtain, using Lemma 1.2 and (0),

$$\begin{aligned} a &= (a_1 a_2 a_3 b)^3 \overline{d_1} \overline{d_2} \\ &= [(a_1 a_2)(b a_1 a_2 b)(a_3 b a_1 a_2 b a_3)][(a'_1 a'_2)(\overline{a_3} \overline{b} \overline{a_1} \overline{a_2} \overline{b} \overline{a_3})(\overline{c} \overline{b} \overline{a_1} \overline{a_2} \overline{b} \overline{c})] \\ &= a_1 a_2 a'_1 a'_2 [b a_1 a_2 b \overline{c} \overline{b} \overline{a_1} \overline{a_2} \overline{b}] \end{aligned}$$

and, similarly,

$$a' = (a'_1 a'_2 a'_3 b')^3 \overline{d'_1} \overline{d'_2} = a'_1 a'_2 a_1 a_2 [b' a'_1 a'_2 b' \overline{c} \overline{b'} \overline{a'_1} \overline{a'_2} \overline{b'}] \overline{c}.$$

But one can check that $\tau_\beta \tau_{\alpha_1} \tau_{\alpha_2} \tau_\beta(\gamma) = \tau_{\beta'} \tau_{\alpha'_1} \tau_{\alpha'_2} \tau_{\beta'}(\gamma)$. So, if γ' denotes this curve, one has (by Corollary 1.4)

$$a = a_1 a_2 a'_1 a'_2 \overline{c'} \overline{c} = a'.$$

Lemma 2.7. *Let δ be a separating curve in Σ such that $\Sigma \setminus \delta$ has a component of genus greater than two. Let h be a diffeomorphism of Σ which is decomposed into a product of twists along non-separating curves. Then, if $\delta' = h(\delta)$ and m is the word associated to h in G' , one has $\psi(\delta') = m\psi(\delta)\overline{m}$.*

Proof. Let $E = (\alpha_1, \alpha_2, \alpha_3, \beta)$ be a star such that $\psi(\delta) = (a_1 a_2 a_3 b)^3 \overline{d_1} \overline{d_2}$ (Figure 21). Then, if $E' = (\alpha'_1, \alpha'_2, \alpha'_3, \beta')$ is the image of E by h , one has $\psi(\delta') = (a'_1 a'_2 a'_3 b')^3 \overline{d'_1} \overline{d'_2}$. Furthermore, Corollary 1.4 gives $a'_k = m(a_k)$, $b' = m(b)$ and $d'_k = m(d_k)$, which imply the required result. \square

Second case. The two components of $\Sigma \setminus \alpha$ have genus one. Consider the curves shown in Figure 22 and define $\psi(\alpha)$ by $\psi(\alpha) = (a_1 b a_2)^4 \overline{\psi(\delta)}$ (according to the first case, $\psi(\delta)$ is well defined). We have to show that $\psi(\alpha)$ is independent of the choice

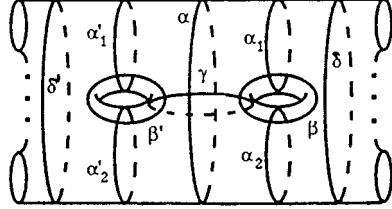


FIGURE 24

of the chain. So, let $(\alpha'_1, \beta', \alpha'_2)$ be a second chain and denote $(a_1 b a_2)^4 \overline{\psi(\delta)}$ by a and $(a'_1 b' a'_2)^4 \overline{\psi(\delta')}$ by a' .

If the two chains lie in the same component Σ_1 of $\Sigma \setminus \alpha$, there is a diffeomorphism h of Σ_1 which sends one of the two chains to the other one fixing α . By Lemma 2.6, one can decompose h into a product of twists along non-separating curves or boundary curves. But since a twist on a boundary curve is central, one can suppose that h is a product of twists along only non-separating curves. Extend h to Σ by identity. Then, if m is the word in G' corresponding to the decomposition of h , one has by Corollary 1.4 and Lemma 2.7 $a' = m(a)$. At last, since all the curves involved in the decomposition of m are disjoint from α , m commutes with a by Corollary 1.4.

We now suppose that the chains lie in distinct components of $\Sigma \setminus \alpha$ (Figure 24). We set

$$\begin{aligned} X_1 &= a_1 a_2, \quad X_2 = b X_1 b, \quad X_3 = c X_2 c, \quad X'_1 = a'_1 a'_2, \\ X'_2 &= b' X'_1 b', \quad \text{and} \quad X'_3 = c X'_2 c. \end{aligned}$$

Then, one has, using Lemma 1.2, $a = X_1^2 X_2^2 \overline{\psi(\delta)}$ with $\psi(\delta) = X_1 X_2 X_3 \overline{a'_1 a'_2}$ and $a' = X'_1^2 X'_2^2 \overline{\psi(\delta')}$ with $\psi(\delta') = X'_1 X'_2 X'_3 \overline{a'_1 a'_2}$. Thus,

$$a = (X_1 X_2)^2 \overline{X_1 X_2 X_3} X'_1 = X_1 X_2 \overline{c} \overline{X_2} \overline{c} X'_1 \quad \text{and} \quad a' = X'_1 X'_2 \overline{c} \overline{X'_2} \overline{c} X_1.$$

But since $\tau_\beta \tau_{\alpha_1} \tau_{\alpha_2} \tau_\beta(\gamma) = \tau_{\beta'} \tau_{\alpha'_1} \tau_{\alpha'_2} \tau_{\beta'}(\gamma)$, by Corollary 1.4 we have $X_2(c) = X'_2(c)$. Thus, we obtain $a = X_1 X'_2 \overline{c} \overline{X'_2} \overline{c} X'_1 = a'$.

We have proved that $\psi: \mathcal{M} \rightarrow G'$ is well defined. It remains to prove that ψ is a homomorphism. To do this, it is enough to show that the relations (I) and (IV) are mapped by ψ into true relations in G' .

The braids. Let λ and μ be two curves in Σ , $\nu = \tau_\mu(\lambda)$. We have to prove that $\psi(\nu) = \psi(\mu)\psi(\lambda)\psi(\mu)^{-1}$. This is shown by Corollary 1.4 if λ does not separate. Otherwise, if a component of $\Sigma \setminus \lambda$ has genus greater than two, the relation is still true by Lemma 2.7.

Finally, let us suppose that λ separates and that the two components of $\Sigma \setminus \lambda$ have genus one. Let $C(\alpha_1, \beta, \alpha_2)$ be a chain such that $\psi(\lambda) = (a_1 b a_2)^4 \overline{\psi(\delta)}$ (see Figure 22) and denote by $\alpha'_1, \beta', \alpha'_2$ and δ' the respective image of $\alpha_1, \beta, \alpha_2$ and δ under τ_μ . Then, one has $\psi(\nu) = (a'_1 b' a'_2)^4 \overline{\psi(\delta')}$.

But there are some non-separating curves $\varepsilon_1, \dots, \varepsilon_p$ such that $\psi(\mu)$ is equal to the product $e_1 \dots e_p$. Thus, by Corollary 1.4, one has $a'_k = \psi(\mu)(a_k)$, $b' = \psi(\mu)(b)$ and, by Lemma 2.7, $\psi(\delta') = \psi(\mu)(\psi(\delta))$. So, we obtain $\psi(\nu) = \psi(\mu)\psi(\lambda)\psi(\mu)^{-1}$.

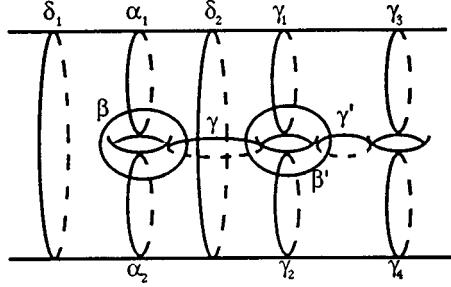


FIGURE 25

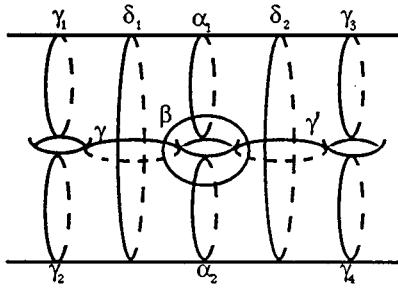


FIGURE 26

Remark. Since the braids relations are mapped by ψ into true relations, one has in G' , for any disjoint s.c.c. α and β in Σ , the relation

$$(B) \quad \psi(\alpha)\psi(\beta) = \psi(\beta)\psi(\alpha).$$

Lemma 2.8. *The chain relations are satisfied in G' .*

Proof. Let $(\alpha_1, \beta, \alpha_2)$ be a chain C , and δ_1 and δ_2 its boundary components. If δ_1 and δ_2 do not separate, the relation is satisfied by (II)' and Theorem 1.6. So, we suppose that δ_1 and δ_2 separate in Σ . If g is equal to two, the relation is satisfied by definition of $\psi(\delta_k)$. And when $g \geq 3$, two different cases occur, which are described by Figures 25 and 26 (we have to consider the second case only if $g=3$).

First case (Figure 25). The definition of $\psi(\delta_k)$ gives $\psi(\delta_1) = X_1 X_2 X_3 \overline{Y_1}$ and $\psi(\delta_2) = Y_1 Y_2 Y'_3 \overline{c_3} \overline{c_4}$, where

$$\begin{aligned} X_1 &= a_1 a_2, & X_2 &= b X_1 b, & X_3 &= c X_2 c, \\ Y_1 &= c_1 c_2, & Y_2 &= b' Y_1 b', & Y'_3 &= c' Y_2 c'. \end{aligned}$$

Furthermore, if $Y_3 = c Y_2 c$, one has by Lemma 1.7 $Y_3 Y'_3 = a_1 a_2 c_3 c_4$. Thus, we obtain

$$\begin{aligned} \psi(\delta_1)\psi(\delta_2) &= X_1 X_2 X_3 Y_2 Y'_3 \overline{c_3} \overline{c_4} = (X_1 X_2 X_3 Y_2)(\overline{Y_3} X_1) \\ &= X_1^2 X_2^2 \overline{X_2} c X_2 c Y_2 \overline{c} \overline{Y_2} \overline{c}. \end{aligned}$$

At last, since $\overline{X_2}(c) = c Y_2(c)$ (Corollary 1.4), we have $X_1^2 X_2^2 = \psi(\delta_1)\psi(\delta_2)$.

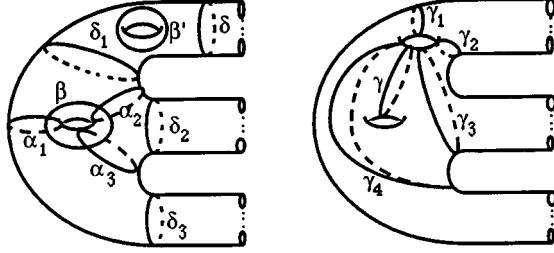


FIGURE 27

Second case (Figure 26). Set $X_1 = a_1 a_2$, $X_2 = b X_1 b$, $X_3 = c X_2 c$ and $X'_3 = c' X_2 c'$. Then, by definition, we have

$$\psi(\delta_1) = X_1 X_2 X'_3 \overline{c_3} \overline{c_4} \quad \text{and} \quad \psi(\delta_2) = X_1 X_2 X_3 \overline{c_1} \overline{c_2}$$

and so

$$\psi(\delta_1) \psi(\delta_2) = X_1^2 X_2^2 X'_3 X_3 \overline{c_1} \overline{c_2} \overline{c_3} \overline{c_4}.$$

Then, using the equality $X'_3 X_3 = c_1 c_2 c_3 c_4$ (given by Lemma 1.7), one obtains the required result. \square

The stars. Let $(\alpha_1, \alpha_2, \alpha_3, \beta)$ be a star E in Σ , and let δ_1, δ_2 , and δ_3 be the boundary components. If all these curves do not separate Σ , E is satisfied in G' by (II)', (III)' and Theorem 1.6. If only one curve δ_i separates, E is satisfied by definition of $\psi(\delta_i)$. So, let us suppose that δ_1, δ_2 and δ_3 separate Σ . Since $g \geq 2$, one can suppose that the component of $\Sigma \setminus \delta_1$ which does not contain δ_2 is of genus greater than or equal to one. Consider the curves shown on Figure 27, and set

$$X_1 = a_3 c, \quad X_2 = b X_1 b, \quad X_3 = a_1 X_2 a_1, \quad X'_3 = a_2 X_2 a_2,$$

$$Y_1 = c_1 c_2, \quad Y_2 = b' Y_1 b', \quad Y_3 = c Y_2 c.$$

By definition of $\psi(\delta_2)$, $\psi(\delta_3)$ and $\psi(\delta)$, one has

$$X_1 X_2 X_3 = c_1 c_3 \psi(\delta_3), \quad X_1 X_2 X'_3 = c_2 c_4 \psi(\delta_2), \quad Y_1 Y_2 Y_3 = a_1 a_2 \psi(\delta)$$

and Lemma 2.8 gives $X_1^2 X_2^2 = c_3 c_4$ and $Y_1^2 Y_2^2 = \psi(\delta_1) \psi(\delta)$. From these relations we obtain

$$\begin{aligned} \psi(\delta_2) \psi(\delta_3) &= X_1 X_2 X'_3 \overline{c_2} \overline{c_4} X_1 X_2 X_3 \overline{c_1} \overline{c_3} \\ &= X_1^2 X_2^2 \overline{c_1} \overline{c_2} \overline{c_3} \overline{c_4} X_3 X'_3 \quad \text{by (B) and Lemma 1.2} \\ &= \overline{c_1} \overline{c_2} (a_1 b a_3 c b a_1) (a_2 b a_3 c b a_2) \\ &= \overline{c_1} \overline{c_2} (c b a_1 a_2 b c) (a_3 b a_1 a_2 b a_3) \quad \text{by (B),} \\ \psi(\delta_1) \psi(\delta_2) \psi(\delta_3) &= (Y_1^2 Y_2^2 \psi(\delta)^{-1}) \overline{c_1} \overline{c_2} (c b a_1 a_2 b c) (a_3 b a_1 a_2 b a_3) \\ &= Y_1^2 Y_2^2 (\overline{Y_1} \overline{Y_2} \overline{Y_3} a_1 a_2) \overline{Y_1} (c b a_1 a_2 b c) (a_3 b a_1 a_2 b a_3) \\ &= a_1 a_2 Y_2 \overline{c} \overline{Y_2} \overline{c} (c b a_1 a_2 b c) (a_3 b a_1 a_2 b a_3) \quad \text{by (0).} \end{aligned}$$

Now, set $Z_1 = a_1 a_2$, $Z_2 = b Z_1 b$ and $Z_3 = a_3 Z_2 a_3$. Then, using Corollary 1.4 one can check that $Y_2(c) = Z_2(c)$. Thus, we obtain

$$\psi(\delta_1) \psi(\delta_2) \psi(\delta_3) = Z_1 [Z_2 \overline{c} \overline{Z_2} \overline{c}] [c Z_2 c] Z_3 = Z_1 Z_2 Z_3.$$

Lemma 1.2 shows that this last relation is the star E .

This proves that ψ is a homomorphism. One has clearly $\psi \circ \varphi = \text{Id}_{G'}$ so φ is a monomorphism from G' to \mathcal{M} . Since φ is onto, this concludes the proof of Theorem B.

3. MAPPING CLASS GROUP WITH p_1 -STRUCTURE

A Topological Quantum Field Theory is a functor from a cobordism category to a category of modules with suitable properties (see for instance [A1]). When we consider the cobordism category of 2-dimensional surfaces and 3-dimensional cobordisms, this functor produces projective representations of mapping class groups. One can linearize this representation by considering a certain central extension (G. Masbaum and J. Roberts describe some of these extensions in [M-R]).

In [BHMV2], the authors construct a TQFT V_p from the Kauffman bracket. In order to solve the framing anomaly, all the manifolds are equipped with p_1 -structure. The central extension obtained this way is the mapping class group with p_1 -structure and the goal of this section is to give a presentation of this group.

3.1. Definitions.

Definition 3.1. Let X be the homotopy fiber of the map $p_1 : BO \rightarrow K(\mathbb{Z}, 4)$ corresponding to the first Pontryagin class of the universal stable bundle γ over BO . A p_1 -structure on a manifold M is a lifting of a classifying map of the stable tangent bundle of M to X .

Definition 3.2. A *homotopy* between two p_1 -structures Φ_0 and Φ_1 on a manifold M is a p_1 -structure Φ on $M \times I$ such that $\Phi|_{M \times 0}$ and $\Phi|_{M \times 1}$ coincide respectively with Φ_0 and Φ_1 .

If N is a submanifold of M , one can define a homotopy rel N in the obvious way.

Remarks. (i) A p_1 -structure up to homotopy is the analogue of a spin structure, where the second Stiefel-Whitney class w_2 is replaced by the first Pontryagin class p_1 .

(ii) We consider actual p_1 -structure rather than the homotopy class of such a structure. This allows manifolds with p_1 -structure to be glued along parts of their boundary.

For the relevant classical algebraic topology or structure on manifolds, see for instance [S] or [Sw]. The following fact is easily proven using obstruction theory.

Proposition 3.3. *If M is a manifold and N a submanifold of M , then, the set of homotopy class rel N of p_1 -structure on M is affinely isomorphic to $H^3(M, N; \mathbb{Z})$.*

Thus, manifolds of dimension less than or equal to two admit a unique p_1 -structure up to homotopy, and the set of homotopy class rel boundary of p_1 -structure on an oriented, compact, connected 3-manifold is affinely isomorphic to \mathbb{Z} .

Now, let Σ be an oriented, connected surface and φ a given p_1 -structure on Σ . We shall define $\tilde{\mathcal{M}}_\Sigma$, the mapping class group with p_1 -structure. For $f \in \mathcal{M}_\Sigma$, we provide $\partial(\Sigma \times I)$ with the following p_1 -structure: $\varphi \circ f$ on $\Sigma \times 0$ and φ on $(\Sigma \times 1 \cup \partial\Sigma \times I)$. This can be extended to $\Sigma \times I$, and P_f , the set of homotopy class of such extensions, is affinely isomorphic to \mathbb{Z} .

Definition 3.4. The *mapping class group with p_1 -structure* is the set of all pairs (f, F) where $f \in \mathcal{M}_\Sigma$ and $F \in P_f$, together with the obvious composition.

Remark. Atiyah ([A2]) has previously defined this group in a different way.

The forgetful map μ is an epimorphism from $\tilde{\mathcal{M}}_\Sigma$ to \mathcal{M}_Σ , with kernel \mathbb{Z} . Let us denote by u a generator of $\text{Ker } \mu$. Since to modify a given p_1 -structure on $\Sigma \times I$, it is enough to change it in a small 3-ball, u is in the center of $\tilde{\mathcal{M}}_\Sigma$. So, we obtain the following central extension of \mathcal{M}_Σ :

$$(**) \quad 0 \longrightarrow \mathbb{Z} \longrightarrow \tilde{\mathcal{M}} \xrightarrow{\mu} \mathcal{M} \longrightarrow 1.$$

Note that this extension is natural in the following sense: if Σ' is a subsurface of Σ , then we have the following commutative diagram, where i_* and i_\sharp are the obvious morphisms induced by the inclusion i of Σ' in Σ :

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \tilde{\mathcal{M}}_{\Sigma'} & \xrightarrow{\mu'} & \mathcal{M}_{\Sigma'} \longrightarrow 1 \\ & & \parallel & & \downarrow i_* & & \downarrow i_\sharp \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \tilde{\mathcal{M}}_\Sigma & \xrightarrow{\mu} & \mathcal{M}_\Sigma \longrightarrow 1. \end{array}$$

3.2. Presentation of $\tilde{\mathcal{M}}$. Let τ_α be a Dehn twist. We shall construct a canonical lifting $\tilde{\tau}_\alpha = (\tau_\alpha, A)$. To do this, consider a neighborhood $V \approx S^1 \times I_1$ ($I_1 = [0, 1]$) of α in Σ and define A on $(\Sigma \setminus V) \times I$ by $A(\sigma, t) = \varphi(\sigma)$. We have to extend to $V \times I$ the p_1 -structure on $\partial(V \times I)$ which is equal to $\varphi \circ \tau_\alpha$ on $V \times 0$ and to φ on $(V \times 1 \cup (\partial V) \times I)$.

Now, we paste a copy of $D^2 \times I_1$ to V identifying $S^1 \times I_1$ with V .

- * τ_α is a diffeomorphism of $S^1 \times I_1$ which can be extended to $D^2 \times I_1$.
- * Since $H^p(D^2 \times I_1, S^1 \times I_1; \mathbb{Z}) = 0$ for $p = 3$ or 4 , there is one and only one way to extend $\varphi|_V$ to $D^2 \times I_1$ (up to homotopy).

So, we can define a p_1 -structure Φ on $D^2 \times \partial(I_1 \times I)$ which extends the given one on $\partial(V \times I) = S^1 \times \partial(I_1 \times I)$ as follow:

$$\begin{aligned} \Phi(x, \tau, t) &= \varphi(x, \tau) && \text{for } (x, \tau, t) \in D^2 \times \partial I_1 \times I, \\ \Phi(x, \tau, 0) &= \varphi \circ \tau_\alpha(x, \tau) && \text{for } (x, \tau) \in D^2 \times I_1, \\ \Phi(x, \tau, 1) &= \varphi(x, \tau) && \text{for } (x, \tau) \in D^2 \times I_1. \end{aligned}$$

Since $H^p(D^2 \times I_1 \times I, D^2 \times \partial(I_1 \times I); \mathbb{Z}) = 0$ when p is equal to four or three, there is one and only one way to extend Φ to $D^2 \times I_1 \times I$ up to homotopy. We will denote this extension by π_α , and we define A on $V \times I$ by $A(\sigma, t) = \pi_\alpha(\sigma, t)$. This gives a well defined p_1 -structure A on $\Sigma \times I$, and we will denote by $\tilde{\tau}_\alpha$ the element (τ_α, A) of $\tilde{\mathcal{M}}$.

Remark. It is easy to see that when α bounds a disc in Σ , then $\tilde{\tau}_\alpha = 1_{\tilde{\mathcal{M}}}$.

Since Dehn twists generate \mathcal{M} , the set of all $\tilde{\tau}_\alpha$ together with the generator u of \mathbb{Z} generate $\tilde{\mathcal{M}}$. Because u is central in $\tilde{\mathcal{M}}$, one has the relation $u\tilde{\tau}_\alpha = \tilde{\tau}_\alpha u$ for any s.c.c. α in Σ . Theorems 3.7, 3.8 and 3.9 below tell us how braids, lanterns, stars and chains are lifted to $\tilde{\mathcal{M}}$ by the section $s : \mathcal{M} \rightarrow \tilde{\mathcal{M}}$, $s(\tau_\alpha) = \tilde{\tau}_\alpha$. Using these results, we obtain from Theorems A and B the two following presentations of $\tilde{\mathcal{M}}$.

Notation. If $m \in \mathcal{M}$ is a word $\tau_{\alpha_1} \dots \tau_{\alpha_p}$, we will denote by \tilde{m} the corresponding word $\tilde{\tau}_{\alpha_1} \dots \tilde{\tau}_{\alpha_p} \in \tilde{\mathcal{M}}$.

Theorem 3.5. *For any oriented surface Σ , $\tilde{\mathcal{M}}_\Sigma$ has the following presentation:*

generators: $\{\tilde{\tau}_\alpha / \alpha \text{ s.c.c. in } \Sigma\} \cup \{u\}$ (if α bounds a disc in Σ , then $\tilde{\tau}_\alpha = 1$) .

relations: (1) For any s.c.c. α in Σ , $\tilde{\tau}_\alpha u = u \tilde{\tau}_\alpha$.

(2) $\tilde{T} = 1$ for any braids T_0 , T_1 and T_{2_0} .

(3) $\tilde{C} = u^{12}$ for any chain C .

(4) $\tilde{L} = 1$ for any lantern L .

If $g \geq 1$, one can replace (3) and (4) by

(5) $\tilde{E} = u^{12}$ for any star E .

With the non-separating curves, we obtain

Theorem 3.6. *For any oriented surface Σ of genus greater than or equal to two, $\tilde{\mathcal{M}}_\Sigma$ has the following presentation:*

generators: $\{\tilde{\tau}_\alpha / \alpha \text{ non-separating s.c.c. in } \Sigma\} \cup \{u\}$

relations: (1)' For any non-separating s.c.c. α in Σ , $\tilde{\tau}_\alpha u = u \tilde{\tau}_\alpha$.

(2)' $\tilde{T} = 1$ for any braids T_0 and T_1 between only non-separating curves.

(3)' $\tilde{C}_0 = u^{12}$ for one chain C_0 with non-separating curves.

(4)' $\tilde{L}_0 = 1$ for one lantern L_0 with non-separating curves.

When $g \geq 3$, one can replace (3)' or (4)' by

(5)' $\tilde{E}_0 = u^{12}$ for one star E_0 (with non-separating curves).

3.3. Lifting of the braid relations. The following theorem proves that the braid relations are still true in $\tilde{\mathcal{M}}_{g,r}$, where g and r are non-negative.

Theorem 3.7. *Let α be a s.c.c. in Σ , $m = (f, F) \in \tilde{\mathcal{M}}$ and $\gamma = f(\alpha)$. Then, in $\tilde{\mathcal{M}}$ we have the relation*

$$\tilde{\tau}_\gamma = m \tilde{\tau}_\alpha \overline{m} .$$

Proof. Denote the element $m \tilde{\tau}_\alpha \overline{m}$ by (τ_γ, Φ) (see Figure 28) and consider a neighborhood V of γ in Σ . $V' = f^{-1}(V)$ is a neighborhood of α .

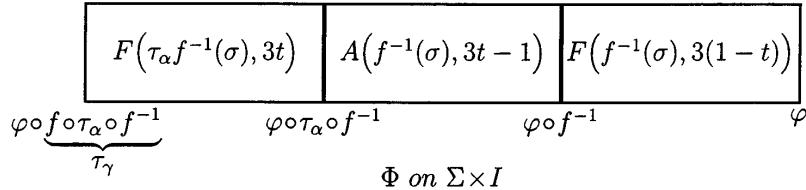


FIGURE 28

First step. We shall homotope Φ rel $\partial(\Sigma \times I)$ to a p_1 -structure Ψ on $\Sigma \times I$ such that $\Psi(\sigma, t) = \varphi(\sigma)$ for $\sigma \notin V$. To do this, we have to define a p_1 -structure H on $\Sigma \times I \times I$ such that:

$$H|_{\Sigma \times I \times 0}(\sigma, t, 0) = \Phi(\sigma, t) , \quad H|_{(\Sigma \setminus V) \times I \times 1}(\sigma, t, 1) = \varphi(\sigma) ,$$

$$H|_{\Sigma \times 0 \times I}(\sigma, 0, \theta) = \varphi \circ \tau_\gamma(\sigma) , \quad H|_{\Sigma \times 1 \times I}(\sigma, 1, \theta) = \varphi(\sigma) = H|_{\partial \Sigma \times I \times I}(\sigma, t, \theta) .$$

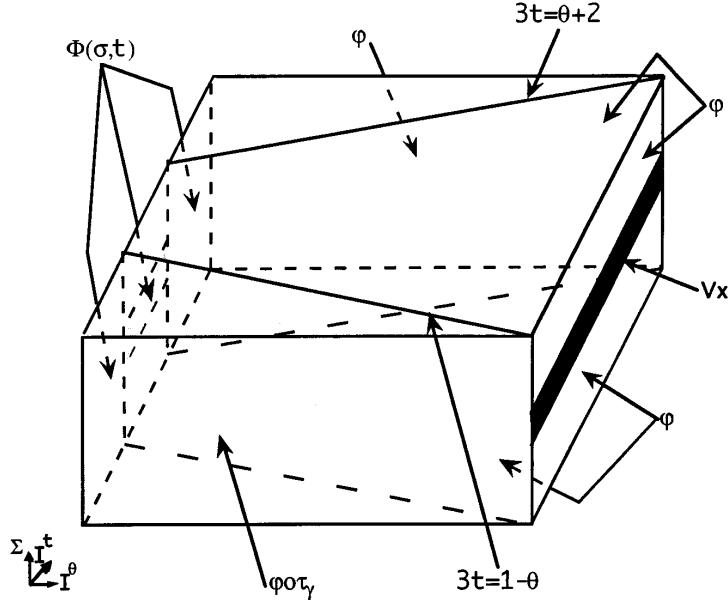


FIGURE 29

First, we define

$$H(\sigma, t, \theta) = \begin{cases} F(\tau_\alpha f^{-1}(\sigma), 3t) & \text{if } 3t \leq 1 - \theta, \\ F(f^{-1}(\sigma), 3(1-t)) & \text{if } 3t \geq 2 + \theta, \\ F(f^{-1}(\sigma), 1 - \theta) & \text{if } 1 - \theta \leq 3t \leq 2 + \theta \text{ and } \sigma \notin V. \end{cases}$$

Then, it remains us to extend H to $W = \{(\sigma, t, \theta) \in V \times I \times I \text{ such that } 1 - \theta \leq 3t \leq 2 + \theta\}$, which is diffeomorphic to $V \times I \times I$ (see Figure 29; we will identify W with $V \times I \times I$). H is given on $M = (\partial(V \times I) \times I) \cup (V \times I \times 0)$. Since $H^4(W, M) = H^3(W, M) = 0$, there is one and only one way to extend H to W up to homotopy. We define Ψ by $\Psi(\sigma, t) = H(\sigma, t, 1)$.

Second step. To show that $(\tau_\gamma, \Psi) = \tilde{\tau}_\gamma$, we only need to look at the p_1 -structure on $V \times I$. So, from now on, φ is considered as a p_1 -structure on V and V' , and f is considered as a diffeomorphism from V' to V . Consider two copies of $D^2 \times I_1$ (one is denoted $D'^2 \times I_1$) and paste them on V and V' . Extend f to a diffeomorphism from $D'^2 \times I_1$ to $D^2 \times I_1$, and φ to a p_1 -structure on $D^2 \times I_1$ and $D'^2 \times I_1$. Then, we define a p_1 -structure Ξ on $\partial(D^2 \times I_1 \times I)$ by

$$\Xi(x, \tau, t) = \begin{cases} \Psi(x, \tau, t) & \text{if } x \in S^1, \\ \varphi \circ \tau_\gamma(x, \tau) & \text{if } t = 0, \\ \varphi(x, \tau) & \text{if } t = 1 \text{ or } \tau \in \{0, 1\}. \end{cases}$$

To prove that $(\tau_\gamma, \Psi) = \tilde{\tau}_\gamma$, it is enough to show that Ξ can be extended to $D^2 \times I_1 \times I$ (see definition of $\tilde{\tau}_\gamma$ in 3.2). To do this, we shall find a p_1 -structure K on $D^2 \times I_1 \times I \times I$ such that $K|_{\partial(D^2 \times I_1 \times I) \times 1}$ coincides with Ξ . Then, $K|_{D^2 \times I_1 \times I \times 1}$ will be the extension of Ξ .

We recall that F is a p_1 -structure on $\Sigma \times I$. We shall extend it to $D'^2 \times I_1 \times I$. We consider its restriction to $V' \times I = S^1 \times I_1 \times I$, which we will also denote F .

Since $H^4(D^2 \times I_1 \times I, (S^1 \times I_1 \times I) \cup (D^2 \times I_1 \times \partial I)) = 0$, there exists a p_1 -structure \hat{F} on $D'^2 \times I_1 \times I$ such that $\hat{F}|_{S^1 \times I_1 \times I} = F$, $\hat{F}|_{D'^2 \times I_1 \times 0} = \varphi \circ f$ and $\hat{F}|_{D'^2 \times I_1 \times 1} = \varphi$.

Now, we can define K on $M' = \overline{\partial(D^2 \times I_1 \times I \times I)} \setminus D^2 \times I \times I \times 1$ by

$$K(x, \tau, t, \theta) = \begin{cases} \pi_\alpha(f^{-1}(x, \tau), t) & \text{if } \theta = 0, \\ H(x, \tau, t, \theta) & \text{if } x \in S^1 \text{ (with } W \equiv S^1 \times I_1 \times I \times I\text{)}, \\ \hat{F}(f^{-1}(x, \tau), 1 - \theta) & \text{if } t = 1 \text{ or } \tau \in \{0, 1\}, \\ \hat{F}(\tau_\alpha f^{-1}(x, \tau), 1 - \theta) & \text{if } t = 0. \end{cases}$$

One can check that K coincides with Ξ on $\partial(D^2 \times I_1 \times I) \times 1$; since when $p=3$ or 4 we have $H^p(D^2 \times I_1 \times I \times I, M') = 0$ K can be extended in a unique way up to homotopy to $D^2 \times I_1 \times I \times I$. \square

3.4. Lifting of the lantern relations.

Theorem 3.8. *Let Σ be a sphere with four holes. If $\alpha_1, \dots, \alpha_4, \beta_1, \beta_2$ and β_3 are the curves shown in Figure 1, then one has in $\tilde{\mathcal{M}}_\Sigma$:*

$$\tilde{\tau}_{\alpha_1} \tilde{\tau}_{\alpha_2} \tilde{\tau}_{\alpha_3} \tilde{\tau}_{\alpha_4} = \tilde{\tau}_{\beta_3} \tilde{\tau}_{\beta_2} \tilde{\tau}_{\beta_1}.$$

Consequence. Since the extension $(**)$ is natural, the lantern relations will be true in $\tilde{\mathcal{M}}_\Sigma$ for any surface Σ .

Proof. Since $\tau_{\alpha_1} \tau_{\alpha_2} \tau_{\alpha_3} \tau_{\alpha_4} = \tau_{\beta_3} \tau_{\beta_2} \tau_{\beta_1}$, there exists an integer n such that

$$\tilde{\tau}_{\alpha_1} \tilde{\tau}_{\alpha_2} \tilde{\tau}_{\alpha_3} \tilde{\tau}_{\alpha_4} = \tilde{\tau}_{\beta_3} \tilde{\tau}_{\beta_2} \tilde{\tau}_{\beta_1} u^n.$$

Consider an embedding of Σ in a surface Σ' such that α_4 bounds a disc in Σ' . Then, we have $\tilde{\tau}_{\alpha_4} = 1$ and β_k is isotopic to α_k , thus $\tilde{\tau}_{\beta_k} = \tilde{\tau}_{\alpha_k}$ and $\tilde{\tau}_{\alpha_1} \tilde{\tau}_{\alpha_2} \tilde{\tau}_{\alpha_3} = \tilde{\tau}_{\alpha_3} \tilde{\tau}_{\alpha_2} \tilde{\tau}_{\alpha_1} u^n$, which implies that $n=0$. \square

3.5. Lifting of the star and chain relations.

Theorem 3.9. *Let Σ be a surface of genus one with three holes. If $\alpha_1, \alpha_2, \alpha_3, \beta, \delta_1, \delta_2$ and δ_3 are the curves shown in Figure 2, then, in $\tilde{\mathcal{M}}_\Sigma$, we have the relation:*

$$(\tilde{\tau}_{\alpha_1} \tilde{\tau}_{\alpha_2} \tilde{\tau}_{\alpha_3} \tilde{\tau}_\beta)^3 = \tilde{\tau}_{\delta_1} \tilde{\tau}_{\delta_2} \tilde{\tau}_{\delta_3} u^{12}.$$

Consequence. For any surface Σ and any stars E or chain C in Σ , in $\tilde{\mathcal{M}}_\Sigma$ we have the relation $\tilde{E} = \tilde{C} = u^{12}$.

Proof. Since the star relations are verified in \mathcal{M} , there exists an integer n such that $(\tilde{\tau}_{\alpha_1} \tilde{\tau}_{\alpha_2} \tilde{\tau}_{\alpha_3} \tilde{\tau}_\beta)^3 = \tilde{\tau}_{\delta_1} \tilde{\tau}_{\delta_2} \tilde{\tau}_{\delta_3} u^n$. Consider an embedding of Σ in $S^1 \times S^1$ and denote by α the common image of α_1, α_2 and α_3 . Since each of the three curves δ_1, δ_2 and δ_3 bounds a disc in $S^1 \times S^1$, the relation becomes

$$(\star) \quad (\tilde{\tau}_\alpha \tilde{\tau}_\beta \tilde{\tau}_\alpha)^4 = u^n.$$

In order to prove that $n = 12$, we shall use the action of $\tilde{\mathcal{M}}_{S^1 \times S^1}$ on $V_p = V_p(S^1 \times S^1)$ induced by the TQFT developed in [BHMV2].

First, let us describe V_p . We suppose that $p \geq 4$ and we consider the ring k_p define by

$$k_p = \mathbb{Z}[\frac{1}{p}, A, \kappa]/(\varphi_{2p}(A), \kappa^6 - u)$$

where $\varphi_{2p}(A)$ is the $2p$ -th cyclotomic polynomial and $u = A^{-6 - \frac{p(p+1)}{2}}$. Let us denote by $K(D^2 \times S^1)$ (resp. $K(S^1 \times D^2)$) the Jones-Kauffman module of $D^2 \times S^1$ (resp. $S^1 \times D^2$) over k_p . There is a bilinear form

$$\langle , \rangle : K(D^2 \times S^1) \times K(-S^1 \times D^2) \longrightarrow k_p$$

which associates to a pair of banded links L, L' in $D^2 \times S^1$ and $S^1 \times D^2$ the Kauffman bracket of $L \cup L'$ in S^3 , using the standard decomposition

$$D^2 \times S^1 \underset{S^1 \times S^1}{\cup} S^1 \times (-D^2)$$

of S^3 . Then, V_p is the quotient of $K(D^2 \times S^1)$ by the left kernel. It is shown in [BHMV1] that V_p is a k_p -module of dimension $[(p-1)/2]$.

The Dehn twist τ_α induces an obvious endomorphism t on $K(D^2 \times S^1)$ which descends to V_p . Similarly, τ_β induces an endomorphism on $K(-S^1 \times D^2)$. Considering its adjoint, we obtain a new endomorphism t' on V_p . Then, the actions of $\tilde{\tau}_\alpha$ and $\tilde{\tau}_\beta$ on V_p coincide respectively with t and t' . By using methods and results of [BHMV1], one can check that $(tt't)^2 = -A^{-6}A^{-p(p-1)/2} \text{Id}_{V_p}$ when p is odd. So, since u acts as the multiplication by the scalar κ , $(*)$ implies $A^{-12}A^{-p(p-1)} = \kappa^n$ in k_p . Suppose now that $p \equiv -1 \pmod{4}$. Then, $\kappa^n = \kappa^{12}$ and, since $\kappa^6 = A^{-6}$, we obtain $1 = \kappa^{6(n-12)} = A^{-6(n-12)}$. Moreover, since A is a primitive $2p$ -th root of unity, one has $-6(n-12) \equiv 0 \pmod{2p}$ for all integers $p \geq 4$ such that $p \equiv -1 \pmod{4}$. This proves the required equality $n=12$. \square

4. ON CENTRAL EXTENSIONS OF \mathcal{M}_Σ

Let Σ be a surface of genus $g \geq 0$ with r boundary components ($r \geq 0$). Recall that S_Σ is a group generated by $\{a_\alpha / \alpha \text{ a non-separating s.c.c. in } \Sigma\}$ and defined by the relations T_0 and T_1 . Let \mathcal{T}_Σ be the subgroup of $\widetilde{\mathcal{M}}_\Sigma$ generated by $\{\tilde{\tau}_\alpha / \alpha \text{ s.c.c. in } \Sigma\} \cup \{u^{12}\}$. In this section, we shall prove the following theorem:

Theorem 4.1. (i) *Considering the restriction of μ to \mathcal{T}_Σ , we have the natural central extension*

$$(*) \quad 0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{T}_\Sigma \xrightarrow{\mu} \mathcal{M}_\Sigma \longrightarrow 1.$$

(ii) *If $g \geq 2$, \mathcal{T}_Σ is generated by $\{\tilde{\tau}_\alpha / \alpha \text{ non-separating s.c.c. in } \Sigma\}$.*

(iii) *One can define natural homomorphisms $\varphi : S_\Sigma \rightarrow \mathcal{T}_\Sigma$ and $\psi : S_\Sigma \rightarrow \mathbb{Z}$ by $\varphi(a_\alpha) = \tilde{\tau}_\alpha$ and $\psi(a_\alpha) = 1$.*

When $g \geq 3$, φ and ψ induce an isomorphism between S_Σ and $\mathcal{T}_\Sigma \times \mathbb{Z}$.

When $g=2$, φ is an isomorphism from S_Σ to \mathcal{T}_Σ .

(iv) *When $g \geq 3$, the extension $(*)$ is the universal central extension of \mathcal{M}_Σ .*

Clearly, this theorem proves Theorem C (the functorial properties are obvious) and implies the following two corollaries:

Corollary 4.2. *The kernel of the quotient map $S_\Sigma \rightarrow \mathcal{M}_\Sigma$ is isomorphic to \mathbb{Z} (generated by C_0) when $g=2$ and to $\mathbb{Z} \oplus \mathbb{Z}$ (generated by L_0 and C_0) when $g \geq 3$.*

Corollary 4.3. *Suppose that $g \geq 3$. Then, \mathcal{T}_Σ admits a presentation with generators $\{a_\alpha / \alpha \text{ non-separating s.c.c. in } \Sigma\}$ and the relations (I)' and (II)' (braids T_0 and T_1 and a lantern L with only non-separating curves).*

Proof of Theorem 4.1. (i) is clear since the kernel of $\mu|_{\mathcal{T}_\Sigma}$ is generated by u^{12} .

To prove (ii), first one can see, by considering a chain C without separating curves and using Theorem 3.9, that u^{12} is in the subgroup of $\widetilde{\mathcal{M}}_\Sigma$ generated by $\{\tilde{\tau}_\alpha / \alpha \text{ a non-separating s.c.c. in } \Sigma\}$. Then, we can conclude exactly as in the proof of Lemma 2.5.

Since braid relations are verified in $\widetilde{\mathcal{M}}_\Sigma$, the map φ is a morphism. Furthermore, (ii) implies that φ is onto when $g \geq 2$. Now, we shall determine $\text{Ker } \varphi$. Denote by $q : S_\Sigma \rightarrow \mathcal{M}_\Sigma$ the quotient map given by Theorem B. First, we suppose that $g \geq 3$. Then, $\text{Ker } q$ is generated by two elements C_0 and L_0 coming from chains and lanterns. So, considering the commutative diagram

$$\begin{array}{ccc} S_\Sigma & \xrightarrow{q} & \mathcal{M}_\Sigma \\ \varphi \searrow & & \swarrow \mu \\ & \mathcal{T}_\Sigma & \end{array}$$

we see that $\text{Ker } \varphi \subset \langle C_0, L_0 \rangle$. Since this group is abelian (Theorem 1.6) and $\varphi(L_0) = 1$ and $\varphi(C_0) = u^{12}$, we have $\text{Ker } \varphi = \langle L_0 \rangle$. But since ψ is the abelianized morphism and $\psi(L_0) = 1$, we have $\langle L_0 \rangle = \mathbb{Z}$ and we obtain the following exact sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow S_\Sigma \xrightarrow{\varphi} \mathcal{T}_\Sigma \longrightarrow 1.$$

Furthermore, since $\psi : S_\Sigma \rightarrow \mathbb{Z}$ is a section, this sequence splits. When $g=2$, $\text{Ker } q$ is cyclic generated by C_0 . Thus, φ is an isomorphism.

(iv) was shown by Harer in [H]. □

5. THE GENUS ONE CASE

The goal of this section is to prove Lemma 2.6 and the fact that there is no epimorphism from $S_{1,r}$ to $\mathcal{M}_{1,r}$ if $r \geq 2$. In fact, we shall prove stronger results.

Lemma 5.1. *Considering the curves shown on Figure 30, $\mathcal{M}_{1,r}$ is generated by the twists τ_{α_i} , τ_β and τ_{δ_j} for $j \geq 2$.*

Proof. We shall use an inductive argument. Capping off δ_r by a disc with center p , one can obtain the following two exact sequences (see [B] and [H]):

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f_1} \mathcal{M}_{1,r} \xrightarrow{f_2} \mathcal{N}_{1,r-1} \longrightarrow 1,$$

$$0 \longrightarrow \pi_1(\Sigma_{1,r-1}, p) \xrightarrow{f_3} \mathcal{N}_{1,r-1} \xrightarrow{f_4} \mathcal{M}_{1,r-1} \longrightarrow 1,$$

where $\mathcal{N}_{1,r-1}$ is the group of isotopy classes of orientation preserving diffeomorphisms of $\Sigma_{1,r-1}$ which are the identity on the boundary and which leave fixed the base point p . The four morphisms f_k are defined as follows:

$$\rightarrow f_1(1) = \delta_r.$$

$\rightarrow f_2$ is obtained by extending each map over the disc by the identity.

\rightarrow If α is a loop based on p and α' and α'' the two boundary components of a neighborhood of α , then $f_3(\alpha) = \tau_{\alpha'} \tau_{\alpha''}^{-1}$ (here and below, we will denote by the same letter a curve and its homotopy class).

$\rightarrow f_4$ is defined by forgetting the base point p .

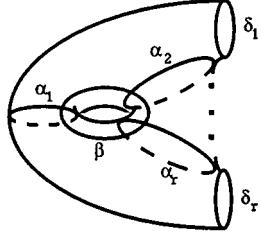


FIGURE 30

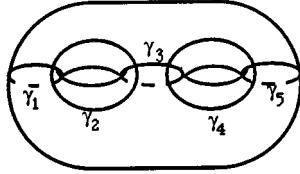


FIGURE 31

It is known that $\mathcal{M}_{1,1}$ is generated by τ_{α_1} and τ_β . Suppose inductively that $\mathcal{M}_{1,r-1}$ is generated by $\tau_{\alpha_1}, \dots, \tau_{\alpha_{r-1}}, \tau_\beta, \tau_{\delta_2}, \dots, \tau_{\delta_{r-1}}$. Then, $\mathcal{N}_{1,r-1}$ is generated by these twists and $\text{Im } f_3$. Furthermore, $\pi_1(\Sigma_{1,r-1}, p)$ is generated by the homotopy classes of the curves $\alpha_1, \dots, \alpha_{r-1}$ and β . If we denote by α'_i the curve $\tau_\beta \tau_{\alpha_1} \tau_{\alpha_r}^{-1} \tau_\beta^{-1} (\alpha_i)$, then one can check that f_3 maps α_i into $\tau_{\alpha_i} \tau_{\alpha'_i}^{-1}$, and if $\beta' = \tau_{\alpha_1} \tau_\beta (\alpha_r)$, then $f_3(\beta) = \tau_\beta \tau_{\beta'}^{-1}$. Thus, since $\tau_{\alpha'_i} = \tau_\beta \tau_{\alpha_1} \tau_{\alpha_r}^{-1} \tau_\beta^{-1} (\tau_{\alpha_i})$ and $\tau_{\beta'} = \tau_{\alpha_1} \tau_\beta (\tau_{\alpha_r})$, the group $\mathcal{N}_{1,r-1}$ is generated by $\tau_{\alpha_1}, \dots, \tau_{\alpha_{r-1}}, \tau_\beta, \tau_{\delta_2}, \dots, \tau_{\delta_{r-1}}$ and τ_{α_r} .

Finally, we obtain the required result for $\mathcal{M}_{1,r}$ using the first exact sequence. \square

Proposition 5.2. *Suppose that $r \geq 2$. Then, the twists along non-separating curves and $r-2$ boundary curves do not generate $\mathcal{M}_{1,r}$.*

Proof. Referring to Figure 30, suppose that τ_{δ_1} and τ_{δ_2} can be decomposed into a product of twists along non-separating curves or the curves $\delta_3, \dots, \delta_r$. First, let α be a non-separating curve in $\Sigma = \Sigma_{1,r}$. There is a diffeomorphism h of Σ which sends α_1 on α . Lemma 5.1 tells us that h can be decomposed into a product of twists along the curves $\alpha_1, \dots, \alpha_r, \beta, \delta_2, \dots, \delta_r$. One has $\tau_\alpha = h \tau_{\alpha_1} \bar{h}$; since Dehn twists along boundary curves are central in $\mathcal{M}_{1,r}$, one still has $\tau_\alpha = h' \tau_{\alpha_1} \bar{h}'$, where h' is obtained from h by removing the boundary twists in its decomposition. Thus, one can suppose that τ_{δ_1} and τ_{δ_2} are decomposed into a product of twists along the curves $\alpha_1, \dots, \alpha_r, \beta, \delta_3, \dots, \delta_r$.

Now, let $\varphi : \mathcal{M}_{1,r} \rightarrow \mathcal{M}_{2,0}$ be the morphism obtained by capping off each of the boundary components δ_j for $j \geq 3$ with a disc and gluing a cylinder onto δ_1 and δ_2 . Then, if $\gamma_1, \dots, \gamma_5$ are the curves shown in Figure 31, one has $\varphi(\tau_\beta) = \tau_{\gamma_2}$, $\varphi(\tau_{\alpha_i}) = \tau_{\gamma_1}$ for $i \neq 2$, $\varphi(\tau_{\alpha_2}) = \tau_{\gamma_3}$, $\varphi(\tau_{\delta_1}) = \varphi(\tau_{\delta_2}) = \tau_{\gamma_5}$ and $\varphi(\tau_{\delta_j}) = 1$ for $j \geq 3$. Thus, our hypothesis implies that τ_{γ_5} can be decomposed into a product of twists along the three curves γ_1, γ_2 and γ_3 . Then, since $\tau_{\gamma_1}, \dots, \tau_{\gamma_5}$ generate

$\mathcal{M}_{2,0}$, one only needs $\tau_{\gamma_1}, \dots, \tau_{\gamma_4}$ to generate $\mathcal{M}_{2,0}$. This is false according to Humphries ([Hu]). \square

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