SHARP UPPER BOUND FOR THE FIRST NON-ZERO NEUMANN EIGENVALUE FOR BOUNDED DOMAINS IN RANK-1 SYMMETRIC SPACES

A. R. AITHAL AND G. SANTHANAM

Abstract. In this paper, we prove that for a bounded domain $\Omega$ in a rank-1 symmetric space, the first non-zero Neumann eigenvalue $\mu_1(\Omega) \leq \mu_1(B(r_1))$ where $B(r_1)$ denotes the geodesic ball of radius $r_1$ such that $\text{vol}(\Omega) = \text{vol}(B(r_1))$ and equality holds iff $\Omega = B(r_1)$. This result generalises the works of Szego, Weinberger and Ashbaugh-Benguria for bounded domains in the spaces of constant curvature.

1. Introduction and Statement of theorems

In this paper we study the Neumann eigenvalue problem

\[ \Delta u = \mu u \quad \text{in} \quad \Omega, \quad \nu.u \equiv 0 \quad \text{on} \quad \partial \Omega, \]

where $\Omega$ is a bounded domain in a rank-1 symmetric space, $\partial \Omega$ is the boundary of $\Omega$, $\nu$ is the outward normal to $\Omega$ and $\nu.u$ denotes the directional derivative of $u$ in the direction $\nu$.

In 1954, Szego [6] proved that for all simply connected domains of given area in $\mathbb{R}^2$, the maximum of the first non-zero Neumann eigenvalue is attained for a ball. Later, Weinberger [7] extended this result for bounded domains in $\mathbb{R}^n$ for all $n \geq 2$.

Recently Ashbaugh and Benguria [1] have studied the problem (1) for a domain contained in a hemisphere of the Euclidean sphere $S^n$. For such a domain $\Omega$ they have proved that $\mu_1(\Omega) \leq \mu_1(r_1)$ where $B(r_1)$ denotes a geodesic ball of radius $r_1$ such that $\text{vol}(\Omega) = \text{vol}(B(r_1))$ and the equality holds iff $\Omega$ is a geodesic ball. They also show, using the methods of [7], that a similar result is also true for real hyperbolic space $\mathbb{H}^n$.

In this paper, we consider bounded domains in the remaining rank-1 symmetric spaces. If $\Omega$ is a domain in a rank-1 symmetric space of compact type, then we have a restriction on the size of the domain $\Omega$ viz., that $\Omega$ is contained in a geodesic ball of radius $\frac{i(M)}{4}$, where $i(M)$ denotes the injectivity radius of $(M, g)$. We prove the following theorems.

Theorem 1. Let $\Omega$ be a domain contained in a geodesic ball of radius $\frac{i(M)}{4}$ in a rank-1 symmetric space $(M^n, ds^2)$ of compact type, where $ds^2$ denotes the canonical
Riemannian metric on $M^n$ with sectional curvature $1 \leq K_M \leq 4$. Then

$$\mu_1(\Omega) \leq \mu_1(B(r_1)) := \mu_1(r_1)$$

where $B(r_1)$ is a geodesic ball of radius $r_1$ having the same volume as that of $\Omega$. Further the equality holds iff $\Omega$ is a geodesic ball.

**Theorem 2.** Let $\Omega$ be a bounded domain in a rank-1 symmetric space $(M^n, ds^2)$ of non-compact type, where $ds^2$ denotes the canonical Riemannian metric on $M^n$ with sectional curvature $-4 \leq K_M \leq -1$. Then

$$\mu_1(\Omega) \leq \mu_1(B(r_1)) = \mu(r_1)$$

where $B(r_1)$ denotes a geodesic ball of radius $r_1$ having the same volume as that of $\Omega$. Further the equality holds iff $\Omega$ is a geodesic ball.

As mentioned earlier, for the symmetric spaces of constant sectional curvature, the above results have been established in [7] and [1]. See also the concluding remarks in section 5.

The crucial step in the works of [7] and [1] is what has come to be known as the *centre of mass theorem*. In this paper we formulate this in a more geometric and conceptual way and present a simple proof. After decomposing the Laplacian $\Delta$ and conceptual way and present a simple proof. After decomposing the Laplacian $\Delta$ and identifying the correct test functions, the analytical arguments developed in [7] and [1] carry through.

We refer to [2] and [5] for the basic Riemannian geometry used in this paper.

2. THE CENTRE OF MASS THEOREM FOR DOMAINS IN COMPLETE RIEMANNIAN MANIFOLDS

Let $(M, g)$ be a complete Riemannian manifold. For a point $p \in M$, let us denote by $r(p)$ the convexity radius of $(M, g)$ at $p$. Let $\Omega$ be a domain in $(M, g)$ such that $\Omega$ is contained in $B(p, r(p))$ for some $p \in M$. Let us denote by $C\Omega$ the convex hull of $\Omega$. Let $\exp_q : T_q M \rightarrow M$ be the exponential map and let $X = (x_1, x_2, \ldots, x_n)$ be a system of normal coordinates centred at $q$. We identify $C\Omega$ with $\exp_q^{-1}(C\Omega)$ for each $q \in C\Omega$. We denote $g_q(X, X) = \|X\|_q^2$ for $X \in T_q M$. Our centre of mass theorem is the following.

**Theorem 3.** Let $\Omega$ be a bounded domain in $(M, g)$ contained in $B(q_0, r(q_0))$ for some $q_0 \in M$ and let $G$ be a continuous function on $[0, 2r(q_0)]$ which is positive on $(0, 2r(q_0))$. Then there exists a point $p \in C\Omega$ such that

$$\int_{\Omega} G(\|X\|_p)X dV = 0$$

where $X = (x_1, x_2, \ldots, x_n)$ is a normal coordinate system centred at $p$.

**Proof.** For $q \in C\Omega$, we define

$$v(q) := \int_{\Omega} G(\|X\|_q)X dV$$

where $X = (x_1, x_2, \ldots, x_n)$ is a geodesic normal coordinate system centred at $q$.

Now we shall show that the continuous vector field $v$ points inward along the boundary $\partial C\Omega$ of $C\Omega$. Then the theorem follows from the Brouwer’s fixed point theorem.

Since $C\Omega$ is convex, it is contained in the half space $H_q := \{X \in T_q M : g(X, v(q)) \leq 0\}$ for every $q \in C\Omega$, where $v(q)$ denotes the outward normal to
$\partial C \Omega$ at $q \in \partial C \Omega$. This implies that $g(v(q), v(q)) < 0$ for all $q \in \partial C \Omega$. Thus $v$ points inward along the boundary of $C \Omega$.

We can find a $\delta > 0$ such that $\exp_v(\delta v(q)) \in C \Omega$ for every $q \in C \Omega$. Then the continuous map $f_v : C \Omega \to C \Omega$ defined by

$$f_v(q) := \exp_v(\delta v(q))$$

has a fixed point $p \in C \Omega$ by the Brouwer’s fixed point theorem. Hence $v(p) = 0$. This completes the proof of the theorem. \hfill $\Box$

**Remark.** It is clear from the proof that the centre of mass theorem applies to any bounded domain $\Omega$ in $(M, g)$ such that $C \Omega$ is properly contained in $M$.

### 3. Properties of the first non-zero Neumann eigenvalue for geodesic balls in rank-1 symmetric spaces

Let $(M^n, ds^2)$ denote any one of the following rank-1 symmetric spaces: Complex projective space $CP^n$, quaternionic projective space $HP^n$, the Cayley projective plane $Ca\mathbb{P}^2$ or their non-compact duals. Let $\mathbb{K}$ denote $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $Ca$ and $k = \dim_{\mathbb{K}} \mathbb{K}$. Throughout this paper we will use these notations. Let $\mu_1(r_1)$ denote the first non-zero Neumann eigenvalue for a geodesic ball of radius $r_1$ in $(M^n, ds^2)$.

We begin with the study of $\Delta_M$ in geodesic polar coordinates centred at a point $p \in M$.

$$\Delta_M = -\frac{\partial^2}{\partial r^2} - H(r) \frac{\partial}{\partial r} + \Delta_{S(r)}$$

where $H(r)$ denotes the trace of the second fundamental form of the distance sphere $S(r) := S(p, r)$ and $\Delta_{S(r)}$ denotes the Laplacian of $S(r)$.

Now we will describe $H(r)$ and $\Delta_{S(r)}$. Let $v \in T_p M$ be a unit tangent vector and $\gamma_v(0) = p$ and $\gamma'_v(0) = v$. Let us denote by $J(v, r)$ the Riemannian density function along $\gamma_v(r)$. Since $(M^n, ds^2)$ is a rank-1 symmetric space $J(v, r)$ is independent of $v$ and we write it as $J(r)$. We know that, for $(M^n, ds^2)$ of compact type, for $0 \leq r < \frac{\pi}{2}$

$$J(r) = \sin^{kn-1} r \cos^{k-1} r$$

and for $(M, ds^2)$ of non-compact type, for all $r \geq 0$

$$J(r) = \sinh^{kn-1} r \cosh^{k-1} r.$$

The trace of the second fundamental form $H(r)$ of $S(r)$ is equal to $J'(r)J^{-1}(r)$. Hence

$$H(r) = (kn - 1) \cot r - (k - 1) \tan r$$

for $(M^n, ds^2)$ of compact type and

$$H(r) = (kn - 1) \coth r + (k - 1) \tanh r$$

for $(M^n, ds^2)$ of non-compact type.

As an illustration, we have for $(CP^n, ds^2)$, $J(r) = \sin^{2n-1} r \cos r$ and $H(r) = (2n - 1) \cot r - \tan r$ and for the quaternionic hyperbolic space $(HP^n, ds^2)$, $J(r) = \sinh^{4n-1} r \cosh^3 r$ and $H(r) = (4n - 1) \coth r + 3 \tanh r$. Note that for $Ca\mathbb{P}^2$ we have $n = 2$. Now we study the first non-zero eigenvalue $\lambda_1(S(r))$ of $\Delta_{S(r)}$. 
3.1. \((M^n, ds^2)\) of compact type. We have a natural Riemannian submersion

\[
\Pi : (S(r), ds^2 |_{S(r)}) \rightarrow (M^{n-1}, \sin^2 rds^2)
\]

with totally geodesic fibres, for the distance sphere \(S(r)\) in \((M^n, ds^2)\) with the induced metric \(ds^2 |_{S(r)}\). We always assume that \(0 < r < i(M)\). The fibre of \(\Pi\) containing a point \(\gamma_v(r) = q \in S(r)\), where \(v \in T_pM\) is a unit vector, is \(\mathbb{K}v \cap S(r)\). We can write \(\Delta_{S(r)}\) induced metric \(ds\) with totally geodesic fibres, for the distance sphere submersion \(r<\). Hence for \(r \leq 1\) of \(\Delta_{S(r)}\) where \(\Delta_{S(r)}\) denotes the Laplacian along the fibres of the canonical fibration of the unit sphere \((S^{kn-1}, ds^2)\) with totally geodesic fibres \(S^{k-1}\) and \(\Delta_V := \Delta_{(S^{kn-1}, ds^2)} - \Delta_V\). We rewrite \(\Delta_{S(r)}\) as

\[
\Delta_{S(r)} = \frac{1}{\sin^2 r} \Delta_V + \frac{1}{\sin^2 r} \Delta_{(S^{kn-1}, ds^2)}.
\]

Then we have

\[
\frac{1}{\sin^2 r} \Delta_{H} |_{\Pi^*C^\infty(M^{n-1})} = \Pi^* \Delta_{(M^{n-1}, \sin^2 rds^2)}.
\]

By equation (3), all the eigenfunctions of \(\Delta_{(M^{n-1}, \sin^2 rds^2)}\) are also eigenfunctions of \(\Delta_{S(r)}\) with the same eigenvalues. In particular the first non-zero eigenvalue \(\frac{2kn}{\sin^2 r}\) of \(\Delta_{(M^{n-1}, \sin^2 rds^2)}\) occurs as an eigenvalue of \(\Delta_{S(r)}\).

The Euclidean coordinate functions \(X_i\), for \(1 \leq i \leq kn\), are the first non-zero eigenfunctions of \(\Delta_{(S^{kn-1}, ds^2)}\) corresponding to the first eigenvalue \(kn - 1\). Since the fibres are all totally geodesic, these eigenfunctions restricted to the fibres of \(\Pi\) are also eigenfunctions with eigenvalue \(k - 1\). Hence we get

\[
\Delta_{S(r)} X_i = (\frac{kn - 1}{\sin^2 r} + \frac{k - 1}{\cos^2 r}) X_i
\]

for \(1 \leq i \leq kn\). Now

\[
(\frac{kn - 1}{\sin^2 r} + \frac{k - 1}{\cos^2 r}) < \frac{2kn}{\sin^2 r}
\]

iff

\[
r < \tan^{-1}\left(\sqrt{\frac{kn+1}{k-1}}\right).
\]

Hence for \(r < \tan^{-1}\left(\sqrt{\frac{kn+1}{k-1}}\right)\), \(X_i\), for \(1 \leq i \leq kn\) are the first eigenfunctions of \(\Delta_{S(r)}\) with eigenvalue \(\lambda_1(S(r)) = (\frac{kn - 1}{\sin^2 r} + \frac{k - 1}{\cos^2 r})\).

We remark that \(\lambda_1(S(r))\) is a strictly decreasing function of \(r\) for \(0 \leq \frac{\pi}{4}\). This remark will be used later in section 4.

3.2. \((M^n, ds^2)\) of non-compact type. We will denote by \((M^n)^*\) the compact dual of \(M^n\). As in the compact type, here also, we have a natural Riemannian submersion

\[
\Pi : (S(r), ds^2 |_{S(r)}) \rightarrow ((M^{n-1})^*, \sinh^2 rds^2)
\]
with totally geodesic fibres, for the distance sphere \( S(r) := S(p, r) \) in \((M^n, ds^2)\).

For a point \( q \in S(r) \), the fibre through the point \( q = \gamma_v(r) \), where \( v \in T_pM \) is a unit vector, is \( \mathbb{K}v \cap S(r) \). As before we have

\[
\Delta_{S(r)} = \frac{-1}{\cosh^2 r} \Delta V + \frac{1}{\sinh^2 r} \Delta (s^{kn-1, ds^2})
\]

and the euclidean coordinate functions \( X_i \)'s, for \( 1 \leq i \leq kn \) are eigen functions of \( \Delta_{S(r)} \) with eigenvalue \( \lambda_1(S(r)) = (\frac{kn-1}{\sinh^2 r} - \frac{k-1}{\cosh^2 r}) \). Now \( (\frac{kn-1}{\sinh^2 r} - \frac{k-1}{\cosh^2 r}) \) will be the first non-zero eigenvalue of \( \Delta_{S(r)} \) so long as

\[
\frac{kn-1}{\sinh^2 r} - \frac{k-1}{\cosh^2 r} < \frac{2kn}{\sinh^2 r}
\]

and this inequality holds for all \( r > 0 \). Hence \( \lambda_1(S(r)) = (\frac{kn-1}{\sinh^2 r} - \frac{k-1}{\cosh^2 r}) \) for all \( r > 0 \). Again we remark that \( \lambda_1(S(r)) \) is a strictly decreasing function of \( r \) for \( r > 0 \). See also [3] for further study of Laplacians and Riemannian submersions with totally geodesic fibres.

### 3.3. Now we shall study the first non-zero Neumann eigenvalue \( \mu_1(r_1) \). The first non-zero eigenvalue of problem (1) is, by the separation of variables technique, either the second eigenvalue \( \tau_2 \) of

\[
-\frac{1}{J(r)}Q \frac{\partial}{\partial r}(J(r)Q \frac{\partial f}{\partial r}) = \tau f
\]

where \( f \) is a function defined on \([0, r_1]\) satisfying the boundary conditions \( f(0) \) finite and \( f'(0) = 0 \) or the first eigenvalue \( \mu_1 \) of

\[
-\frac{1}{J(r)}Q \frac{\partial}{\partial r}(J(r)Q \frac{\partial g}{\partial r}) + \lambda_1(S(r))g = \mu g.
\]

where \( g \) is a function defined on \([0, r_1]\) with boundary conditions \( g(0) \) finite and \( g'(0) = 0 \). We note that \( g(0) = 0 \) and also that the first eigenvalue of equation (6) is zero. Since \( g \) is a first eigenfunction of equation (7) and also that \( g(0) = 0 \), \( g \) does not change sign in \((0, r_1)\). We assume that \( g \) is positive in \((0, r_1)\).

Let \( f \) and \( g \) be the eigenfunctions of equation (6) and equation (7) with eigenvalues \( \tau_2 \) and \( \mu_1 \) respectively. Let \( h \) be a non-trivial solution of

\[
-\frac{1}{J(r)}Q \frac{\partial}{\partial r}(J(r)Q \frac{\partial h}{\partial r}) = \mu_1 h.
\]

on \([0, r_1]\). By differentiating the equation (8), we see that \( h' \) satisfies equation (7) with the same eigenvalue \( \mu_1 \). Hence \( h' \) and \( g \) are proportional. We can assume that \( h' = g \). Since \( f \) and \( h \) satisfy the same equation with eigenvalues \( \tau_2 \) and \( \mu_1 \) respectively, we have

\[
Q \frac{\partial}{\partial r}(J(r)(h'f - f'h)) = (\tau_2 - \mu_1)fhJ(r).
\]

Since \( f \) is an eigenfunction corresponding to the second eigenvalue it must change sign in \((0, r_1)\), say at \( a \in (0, r_1) \). We may assume that \( f \) is positive in \((0, a)\) and \( f < 0 \) in \((a, r_1)\). Also we have \( f'(a) < 0 \). Now integrating the equation (9), we get

\[
(\tau_2 - \mu_1) \int_0^afhJ(r)dr = J(r)(h'f - f'h) \big|_0^a = -J(a)f'(a)h(a)
\]
Since $g$ is positive in $(0, r_1)$ and $\mu_1 h(r_1) = g'(r_1) - H(r_1)g(r_1) < 0$, we get $h(r_1) < 0$. Thus, $h' = g$ and $h(r_1) < 0$ together imply that $h \le 0$ in $(0, r_1)$. Now from the equation (10), it follows that $\mu_1 < \tau_2$. Thus we have proved that $\mu_1 = \mu_1(r_1)$.

Now we study the properties of the function $g$ and the function $\mu_1(r_1)$. Let us recall that $g$ satisfies

$$Q \frac{\partial}{\partial r} (J(r)Q \frac{\partial}{\partial r} g) = (\lambda_1(S(r)) - \mu_1(r_1))gJ(r)$$

with boundary conditions $g(0) = 0$ and $g'(r_1) = 0$. Define $\Psi(r) := J(r)g'(r)$. Then $\Psi(0) = 0$ and $\Psi'(r_1) = 0$ and $\Psi'(r) > 0$ near 0. This implies that $\Psi$ increases from zero in the beginning and then increases to zero. In particular $(\lambda_1(S(r)) - \mu_1(r_1))$ must change sign at some point $a \in (0, r_1)$ by the equation (11). Since $\lambda_1(S(r))$ is a strictly decreasing function in $(0, r_1)$, $\Psi'(r) < 0$ in $[a, r_1]$. Hence $\Psi(r) > 0$ and $\mu_1(r_1) > \lambda_1(S(r_1))$. Further, since $\Psi$ is positive in $(0, r_1)$, it follows that $g' > 0$ on $(0, r_1)$. Thus we have proved the following

**Lemma 1.** $g'(r) > 0$ in $(0, r_1)$ and $\mu_1(r_1) > \lambda_1(S(r_1))$.

We note that for $M$ of compact type, we have the restriction $0 < r_1 \leq \frac{\pi}{4}$. Using the lemma we prove the following.

**Proposition 1.** $\mu_1(r_1)$ is a decreasing function of $r_1$.

**Proof.** We set up the prüfer variables $\rho(r)$ and $\theta(r)$ for a $g$ satisfying the Sturm-Liouville system

$$Q \frac{\partial}{\partial r} (P(r)Q \frac{\partial}{\partial r} g) + Q(r)g = 0$$

in $(0, r_1)$ with boundary conditions $g(0) = 0$ and $g'(r_1) = 0$, where $P = J(r)$ and $Q(r) = (\lambda_1(S(r)) - \mu_1(r_1))J(r)$. The variables $\rho(r)$ and $\theta(r) = \theta(r, \mu_1(r_1))$ are defined as $\rho(r) \cos \theta(r) = P(r)Q \frac{\partial}{\partial r} g(r)$ and $g(r) = \rho(r) \sin \theta(r)$. By Lemma 1 in section 7 of [4] we know that $\theta(r, \lambda)$ is an increasing function of $\lambda$ for a fixed $r > 0$. By Lemma 1, $\theta(r, \mu_1(r_1)) \in (0, \frac{\pi}{2})$ for $0 < r < r_1 \leq \frac{\pi}{4}, \mu_1(r_1) > \mu_1(r_2)$. If not, then $\mu_1(r_1) < \mu_1(r_2)$. Hence

$$\frac{\pi}{2} = \theta(r_1, \mu(r_1)) \leq \theta(r_1, \mu_1(r_2)) \in (0, \frac{\pi}{2})$$

which is a contradiction. This completes the proof of the proposition.

**Corollary 1.** For $(M^n, ds^2)$ of compact type, we have $\mu_1(r_1) \geq \mu_1(\frac{\pi}{4}) = \lambda_1(M) = 2k(n + 1)$ for $0 < r_1 \leq \frac{\pi}{4}$.

**Proof.** The function $g(r) = \sin r \cos r$ satisfies the equation (7) with $\mu = 2k(n + 1)$.

4. Proof of Theorem 1

In this section $(M^n, ds^2)$ is of compact type. Let $g$ be the first eigenfunction of the equation (7) on $[0, r_1]$. We define a function $B$ on $[0, r_1]$ by,

$$B(r) = (Q \frac{\partial}{\partial r} g)^2 + \lambda_1(S(r))g^2(r).$$

The following lemma is a main ingredient in the proof of Theorem 1.

**Lemma 2.** $B' \leq 0$ on $[0, r_1]$ for $0 < r_1 \leq \frac{\pi}{4}$. 

Proof. Following [1], we define

\[ q(r) = \sin 2r \frac{g'}{g}. \]

Then

\[ B(r) = \left\{ q^2(r) + 4 \left[ (kn - 1) \cos 2r + (k - 1) \sin^2 r \right] \right\} \frac{g^2}{\sin^2 2r} \]

\[ = \left[ q^2 + 4k(n - 1) \cos 2r + 4(k - 1) \right] \frac{g^2}{\sin^2 2r} \]

and

\[ B'(r) = 2[q q' - 2k(n - 1) \sin 2r] \frac{g^2}{\sin^2 2r} + (q^2 + 4k(n - 1) \cos 2r + 4(k - 1)) \left( \frac{q - 2 \cos 2r}{\sin 2r} \right) \left( \frac{g^2}{\sin^2 2r} \right) \]

The lemma follows once we prove that \( q' \leq 0 \) and \( 0 \leq q \leq 2 \cos 2r \) on \([0, r_1]\). Now we prove the sublemma.

**Sublemma.** \( 0 \leq q \leq 2 \cos 2r \) and \( q' \leq 0 \) on \([0, r_1]\).

**Proof.** We have

\[ q' = \sin 2r \frac{g''}{g} + 2 \cos 2r \frac{g'}{g} - \sin 2r \left( \frac{g'}{g} \right)^2. \]  

Now substituting for \( g'' = -H(r)g' + (\lambda_1(S(r)) - \mu_1(r_1))g \)

in equation (12), we get

\[ q' = -(\mu_1(r_1) - \lambda_1(S(r))) \sin 2r - H(r)q + 2q \cot 2r - \frac{q^2}{\sin 2r}. \]  

We rewrite equation (13) as

\[ q' = -(\mu_1(r_1) - \lambda_1(M) + 4) \sin 2r \]

\[ + \frac{(2 \cos 2r - q) [q + (k(n + 1) - 2) \cos 2r + k(n - 1)]}{\sin 2r}. \]  

From the definition, we have \( q(0) = 2 \) and by an easy computation using the equation (14) we see that \( q'(0) = 0 \). By differentiating the equation (14) and evaluating at \( t = 0 \), using Lemma 1, we get that \( q''(0) \leq -8 \). Further \( q(r_1) = 0 \) and using Lemma 1 and the equation (14) we see that \( q'(r_1) < 0 \).

Now we prove that \( q \leq 2 \cos 2r \) on \([0, r_1]\) using a comparison theorem (see Theorem 7, p. 267 of [4]). Let \( F(r, q) \) denote the right hand side of the equation (14). From the initial values \( q, q' \) and \( q'' \) at \( t = 0 \), it follows that \( q(r) \leq 2 \cos 2r \) for small values of \( r \), say for \( r \in [0, a] \), for some \( a < r_1 \). Now if \( q \geq 2 \cos 2r \) on \([a, a + \epsilon]\) for some \( \epsilon > 0 \), we would have, by the equation (14), for \( r \in [a, a + \epsilon] \)

\[ q'(r) \leq -(\mu_1(r_1) - \lambda_1(M) + 4) \sin 2r \]

\[ < -4 \sin 2r \]

\[ = F(r, \cos 2r). \]
The inequality in the second step above follows from Lemma 1. Now by the comparison theorem cited above, we conclude that \( q \leq 2 \cos 2r \) in \([a, a + \epsilon)\). Thus we have proved that \( q \leq 2 \cos 2r \) on \([0, r_1]\).

To prove that \( q' \leq 0 \) on \([0, r_1]\), we rewrite the equation (14) as

\[
q' = -\mu_1 \sin 2r + \frac{1}{\sin 2r} [2k(n + 1) - 4 - q^2 - k(n - 1)q] + \cot 2r [2k(n - 1) - (k(n + 1) - 4)q]
\]
i.e.,

\[
q' = -\mu_1 \sin 2r + \frac{1}{\sin 2r} [2k(n - 1) + 4 - q^2 - k(n - 1)q] + \cot 2r [2k(n - 1) - (k(n + 1) - 4)q] + 4(k - 2) \frac{(1 - \cos 2r)}{\sin 2r}.
\]

Since \( q \leq 2 \cos 2r \) and \( k \geq 2, 2k(n - 1) + 4 - q^2 - k(n - 1)q \geq 0 \) and \( 2k(n + 1) - 8 - (k(n + 1) - 4)q \geq 0 \). Hence the right hand side \( F(r, q) \) of the above equation is convex in the variable \( r \), as \(-\sin 2r, \frac{1}{\sin 2r}, \cot 2r \) and \( \tan r \) are convex functions for \( 0 < r \leq \frac{\pi}{2} \). As in [1], we conclude that \( q' \leq 0 \) on \([0, r_1]\) for \( 0 < r_1 \leq \frac{\pi}{4} \).

Now to the proof of the theorem. We extend \( g \) to a function \( G \) on \([0, \frac{\pi}{4}]\) by

\[
G(r) = \begin{cases} 
  g(r) & \text{for } 0 \leq r \leq r_1, \\
  g(r_1) & \text{for } r_1 \leq r \leq \frac{\pi}{4}.
\end{cases}
\]

Let \( \Omega \) be a domain in \( M \) contained in a ball of radius \( \frac{\pi}{4} \). Now we apply the centre of mass theorem with weight function \( \frac{G(r)}{r} \) to the domain \( \Omega \). Let \( p \in C\Omega \) be the centre of mass of \( \Omega \). Then, for normal coordinates \( (X_1, X_2, \ldots, X_{kn}) \) centred at \( p \),

\[
\int_{\Omega} \frac{G(r)}{r} X_i dV = 0
\]

for \( 1 \leq i \leq kn \). Now from the Rayleigh-Ritz inequality, we have

\[
\mu_1(\Omega) \leq \int_{\Omega} \left( \nabla (\frac{G X_i}{r}) \right)^2 dV \int_{\Omega} (\frac{G X_i}{r})^2 dV
\]
i.e.,

\[
\mu_1(\Omega) \int_{\Omega} \left( \frac{G X_i}{r} \right)^2 dV \leq \int_{\Omega} \left( \nabla \left( \frac{G X_i}{r} \right) \right)^2 dV.
\]

By summing over \( i = 1, 2, \ldots, kn \) we get

\[
(15) \quad \mu_1(\Omega) \leq \sum_{i=1}^{kn} \int_{\Omega} \left( \nabla \left( \frac{G X_i}{r} \right) \right)^2 dV \int_{\Omega} G^2 dV.
\]

By applying the divergence theorem to the terms in the numerator of the right hand side of the equation (15), we get

\[
\mu_1(\Omega) \leq \frac{\int_{\Omega} (G'^2 + \lambda_1(S(r))G^2) dV}{\int_{\Omega} G^2 dV}.
\]

We denote the function \( G'^2 + \lambda_1(S(r))G^2 \) also by \( B \) on \([0, \frac{\pi}{4}]\). By Lemma 2, \( B \) is a decreasing function on \([0, r_1]\) and since \( \lambda_1(S(r)) \) is a decreasing function on \([r_1, \frac{\pi}{4}]\),
we see that $B$ is a decreasing function on $[0, \frac{\pi}{4}]$. Also $G$ is an increasing function on $[0, \frac{\pi}{4}]$. Following [7], we have

\[ \int_{\Omega} B dV = \int_{\Omega \cap B(r_1)} B dV + \int_{B(r_1) \setminus \Omega \cap B(r_1)} B dV \]

and

\[ \int_{B(r_1)} B dV = \int_{\Omega \cap B(r_1)} B dV + \int_{B(r_1) \setminus \Omega \cap B(r_1)} B dV \]

i.e.,

\[ \int_{\Omega \cap B(r_1)} B dV = \int_{B(r_1)} B dV - \int_{B(r_1) \setminus \Omega \cap B(r_1)} B dV. \]

This implies that

\[ \int_{\Omega} B dV \leq \int_{B(r_1)} B dV - \int_{B(r_1) \setminus \Omega \cap B(r_1)} B dV + B(r_1) \int_{B(r_1) \setminus \Omega \cap B(r_1)} dV. \]

Since \( \text{vol}(B(r_1) \setminus \Omega \cap B(r_1)) = \text{vol}(\Omega \setminus \Omega \cap B(r_1)) \) and $B$ is decreasing,

\[ \int_{\Omega} B dV \leq \int_{B(r_1)} B dV \]

By similar arguments we can prove that

\[ \int_{\Omega} G^2 dV \geq \int_{B(r_1)} G^2 dV. \]

Hence \( \mu_1(\Omega) \leq \mu_1(r_1) \) and equality holds iff \( \Omega = B(p,r_1) \).

5. Proof of Theorem 2

In this section \((M^n, ds^2)\) is of non-compact type. Let \( \mu_1(r_1) \) denote the first non-zero Neumann eigenvalue for the geodesic ball of radius \( r_1 \) for \( r_1 > 0 \). Let $g$ be the eigenfunction satisfying equation (7) on \([0, r_1]\) with eigenvalue \( \mu_1(r_1) \). i.e.,

\[-g'' - ((kn - 1) \coth r + (k - 1) \tanh r) g' + \left( \frac{kn - 1}{\sinh^2 r} - \frac{k - 1}{\cosh^2 r} \right) g = \mu_1(r_1) g \]

with the boundary conditions \( g(0) = 0 \) and \( g'(r_1) = 0 \). We define a function

\[ B(r) = (Q \frac{\partial}{\partial r} g)^2 + \lambda_1(S(r)) g^2(r). \]

Now we verify that $B$ is a decreasing function on \([0, r_1]\).

\[ B'(r) = 2g' g'' + 2gg' \lambda_1(S(r)) \]

\[ -2g^2 \left[ (kn - 1) \frac{\cosh r}{\sinh^3 r} - (k - 1) \frac{\sinh r}{\cosh^3 r} \right]. \]
Now, by substituting for \( g'' \) in equation (16) we get,

\[
\frac{1}{2} B'(r) = -((kn - 1) \coth r + (k - 1) \tanh r) (g')^2 \\
- \left[ (kn - 1) \frac{\cosh r}{\sinh^2 r} - (k - 1) \frac{\sinh r}{\cosh^2 r} \right] g^2 \\
+ 2gg' \left[ \frac{kn - 1}{\sinh^2 r} - \frac{k - 1}{\cosh^2 r} \right] - \mu_1(r_1)gg'.
\]

Now by an easy computation, we get

\[
\frac{1}{2} B'(r) = -\frac{k(n - 1)}{\sinh^3 r} \left[ \left( g' \sinh r - g \right)^2 \cosh r + 2gg' (\cosh r - 1) \sinh r \right] \\
- \frac{2(k - 1)}{\sinh^3 2r} \left[ (g' \sinh 2r - 2g)^2 \cosh 2r + 4gg' (\cosh 2r - 1) \sinh 2r \right] \\
- \mu_1(r_1)gg' \\
\leq 0 \quad \text{by Lemma 1.}
\]

Let \( \Omega \) be a bounded domain in \((M^n, ds^2)\). Let \( B(r_1) \) be a geodesic ball of radius \( r_1 \) in \( M \) such that \( vol(\Omega) = vol(B(r_1)) \). We extend the function \( g \) to a function \( G \) on \([0, \infty)\) by

\[
G(r) = \begin{cases} 
 g(r) & \text{for } 0 \leq r \leq r_1, \\
 g(r_1) & \text{for } r_1 \leq r < \infty.
\end{cases}
\]

Now we apply the centre of mass theorem with the weight function \( \frac{G(r)}{r} \) to the domain \( \Omega \). Let \( p \in C\Omega \) be the centre of mass of \( \Omega \). Then, for normal coordinates \((X_1,X_2, \ldots, X_{kn})\) centred at \( p \),

\[
\int_{\Omega} \frac{G(r)}{r} X_i dV = 0
\]

for \( 1 \leq i \leq kn \). By the Rayleigh-Ritz inequality, we have

\[
\mu_1(\Omega) \leq \frac{\int_{\Omega} \left\| \nabla \left( \frac{G X_i}{r} \right) \right\|^2 dV}{\int_{\Omega} \left( \frac{G X_i}{r} \right)^2 dV}
\]

i.e.,

\[
\mu_1(\Omega) \int_{\Omega} \left( \frac{G X_i}{r} \right)^2 dV \leq \int_{\Omega} \left\| \nabla \left( \frac{G X_i}{r} \right) \right\|^2 dV.
\]

By summing over \( i = 1, 2, 3, \ldots, kn \), we get

\[
\mu_1(\Omega) \leq \frac{\sum_{i=1}^{kn} \int_{\Omega} \left\| \nabla \left( \frac{G X_i}{r} \right) \right\|^2 dV}{\int_{\Omega} G^2 dV}
\]

By applying the divergence theorem to the terms in the numerator of the right hand side of the equation (17), we get

\[
\mu_1(\Omega) \leq \frac{\int_{\Omega} \left( (G')^2 + \lambda_1(S(r))G^2 \right) dV}{\int_{\Omega} G^2 dV}.
\]

We denote the function \((G')^2 + \lambda_1(S(r))G^2\) also by \( B \) on \([0, \infty)\). Since \( B' \leq 0 \), \( B \) is a decreasing function for all \( r > 0 \). Also \( G \) is an increasing function for all \( r > 0 \). As in section 4, we see that \( \mu_1(\Omega) \leq \mu_1(r_1) \) and equality holds iff \( \Omega = B(p, r_1) \).
Concluding Remarks.

1. Our proof of Theorem 1 when applied to \((S^n, ds^2)\) gives the result for a domain contained in a geodesic ball of radius \(\frac{\pi}{4}\). A reflection argument developed in [1], then shows that the theorem is true for a domain contained in a hemisphere of \(S^n\). But this reflection argument cannot be applied to the other symmetric spaces of compact type.

2. The improvement of the size of the domain \(\Omega\) in Theorem 1 depends on the location of the centre of mass of \(\Omega\).

3. In their proof of Theorem 1 for the case of \((S^n, ds^2)\), Ashbaugh and Benguria [1] have used rearrangement of the functions \(B\) and \(G\). As we have shown, this is not needed.

References

5. S. Gallot, D. Hulin and J. Lafontaine, Riemannian Geometry, Universitext, Springer-Verlag, MR 88k:53001

Department of Mathematics, University of Bombay, Vidyanagare, Bombay-400098, India

E-mail address: aithal@mathbu.ernet.in

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay-400-005, India

E-mail address: santhana@math.tifr.res.in