ON BAIRE-1/4 FUNCTIONS

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Abstract. We give descriptions of the spaces $D(K)$ (i.e. the space of differences of bounded semicontinuous functions on $K$) and especially of $B_{1/4}(K)$ (defined by Haydon, Odell and Rosenthal) as well as for the norms which are defined on them. For example, it is proved that a bounded function on a metric space $K$ belongs to $B_{1/4}(K)$ if and only if the $\omega$th-oscillation, $\text{osc}_\omega f$, of $f$ is bounded and in this case $\|f\|_{1/4} = \|f \| + \text{osc}_\omega f \|_\infty$. Also, we classify $B_{1/4}(K)$ into a decreasing family $(S_\xi(K))_{1 \leq \xi < \omega_1}$ of Banach spaces whose intersection is equal to $D(K)$ and $S_1(K) = B_{1/4}(K)$. These spaces are characterized by spreading models of order $\xi$ equivalent to the summing basis of $c_0$, and for every function $f$ in $S_\xi(K)$ it is valid that $\text{osc}_{\xi f}$ is bounded. Finally, using the notion of null-coefficient of order $\xi$ sequence, we characterize the Baire-1 functions not belonging to $S_\xi(K)$.

Introduction

In recent years the study of the first Baire class, $B_1(K)$, of bounded functions on a metric space $K$ led to the definition of interesting subclasses ([H-O-R], [K-L], [F1]). The study of these subclasses revealed significant properties of their elements ([C-M-R], [R2], [F1], [F2]) and provided applications, such as the $c_0$-dichotomy theorem of Rosenthal ([R1]).

Here we study some subclasses of $D(K)$, and especially $B_{1/4}(K)$, of $B_1(K)$. By $D(K)$ is denoted the class of all functions on $K$ which are differences of bounded semicontinuous functions. A classical result of Baire yields that $f \in D(K)$ if and only if there exists a sequence $(f_n)$ of continuous functions on $K$ satisfying

$$\sup_{x \in K} \sum_n |f_n(x)| < \infty \quad \text{and} \quad f = \sum_n f_n.$$  

The class $D(K)$ is a Banach algebra with respect to the $\| \cdot \|_{D}$-norm defined as

$$\|f\|_D = \inf \left\{ \sup_{x \in K} \sum_n |f_n(x)| : (f_n) \subseteq C(K) \text{ satisfying (1)} \right\}.$$  

The subclass $B_{1/4}(K)$ was first defined in [H-O-R] as follows:

$$B_{1/4}(K) = \{ f : K \to \mathbb{R} : \text{there exists } (F_n) \subseteq D(K) \text{ such that } \|F_n - f\|_\infty \to 0 \text{ and } \sup_n \|F_n\|_D < \infty \}.$$
This class is a Banach algebra with respect to the $\| \cdot \|_{1/4}$-norm, given by

$$
\|f\|_{1/4} = \inf \left\{ \sup_n \|F_n\|_D : (F_n) \subseteq D(K) \text{ and } \|F_n - f\|_{\infty} \to 0 \right\}.
$$

In the first section we describe the precise connection between the summing basis \((s_n)\) of \(c_0\) and the normed space \((D(K), \| \cdot \|_D)\): so it is proved in Proposition 1.1 that \(f \in D(K)\) if and only if there is a sequence \((f_n)\) of continuous functions on \(K\) so that \(f_n \to f\) pointwise and there is \(C > 0\) such that

$$
\left\| \sum_{i=1}^k \lambda_i s_i \right\|_{\infty} \leq C \left\| \sum_{i=1}^k \lambda_i s_i \right\|_{\infty}
$$

for every \(k, n_1, \ldots, n_k \in \mathbb{N}\) and scalars \(\lambda_1, \ldots, \lambda_k\).

If this occurs then

$$
\|f\|_D = \inf \left\{ C > 0 : \text{there exists} (f_n) \subseteq C(K) \text{ satisfying (2)} \right\}.
$$

Since for every sequence of continuous functions defined on a compact metric space \(K\) and converging pointwise to a discontinuous function, there exists a subsequence \((f_n)\) and \(\mu > 0\) such that

$$
\mu \left\| \sum_{i=1}^k \lambda_i s_i \right\|_{\infty} \leq \left\| \sum_{i=1}^k \lambda_i f_n \right\|_{\infty}
$$

for \(k, n_1, \ldots, n_k \in \mathbb{N}\) and scalars \(\lambda_1, \ldots, \lambda_k\) ([H-O-R], [R1]), it follows that the functions in \(D(K) \setminus C(K)\) (\(K\) compact) are characterized as pointwise limits of sequences of continuous functions equivalent to the summing basis of \(c_0\) (Remark 1.2).

In the case of \(B_{1/4}(K)\), where \(K\) is a compact metric space, the functions are characterized as pointwise limits of sequences of continuous functions on \(K\) with a property weaker than (2), namely one for which the inequality (2) is valid only for \((n_1, \ldots, n_k)\) in the Schreier family \(\mathcal{F}_1\) (Theorem 2.1). Moreover, if we set

$$
\|f\|_s^1 = \inf \left\{ C > 0 : \text{there exists} (f_n) \subseteq C(K) \text{ such that} \right. \left. \text{pointwise and} \left\| \sum_{i=1}^k \lambda_i f_{n_i} \right\|_{\infty} \leq C \left\| \sum_{i=1}^k \lambda_i s_i \right\|_{\infty} \text{ for every} \right.

\(n_1, \ldots, n_k \in \mathcal{F}_1\) \text{ and scalars} \lambda_1, \ldots, \lambda_k\),

then \(\| \cdot \|_s^1\) is a norm on \(B_{1/4}(K)\) equivalent to the norm \(\| \cdot \|_{1/4}\). This answers in the affirmative a question raised by Haydon, Odell and Rosenthal in [H-O-R]. From this result and (3) we have the characterization of functions in \(B_{1/4}(K) \setminus C(K)\) (\(K\) compact) as pointwise limits of sequences of continuous functions generating spreading models equivalent to the summing basis of \(c_0\).

More generally, we define analogously the spaces \(S_\xi(K)\) and the norms \(\| \cdot \|_\xi\) on them, employing the higher order Schreier family \(\mathcal{F}_\xi\), for \(1 \leq \xi < \omega_1\), as defined by Alspach and Argyros ([A-A]). According to Proposition 3.4, \((S_\xi(K), \| \cdot \|_\xi)\) are Banach spaces, which, for separable metric spaces \(K\), constitute a decreasing hierarchy whose intersection is equal to \(D(K)\) (Theorem 3.8) and of course \(S_1(K) = B_{1/4}(K)\). We further provide alternative descriptions of the spaces \(S_\xi(K), 1 \leq \xi < \omega_1\), and characterize the Baire-1 functions not belonging to \(S_\xi(K)\) (Theorem 3.11), employing the notion of a null-coefficient of order \(\xi\) sequence, defined in [F2].

Because of Mazur’s theorem, \(S_\xi(K)\) is actually a Banach space invariant. That is, if \(X\) is a separable Banach space, \(x^{**} \in X^{**} \setminus X\), and \(K = Ba(X^*, w^*)\), then if
There exist an ordinal $\alpha$ bounded function on an infinite metric space $K$ follows:

Let $\beta > \alpha$. Letting $D \subseteq \mathbb{R}^2$ the author proved the following structural result for relations of a function is given in Theorem 2.9. Rosenthal in [R1] defined for every $K$ on $S$ and it follows that $K,$ $f$ then

$$\|\sum_{i=1}^k \lambda_i x_n\|_{\infty} \leq C \|\sum_{i=1}^k \lambda_i s_i\|$$ for every $(n_1, \ldots, n_k) \in F_\xi$ and scalars $\lambda_1, \ldots, \lambda_k.$

A nice relation between the space $(B_{1/4}(K), \| \cdot \|_{1/4})$ and the transfinite oscillations of a function is given in Theorem 2.9. Rosenthal in [R1] defined for every function $f$ the $\alpha$th-oscillation, $\text{osc}_\alpha f$, of $f$ for every ordinal $\alpha$ (cf. Definition 2.5). In [R2] the author proved the following structural result for $D(K)$: Let $f$ be a real bounded function on an infinite metric space $K$. Then $f \in D(K)$ if and only if there exist an ordinal $\alpha$ such that $\text{osc}_\alpha f$ is bounded and $\text{osc}_\alpha f = \text{osc}_\beta f$ for all $\beta > \alpha$. Letting $\tau$ be the least such $\alpha$, then

$$\|f\|_D = \| |f| + \text{osc}_\tau f\|_{\infty}$$ for all $f \in D(K)$.

We prove an analogous structural result for the case of $B_{1/4}(K)$. Precisely, we have the following theorem: Let $f$ be a real bounded function on a metric space $K$. Then $f \in B_{1/4}(K)$ if and only if $\text{osc}_\tau f$ is bounded. In this case

$$\|f\|_{1/4} = \| |f| + \text{osc}_\omega f\|_{\infty}$$ for $f \in B_{1/4}(K)$.

According to the principal result in [F2], $\text{osc}_\omega f$ is bounded for every function $f$ in $S_\xi(K)$ and every ordinal $\xi$. It is an open problem whether the functions in $S_\xi(K)$ are characterized by this property.

1. Differences of Bounded Semicontinuous Functions

Let $K$ be a metric space. We denote by $C(K)$ the class of continuous functions on $K$ and by $B_1(K)$ the space of bounded first Baire class functions on $K$ (i.e. the pointwise limits of uniformly bounded sequences of continuous functions).

An important subclass of $B_1(K)$ is the class of differences of bounded semicontinuous functions on $K$, denoted by $D(K)$. It is easy to see that

$$D(K) = \{ f \in B_1(K) : f = u - v, \text{ where } u, v \geq 0 \text{ are bounded and lower semicontinuous functions} \}.$$

The class $D(K)$ is a Banach algebra with respect to the norm $\| \cdot \|_D$, defined as follows:

$$\|f\|_D = \inf \{ \|u + v\|_{\infty} : f = u - v \text{ for } u, v \geq 0, \text{ bounded and lower semicontinuous functions} \}.$$

This infimum is attained according to [R1]. A result of Baire gives that

$$D(K) = \{ f \in B_1(K) : \text{there exists } (f_n) \text{ in } C(K) \text{ such that } f = \sum_n f_n \text{ pointwise and } \|\sum_n |f_n|\|_{\infty} < \infty \}$$

and it follows that

$$\|f\|_D = \inf \{ \|\sum_n |f_n|\|_{\infty} : (f_n) \subseteq C(K) \text{ and } f = \sum_n f_n \text{ pointwise} \}.$$
for every \( f \in D(K) \) (see [R2]). It is easy to see that \( \|f\|_\infty \leq \|f\|_D \) for every \( f \in D(K) \) but the two norms are not equivalent in general.

In the following proposition we give the fundamental connection between the summing basis \((s_n)\) of \(c_0\) and the functions in \(D(K)\), as well as between \((s_n)\) and the norm \(\| \cdot \|_D\).

1.1. Proposition. Let \( K \) be a metric space. Then

\[
D(K) = \left\{ f \in B_1(K) : \text{there exists } (f_n) \text{ in } C(K) \text{ and } C > 0 \text{ so that } f_n \to f \text{ pointwise and } \left\| \sum_{i=1}^n \lambda_i f_i \right\|_\infty \leq C \left\| \sum_{i=1}^n \lambda_i s_i \right\| \text{ for all } \lambda_1, \ldots, \lambda_n \in \mathbb{N} \right\},
\]

where \((s_n)\) is the summing basis of \(c_0\). Also, for every \( f \in D(K)\),

\[
\|f\|_D = \|f\|_\infty = \inf \left\{ C > 0 : \text{there exists } (f_n) \subseteq C(K) \text{ such that } f_n \to f \text{ pointwise and } \left\| \sum_{i=1}^n \lambda_i f_i \right\|_\infty \leq C \left\| \sum_{i=1}^n \lambda_i s_i \right\| \text{ for all } \lambda_1, \ldots, \lambda_n \in \mathbb{N} \right\}.
\]

Proof. If \( f \in D(K) \) then there exists a sequence \((g_n)_{n=1}^\infty \) in \(C(K)\) such that \( f = \sum_{n=1}^\infty g_n \) and \( C = \| \sum_{n=1}^\infty \| g_n \|_\infty < \infty \). Set \( f_n = \sum_{i=1}^n g_i \) for every \( n \in \mathbb{N} \). Of course, \( f_n \to f \) pointwise and

\[
\left\| \sum_{i=1}^n \lambda_i f_i \right\|_\infty = \left\| \sum_{i=1}^n (\lambda_1 + \cdots + \lambda_n) g_i \right\|_\infty \leq \left\| \sum_{i=1}^n |g_i| \right\|_\infty \cdot \left\| \sum_{i=1}^n \lambda_i s_i \right\| \leq C \cdot \left\| \sum_{i=1}^n \lambda_i s_i \right\|.
\]

Hence, \( \|f\|_\infty \leq \|f\|_D \) for every \( f \in D(K) \).

On the other hand, if there exist \((f_n)\) in \(C(K)\) and \( C > 0 \) such that \( f_n \to f \) pointwise and \( \| \sum_{i=1}^n \lambda_i f_i \|_\infty \leq C \| \sum_{i=1}^n \lambda_i s_i \| \) for every \( n \in \mathbb{N} \) and scalars \( \lambda_1, \ldots, \lambda_n \), then if we set \( g_0 = 0 \) and \( g_n = f_n - f_{n-1} \) for every \( n \in \mathbb{N} \), we have that \( \sum_{n=1}^\infty g_n = f \).

Also, for \( x \in K \) and \( n \in \mathbb{N} \),

\[
\sum_{i=1}^n |g_i(x)| = \sum_{i=1}^n |f_i - f_{i-1}|(x) = \sum_{i=1}^n |\varepsilon_i(f_i - f_{i-1})(x)| \leq \sum_{i=1}^n (|\varepsilon_i| - |\varepsilon_{i+1}|) f_i(x) \leq C,
\]

where \( \varepsilon_i \in \{-1, 1\} \) so that \( \varepsilon_i(f_i - f_{i-1})(x) \geq 0 \) for every \( i = 1, \ldots, n \) and \( \varepsilon_{n+1} = 0 \).

Hence, we have that \( \|f\|_D \leq \|f\|_\infty \) for every \( f \in D(K) \). \( \square \)

1.2. Remark. It is known ([H-O-R], [R1]) that, for a compact metric space \( K \), every bounded sequence \((f_n)\) in \(C(K)\) converging pointwise to a discontinuous function \( f \) has a basic subsequence \((g_n)\) which dominates the summing basis \((s_n)\) of \(c_0\), i.e. there exists \( \mu > 0 \) such that \( \mu \| \sum_{i=1}^n \lambda_i s_i \| \leq \| \sum_{i=1}^n \lambda_i g_i \|_\infty \) for every \( n \in \mathbb{N} \), \( \lambda_1, \ldots, \lambda_n \in \mathbb{R} \). Hence, for a compact metric space \( K \),

\[
D(K) \setminus C(K) = \left\{ f : K \to \mathbb{R} : \text{there exists } (f_n) \subseteq C(K) \text{ such that } f_n \to f \text{ pointwise and } (f_n) \text{ is equivalent to } (s_n) \right\}.
\]

This result has been proved in [R1] also. Using Mazur’s theorem we have that every uniformly bounded sequence \((f_n)\) converging pointwise to a function \( f \) in \( D(K) \setminus C(K) \) has a convex block subsequence equivalent to \((s_n)\).
2. Baire-1/4 Functions

As we mentioned before, the supremum norm is not equivalent, in general, to the \( \| \cdot \|_D \)-norm in \( D(K) \). The closure of \( D(K) \) in \((B_1(K), \| \cdot \|_\infty)\) has been denoted by \( B_{1/2}(K) \) in [H-O-R]. In the same paper the authors defined the subclass \( B_{1/4}(K) \) of \( B_1(K) \) as follows:

\[
B_{1/4}(K) = \left\{ f \in B_1(K) : \text{there exists } (F_n) \subseteq D(K) \text{ such that } \|F_n - f\|_\infty \to 0 \right\}.
\]

The space \( B_{1/4}(K) \) is complete with respect to the norm

\[
\|f\|_{1/4} = \inf \left\{ \sup_n \|F_n\|_D : (F_n) \subseteq D(K) \text{ and } \|F_n - f\|_\infty \to 0 \right\}.
\]

In the following theorem we will give a characterization of \( B_{1/4}(K) \) and we will define the \( \| \cdot \|_s \)-norm on it, in analogy to \( D(K) \) (Proposition 1.1). We will prove that this norm is equivalent to the \( \| \cdot \|_{1/4} \)-norm answering affirmatively the question raised by Haydon, Odell and Rosenthal in [H-O-R]. The techniques of this proof have been employed before in [F1]. The additional work here is to establish the relation between the norms. For completeness we give the proof in detail. We will use the Schreier family \( F_1 \) which is:

\[
F_1 = \left\{ (n_1, \ldots, n_k) : k \leq n_1 \leq \cdots \leq n_k \in \mathbb{N} \right\}.
\]

2.1. Theorem. Let \( K \) be a compact metric space. Then

\[
B_{1/4}(K) = \left\{ f \in B_1(K) : \text{there exists } (f_n) \text{ in } C(K) \text{ and } C > 0 \text{ so that } f_n \to f \text{ pointwise and } \left| \sum_{i=1}^k \lambda_i f_{n_i} \right|_\infty \leq C \left| \sum_{i=1}^k \lambda_i s_i \right| \text{ for every } (n_1, \ldots, n_k) \in F_1 \text{ and } \lambda_1, \ldots, \lambda_k \right\}.
\]

Also, defining for \( f \in B_{1/4}(K) \)

\[
\|f\|_s = \inf \left\{ C > 0 : \text{there exists } (f_n) \text{ in } C(K) \text{ such that } f_n \to f \text{ pointwise and } \left| \sum_{i=1}^k \lambda_i f_{n_i} \right|_\infty \leq C \left| \sum_{i=1}^k \lambda_i s_i \right| \text{ for every } (n_1, \ldots, n_k) \in F_1 \text{ and } \lambda_1, \ldots, \lambda_k \right\},
\]

\( \| \cdot \|_s \) is a norm on \( B_{1/4}(K) \) equivalent to the norm \( \| \cdot \|_{1/4} \). Moreover,

\[
\|f\|_s \leq \|f\|_{1/4} \leq 4 \|f\|_s \text{ for every } f \in B_{1/4}(K).
\]

Proof. Let \( f \in B_{1/4}(K) \). According to the definition of \( (B_{1/4}(K), \| \cdot \|_{1/4}) \), for every \( \delta > 0 \) there exists a sequence \( (F_m) \) in \( D(K) \) so that \( \|F_m - f\|_{1/4} \to 0 \) and \( \sup_m \|F_m\|_D < \|f\|_{1/4} + \delta \). Let \( M = \|f\|_{1/4} + \delta \) and \( (\epsilon_m) \) a decreasing sequence of positive numbers such that \( \epsilon_m < \frac{\delta}{4M} \) and \( \sum_{i=m+1}^\infty \epsilon_i < \epsilon_m \) for every \( m \in \mathbb{N} \). We can assume that \( \| F_{m+1} - F_m \|_\infty < \epsilon_{m+1} \) for every \( m \in \mathbb{N} \). Hence, for every \( m \in \mathbb{N} \) there exists a sequence \( (g^m_n)_{n=1}^\infty \subseteq C(K) \) converging pointwise to \( F_{m+1} - F_m \) and \( \|g^m_n\|_\infty < \epsilon_{m+1} \) for all \( n \in \mathbb{N} \).

Since \( F_1 \subseteq D(K) \), by Proposition 1.1, there exists a sequence \( (f^m_n) \) in \( C(K) \) converging pointwise to \( F_1 \) and satisfying

\[
\left| \sum_{i=1}^k \lambda_i f^m_i \right|_\infty \leq M \left| \sum_{i=1}^k \lambda_i s_i \right|
\]
for all \( k \in \mathbb{N} \) and scalars \( \lambda_1, \ldots, \lambda_k \). The sequence \((f_n^1 + g_n^1)\) converges pointwise to \( F_2 \). Using Mazur’s theorem and the fact that \( F_2 \in D(K) \), we can find convex block subsequences \((f_n^{1,2}), (g_n^{1,2})\) of \((f_n^1), (g_n^1)\) respectively such that if \( f_n^2 = f_n^{1,2} + g_n^{1,2} \) for every \( n \in \mathbb{N} \) then \( f_n^2 \to F_2 \) pointwise and

\[
\left\| \sum_{i=1}^{k} \lambda_i f_n^i \right\|_{\infty} \leq M \left\| \sum_{i=1}^{k} \lambda_i s_i \right\|
\]

for every \( k \in \mathbb{N} \) and scalars \( \lambda_1, \ldots, \lambda_k \). Now, since \( f_n^2 + g_n^2 \) converges to \( F_3 \), there exist convex block subsequences \((f_n^{2,3}), (g_n^{2,3})\) of \((f_n^2), (g_n^2)\) respectively, such that if \( f_n^3 = f_n^{2,3} + g_n^{2,3} \) for every \( n \in \mathbb{N} \) then \( f_n^3 \to F_3 \) pointwise and

\[
\left\| \sum_{i=1}^{k} \lambda_i f_n^i \right\|_{\infty} \leq M \left\| \sum_{i=1}^{k} \lambda_i s_i \right\|
\]

for every \( k \in \mathbb{N} \) and scalars \( \lambda_1, \ldots, \lambda_k \). Let \((f_n^{1,2,3}), (g_n^{1,2,3})\) be the convex block subsequences of \((f_n^{1,2}), (g_n^{1,2})\) respectively, such that \( f_n^{2,3} = f_n^{1,2,3} + g_n^{1,2,3} \) for every \( n \in \mathbb{N} \). Hence \( f_n^3 = f_n^{1,2,3} + g_n^{1,2,3} \) for every \( n \in \mathbb{N} \). We continue in the obvious way to construct \( f_n^{m-1,m,k} \) and \( g_n^{m-1,m,k} \) for every \( m, k, n \in \mathbb{N} \) with \( m \leq k \), so that \((g_n^{m-1,m,k}), (f_n^{m-1,m,k})\) to be convex block subsequences of \((g_n^{m-1,m,k}), (f_n^{m-1,m,k})\) respectively for every \( m, l, k \in \mathbb{N} \) with \( m \leq l \leq k \) and

(*)

\[
f_n^{m-1,m,k} = f_n^{m-1,m,k} + g_n^{m-1,m,k}
\]

for every \( n, k, m \in \mathbb{N} \) with \( 1 < m \leq k \). Also, for every \( m \in \mathbb{N} \), we construct the sequence \((f_n^m)_{n=1}^{\infty}\) converging pointwise to \( F_m \) and

(**)

\[
\left\| \sum_{i=1}^{k} \lambda_i f_n^m \right\|_{\infty} \leq M \left\| \sum_{i=1}^{k} \lambda_i s_i \right\|
\]

for every \( k \in \mathbb{N} \) and scalars \( \lambda_1, \ldots, \lambda_k \). Finally, we set

\[
h_n^m = f_n^{m-1,m} \quad \text{and} \quad d_n^m = g_n^{m-1,m}
\]

for every \( m, n \in \mathbb{N} \) with \( m \leq n \).

Then, for every \( m \in \mathbb{N}, (h_n^m)_{n=m}^{\infty}, (d_n^m)_{n=m}^{\infty} \) are convex block subsequences of \((f_n^m)_{n=1}^{\infty}, (g_n^m)_{n=1}^{\infty} \) respectively, hence \((h_n^m)_{n=m}^{\infty}\) converges pointwise to \( F_m, \|a_n^m\|_{\infty} < \epsilon_{m+1} \) for every \( m, n \in \mathbb{N} \) with \( m \leq n \) and \((d_n^m)_{n=m}^{\infty}\) converges pointwise to \( F_{m+1} - F_m \). Also, according to (*), we have that

\[
h_n^m = h_n^{m-1} + d_n^{m-1} = h_n^{m-1} + d_n^{m-1} + \cdots + d_n^{m-1}
\]

for every \( n, m, l \in \mathbb{N} \) with \( l < m \leq n \).

We set \( h_n = h_n^m \) for every \( n \in \mathbb{N} \). Thus \( h_n = h_n^m + d_n^m + \cdots + d_n^{m-1} \) for every \( m, n \in \mathbb{N} \) with \( m < n \). It is easy to prove that \((h_n)\) converges pointwise to \( f \). If \((n_1, \ldots, n_k) \in F_1 \) and \( \lambda_1, \lambda_2, \ldots, \lambda_k \) are scalars then

\[
\left\| \sum_{i=1}^{k} \lambda_i h_{n_i} \right\|_{\infty} \leq \left\| \sum_{i=1}^{k} \lambda_i h_{n_i}^k \right\|_{\infty} + \left\| \sum_{i=1}^{k} \lambda_i (d_{n_i}^1 + \cdots + d_{n_i}^{m_i-1}) \right\|_{\infty}.
\]

First, since \((h_n^k)_{n=k}^{\infty}\) is a convex block subsequence of \((f_n^k)_{n=1}^{\infty}\), we have from (**) that

\[
\left\| \sum_{i=1}^{k} \lambda_i h_{n_i}^k \right\|_{\infty} \leq M \left\| \sum_{i=1}^{k} \lambda_i s_i \right\|
\]
Secondly,\[
\left\| \sum_{i=1}^{k} \lambda_i \left( d_{n_i}^k + \cdots + d_{n_i}^{m_i-1} \right) \right\|_\infty \leq \varepsilon_k \cdot \sum_{i=1}^{k} |\lambda_i| \leq 2k\varepsilon_k \sum_{i=1}^{k} \lambda_i s_i < \delta \sum_{i=1}^{k} \lambda_i s_i.
\]
Hence\[
\left\| \sum_{i=1}^{k} \lambda_i h_{n_i} \right\|_\infty \leq (\|f\|_{1/4} + 2\delta) \cdot \sum_{i=1}^{k} \lambda_i s_i.
\]
This gives\[
\|f\|_s \leq \|f\|_{1/4} + 2\delta \text{ for every } \delta > 0
\]
and finally\[
\|f\|_s \leq \|f\|_{1/4} \text{ for every } f \in B_{1/4}(K).
\]
On the other hand, let \((f_n)\) be a sequence in \(C(K)\) converging pointwise to \(f\) and \(C > 0\) such that\[
\left\| \sum_{i=1}^{k} \lambda_i f_{n_i} \right\|_\infty \leq C \left\| \sum_{i=1}^{k} \lambda_i s_i \right\|
\]
for every \((n_1, \ldots, n_k) \in F_1\) and scalars \(\lambda_1, \ldots, \lambda_k\). According to a characterization of functions in \(B_{1/4}(K)\) given by Haydon, Odell and Rosenthal in [H-O-R], a function \(f\) belongs to \(B_{1/4}(K)\) if for \(\varepsilon > 0\) there exists a sequence \((g_{n_i})_{n=0}^\infty\) in \(C(K)\) with \(g_0 = 0\), converging pointwise to \(f\) and such that for every subsequence \((g_{n_i})\) of \((g_n)\) and \(x \in K\) to have\[
\sum_{j \in B((n_i),x)} |g_{n_{j+1}}(x) - g_{n_j}(x)| \leq M,
\]
where\[
B((n_i),x) = \{ j \in N : |g_{n_{j+1}}(x) - g_{n_j}(x)| \geq \varepsilon \}.
\]
In this case, it is easy to see that \(\|f\|_{1/4} \leq 4M\).

For \(\varepsilon > 0\), let \(m\) be an integer such that \(m > C/\varepsilon\). Set \(g_n = f_{2m+n}\) for every \(n \in N\). Then, for every strictly increasing sequence \((n_i)\) in \(N\) and \(x \in K\) we claim that \(#B((n_i),x) < m\). Indeed, if \(j_1, \ldots, j_m \in B((n_i),x)\), then\[
m \cdot \varepsilon \leq \sum_{i=1}^{m} |g_{n_{j_{i+1}}} - g_{n_{j_{i}}}| = \sum_{i=1}^{m} \varepsilon_j |f_{2m+n_{j_{i+1}}} - f_{2m+n_{j_{i}}}|(x) \leq C,
\]
where \(\varepsilon_1, \ldots, \varepsilon_m \in (-1,1)\), so that \(\varepsilon_j |f_{2m+n_{j_{i+1}}} - f_{2m+n_{j_{i}}}|(x) \geq 0\), a contradiction. Hence \(#B((n_i),x) < m\) and thus\[
\sum_{j \in B((n_i),x)} |g_{n_{j+1}}(x) - g_{n_j}(x)| \leq C.
\]
Hence \(f \in B_{1/4}(K)\) and \(\|f\|_{1/4} \leq 4 \|f\|_s\).

2.2. Remark. It is easy to prove (see [F1]) that a sequence \((x_n)\) in a Banach space \(X\) has a subsequence generating a spreading model equivalent to the summing basis \((s_n)\) if and only if it has a subsequence \((y_n)\) with the following property:
there exist $\mu, C > 0$ such that
\[
\mu \left\| \sum_{i=1}^{k} \lambda_i s_i \right\| \leq \left\| \sum_{i=1}^{k} \lambda_i y_{n_i} \right\| \leq C \left\| \sum_{i=1}^{k} \lambda_i s_i \right\|
\]
for every $(n_1, \ldots, n_k) \in F_1$ and scalars $\lambda_1, \ldots, \lambda_k$.
Hence, it follows from the previous theorem and Remark 1.2, for a compact metric space $K$ that
\[
B_{1/4}(K) \setminus C(K) = \left\{ f \in B_1(K) : \text{there exists } (f_n) \subseteq C(K) \text{ such that } f_n \to f \right\}
\]
pointwise and $(f_n)$ generates spreading model equivalent to $(s_n)$.

This result has been proved in [F1] also. Furthermore, it has been proved in [H-O-R] that every uniformly bounded sequence $(f_n)$ in $C(K)$ converging pointwise to a function in $B_{1/4}(K) \setminus C(K)$ has a convex block subsequence generating a spreading model equivalent to $(s_n)$.

In the following proposition we will give another description of $B_{1/4}(K)$ and we will prove the equality of the norm $\| \cdot \|_1$ with a norm on $B_{1/4}(K)$ analogous to the $\| \cdot \|_D$-norm on $D(K)$.

**2.3. Proposition.** For every compact metric space $K$, a function $f : K \to \mathbb{R}$ belongs to $B_{1/4}(K)$ if and only if there exists $(f_n)$ in $C(K)$ such that $f = \sum_{n=1}^{\infty} f_n$ pointwise and for $n_0 = f_0 = 0$,
\[
\sup \left\{ \left\| \sum_{i=1}^{k} |f_{n_{i-1}+1} + \cdots + f_{n_i}| \right\|_\infty : (n_1, \ldots, n_k) \in F_1 \right\} < \infty.
\]
Also, for every $f \in B_{1/4}(K)$ we have
\[
\|f\|_1 = \|f\|_D = \inf \left\{ \sup \left\{ \left\| \sum_{i=1}^{k} |f_{n_{i-1}+1} + \cdots + f_{n_i}| \right\|_\infty : (n_1, \ldots, n_k) \in F_1 \right\} : (f_n) \subseteq C(K) \text{ with } f = \sum_{n=1}^{\infty} f_n \right\}.
\]

**Proof.** If $f \in B_{1/4}(K)$ then for every $\epsilon > 0$, from the previous theorem, there exists $(g_n)_{n=0}^{\infty} \subseteq C(K)$, $g_0 = 0$, such that $g_n \to f$ pointwise and
\[
\left\| \sum_{i=1}^{k} \lambda_i g_{n_i} \right\|_\infty \leq (\|f\|_1^1 + \epsilon) \left\| \sum_{i=1}^{k} \lambda_i s_i \right\|
\]
for every $(n_1, \ldots, n_k) \in F_1$ and scalars $\lambda_1, \ldots, \lambda_k$. Set $f_n = g_n - g_{n-1}$ for every $n \in \mathbb{N}$. Then $f = \sum_{n=1}^{\infty} f_n$ pointwise. Also, for $(n_1, \ldots, n_k) \in F_1$ and $x \in K$ we have
\[
\sum_{i=1}^{k} |f_{n_{i-1}+1} + \cdots + f_{n_i}|(x) = \sum_{i=1}^{k} \varepsilon_i (f_{n_{i-1}+1} + \cdots + f_{n_i})(x)
\]
\[
= \sum_{i=1}^{k} \varepsilon_i (g_{n_i} - g_{n_{i-1}})(x) = \sum_{i=1}^{k} (\varepsilon_{i} - \varepsilon_{i+1}) g_{n_i}(x) \leq |f|_1 + \epsilon,
\]
where \( \varepsilon_i \in \{-1, 1\} \) so that \( \varepsilon_i(f_{n_i-1+1} + \cdots + f_{n_i}) (x) \geq 0 \) for all \( i = 1, \ldots, k \) and \( \varepsilon_{k+1} = 0 \). This gives that \( \|f\|_D^2 \leq \|f\|_4 \) for every \( f \in B_{1/4}(K) \).

On the other hand, let \( (g_n) \subseteq C(K) \) and \( C > 0 \) be such that \( f = \sum_{n=1}^{\infty} g_n \) pointwise and

\[
\left\| \sum_{i=1}^{k} |g_{n_i-1+1} + \cdots + g_{n_i}| \right\|_\infty \leq C \quad (n_0 = g_0 = 0)
\]

for every \( (n_1, \ldots, n_k) \in \mathcal{F}_1 \). Set \( f_n = \sum_{i=1}^{n} g_i \) for every \( n \in \mathbb{N} \). Of course \( f_n \to f \) pointwise. Also, for \( (n_1, \ldots, n_k) \in \mathcal{F}_1 \), \( x \in K \) and scalars \( \lambda_1, \ldots, \lambda_k \) we have

\[
\left| \sum_{i=1}^{k} \lambda_i f_{n_i} \right| (x) = \left| \sum_{i=1}^{k} \lambda_i (g_1 + \cdots + g_{n_i}) \right| (x)
= \left| \sum_{i=1}^{k} (\lambda_i + \cdots + \lambda_k) \cdot (g_{n_i-1+1} + \cdots + g_{n_i}) \right| (x)
\leq \left| \sum_{i=1}^{k} \sum_{j=i}^{k} \lambda_j \cdot (g_j \cdot \sum_{j=n_{i-1}+1}^{n_i} g_j) \right| (x)
\leq \left| \sum_{i=1}^{k} \lambda_i s_i \right| \cdot \left| \sum_{i=1}^{k} \sum_{j=n_{i-1}+1}^{n_i} g_j \right| (x) \leq C \cdot \left| \sum_{i=1}^{k} \lambda_i s_i \right|.
\]

Hence \( f \in B_{1/4}(K) \) and \( \|f\|_4 \leq \|f\|_D \). This completes the proof. \( \square \)

2.4. **Corollary.** For every compact metric space \( K \), a function \( f : K \to \mathbb{R} \) belongs to \( B_{1/4}(K) \) if and only if there exists \( (f_n) \) in \( C(K) \) such that \( f_n \to f \) pointwise and for \( n_0 = f_0 = 0 \),

\[
\sup \left\{ \left\| \sum_{i=1}^{k} f_{n_i} - f_{n_{i-1}} \right\|_\infty : (n_1, \ldots, n_k) \in \mathcal{F}_1 \right\} < \infty.
\]

Also, for every \( f \in B_{1/4}(K) \) we have

\[
\|f\|_4 \leq \inf \left\{ \sup \left\{ \left\| \sum_{i=1}^{k} f_{n_i} - f_{n_{i-1}} \right\|_\infty : (n_1, \ldots, n_k) \in \mathcal{F}_1 \right\} : \right. \]

\where \( (f_n) \subseteq C(K) \) and \( f_n \to f \) pointwise \}. \]

In the following theorem we will give a characterization of the functions in \( B_{1/4}(K) \) and also an identity for \( \|f\|_{1/4} \), where \( f \) is in \( B_{1/4}(K) \), using the transfinitely oscillations of \( f \), which have been defined by H. Rosenthal in [R1]. We recall this definition.

2.5. **Definition.** [R1] Let \( K \) be a metric space. One defines the upper semicontinuous envelope \( Ug \) of an extended real valued function \( g : K \to [-\infty, +\infty] \) as follows:

\[
Ug = \inf \{ h : K \to [-\infty, +\infty] : h \text{ is continuous and } h \geq g \}.
\]
It is easy to see that for \( x \in K \)
\[
\mathcal{U}g(x) = \lim_{y \to x} g(y) = \max \{ L \in [-\infty, +\infty] : \exists x_n \to x, g(x_n) \to L \}
= \inf \left\{ \sup_{y \in U} g(y) : U \text{ is a neighbourhood of } x \right\}.
\]

In [R1] the author associates with each bounded function \( f : K \to \mathbb{R} \) a transfinite increasing family \( (\text{osc}_\alpha f)_{1 \leq \alpha} \) of upper semicontinuous functions which are called \( \alpha \)-th oscillations of \( f \). They have been defined by induction as follows:
\[
\text{osc}_0 f = 0.
\]

If \( \text{osc}_\alpha f \) has been defined, then for every \( x \in K \)
\[
\text{osc}_{\alpha+1} f(x) = \lim_{y \to x} \left( |f(y) - f(x)| + \text{osc}_\alpha f(y) \right)
\]
and consequently
\[
\text{osc}_{\alpha+1} f = \mathcal{U} \text{osc}_{\alpha+1} f.
\]

If \( \alpha \) is a limit ordinal and \( \text{osc}_\beta f \) has been defined for all \( \beta < \alpha \) then
\[
\text{osc}_\alpha f = \sup_{\beta < \alpha} \text{osc}_\beta f
\]
and consequently
\[
\text{osc}_\alpha f = \mathcal{U} \text{osc}_\alpha f.
\]

According to [R2], a bounded function \( f : K \to \mathbb{R} \) is in \( D(K) \) if and only if \( \text{osc}_\alpha f \) is a bounded function for every ordinal \( \alpha \). In this case there exists an ordinal \( \alpha \) so that \( \text{osc}_\alpha f \) is bounded and \( \text{osc}_\alpha f = \text{osc}_\beta f \) for all \( \beta > \alpha \). Moreover, letting \( \tau \) be the least such \( \alpha \),
\[
\|f\|_D = \|f| + \text{osc}_\tau f\|_\infty.
\]

We will prove an analogous structural result for \( B_{1/4}(K) \). Precisely, we will prove that a bounded function \( f \) is in \( B_{1/4}(K) \) if and only if \( \text{osc}_\omega f \) is bounded and when this occurs then
\[
\|f\|_{1/4} = \|f| + \text{osc}_\omega f\|_\infty.
\]

Before the proof of this theorem we will give three lemmas. In the first lemma we list some elementary relations which are used in the sequel.

2.6. Lemma. Let \( f, g \) be bounded functions on a metric space \( K \) and \( \alpha \) an ordinal number.
(1) If \( f \leq g \) then \( \mathcal{U}f \leq \mathcal{U}g \).
(2) \( \mathcal{U}(f + g) \leq \mathcal{U}f + \mathcal{U}g \).
(3) \( \mathcal{U}(f - \mathcal{U}g) = \mathcal{U}(\mathcal{U}f - \mathcal{U}g) \leq \mathcal{U}(f - g) \).
(4) \( \mathcal{U}f = f \) if and only if \( f \) is upper semicontinuous.
(5) \( \text{osc}_\alpha f \) is an upper semicontinuous \([0, +\infty]\)-valued function on \( K \).
(6) \( \text{osc}_\alpha tf = |t|\text{osc}_\alpha f \) for every \( t \in \mathbb{R} \).
(7) \( \text{osc}_\alpha (f + g) \leq \text{osc}_\alpha f + \text{osc}_\alpha g \).
(8) \( \text{osc}_\alpha (f + g) = \text{osc}_\alpha f \) if \( g \) is a continuous function on \( K \).
(9) If \( \text{osc}_\alpha f \) is bounded then \( \mathcal{U}(\text{osc}_\alpha f \pm f) \leq \text{osc}_{\alpha+1} f \pm f \).
Thus it is proved that
\[ U(\text{osc}_\alpha f + f)(x) = \lim_{n \to \infty} \text{osc}_\alpha f(y_n) + f(y_n). \]

Since the functions \( f \) and \( \text{osc}_\alpha f \) are bounded, we may assume without loss of
generality that the limits
\[ \lim_{n \to \infty} \text{osc}_\alpha f(y_n), \lim_{n \to \infty} |f(y_n) - f(x)|, \lim_{n \to \infty} f(y_n) \]
all exist. We then have that
\[
\text{osc}_{\alpha + 1} f(x) \geq \lim_{n \to \infty} \left( |f(y_n) - f(x)| + \text{osc}_\alpha f(y_n) \right) \\
= \lim_{n \to \infty} |f(y_n) - f(x)| + \lim_{n \to \infty} \text{osc}_\alpha f(y_n) \\
\geq \lim_{n \to \infty} \left( \text{osc}_\alpha f(y_n) + f(y_n) \right) - f(x) \\
= U(\text{osc}_\alpha f + f)(x) - f(x).
\]

Thus it is proved that \( U(\text{osc}_\alpha f + f) \leq \text{osc}_{\alpha + 1} f + f \). If instead of \( f \) we use \(-f\), we
have that \( U(\text{osc}_\alpha f - f) \leq \text{osc}_{\alpha + 1} f - f \), since \( \text{osc}_\alpha f = \text{osc}_\alpha (-f) \).

2.7. **Lemma.** Let \( f : K \to \mathbb{R} \) be a bounded function. For every \( n \in \mathbb{N} \) we have that
\[ U(\text{osc}_{n+2} f - \text{osc}_{n+1} f) \leq U(\text{osc}_{n+1} f - \text{osc}_n f). \]

**Proof.** Using (3) of the previous lemma, we have that
\[
U(\text{osc}_{n+2} f - \text{osc}_{n+1} f) = U(\text{osc}_{n+2} f - \text{osc}_{n+1} f) \\
\leq U(\text{osc}_{n+2} f - \text{osc}_{n+1} f), \text{ for every } n \in \mathbb{N}.
\]

Hence it is sufficient to prove that
\[ U(\text{osc}_{n+2} f - \text{osc}_{n+1} f) \leq U(\text{osc}_{n+1} f - \text{osc}_n f) \text{ for every } n \in \mathbb{N}. \]

By (1) and (4) of the previous lemma, the proof of this lemma will be complete as soon as we prove that
\[ \text{osc}_{n+1} f - \text{osc}_{n+1} f \leq U(\text{osc}_{n+1} f - \text{osc}_n f) \text{ for every } n \in \mathbb{N}. \]

Case \( n = 0 \). We have for \( x \in K \),
\[
\text{osc}_1 f(x) - \text{osc}_1 f(x) = \lim_{y \to x} \left( \text{osc}_1 f(y) + |f(y) - f(x)| \right) - \lim_{y \to x} |f(y) - f(x)| \\
\leq \lim_{y \to x} \text{osc}_1 f(y) = U(\text{osc}_1 f)(x) = \text{osc}_1 f(x)
\]
(since \( \text{osc}_1 f \) is upper semicontinuous).

In general for \( n > 0, n \in \mathbb{N} \), we have for \( x \in K \),
\[
\text{osc}_{n+2} f(x) - \text{osc}_{n+1} f(x) \\
= \lim_{y \to x} \left( \text{osc}_{n+1} f(y) + |f(y) - f(x)| \right) - \lim_{y \to x} (\text{osc}_n f(y) + |f(y) - f(x)|) \\
\leq \lim_{y \to x}(\text{osc}_{n+1} f(y) - \text{osc}_n f(y)) = U(\text{osc}_{n+1} f - \text{osc}_n f)(x).
\]

This completes the proof.

The following lemma was proved by A. Louveau ([F-L]). For completeness we give the proof.
2.8. Lemma. [F-L] Let \((g_n)_{n=1}^\infty\) be a sequence of bounded, upper semicontinuous functions on a metric space \(K\) with \(g_0 = 0\). If the sequence \((\mathcal{U}(g_{n+1} - g_n))_{n=0}^\infty\) is decreasing, then \(\mathcal{U}(g_{n+1} - g_n) \leq \frac{1}{n+1} \cdot g_{n+1}\) for every \(n \in \mathbb{N}\).

Proof. For \(n = 0\), it reduces to \(\mathcal{U}g_1 \leq g_1\), which is trivial since \(g_1\) is upper semicontinuous. Suppose we know it for \(n\). For the induction step, it suffices, since \(g_{n+2}\) is use, to prove:

\[
g_{n+2} - g_{n+1} \leq \frac{g_{n+2} + g_{n+1}}{n+2},
\]

i.e., \(g_{n+2} \leq \frac{g_{n+2} + g_{n+1}}{n+2} + g_{n+1}\).

But since \(1 = \frac{1}{n+2} + \frac{n+1}{n+2}\), it suffices to show

\[
\frac{n+1}{n+2} g_{n+2} \leq g_{n+1}, \quad \text{i.e.,} \quad g_{n+2} \leq \frac{n+1}{n+2} g_{n+1} = g_{n+1} + \frac{1}{n+1} g_{n+1}.
\]

But this follows immediately from the induction step. \(\square\)

2.9. Theorem. Let \(K\) be a metric space. Then

\[
\|f\|_{1/4} = \|f + \hat{\text{osc}}_f\|_{\infty}
\]

for all \(f \in B_{1/4}(K)\).

Proof. Suppose \(f \in B_{1/4}(K)\). It follows from the definition of \(B_{1/4}(K)\) that for every \(\epsilon > 0\) one has a sequence \((g_n)\) in \(D(K)\) with \(\|g_n - f\|_{\infty} \to 0\) and \(\sup_n \|g_n\|_D < \|f\|_{1/4} + \epsilon\). Set \(\epsilon_n = \|g_n - f\|_{\infty}\). Then by induction on \(k\),

\[
\text{osc}_k f \leq \text{osc}_k g_n + 2k\epsilon_n \text{ for every } k, n \in \mathbb{N}.
\]

Hence

\[
|f| + \text{osc}_k f \leq |g_n| + \epsilon_n + \text{osc}_k g_n + 2k\epsilon_n
\]

\[
\leq |g_n| + \text{osc}_k g_n + (2k + 1)\epsilon_n
\]

\[
\leq \|g_n\|_D + (2k + 1)\epsilon_n \text{ for every } k, n \in \mathbb{N}.
\]

Letting first \(n \to \infty\) and then \(k \to +\infty\), we get

\[
|f| + \hat{\text{osc}}_f \leq \sup_n \|g_n\|_D \leq \|f\|_{1/4} + \epsilon.
\]

Since \(\epsilon\) is arbitrary, we have that

\[
\|f| + \hat{\text{osc}}_f\|_{\infty} \leq \|f\|_{1/4}
\]

and of course that \(\hat{\text{osc}}_f\) and, consequently, \(\text{osc}_f\) are bounded functions.

On the other hand, let \(f : K \to \mathbb{R}\) be a bounded function with \(\text{osc}_f\) also bounded. Set

\[
g_n = \frac{\lambda_n - \mathcal{U} (\text{osc}_n f - f)}{2} - \frac{\lambda_n - \mathcal{U} (\text{osc}_n f + f)}{2},
\]

where \(\lambda_n = \|f| + \text{osc}_n f\|_{\infty}\) for every \(n \in \mathbb{N}\). Then \(g_n \in D(K)\) and

\[
\|g_n\|_D \leq \|\lambda_n - \frac{1}{2} \mathcal{U} (\text{osc}_n f - f) - \frac{1}{2} (\text{osc}_n f + f)\|_{\infty} \leq \lambda_n \leq \|f| + \text{osc}_f\|_{\infty} \text{ for every } n \in \mathbb{N}.
\]

The first inequality holds for every \(n \in \mathbb{N}\), since from (1), (2) and (4) of Lemma 2.6 we have

\[
\mathcal{U} (\text{osc}_n f - f) + \mathcal{U} (\text{osc}_n f + f) \geq 2\mathcal{U} (\text{osc}_n f) = 2\text{osc}_n f \geq 0 \text{ and}
\]
\[ \lambda_n - \frac{1}{2} \| \text{osc}_{n} f + f \| - \frac{1}{2} \| \text{osc}_{n} f - f \| \geq \lambda_n - \| \text{osc}_{n} f + |f| \| \geq \lambda_n - \| \text{osc}_{n} f + |f| \| \infty \\
= \lambda_n - \| \text{osc}_{n} f + |f| \| \infty = 0. \]

If we could prove that \( \| g_n - f \| \infty \rightarrow 0 \), then we would have that \( f \in B_{1/4}(K) \) and \( \| f \|_{1/4} \leq \| f \| + \| \text{osc}_{\omega} f \| \infty \). Now, according to (9) of Lemma 2.6,

\[ g_n - f = \frac{1}{2} \| \text{osc}_{n} f + f \| - \frac{1}{2} \| \text{osc}_{n} f - f \| - f \leq \frac{1}{2} (\| \text{osc}_{n+1} f + f \| - \| \text{osc}_{n+1} f - f \| - f) \]

Thus \( \| g_n - f \| \infty \leq \frac{1}{n+1} \| \text{osc}_{\omega} f \| \infty \). This finishes the proof of the theorem.

**Remark.** Using the invariants \( (f_\alpha)_{1 \leq \alpha} \) which have been introduced by Kechris and Louveau in [KL] and which are similar to the \( \alpha^{10} \)-oscillations of the function \( f \), we proved with Louveau ([FL]) that a bounded function \( f \) is in \( B_{1/4}(K) \) if and only if \( f_\omega \) is bounded and in this case

\[ \frac{1}{3} \| f_\omega \| \infty \leq \| f \|_{1/4} \leq 4 \| f_\omega \| \infty + 5 \| f \| \infty. \]

But the previous theorem shows that the transfinite oscillations appear to be more appropriate than the \( f_\alpha \)'s.

After proving this theorem, I learned that H. Rosenthal ([R2]) had an analogous result. Precisely, he proved in [R2] that \( f \) belongs to \( B_{1/4}(K) \) (case \( f : K \rightarrow C \)) if and only if \( \text{osc}_\omega f \) is bounded and when this occurs and \( f \) is real valued,

\[ \frac{1}{2} (\| f \| \infty + \| \text{osc}_\omega f \| \infty) \leq \| f \|_{1/4} \leq \| f \| \infty + 3 \| \text{osc}_\omega f \| \infty. \]

3. A Classification of \( B_{1/4}(K) \)

We will define a classification of \( B_{1/4}(K) \), where \( K \) is a separable metric space, into a decreasing hierarchy \( (S_{\xi}(K))_{1 \leq \xi \leq \omega_1} \) of Banach spaces whose intersection is equal to \( D(K) \). The functions in \( S_{\xi}(K) \) have a property stronger than the one of the functions in \( B_{1/4}(K) \) which is described in Proposition 2.3. Precisely, the
families $\mathcal{F}_\xi$, which have been defined by D. Alspach and S. Argyros in [A-A], are used instead of the Schreier family $\mathcal{F}_1$. We quote the definition of the $\mathcal{F}_\xi$’s.

3.1. Definition ([A-A]). For every limit ordinal $\xi$, let $(\xi_n)$ be a sequence of ordinal numbers strictly increasing to $\xi$. Then $\mathcal{F}_0 = \{\{n\} : n \in \mathbb{N}\}$.

Suppose that $\mathcal{F}_\xi$ is defined, then

$$\mathcal{F}_{\xi+1} = \{F \subseteq \mathbb{N} : F \subseteq F_1 \cup \cdots \cup F_n \text{ with } \{n\} < F_1 < \cdots < F_n \text{ and } F_i \in \mathcal{F}_\xi \text{ for all } i = 1, \ldots, n\}.$$ 

If $\xi$ is a limit ordinal, $\mathcal{F}_\xi = \{F \subseteq \mathbb{N} : F \in \mathcal{F}_\xi, \text{ and } \{n\} \leq F\}$. Using the families $\mathcal{F}_\xi$, for every ordinal $\xi$, we extended the notion of spreading model in [F2] as follows:

3.2. Definition ([F2]). Let $X$ be a Banach space, $\xi$ an ordinal number and $(x_n)$ a sequence in $X$. We say that $(x_n)$ generates spreading model of order $\xi$ equivalent to a basic sequence $(e_n)$ if there exist $\mu > 0$ and $C > 0$ such that:

$$\mu \left\| \sum_{i=1}^{k} \lambda_i e_{n_i} \right\| \leq C \left\| \sum_{i=1}^{k} \lambda_i x_{n_i} \right\|$$

for every $(n_1, \ldots, n_k) \in \mathcal{F}_\xi$ and scalars $\lambda_1, \ldots, \lambda_k$.

Now we will define the spaces $S_\xi(K)$ for every ordinal $\xi$, which are characterized by spreading models of order $\xi$ equivalent to the summing basis $(s_n)$ of $c_0$.

3.3. Definition. Let $K$ be a metric space and $\xi$ an ordinal number. We define the space

$$S_\xi(K) = \left\{ f : K \rightarrow \mathbb{R} : \text{there exists } (f_n) \subseteq C(K) \text{ and } C > 0 \text{ such that } f_n \rightarrow f \text{ pointwise and } \left\| \sum_{i=1}^{k} \lambda_i f_{n_i} \right\|_\infty \leq C \left\| \sum_{i=1}^{k} \lambda_i s_i \right\| \text{ for every } (n_1, \ldots, n_k) \in \mathcal{F}_\xi \text{ and scalars } \lambda_1, \ldots, \lambda_k \right\}$$

and the norm $\| \cdot \|_\xi$ on it as follows:

$$\|f\|_\xi = \inf \left\{ C > 0 : \text{there exists } (f_n) \subseteq C(K) \text{ such that } f_n \rightarrow f \text{ pointwise and } \left\| \sum_{i=1}^{k} \lambda_i f_{n_i} \right\|_\infty \leq C \left\| \sum_{i=1}^{k} \lambda_i s_i \right\| \text{ for every } (n_1, \ldots, n_k) \in \mathcal{F}_\xi \text{ and scalars } \lambda_1, \ldots, \lambda_k \right\}$$

If $K$ is a compact metric space, it is easy to prove (see Remark 1.2) that

$$S_\xi(K) \setminus C(K) = \left\{ f : K \rightarrow \mathbb{R} : \text{there exists } (f_n) \text{ in } C(K) \text{ such that } f_n \rightarrow f \text{ pointwise and } (f_n) \text{ generates spreading model of order } \xi \text{ equivalent to } (s_n) \right\}.$$ 

Of course, $S_1(K) = B_{1/4}(K)$ for a compact metric space $K$. Also, for every ordinal number $\xi$, $S_\xi(K)$ is a linear subspace of $B_1(K)$. Although the family $(\mathcal{F}_\xi)_{\xi \leq \xi}$ is not increasing, it has the property: for every $1 \leq \beta < \xi$, there exists $n_0 = n_0(\beta, \xi)$ in $\mathbb{N}$ such that if $A \in \mathcal{F}_\beta$ and $\{n_0\} \leq A$ then $A \in \mathcal{F}_{\xi}$. Hence, it is easy to prove that the family $(S_\xi(K))_{\xi \leq \xi}$ is decreasing and, also, $\|f\|^2_\beta \leq \|f\|_\xi^2$ for every $1 \leq \beta < \xi$ and $f$ in $S_1(K)$.

3.4. Proposition. For every ordinal number $\xi$, $(S_\xi(K), \| \cdot \|_\xi)$ is a Banach space.
Proposition. For every metric space \( K \) and ordinal number \( \xi \), a function \( f : K \to \mathbb{R} \) belongs to \( S_\xi(K) \) if and only if there exists \( (f_n) \) in \( C(K) \) such that \( f = \sum_{n=1}^{\infty} f_n \) pointwise and for \( n_0 = f_0 = 0 \),

\[
\sup \left\{ \left\| \sum_{i=1}^{k} |f_{n_{i-1}+1} + \cdots + f_{n_i}| \right\|_\infty : (n_1, \ldots, n_k) \in \mathcal{F}_\xi \right\} < \infty.
\]
Also, for every \( f \in S_\xi(K) \),
\[
\|f\|_\xi = \inf \left\{ \sup \left\{ \left\| \sum_{i=1}^{k} f_{n_i} - f_{n_{i-1}} \right\|_\infty : (n_1, \ldots, n_k) \in \mathcal{F}_\xi \right\} : \right.
\]
for every \((f_n)\) in \(C(K)\) with \(f = \sum_{n} f_n\) pointwise.

Proof. The proof is analogous to the proof of Proposition 2.3. \(\square\)

3.6. Corollary. For every metric space \( K \) and ordinal number \( \xi \), a function \( f : K \to \mathbb{R} \) belongs to \( S_\xi(K) \) if and only if there exists \((f_n)\) in \(C(K)\) such that \(f_n \to f\) pointwise and for \(n_0 = f_0 = 0\),
\[
\sup \left\{ \left\| \sum_{i=1}^{k} f_{n_i} - f_{n_{i-1}} \right\|_\infty : (n_1, \ldots, n_k) \in \mathcal{F}_\xi \right\} < \infty.
\]

Also, for every \( f \in S_\xi(K) \),
\[
\|f\|_\xi = \inf \left\{ \sup \left\{ \left\| \sum_{i=1}^{k} f_{n_i} - f_{n_{i-1}} \right\|_\infty : (n_1, \ldots, n_k) \in \mathcal{F}_\xi \right\} : \right.
\]
for every \((f_n) \subseteq C(K)\) with \(f_n \to f\) pointwise.

From a result in [F2], we have the following connection between the functions in \( S_\xi(K) \) and the transfinite oscillations.

3.7. Theorem ([F2]). Let \( K \) be a metric space and \( \xi \) an ordinal number. Then
\[
S_\xi(K) \subseteq \left\{ f : K \to \mathbb{R} : \text{osc}_{\cdot,\xi} f \text{ is bounded} \right\}. \]

Proof. It follows from the proof of Theorem 9 in [F2] that, for every function \( f \) in \( S_\xi(K) \), the function \( u_{\alpha}(f) \) is bounded (the functions \( u_{\alpha}(f) \), were introduced in [R1] and are similar to the \( \alpha^{th} \) oscillations of \( f \)). But, as it is proved in [R1],
\[
\text{osc}_{\alpha} f \leq u_{\alpha}(f) + u_{\alpha}(-f)
\]
for every ordinal number \( \alpha \). Hence, \( \text{osc}_{\cdot,\xi} f \) is bounded.

This theorem yields immediately the following result. \(\square\)

3.8. Theorem. Let \( K \) be a separable metric space. The intersection of all the classes \( S_\xi(K) \), \(1 \leq \xi < \omega_1\), is equal to \( D(K) \).

Proof. It follows from the previous theorem and the fact that \( f \) belongs to \( D(K) \) if and only if \( \text{osc}_{\alpha} f \) is bounded for every countable ordinal \( \alpha \) ([R1]). \(\square\)

In [F2] we defined for every ordinal \( \xi \) the notion of a null-coefficient of order \( \xi \) (\( \xi \)-n.c.) sequence in a Banach space and we proved that every bounded, Baire-1 function \( f \) with \( \text{osc}_{\cdot,\xi} f \) unbounded has the property that every bounded sequence of continuous functions converging pointwise to \( f \) is null-coefficient of order \( \xi \). We will prove in the sequel that this property characterizes the functions in \( B_1(K) \setminus S_\xi(K) \).

3.9. Definition ([F2]). A sequence \((x_n)\) in a Banach space is called null-coefficient of order \( \xi \) (\( \xi \)-n.c.), where \( \xi \) is an ordinal number, if whenever the scalars \((c_n)\) satisfy:
\[
\sup \left\{ \left\| \sum_{i=1}^{k} c_{n_{2i}} (x_{n_{2i}} - x_{n_{2i-1}}) \right\| : (n_1, \ldots, n_{2k}) \in \mathcal{F}_\xi \right\} < \infty
\]
the sequence \((c_n)\) converges to 0.
3.10. **Proposition.** Let $\xi$ be an ordinal number, and $(x_n)$ a weak-Cauchy and non-weakly convergent sequence in a Banach space. Then $(x_n)$ is not null-coefficient of order $\xi$ if and only if it has a subsequence with spreading model of order $\xi$ equivalent to the summing basis of $c_0$.

**Proof.** If $(x_n)$ is not null-coefficient of order $\xi$ then there exists a bounded sequence of scalars $(c_n)$ such that $(c_n)$ is not null-converging and

$$(*) \quad \left\| \sum_{i=1}^{k} c_{n_{2i}} (x_{n_{2i}} - x_{n_{2i-1}}) \right\| \leq 1$$

for every $(n_1, \ldots, n_{2k}) \in \mathcal{F}_\xi$.

So we can find $\epsilon > 0$ and a subsequence $(c_{n_i})$ of $(c_n)$ such that $c_{n_i} > \epsilon$ for every $t \in \mathbb{N}$ (otherwise replace $c_n$ by $-c_n$).

Consider $x_n, n \in \mathbb{N}$, as elements of $C(K)$, where $K$ is the unit ball of the dual of $X = [x_n]$, the closed subspace generated by $(x_n)$, with respect to the weak* topology. Since $(x_n)$ converges with respect to the $w^*$-topology to a function $x^{**} \in X^{**} \setminus X$ (Remark 1.2) there exists a subsequence $(x_{n_s})$ of $(x_n)$ and $\mu > 0$ such that

$$\mu \left\| \sum_{i=1}^{k} \lambda_i s_i \right\| \leq \left\| \sum_{i=1}^{k} \lambda_i x_{n_{t_i}} \right\|$$

for every $k \in \mathbb{N}$ and scalars $\lambda_1, \ldots, \lambda_k$. Set $y_s = x_{n_s}$ and $c_{n_s} = a_s$ for every $s \in \mathbb{N}$.

We will prove that the subsequence $(y_s)$ of $(x_n)$ has spreading model of order $\xi$ equivalent to the summing basis $(s_n)$ of $c_0$. Indeed, for every $(s_1, \ldots, s_k) \in \mathcal{F}_\xi$ and $x \in K$ we have $y_{s_0} = y_0 = 0$ and

$$\sum_{i=1}^{k} |y_{s_i} - y_{s_{i-1}}|(x) \leq \frac{1}{\epsilon} \sum_{i=1}^{k} a_{s_i} |y_{s_i} - y_{s_{i-1}}|(x)$$

$$= \frac{1}{\epsilon} \left( \sum_{i=1}^{k} a_{s_i} \cdot \varepsilon_{s_i} (y_{s_i} - y_{s_{i-1}}) \right)(x) \quad \text{(where } \varepsilon_{s_i} \in \{-1, 1\}\text{)}$$

$$\leq \frac{1}{\epsilon} a_{s_1} \|y_{s_1}\| + \frac{1}{\epsilon} \sum_{i=2}^{k} a_{s_i} (y_{s_i} - y_{s_{i-1}}) \quad \left\| \sum_{i=2}^{k} a_{s_{1{i}}} (y_{s_{1{i}}} - y_{s_{1{i-1}}}) \right\| (x)$$

$$+ \frac{1}{\epsilon} \sum_{i=2}^{k} a_{s_{1{i}}} (y_{s_{1{i}}} - y_{s_{1{i-1}}}) \quad \left\| \sum_{i=2}^{k} a_{s_{1{i}}} (y_{s_{1{i}}} - y_{s_{1{i-1}}}) \right\| (x)$$

$$+ \frac{1}{\epsilon} \sum_{i=even}^{\infty} a_{s_{1{i}}} (y_{s_{1{i}}} - y_{s_{1{i-1}}}) \quad \left\| \sum_{i=even}^{\infty} a_{s_{1{i}}} (y_{s_{1{i}}} - y_{s_{1{i-1}}}) \right\| (x) \leq \frac{4}{\epsilon} + \frac{1}{\epsilon} \cdot ||(c_n)||_\infty \cdot ||(x_n)\|_\infty = C.$$
Finally, for every \((s_1, \ldots, s_k) \in \mathcal{F}_\xi\) and scalars \(\lambda_1, \ldots, \lambda_k\) we have
\[
\left\| \sum_{i=1}^{k} \lambda_i y_{s_i} \right\| = \left\| \sum_{i=1}^{k} (\lambda_1 + \cdots + \lambda_k)(y_{s_i} - y_{s_{i-1}}) \right\| \leq C \left\| \sum_{i=1}^{k} \lambda_i s_i \right\|,
\]
which completes the proof. \(\Box\)

3.11. **Theorem.** Let \(K\) be a metric space and \(\xi\) an ordinal number. Then
\[
B_1(K) \setminus S_\xi(K) = \left\{ f \in B_1(K) : \text{every bounded sequence } (f_n) \text{ in } C(K) \text{ converging pointwise to } f \text{ is null-coefficient of order } \xi \right\}.
\]

**Proof.** Let \(f \in B_1(K) \setminus S_\xi(K)\) and a bounded sequence \((f_n)\) in \(C(K)\) converging pointwise to \(f\). Then \((f_n)\) is null-coefficient of order \(\xi\). Indeed, if \((f_n)\) is not \(\xi\)-n.c., then according to the proof of the previous proposition, we can find a subsequence \((g_n)\) of \((f_n)\) and \(C > 0\) such that
\[
\left\| \sum_{i=1}^{k} |f_{n_i} - f_{n_{i-1}}| \right\|_\infty \leq C
\]
for all \((n_1, \ldots, n_k) \in \mathcal{F}_\xi\). Hence, it follows from Corollary 3.6 that \(f \in S_\xi(K)\), a contradiction.

On the other hand, if \(f \in S_\xi(K)\) then there exists a sequence \((f_n) \subseteq C(K)\) converging pointwise to \(f\) and \(C > 0\) such that
\[
\left\| \sum_{i=1}^{k} |f_{n_i} - f_{n_{i-1}}| \right\|_\infty \leq C
\]
for every \((n_1, \ldots, n_k) \in \mathcal{F}_\xi\), according to Corollary 3.6. Thus, if \(c_n = 1\) for every \(n \in \mathbb{N}\), we have
\[
\left\| \sum_{i=1}^{k} (f_{n_{2i}} - f_{n_{2i-1}}) \right\|_\infty \leq \left\| \sum_{i=1}^{k} |f_{n_{2i}} - f_{n_{2i-1}}| \right\|_\infty \leq \sum_{i=1}^{2k} |f_{n_i} - f_{n_{i-1}}| \leq C
\]
for every \((n_1, \ldots, n_{2k}) \in \mathcal{F}_\xi\). Hence \((f_n)\) is not null-coefficient of order \(\xi\).

This completes the proof. \(\Box\)

3.12. **Corollary.** Let \(K\) be a compact metric space. Then
\[
B_1(K) \setminus B_{1/4}(K) = \left\{ f \in B_1(K) : \text{every bounded sequence } (f_n) \text{ in } C(K) \text{ converging pointwise to } f \text{ is null-coefficient of order } 1 \right\}.
\]

**References**


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