ON THE STRONG EQUALITY BETWEEN SUPERCOMPACTNESS AND STRONG COMPACTNESS

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ABSTRACT. We show that supercompactness and strong compactness can be equivalent even as properties of pairs of regular cardinals. Specifically, we show that if $V \models ZFC + GCH$ is a given model (which in interesting cases contains instances of supercompactness), then there is some cardinal and cofinality preserving generic extension $V[G] \models ZFC + GCH$ in which, (a) (preservation) for $\kappa \leq \lambda$ regular, if $V \models "\kappa$ is $\lambda$ supercompact", then $V[G] \models "\kappa$ is $\lambda$ supercompact" and so that, (b) (equivalence) for $\kappa \leq \lambda$ regular, $V[G] \models "\kappa$ is $\lambda$ strongly compact" iff $V[G] \models "\kappa$ is $\lambda$ supercompact", except possibly if $\kappa$ is a measurable limit of cardinals which are $\lambda$ supercompact.

0. Introduction and Preliminaries

It is a well known fact that the notion of strongly compact cardinal represents a singularity in the hierarchy of large cardinals. The work of Magidor [Ma1] shows that the least strongly compact cardinal and the least supercompact cardinal can coincide, but also, the least strongly compact cardinal and the least measurable cardinal can coincide. The work of Kimchi and Magidor [KiM] generalizes this, showing that the class of strongly compact cardinals and the class of supercompact cardinals can coincide (except by results of Menas [Me] and [A] at certain measurable limits of supercompact cardinals), and the first $n$ strongly compact cardinals (for $n$ a natural number) and the first $n$ measurable cardinals can coincide. Thus, the precise identity of certain members of the class of strongly compact cardinals cannot be ascertained vis à vis the class of measurable cardinals or the class of supercompact cardinals.

An interesting aspect of the proofs of both [Ma1] and [KiM] is that in each result, all “bad” instances of strong compactness are not obliterated. Specifically, in each model, since the strategy employed in destroying strongly compact cardinals which aren’t also supercompact is to make them non-strongly compact after a certain point either by adding a Prikry sequence or a non-reflecting stationary set of ordinals of the appropriate cofinality, there may be cardinals $\kappa$ and $\lambda$ so that $\kappa$ is $\lambda$ strongly

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compact yet $\kappa$ isn’t $\lambda$ supercompact. Thus, whereas it was proven by Kimchi and Magidor that the classes of strongly compact and supercompact cardinals can coincide (with the exceptions noted above), it was not known whether a “local” version of this was possible, i.e., if it were possible to obtain a model in which, for the class of pairs $(\kappa, \lambda)$, $\kappa$ is $\lambda$ strongly compact iff $\kappa$ is $\lambda$ supercompact. This is more delicate.

The purpose of this paper is to answer the above question in the affirmative. Specifically, we prove the following

**Theorem.** Suppose $V \models \text{ZFC + GCH}$ is a given model (which in interesting cases contains instances of supercompactness). There is then some cardinal and cofinality preserving generic extension $V[G] \models \text{ZFC + GCH}$ in which:

(a) (Preservation) For $\kappa \leq \lambda$ regular, if $V \models \text{“}\kappa$ is $\lambda$ supercompact”, then $V[G] \models \text{“}\kappa$ is $\lambda$ supercompact”. The converse implication holds except possibly when $\kappa = \sup\{\delta < \kappa : \delta$ is $\lambda$ supercompact$\}$.

(b) (Equivalence) For $\kappa \leq \lambda$ regular, $V[G] \models \text{“}\kappa$ is $\lambda$ strongly compact” iff $V[G] \models \text{“}\kappa$ is $\lambda$ supercompact”, except possibly if $\kappa$ is a measurable limit of cardinals which are $\lambda$ supercompact.

Note that the limitation given in (b) above is reasonable, since trivially, if $\kappa$ is measurable, $\kappa < \lambda$, and $\kappa = \sup\{\delta < \kappa : \delta$ is either $\lambda$ supercompact or $\lambda$ strongly compact$\}$, then $\kappa$ is $\lambda$ strongly compact. Further, it is a theorem of Menas [Me] that under GCH, for $\kappa$ the first, second, third, or $\omega$th for $\alpha < \kappa$ measurable limit of cardinals which are $\kappa^+$ strongly compact or $\kappa^+$ supercompact, $\kappa$ is $\kappa^+$ strongly compact yet $\kappa$ isn’t $\kappa^+$ supercompact. Thus, if there are sufficiently large cardinals in the universe, it will never be possible to have a complete coincidence between the notions of $\kappa$ being $\lambda$ strongly compact and $\kappa$ being $\lambda$ supercompact for a $\lambda$ regular cardinal.

Note that in the statement of our Theorem, we do not mention what happens if $\lambda > \kappa$ is a singular cardinal. This is since the behavior when $\lambda > \kappa$ is a singular cardinal is provable in $\text{ZFC + GCH}$ (which implies any limit cardinal is a strong limit cardinal). Specifically, if $\lambda > \kappa$ is so that $\text{cof}(\lambda) < \kappa$, then by a theorem of Magidor [Ma3], $\kappa$ is supercompact iff $\kappa$ is $\lambda^+$ supercompact, so automatically, by clause (a) of our Theorem, $\lambda$ supercompactness is preserved between $V$ and $V[G]$. Also, if $\lambda > \kappa$ is so that $\text{cof}(\lambda) < \kappa$, then by a theorem of Solovay [SRK], $\kappa$ is $\lambda$ strongly compact iff $\kappa$ is $\lambda^+$ strongly compact, so by clause (b) of our Theorem, it can never be the case that $V[G] \models \text{“}\kappa$ is $\lambda$ strongly compact” unless $V[G] \models \text{“}\kappa$ is $\lambda$ supercompact” as well. Further, if $\lambda > \kappa$ is so that $\lambda > \text{cof}(\lambda) \geq \kappa$, then it is not too difficult to see (and will be shown in Section 2) that if $\kappa$ is $\lambda^+$ strongly compact or $\lambda^+$ supercompact for all $\lambda' < \lambda$, then $\kappa$ is $\lambda$ strongly compact, and there is no reason to believe $\kappa$ must be $\lambda$ supercompact. In fact, it is a theorem of Magidor [Ma4] (irrespective of GCH) that if $\mu$ is a supercompact cardinal, there will always be many cardinals $\kappa, \lambda < \mu$ so that $\lambda > \kappa$ is a singular cardinal of cofinality $\geq \kappa$, $\lambda$ is $\lambda$ strongly compact, $\kappa$ is $\lambda^+$ supercompact for all $\lambda' < \lambda$, yet $\kappa$ isn’t $\lambda$ supercompact. Thus, there can never be a complete coincidence between the notions of $\kappa$ being $\lambda$ strongly compact and $\kappa$ being $\lambda$ supercompact if $\lambda > \kappa$ is an arbitrary cardinal, assuming there are supercompact cardinals in the universe.

The structure of this paper is as follows. Section 0 contains our introductory comments and preliminary material concerning notation, terminology, etc. Section 1 defines and discusses the basic properties of the forcing notion used in the iteration
we employ to construct our final model. Section 2 gives a complete statement and proof of the theorem of Magidor mentioned in the above paragraph and proves our Theorem in the case for which there is one supercompact cardinal $\kappa$ in the universe which contains no strongly inaccessible cardinals above it. Section 3 shows how the ideas of Section 2 can be used to prove the Theorem in the general case. Section 4 contains our concluding remarks.

Before beginning the material of Section 1, we briefly mention some preliminary information. Essentially, our notation and terminology are standard, and when this is not the case, this will be clearly noted. We take this opportunity to mention the universe obtained by forcing with $P$. If $x \in V[G]$, then $x$ will be a term in $V$ for $x$. We may, from time to time, confuse terms with the sets they denote and write $x$ when we actually mean $\bar{x}$, especially when $x$ is some variant of the generic set $G$.

If $\kappa$ is a cardinal, then for $P$ a partial ordering, $P$ is $(\kappa, \infty)$-distributive if for any sequence $\langle D_\alpha : \alpha < \kappa \rangle$ of dense open subsets of $P$, $D = \bigcap_{\alpha < \kappa} D_\alpha$ is a dense open subset of $P$. $P$ is $\kappa$-closed if given a sequence $\langle p_\alpha : \alpha < \kappa \rangle$ of elements of $P$ so that $\beta < \gamma < \kappa$ implies $p_\beta \leq p_\gamma$ (an increasing chain of length $\kappa$), then there is some $\alpha < \kappa$, $p_\alpha \leq p$ for all $\alpha < \kappa$. $P$ is $\kappa$-directed closed if for every cardinal $\delta < \kappa$ and every directed set $\langle p_\alpha : \alpha < \delta \rangle$ of elements of $P$ (where $\langle p_\alpha : \alpha < \delta \rangle$ is directed if for every two distinct elements $p_\alpha, p_\beta \in \langle p_\alpha : \alpha < \delta \rangle$, $p_\alpha$ and $p_\beta$ have a common upper bound) there is an upper bound $p \in P$. $P$ is $\kappa$-strategically closed if in the two person game in which the players construct an increasing sequence $\langle p_\alpha : \alpha < \kappa \rangle$, where player I plays odd stages and player II plays even and limit stages, then player II has a strategy which ensures the game can always be continued. $P$ is $\kappa$-strategically closed if $P$ is $\delta$-strategically closed for all cardinals $\delta < \kappa$. $P$ is $\kappa$-directed closed if $P$ is $\kappa$-strategically closed if in the two person game in which the players construct an increasing sequence $\langle p_\alpha : \alpha < \kappa \rangle$, where player I plays odd stages and player II plays even and limit stages, then player II has a strategy which ensures the game can always be continued. Note that trivially, if $P$ is $\kappa$-closed, then $P$ is $\kappa$-strategically closed and $\kappa^+$-strategically closed. The converse of both of these facts is false.

For $\kappa$ a regular cardinal, two partial orderings to which we will refer quite a bit are the standard partial orderings $Q_\lambda^\kappa$ for adding a Cohen subset to $\kappa^+$ using conditions having support $\kappa$ and $Q_\lambda^\kappa$ for adding $\kappa^+$ many Cohen subsets to $\kappa$ using conditions having support $< \kappa$. The basic properties and explicit definitions of these partial orderings may be found in [J].

Finally, we mention that we are assuming complete familiarity with the notions of strong compactness and supercompactness. Interested readers may consult [SRK] or [KaM] for further details. We note only that all elementary embeddings witnessing the $\lambda$ supercompactness of $\kappa$ are presumed to come from some fine, $\kappa$-complete, normal ultrafilter $U$ over $P_\kappa(\lambda) = \{x \subseteq \lambda : |x| < \kappa\}$. Also, where appropriate, all
ultralogies via a supercompact ultrafilter over $P_\kappa(\lambda)$ will be confused with their transitive isomorphs.

1. The Forcing Conditions

In this section, we describe and prove the basic properties of the forcing conditions we shall use in our later iteration. Let $\delta < \lambda$, $\lambda \geq \aleph_1$ be regular cardinals in our ground model $V$.

We define three notions of forcing. Our first notion of forcing $P_{\delta,\lambda}^0$ is just the standard notion of forcing for adding a non-reflecting stationary set of ordinals of cofinality $\delta$ to $\lambda^+$. Specifically, $P_{\delta,\lambda}^0 = \{ p : \text{For some } \alpha < \lambda^+, p : \alpha \rightarrow \{0,1\} \text{ is a characteristic function of } S_p, \text{ a subset of } \alpha \text{ not stationary at its supremum nor having any initial segment which is stationary at its supremum, so that } \beta \in S_p \text{ implies } \beta > \delta \text{ and } \text{cof}(\beta) = \delta \}$, ordered by $q \geq p$ iff $q \supseteq p$ and $S_p = S_q \cup \text{sup}(\{ S_p \})$, i.e., $S_q$ is an end extension of $S_p$. It is well-known that for $G$-$V$-generic over $P_{\delta,\lambda}^0$ (see [Bu] or [KLM]), in $V[G]$, a non-reflecting stationary set $S = S[G] = \bigcup \{ S_p : p \in G \} \subseteq \lambda^+$ of ordinals of cofinality $\delta$ has been introduced, the bounded subsets of $\lambda^+$ are the same as those in $V$, and cardinals, cofinalities, and GCH have been preserved. It is also virtually immediate that $P_{\delta,\lambda}^0$ is $\delta$-directed closed.

Work now in $V_1 = V^{P_{\delta,\lambda}^0}$, letting $\dot{S}$ be a term always forced to denote the above set $S$. $P_{\delta,\lambda}^0[S]$ is the standard notion of forcing for introducing a club set $C$ which is disjoint to $S$ (and therefore makes $S$ non-stationary). Specifically, $P_{\delta,\lambda}^0[S] = \{ p : \text{For some successor ordinal } \alpha < \lambda^+, p : \alpha \rightarrow \{0,1\} \text{ is a characteristic function of } C_p, \text{ a club subset of } \alpha, \text{ so that } C_p \cap S = \emptyset \}$, ordered by $q \geq p$ iff $C_q$ is an end extension of $C_p$. It is again well-known (see [MS]) that for $H$-$V_1$-generic over $P_{\delta,\lambda}^0[S]$, a club set $C = C[H] = \bigcup \{ C_p : p \in H \} \subseteq \lambda^+$ which is disjoint to $S$ has been introduced, the bounded subsets of $\lambda^+$ are the same as those in $V_1$, and cardinals, cofinalities, and GCH have been preserved.

Before defining in $V_1$ the partial ordering $P_{\delta,\lambda}^1[S]$ which will be used to destroy strong compactness, we first prove two preliminary lemmas.

**Lemma 1.** $\vdash_{P_{\delta,\lambda}^0}[\dot{S}]”, i.e., $V_1 \models \text{“There is a sequence } \langle x_\alpha : \alpha \in S \rangle \text{ so that for each } \alpha \in S, x_\alpha \subseteq \alpha \text{ is cofinal in } \alpha, \text{ and for any } A \in [\lambda^+]^{\lambda^+}, \{ \alpha \in S : x_\alpha \subseteq A \} \text{ is stationary”}.

**Proof of Lemma 1.** Since $V \models \text{GCH}$ and $V$ and $V_1$ contain the same bounded subsets of $\lambda^+$, we can let $\langle y_\alpha : \alpha < \lambda^+ \rangle \in V$ be a listing of all elements $x \in ([\lambda^+]^{\delta})^V = ([\lambda^+]^{\delta})^V_i$ so that each $x \in [\lambda^+]^{\delta}$ appears on this list $\lambda^+$ times at ordinals of cofinality $\delta$, i.e., for any $x \in [\lambda^+]^{\delta}$, $\lambda^+ = \sup \{ \alpha < \lambda^+ : \text{cof}(\alpha) = \delta \}$ and $y_\alpha = x$. This then allows us to define $\langle x_\alpha : \alpha \in S \rangle$ by letting $x_\alpha$ be $y_\beta$ for the least $\beta \in S - (\alpha + 1)$ so that $y_\beta \subseteq \alpha$ and $y_\beta$ is unbounded in $\alpha$. By genericity, each $x_\alpha$ is well-defined.

Now let $p \in P_{\delta,\lambda}^0$ be so that $p\vdash \check{\dot{A}} \in [\lambda^+]^{\lambda^+}$ and $\dot{K} \subseteq \lambda^+$ is club”. We show that for some $r \geq p$ and some $\zeta < \lambda^+$, $r\vdash \check{\zeta} \in \dot{K} \cap \dot{S}$ and $\dot{x}_\zeta \subseteq \dot{A}^\zeta$. To do this, we inductively define an increasing sequence $\langle p_\alpha : \alpha < \delta \rangle$ of elements of $P_{\delta,\lambda}^0$ and increasing sequences $\langle \beta_\alpha : \alpha < \delta \rangle$ and $\langle \gamma_\alpha : \alpha < \delta \rangle$ of ordinals $\leq \lambda^+$ so that $\beta_0 \leq \gamma_0 \leq \beta_1 \leq \gamma_1 \leq \cdots \leq \beta_\alpha \leq \gamma_\alpha \leq \cdots (\alpha < \delta)$. We begin by letting $p_0 = p$ and $\beta_0 = \gamma_0 = 0$. For $\eta = \alpha + 1 < \delta$ a successor, let $p_\eta \geq p_\alpha$ and $\beta_\eta \leq \gamma_\eta$,
\( \beta_\eta \geq \max(\beta_\alpha, \gamma_\alpha, \sup(\text{dom}(p_\alpha))) + 1 \) be so that \( p_\eta \models \\{ \beta_\eta \in \mathbf{A} \text{ and } \gamma_\eta \in \mathbf{K} \} \). For \( \rho < \delta \) a limit, let \( p_\rho = \bigcup_{\alpha < \rho} p_\alpha \), \( \beta_\rho = \bigcup_{\alpha < \rho} \beta_\alpha \), and \( \gamma_\rho = \bigcup_{\alpha < \rho} \gamma_\alpha \). Note that since \( \rho < \delta, p_\rho \) is well-defined, and since \( \delta < \lambda^+ \), \( \beta_\rho, \gamma_\rho < \lambda^+ \). Also, by construction, \( \bigcup_{\alpha < \delta} \beta_\alpha = \bigcup_{\alpha < \delta} \gamma_\alpha = \bigcup_{\alpha < \delta} \sup(\text{dom}(p_\alpha)) < \lambda^+ \). Call \( \zeta \) this common sup. We thus have that \( q = \bigcup_{\alpha < \delta} p_\alpha \cup \{ \zeta \} \) is a well-defined condition, so that \( q \models \{ \beta_\alpha : \alpha \in \delta - \{ 0 \} \} \subseteq \mathbf{A} \) and \( \zeta \in \mathbf{K} \cap \mathbf{S}^\ast \).

To complete the proof of Lemma 1, we know that as \( \langle \beta_\alpha : \alpha \in \delta - \{ 0 \} \rangle \in V \) and as each \( \eta \in \{ y_\alpha : \alpha < \lambda^+ \} \) must appear \( \lambda^+ \) times at ordinals of cofinality \( \delta \), we can find some \( \eta \in (\zeta, \lambda^+) \) so that \( \text{cof}(\eta) = \delta \) and \( \langle \beta_\alpha : \alpha \in \delta - \{ 0 \} \rangle \models y_\eta \). If we let \( r \geq q \) be so that \( r \models \{ \mathcal{S} \cap [\zeta, \eta] = \{ \zeta, \eta \} \} \), then \( r \models \{ x_\zeta = y_\eta = \langle \beta_\alpha : \alpha \in \delta - \{ 0 \} \rangle \} \). This proves Lemma 1. \( \square \) Lemma 1

We fix now in \( V_1 \) a \( \blacktriangle(S) \) sequence \( X = \langle x_\alpha : \alpha \in S \rangle \).

**Lemma 2.** Let \( S' \) be an initial segment of \( S \) so that \( S' \) is not stationary at its supremum nor has any initial segment which is stationary at its supremum. There is then a sequence \( \langle y_\alpha : \alpha \in S' \rangle \) so that for every \( \alpha \in S' \), \( y_\alpha \subseteq x_\alpha \), \( x_\alpha - y_\alpha \) is bounded in \( \alpha \), and if \( \alpha_1 \neq \alpha_2 \in S' \), then \( y_{\alpha_1} \cap y_{\alpha_2} = \emptyset \).

**Proof of Lemma 2.** We define by induction on \( \alpha \leq \alpha_0 = \sup S' + 1 \) a function \( h_\alpha \) so that \( \text{dom}(h_\alpha) = S' \cap \alpha \), \( h_\alpha(\beta) < \beta \), and \( \langle x_\beta - h_\alpha(\beta) : \beta \in S' \cap \alpha \rangle \) is pairwise disjoint. The sequence \( \langle x_\beta - h_\alpha(\beta) : \beta \in S' \rangle \) will be our desired sequence.

If \( \alpha = 0 \), then we take \( h_\alpha \) to be the empty function. If \( \alpha = \beta + 1 \) and \( \beta \notin S' \), then we take \( h_\alpha = h_\beta \). If \( \alpha = \beta + 1 \) and \( \beta \in S' \), then we notice that since each \( x_\gamma \in X \) has order type \( \delta \) and is cofinal in \( \gamma \), for all \( \gamma \in S' \cap \beta \), \( x_\beta \cap \gamma \) is bounded in \( \gamma \). This allows us to define a function \( h_\alpha \) having domain \( S' \cap \alpha \) by \( h_\alpha(\beta) = 0 \), and for \( \gamma \in S' \cap \beta \), \( h_\alpha(\gamma) = \min\{ \rho : \rho < \gamma, \rho \geq \beta_\gamma(\gamma), \text{ and } x_\beta \cap \gamma \subseteq \rho \} \).

By the next to last sentence and the induction hypothesis on \( h_\beta \), \( h_\alpha(\gamma) < \gamma \). And, if \( \gamma_1 < \gamma_2 \in S' \cap \alpha \), then \( \gamma_2 < \beta \), \( (x_{\gamma_1} - h_\alpha(\gamma_1)) \cap (x_{\gamma_2} - h_\alpha(\gamma_2)) \subseteq (x_{\gamma_1} - h_\beta(\gamma_1)) \cap (x_{\gamma_2} - h_\beta(\gamma_2)) \subseteq \emptyset \) by the induction hypothesis on \( h_\beta \). If \( \gamma_2 = \beta \), then \( (x_{\gamma_1} - h_\alpha(\gamma_1)) \cap (x_{\gamma_2} - h_\alpha(\gamma_2)) = (x_{\gamma_1} - h_\alpha(\gamma_1)) \cap x_{\gamma_2} = \emptyset \) by the definition of \( h_\alpha(\gamma_1) \). The sequence \( \langle x_\gamma - h_\alpha(\gamma) : \gamma \in S' \cap \alpha \rangle \) is thus as desired.

If \( \alpha \) is a limit ordinal, then as \( S' \) is non-stationary at its supremum nor has any initial segment which is stationary at its supremum, we can let \( \langle \beta_\gamma : \gamma < \text{cof}(\alpha) \rangle \) be a strictly increasing, continuous sequence having sup \( \alpha \) so that for all \( \gamma < \text{cof}(\alpha) \), \( \beta_\gamma \notin S' \). Thus, if \( \rho \in S' \cap \alpha \), then \( \{ \beta_\gamma : \beta_\gamma < \rho \} \) is bounded in \( \rho \), meaning we can find some largest \( \gamma \) so that \( \beta_\gamma < \rho \). It is also the case that \( \rho < \beta_{\gamma + 1} \). This allows us to define \( h_\alpha(\rho) = \max\{ h_{\beta_{\gamma + 1}}(\rho), \beta_\gamma \} \) for the \( \gamma \) just described. It is still the case that \( h_\alpha(\rho) < \rho \). And, if \( \rho_1, \rho_2 \in \langle \beta_\gamma, \beta_{\gamma + 1} \rangle \), then \( (x_{\rho_1} - h_\alpha(\rho_1)) \cap (x_{\rho_2} - h_\alpha(\rho_2)) \subseteq (x_{\rho_1} - h_{\beta_{\gamma + 1}}(\rho_1)) \cap (x_{\rho_2} - h_{\beta_{\gamma + 1}}(\rho_2)) = \emptyset \) by the definition of \( h_{\beta_{\gamma + 1}} \). If \( \rho_1 \in \langle \beta_\gamma, \beta_{\gamma + 1} \rangle \), \( \rho_2 \in \langle \beta_{\sigma}, \beta_{\sigma + 1} \rangle \) with \( \gamma < \sigma \), then \( (x_{\rho_1} - h_\alpha(\rho_1)) \cap (x_{\rho_2} - h_\alpha(\rho_2)) \subseteq x_{\rho_1} \cap (x_{\rho_2} - \beta_\sigma) \subseteq \rho_1 - \beta_\sigma \subseteq \rho_1 - \beta_{\gamma + 1} = \emptyset \). Thus, the sequence \( \langle x_\rho - h_\alpha(\rho) : \rho \in S' \cap \alpha \rangle \) is again as desired. This proves Lemma 2. \( \square \) Lemma 2

At this point, we are in a position to define in \( V_1 \) the partial ordering \( P_{1.\lambda}^1[S] \), which will be used to destroy strong compactness. \( P_{1.\lambda}^1[S] \) is now the set of all \( 4 \)-tuples \( \langle w, \alpha, \bar{r}, Z \rangle \) satisfying the following properties.
Assume to the contrary that

Proof of Lemma 3.

Since we will only be concerned in general with the case when \( \kappa \) is strongly inaccessible and \( \delta < \kappa < \lambda \), we assume without loss of generality that this is the case throughout the rest of the paper.

Lemma 3. \( V_{1}^{P_{\delta,\lambda}[S]} \models \text{“} \kappa \text{ is not } \lambda \text{ strongly compact} \text{”} \) if \( \delta < \kappa < \lambda \).

Remark. Since we will only be concerned in general with the case when \( \kappa \) is strongly inaccessible and \( \delta < \kappa < \lambda \), we assume without loss of generality that this is the case throughout the rest of the paper.

Proof of Lemma 3. Assume to the contrary that \( V_{1}^{P_{\delta,\lambda}[S]} \models \text{“} \kappa \text{ is } \lambda \text{ strongly compact} \text{”} \), and by our earlier remarks, let \( p \models \text{“} D \text{ is a } \kappa \text{-additive uniform ultrafilter over } \lambda \text{”} \). We show that \( p \) can be extended to a condition \( q \) so that for some ordinal \( \alpha^q < \lambda \) and some \( \delta \) sequence \( \langle s_i : i < \delta \rangle \) of \( D \) measure 1 sets, \( q \models \text{“} \bigcap_{i < \delta} s_i \subseteq \alpha^q \text{”} \), an immediate contradiction.
We use a \(\Delta\)-system argument to establish this. First, for \(G_1, V_1\)-generic over \(P_{\lambda,\lambda}[S]\) and \(i < \lambda^+,\) let \(r_i^* = \bigcup\{r_i^p : 3p = \langle w^p, \alpha^p, \bar{r}^p, Z^p \rangle \in G_1[r_i^p \in \bar{r}^p]\}\). It is the case that \(\forces p_{\lambda,\lambda}[S]^{\bar{r}^p} r_i^* : \lambda \rightarrow \{0,1\}\) is a function whose domain is all of \(\lambda^+\).

To see this, for \(p = \langle w^p, \alpha^p, \bar{r}^p, Z^p \rangle\), since \(|Z^p| < \lambda,\) \(w^p \in [\lambda^+]^{<\lambda}\), and \(z \in Z^p\) implies \(z \in [\lambda^+]^{<\lambda}\), the condition \(q = \langle w^q, \alpha^q, \bar{r}^q, Z^q \rangle\) given by \(\alpha^q = \alpha^p, Z^q = Z^p, w^q = w^p \cup \bigcup\{z : z \in Z^p\}\), and \(\bar{r}^q = \langle r_i^q : i \in w^q \rangle\) defined by \(r_i^q = r_i^p\) if \(i \in w^p\) and \(r_i^q\) is the empty function if \(i \in w^q - w^p\) is a well-defined condition. (This just means we may as well assume that for \(p = \langle w^p, \alpha^p, \bar{r}^p, Z^p \rangle,\) \(z \in Z^p\) implies \(z \subseteq w^p\).) Further, since \(|Z^p| < \lambda,\) \(\bigcup\{\beta : \exists z \in Z^p[z = x_\beta]\} = \gamma < \lambda^+\). Therefore, if \(\gamma' \in (\gamma, \lambda^+)\) and \(S' \subseteq \gamma'\) is so that \(sup S' = \gamma'\) and \(S'\) is an initial segment of \(S\) so that \(S'\) is not stationary at its supremum nor has any initial segment which is stationary at its supremum, then by Lemma 2, there is a sequence \((y_\beta : \beta \in S')\) so that for every \(\beta \in S',\) \(y_\beta \subseteq x_\beta, x_\beta - y_\beta\) is bounded in \(\beta,\) and if \(\beta_1 \neq \beta_2 \in S',\) then \(y_{\beta_1} \cap y_{\beta_2} = \emptyset\).

This means that if \(z \in Z^p\) and \(z = x_\beta\) for some \(\beta,\) then \(y_\beta \subseteq w^q\).

Choose now for \(\beta \in S'\) sets \(y_\beta^1\) and \(y_\beta^2\) so that \(y_\beta = y_\beta^1 \cup y_\beta^2, y_\beta^1 \cap y_\beta^2 = \emptyset,\) and \(|y_\beta^1| = |y_\beta^2| = \delta.\) If \(\rho \in (\alpha^q, \lambda),\) then for each \(\beta \in S'\) \(z \in Z^q\) and for each \(r_i^q \in \bar{r}^q\) such that \(i \in y_\beta,\) we can extend \(r_i^q\) to \(r_i'' : \rho \rightarrow \{0,1\}\) by letting \(r_i''|\alpha^q = r_i^q|\alpha^q,\) and for \(\alpha \in (\alpha^q, \rho), r_i''|\alpha = 0\) if \(i \in y_\beta^1\) and \(r_i''|\alpha = 1\) if \(i \in y_\beta^2\). For \(i \in w^q\) so that there is no \(\beta\) with \(x_\beta \in Z^q\) and \(i \in y_\beta,\) we extend \(r_i^q\) to \(r_i' : \rho \rightarrow \{0,1\}\) by letting \(r_i'|\alpha^q = r_i^q|\alpha^q,\) and for \(\alpha \in (\alpha^q, \rho), r_i'|\alpha = 0.\) If we let \(\bar{s} = \langle r_i' : i \in w^q\rangle,\) then \(\bar{s} = \langle r_i'' : i \in w^q\rangle\) can be verified to be such that \(\bar{s}\) is well-defined and \(t \geq q.\)

We have therefore shown by density that \(\forces p_{\lambda,\lambda}[S]^{\bar{r}^p} r_i^* : \lambda \rightarrow \{0,1\}\) is a function whose domain is all of \(\lambda^+\). Thus, we can let \(r_i^\ell = \{\alpha < \lambda : r_i^\ell(\alpha) = \ell\}\) for \(\ell \in \{0,1\}\).

For each \(i < \lambda^+,\) pick \(p_i = \langle w^{p_i}, \alpha^{p_i}, \bar{r}^{p_i}, Z^{p_i} \rangle \geq p\) so that \(p_i \forces \langle \bar{r}^{p_i}^\ell(i) \in \bar{D}^\ell \rangle\) for some \(\ell(i) \in \{0,1\}\). This is possible since \(\forces p_{\lambda,\lambda}[S]^{\bar{r}^p} r_i^* : \lambda \rightarrow \{0,1\}\).

Without loss of generality, by extending \(p_i\) if necessary, we can assume that \(i \in w^{p_i}\). Thus, since each \(w^{p_i} \in [\lambda^+]^{<\lambda}\), we can find some stationary \(A \subseteq \{i < \lambda^+ : \text{cof}(i) = \lambda\}\) so that \(\{w^j : i \in A\}\) forms a \(\Delta\)-system, i.e., that for \(i \neq j \in A, w^{p_i} \cap w^{p_j}\) is some constant value \(w\) which is an initial segment of both. (Note we can assume that for \(i \in A, w_i \cap i = w,\) and for some fixed \(\ell(i) \in \{0,1\},\) for every \(i \in A, p_i \forces \langle \bar{r}^{p_i^\ell(i)} \in \bar{D}^{\ell(i)} \rangle.)\) Also, by clause 4) of the definition of the forcing, \(|Z^{p_i}| < \lambda\) for each \(i < \lambda^+\). Therefore, \(Z^{p_i} \in [\lambda^+]^{<\lambda},\) so as \(|\lambda^+| = \lambda^+\) by GCH, the same sort of \(\Delta\)-system argument allows us to assume in addition that for all \(i \in A, Z^{p_i} \cap \mathcal{P}(w)\) is some constant value \(Z.\) Further, since each \(\alpha^{p_i} < \lambda,\) we can assume that \(\alpha^{p_i}\) is some constant \(\alpha^0\) for each \(i \in A.\) Then, since any \(\bar{r}^{p_i} = \langle r_j : j \in w^{p_i} \rangle \) for \(i \in A\) is composed of a sequence of functions from \(\alpha^0\) to 2, \(\alpha^0 < \lambda,\) and \(w < \lambda,\) GCH allows us to conclude that for \(i \neq j \in A, \bar{r}^{p_j}|w = \bar{r}^{p_i}|w\). And, since \(i \in w^{p_i},\) we know that we can also assume (by thinning \(A\) if necessary) that \(B = \{\sup(w^{p_i}) : i \in A\}\) is so that \(i < j \in A\) implies \(i < \sup(w^{p_i}) < \min(w^{p_j} - w) \leq \sup(w^{p_i}).\) We know in addition by the choice of \(X = \langle x_\beta : \beta \in S'\rangle\) that for some \(\gamma \in S, x_\gamma \subseteq A.\) Let \(x_\gamma = \{i_\beta : \beta < \delta\}.

We are now in a position to define the condition \(q\) referred to earlier. We proceed by defining each of the four coordinates of \(q.\) First, let \(w^q = \bigcup_{\beta < \delta} w^{p_i^\beta}.\) As \(\lambda\) and \(\lambda^+\) are regular, \(\delta < \lambda,\) and each \(w^{p_i^\beta} \in [\lambda^+]^{<\lambda}, \ w^q\) is well-defined and in \([\lambda^+]^{<\lambda}.\)

Second, let \(\alpha^q = \alpha^0.\) Third, let \(\bar{r}^q = \langle r_i^q : i \in w^q \rangle\) be defined by \(r_i^q = r_i^{p_i^\beta}\) if \(i \in w^{p_i^\beta} .\) The property of the \(\Delta\)-system that \(i \neq j \in A\) implies \(\bar{r}^{p_i}|w = \bar{r}^{p_j}|w\) tells
us $r^q$ is well defined. Finally, to define $Z^q$, let $Z^q = \bigcup_{\beta < \delta} Z^i_\beta \cup \{i_\beta : \beta < \delta\}$. By the last three sentences in the preceding paragraph and our construction, $\{i_\beta : \beta < \delta\}$ generates a new set which can be included in $Z^q$, and $Z^q$ is well-defined.

We claim now that $q \geq p$ so that $q \models \forall \beta < \delta r^i_\beta \subseteq \alpha^p$. To see this, assume the claim fails. This means that for some $q^1 \geq q$ and some $\alpha^q \leq \eta < \lambda$, $q^1 \models \forall \beta < \delta r^i_\beta \subseteq \alpha^q$, without loss of generality, since $q^1$ can always be extended if necessary, we can assume that $\eta < \alpha^q$. But then, by the definition of $\leq$, for many $\beta < \delta$, $q^1 \models \forall \beta < \delta \forall \pi \exists \eta \beta \eta \notin r^i_\beta$, an immediate contradiction. Thus, $q^1 \models \forall \beta < \delta r^i_\beta \subseteq \alpha^q$, which, since $\delta < \kappa$, contradicts that $q \models \forall \beta < \delta r^i_\beta \subseteq \beta^\kappa$. This proves Lemma 3. \end{proof}

Recall we mentioned prior to the proof of Lemma 3 that $P^1_{\delta,\lambda}[S]$ is designed so that a further forcing with $P^2_{\delta,\lambda}[S]$ will resuscitate the $\lambda$ supercompactness of $\kappa$, assuming the correct iteration has been done. That this is so will be shown in the next section. In the meantime, we give an idea of why this will happen by showing that the forcing $P^0_{\delta,\lambda} * (P^1_{\delta,\lambda}[S] \times P^2_{\delta,\lambda}[S])$ is rather nice. Specifically, we have the following lemma.

\begin{lemma}
$P^0_{\delta,\lambda} * (P^1_{\delta,\lambda}[S] \times P^2_{\delta,\lambda}[S])$ is equivalent to $Q^\lambda_{\delta} * Q^\lambda_{\delta}$.
\end{lemma}

\begin{proof}
Let $G$ be $V$-generic over $P^0_{\delta,\lambda} * (P^1_{\delta,\lambda}[S] \times P^2_{\delta,\lambda}[S])$, with $G^0_{\delta,\lambda}$, $G^1_{\delta,\lambda}$, and $G^2_{\delta,\lambda}$ the projections onto $P^0_{\delta,\lambda}$, $P^1_{\delta,\lambda}[S]$, and $P^2_{\delta,\lambda}[S]$ respectively. Each $G^i_{\delta,\lambda}$ is appropriately generic. So, since $P^1_{\delta,\lambda}[S] \times P^2_{\delta,\lambda}[S]$ is a product in $V[G^0_{\delta,\lambda}]$, we can rewrite the forcing in $V[G^0_{\delta,\lambda}]$ as $P^2_{\delta,\lambda}[S] \times P^1_{\delta,\lambda}[S]$ and rewrite $V[G]$ as $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}][G^1_{\delta,\lambda}]$.

It is well-known (see [MS]) that the forcing $P^0_{\delta,\lambda} \ast P^2_{\delta,\lambda}[S]$ is equivalent to $Q^\lambda_{\delta}$. That is, this is easily seen from the fact that $P^0_{\delta,\lambda} \ast P^2_{\delta,\lambda}[S]$ is non-trivial, has cardinality $\lambda^+$, and is such that $D = \{(p, q) \in P^0_{\delta,\lambda} \ast P^2_{\delta,\lambda}[S] : $ for some $\alpha, \text{dom}(p) = \alpha + 1, p \models \forall \alpha \in \beta^\alpha \notin \beta^\alpha \times \beta^\alpha \text{ and } q \models \forall \alpha \in \beta^\alpha \} $ is dense in $P^0_{\delta,\lambda} \ast P^2_{\delta,\lambda}[S]$ and is $\lambda$-closed. This easily implies the desired equivalence. Thus, $V$ and $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}][G^1_{\delta,\lambda}]$ have the same cardinals and cofinalities, and the proof of Lemma 4 will be complete once we show that in $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}]$, $P^1_{\delta,\lambda}[S]$ is equivalent to $Q^\lambda_{\delta}$.

To this end, working in $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}]$, we first note that as $S \subseteq \lambda^+$ is now a non-stationary set of all of whose initial segments are non-stationary, by Lemma 2, for the sequence $\langle x_\beta : \beta \in S \rangle$, there must be a sequence $\langle y_\beta : \beta \in S \rangle$ so that for every $\beta \in S$, $y_\beta \subseteq x_\beta$, $x_\beta - y_\beta$ is bounded in $\beta$, and if $\beta_1 \neq \beta_2 \in S$, then $y_{\beta_1} \cap y_{\beta_2} = \emptyset$. Given this fact, it is easy to observe that $P^1 = \{\langle w, \alpha, \vec{r}, Z \rangle \in P^1_{\delta,\lambda} : $ for every $\beta \in S$, either $y_\beta \subseteq w$ or $y_\beta \cap w = \emptyset \} $ is dense in $P^1_{\delta,\lambda}[S]$. To show this, given $\langle w, \alpha, \vec{r}, Z \rangle \in P^1_{\delta,\lambda}[S]$, $\vec{r} = \langle r_i : i \in w \rangle$, let $Y_w = \{y \in \langle y_\beta : \beta \in S \rangle : y \cap w \neq \emptyset \}$. As $|w| < \lambda$ and $y_{\beta_1} \cap y_{\beta_2} = \emptyset$ for $\beta_1 \neq \beta_2 \in S$, $|Y_w| < \lambda$. Hence, as $|y| = \delta < \lambda$ for $y \in Y_w$, $|y| < \lambda$ for $w' = w \cup (\bigcup Y_w)$. This means $\langle w', \alpha, \vec{r'}, Z \rangle$ for $\vec{r'} = \langle r'_i : i \in w' \rangle$ defined by $r'_i = r_i$ if $i \in w$ and $r'_i$ is the empty function if $i \in w' - w$ is a well-defined condition extending $\langle w, \alpha, \vec{r}, Z \rangle$. Thus, $P^1$ is dense in $P^1_{\delta,\lambda}[S]$, so to analyze the forcing properties of $P^1_{\delta,\lambda}[S]$, it suffices to analyze the forcing properties of $P^1$. 

For $\beta \in S$, let $Q_\beta = \{(w, \alpha, \tilde{r}, Z) \in P^1 : w = y_\beta\}$, and let $Q' = \{(w, \alpha, \tilde{r}, Z) \in P^1 : w \subseteq \lambda^+ - \bigcup_{\beta \in S} y_\beta\}$. Let $Q''$ be those elements of $\prod_{\beta \in S} Q_\beta \times Q'$ of support $\alpha < \lambda$ under the product ordering. Adopting the notation of Lemma 3, given $p = \langle q_\beta : \beta \in A, q \rangle \in Q''$ where $A \subseteq S$ and $|A| < \lambda$, as $|A| < \lambda$ and $\alpha$ is regular, $\alpha = \sup\{\alpha^{q_\beta} : \beta \in A\} \cup \alpha^q < \lambda$, so without loss of generality, each $q_\beta$ and $q$ can be extended to conditions $q'_\beta$ and $q'$ so that $\alpha$ occurs in $q'_\beta$ and $q'$. This means $Q = \{p = \langle q_\beta : \beta < \gamma < \lambda \rangle \in Q' : \alpha^{q_\beta} = \alpha^{q'_\beta} \text{ for } \beta \neq \gamma\}$ different coordinates of $p$ can be extended to conditions $q'_\beta$ and $q'$ so that $\alpha$ occurs in $q'_\beta$ and $q'$. Thus, for $\beta \in S$, $Q_\beta[S]$ is forcing equivalent to adding a Cohen subset to $\lambda$. This last coordinate and change in the ordering are necessary to destroy the strong compactness of $\kappa$ when forcing with $P^1_{\delta,\lambda}[S]$. Once the fact $S$ is stationary has been destroyed by forcing with $P^2_{\delta,\lambda}[S]$, Lemma 4 shows that this last coordinate $Z^p$ of a condition $p \in P^1_{\delta,\lambda}[S]$ and change in the ordering in a sense become irrelevant.

It is clear from Lemma 4 that $P^0_{\delta,\lambda} \ast P^1_{\delta,\lambda}[\hat{S}]$, being equivalent to $Q^0_{\delta,\lambda} \ast Q^1_{\delta,\lambda}$, preserves GCH, cardinals, and cofinalities, and has a dense subset which is $< \lambda$-closed and satisfies $\lambda^{++}$-c.c. Our next lemma shows that the forcing $P^0_{\delta,\lambda} \ast P^1_{\delta,\lambda}[\hat{S}]$ is also rather nice.

**Lemma 5.** $P^0_{\delta,\lambda} \ast P^1_{\delta,\lambda}[\hat{S}]$ preserves GCH, cardinals, and cofinalities, is $< \lambda$-strategically closed, and is $\lambda^{++}$-c.c.

**Proof of Lemma 5.** Let $G' = G^0_{\delta,\lambda} \ast G^1_{\delta,\lambda}$ be $V$-generic over $P^0_{\delta,\lambda} \ast P^1_{\delta,\lambda}[\hat{S}]$, and let $G^2_{\delta,\lambda}$ be $V[G']$-generic over $P^2_{\delta,\lambda}[S]$. Thus, $G' \ast G^2_{\delta,\lambda} = G$ is $V$-generic over $P^0_{\delta,\lambda} \ast (P^1_{\delta,\lambda}[\hat{S}] \ast P^2_{\delta,\lambda}[S]) = P^0_{\delta,\lambda} \ast (P^1_{\delta,\lambda}[\hat{S}] \times P^2_{\delta,\lambda}[S])$. By Lemma 4, $V[G] \models$ GCH and
has the same cardinals and cofinalities as \( V \), so since \( V[G'] \subseteq V[G] \) forcing with \( P^0_{\delta,\lambda} \ast P^1_{\delta,\lambda}[\hat{S}] \) over \( V \) preserves GCH, cardinals, and cofinalities.

We now show the \( \lambda \)-strategic closure of \( P^0_{\delta,\lambda} \ast P^1_{\delta,\lambda}[\hat{S}] \). We first note that \( (P^0_{\delta,\lambda} \ast P^1_{\delta,\lambda}[\hat{S}]) \ast P^2_{\delta,\lambda}[\hat{S}] = P^0_{\delta,\lambda} \ast (P^1_{\delta,\lambda}[\hat{S}] \ast P^2_{\delta,\lambda}[\hat{S}]) \) has by Lemma 4 a dense subset which is < \( \lambda \)-closed, the desired fact follows from the more general fact that if \( P \ast Q \) is a partial ordering with a dense subset \( R \) so that \( R \) is < \( \lambda \)-closed, then \( P \) is < \( \lambda \)-strategically closed. To show this more general fact, let \( \gamma \leq \lambda \) be a cardinal. Suppose I and II play to build an increasing chain of elements of \( P \), with \( \langle p_\beta : \beta \leq \alpha + 1 \rangle \) enumerating all plays by I and II through an odd stage \( \alpha + 1 \) and \( \langle q_\beta : \beta < \alpha + 1 \rangle \) enumerating a set of auxiliary plays by II which have been chosen so that \( \langle p_\beta, q_\beta \rangle : \beta < \alpha + 1 \) and \( \beta \) is even or a limit ordinal) enumerates an increasing chain of elements of the dense subset \( R \subseteq P \ast Q \). At stage \( \alpha + 2 \), II chooses \( \langle p_{\alpha+2}, q_{\alpha+2} \rangle \) so that \( \langle p_{\alpha+2}, q_{\alpha+2} \rangle \in R \) and so that \( \langle p_{\alpha+2}, q_{\alpha+2} \rangle \geq \langle p_{\alpha+1}, q_{\alpha} \rangle ; \) this makes sense, since inductively, \( \langle p_{\alpha}, q_{\alpha} \rangle \in R \subseteq P \ast Q \), so as I has chosen \( p_{\alpha+1} \geq p_{\alpha}, \langle p_{\alpha+1}, q_{\alpha} \rangle \in P \ast Q \).

By the < \( \lambda \)-closure of \( R \), at any limit stage \( \eta \leq \gamma \), II can choose \( \langle p_\eta, q_\eta \rangle \) so that \( \langle p_\eta, q_\eta \rangle \) is an upper bound to \( \langle p_\beta, q_\beta \rangle : \beta < \eta \) and \( \beta \) is even or a limit ordinal). The preceding yields a winning strategy for II, so \( P \) is < \( \lambda \)-strategically closed.

Finally, to show \( P^0_{\delta,\lambda} \ast P^1_{\delta,\lambda}[\hat{S}] \) is \( \lambda^{+\ast} \)-c.c., we simply note that this follows from the general fact about iterated forcing (see [Ba]) that if \( P \ast Q \) satisfies \( \lambda^{+\ast} \)-c.c., then \( P \) satisfies \( \lambda^{+\ast} \)-c.c. (Here, \( P = P^0_{\delta,\lambda} \ast P^1_{\delta,\lambda}[\hat{S}] \) and \( Q = P^2_{\delta,\lambda}[\hat{S}] \).) This proves Lemma 5. \( \square \) Lemma 5.

We remark that \( \lceil P^0_{\delta,\lambda} \ast P^1_{\delta,\lambda}[\hat{S}] \rceil \) is \( \lambda^+ \)-c.c., for if \( A = \langle p_\alpha : \alpha < \lambda^+ \rangle \) were a size \( \lambda^+ \) antichain of elements of \( P^1_{\delta,\lambda}[\hat{S}] \) in \( V[G^0_{\delta,\lambda}] \), then as \( V[G^0_{\delta,\lambda}] \) and \( V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}] \)

have the same cardinals, \( A \) would be a size \( \lambda^+ \) antichain of elements of \( P^1_{\delta,\lambda}[\hat{S}] \) in \( V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}] \). By Lemma 4, in this model, a dense subset of \( P^1_{\delta,\lambda}[\hat{S}] \) is isomorphic to \( Q^1_{\delta,\lambda} \), which has the same definition in either \( V[G^0_{\delta,\lambda}] \) or \( V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}] \) (since \( P^0_{\delta,\lambda} \) is \( \lambda \)-strategically closed and \( P^0_{\delta,\lambda} \ast P^2_{\delta,\lambda}[\hat{S}] \) is \( \lambda \)-closed) and so is \( \lambda^+ \)-c.c. in either model.

We conclude this section with a lemma which will be used later in showing that it is possible to extend certain elementary embeddings witnessing the appropriate degree of supercompactness.

**Lemma 6.** For \( V_1 = V[P^0_{\delta,\lambda}] \), the models \( V_1^{P^1_{\delta,\lambda}[\hat{S}] \times P^2_{\delta,\lambda}[\hat{S}]} \) and \( V_1^{P^1_{\delta,\lambda}[\hat{S}]} \) contain the same \( \lambda \) sequences of elements of \( V_1 \).

**Proof of Lemma 6.** By Lemma 4, since \( P^0_{\delta,\lambda} \ast P^2_{\delta,\lambda}[\hat{S}] \) is equivalent to the forcing \( Q^0_{\delta,\lambda} \) and \( V \subseteq V[P^0_{\delta,\lambda}] \subseteq V[P^0_{\delta,\lambda} \ast P^2_{\delta,\lambda}[\hat{S}]} \), the models \( V \), \( V[P^0_{\delta,\lambda}] \), and \( V[P^0_{\delta,\lambda} \ast P^2_{\delta,\lambda}[\hat{S}]} \) all contain the same \( \lambda \) sequences of elements of \( V \). Thus, since a \( \lambda \) sequence of elements of \( V_1 = V[P^0_{\delta,\lambda}] \) can be represented by a \( \lambda \)-term which is actually a function \( h : \lambda \rightarrow V \), it immediately follows that \( V[P^0_{\delta,\lambda}] \) and \( V[P^0_{\delta,\lambda} \ast P^2_{\delta,\lambda}[\hat{S}]} \) contain the same \( \lambda \) sequences of elements of \( V[P^0_{\delta,\lambda}] \).

Now let \( f : \lambda \rightarrow V_1 \) be so that \( f \in (V[P^0_{\delta,\lambda} \ast P^2_{\delta,\lambda}[\hat{S}])^{P^1_{\delta,\lambda}[\hat{S}]} = V_1^{P^1_{\delta,\lambda}[\hat{S}] \times P^2_{\delta,\lambda}[\hat{S}]} \), and let \( g : \lambda \rightarrow V_1, g \in V[P^0_{\delta,\lambda} \ast P^2_{\delta,\lambda}[\hat{S}]} \) be a term for \( f \). By the previous paragraph, \( g \in V[P^0_{\delta,\lambda}] \).

Since Lemma 4 shows that \( P^1_{\delta,\lambda}[\hat{S}] \) is \( \lambda^+ \)-c.c. in \( V[P^0_{\delta,\lambda} \ast P^2_{\delta,\lambda}[\hat{S}]} \), for each \( \alpha < \lambda \), the antichain \( A_\alpha \) defined in \( V[P^0_{\delta,\lambda} \ast P^2_{\delta,\lambda}[\hat{S}]} \) by \( \{ p \in P^1_{\delta,\lambda}[\hat{S}] : p \) decides a value for \( g(\alpha) \} \) is
so that $V^{P^{0}_{\delta,\lambda}}[S]\models "|A_{\alpha}| \leq \lambda."$ Hence, by the preceding paragraph, since $A_{\alpha}$ is a set of elements of $V^{P^{0}_{\delta,\lambda}}$, $A_{\alpha} \in V^{P^{0}_{\delta,\lambda}}$ for each $\alpha < \lambda$. Therefore, again by the preceding paragraph, the sequence $\langle A_{\alpha} : \alpha < \lambda \rangle \in V^{P^{0}_{\delta,\lambda}}$. This just means that the term $g \in V^{P^{0}_{\delta,\lambda}}$ can be evaluated in $V^{P^{1}_{\delta,\lambda}}$, i.e., $f \in V^{P^{1}_{\delta,\lambda}}[S]$. This proves Lemma 6. \qed Lemma 6

2. THE CASE OF ONE SUPERCOMPACT CARDINAL WITH NO LARGER INACCESSIBLES

In this section, we give a proof of our Theorem, starting from a model $V$ for "$\text{ZFC} + \text{GCH} + \text{There is one supercompact cardinal } \kappa \text{ and no } \lambda > \kappa \text{ is inaccessible}". Before defining the forcing conditions used in the proof of this version of our Theorem, we first give a proof of the theorem of Magidor mentioned in Section 0 which shows that if there is a supercompact cardinal, then there always must be cardinals $\delta < \lambda$ so that $\delta$ is $\lambda$ strongly compact yet $\delta$ isn't $\lambda$ supercompact.

Lemma 7 (Magidor [Ma4]). Suppose $\kappa$ is a supercompact cardinal. Then $B = \{\delta < \kappa : \delta \text{ is } \lambda_{\delta} \text{ strongly compact for } \lambda_{\delta} \text{ the least singular strong limit cardinal } > \delta \text{ of cofinality } \delta, \delta \text{ is not } \lambda_{\delta} \text{ supercompact, yet } \delta \text{ is } \alpha \text{ supercompact for all } \alpha < \lambda_{\delta} \}$ is unbounded in $\kappa$.

Proof of Lemma 7. Let $\lambda_{\kappa} > \kappa$ be the least singular strong limit cardinal of cofinality $\kappa$, and let $j : V \rightarrow M$ be an elementary embedding witnessing the $\lambda_{\kappa}$ supercompactness of $\kappa$ with $j(\kappa)$ minimal. As $j(\kappa)$ is least, $M \models "\kappa \text{ is not } \lambda_{\kappa} \text{ supercompact}"$. As $M^{\lambda_{\kappa}} \subseteq M$ and $\lambda_{\kappa}$ is a strong limit cardinal, $M \models "\kappa \text{ is } \alpha \text{ supercompact for all } \alpha < \lambda_{\kappa}"$.

Let $\mu \in V$ be a $\kappa$-additive measure over $\kappa$, and let $\langle \lambda_{\alpha} : \alpha < \lambda_{\kappa} \rangle$ be a sequence of cardinals cofinal in $\lambda_{\kappa}$ in both $V$ and $M$. As $M^{\lambda_{\kappa}} \subseteq M$ and $\lambda_{\kappa}$ is a strong limit cardinal, $\mu \in M$. Also, as $M \models "\kappa \text{ is } \alpha \text{ supercompact for all } \alpha < \lambda_{\kappa}"$, the closure properties of $M$ allow us to find a sequence $\langle \mu_{\alpha} : \alpha < \kappa \rangle \in M$ so that $M \models "\mu_{\alpha} \text{ is a fine, normal, } \kappa \text{-additive ultrafilter over } P_{\kappa}(\lambda_{\alpha})"$. Thus, we can define in $M$ the collection $\mu^{*}$ of subsets of $P_{\kappa}(\lambda_{\alpha})$ by $A \in \mu^{*}$ iff $\alpha < \kappa : A|\lambda_{\alpha} \in \mu_{\alpha}$, where for $A \subseteq P_{\kappa}(\lambda_{\alpha})$, $A|\lambda_{\alpha} = \{p \cap P_{\kappa}(\lambda_{\alpha}) : p \in A\}$. It is easily checked that $\mu^{*}$ defines in $M$ a $\kappa$-additive fine ultrafilter over $P_{\kappa}(\lambda_{\kappa})$. Thus, $M \models "\kappa \text{ is } \alpha \text{ supercompact for all } \alpha < \lambda_{\kappa}, \kappa \text{ is not } \lambda_{\kappa} \text{ supercompact, yet } \kappa \text{ is } \lambda_{\kappa} \text{ strongly compact}"$, so by reflection, the set $B$ of the hypothesis is unbounded in $\kappa$. This proves Lemma 7. \qed Lemma 7

We note that the proof of Lemma 7 goes through if $\lambda_{\kappa}$ becomes the least singular strong limit cardinal $> \delta$ of cofinality $\delta^{+}$, of cofinality $\delta^{++}$, etc. To see this, observe that the closure properties of $M$ and the strong compactness of $\kappa$ ensure that $\kappa^{+}$, $\kappa^{++}$, etc. each carry $\kappa$-additive measures $\mu_{\kappa^{+}}, \mu_{\kappa^{++}}$, etc. which are elements of $M$. These measures may then be used in place of the $\mu$ of Lemma 7 to define the strongly compact measure $\mu^{*}$ over $P_{\kappa}(\lambda_{\kappa})$.

We return now to the proof of our Theorem. Let $\delta = \{\delta_{\alpha} : \alpha \leq \kappa\}$ enumerate the inaccessibles $\leq \kappa$, with $\delta_{\kappa} = \kappa$. Note that since we are in the simple case in which $\kappa$ is the only supercompact cardinal in the universe and has no inaccessibles above it, we can assume each $\delta_{\alpha}$ isn’t $\delta_{\alpha+1}$ supercompact and for the least regular cardinal $\lambda_{\alpha} \geq \delta_{\alpha}$ so that $V \models "\delta_{\alpha} \text{ isn’t } \lambda_{\alpha} \text{ supercompact}"$, $\lambda_{\alpha} < \delta_{\alpha+1}$. (If $\delta$ were the least cardinal so that $\delta$ is $< \beta$ supercompact for $\beta$ the least inaccessible $> \delta$ yet $\delta$ isn’t $\beta$ supercompact, then $V_{\beta}$ would provide the desired model.)
We are now in a position to define the partial ordering \( P \) used in the proof of the theorem. We define a \( \kappa \) stage Easton support iteration \( P_\kappa = \langle P_\alpha, Q_\alpha : \alpha < \kappa \rangle \), and then define \( P = P_{\kappa+1} = P_\kappa * Q_\kappa \) for a certain class partial ordering \( Q_\kappa \) definable in \( V^{P_\kappa} \). The definition is as follows:

1. \( P_0 \) is trivial.
2. Assuming \( P_\alpha \) has been defined for \( \alpha < \kappa \), \( P_{\alpha+1} = P_\alpha * \dot{Q}_\alpha \), with \( \dot{Q}_\alpha \) a term for the full support iteration \( \langle P_{\omega, \lambda}^0 * (P_{\omega, \lambda}^1 [S_\lambda] \times P_{\omega, \lambda}^2 [S_\lambda]) : \delta^+ < \lambda < \lambda_\alpha \rangle \), where \( \dot{S}_\lambda \) is a term for the non-reflecting stationary subset of \( \lambda^+ \) introduced by \( P_{\omega, \lambda}^0 \) for \( \lambda < \lambda_\alpha \) and \( \dot{S}_{\lambda_\alpha} \) is a term for the non-reflecting stationary subset of \( \lambda_\alpha^+ \) introduced by \( P_{\omega, \lambda_\alpha}^0 \).
3. \( Q_\kappa \) is a term for the Easton support iteration of \( \langle P_{\omega, \lambda}^0 * (P_{\omega, \lambda}^1 [S_\lambda] \times P_{\omega, \lambda}^2 [S_\lambda]) : \lambda > \kappa \rangle \), where as before, \( \dot{S}_\lambda \) is a term for the non-reflecting stationary subset of \( \lambda^+ \) introduced by \( P_{\omega, \lambda}^0 \).

The intuitive motivation behind the above definition is that below \( \kappa \) at any inaccessible, we must first destroy and then resurrect all "good" instances of strong compactness, i.e., those which also witness supercompactness, but then destroy the least regular "bad" instance of strong compactness, thus destroying all "bad" instances of strong compactness beyond the least "bad" instance. Since \( \kappa \) is supercompact, it has no "bad" instances of strong compactness, so all instances of \( \kappa \)'s supercompactness are destroyed and then resurrected.

**Lemma 8.** For \( G \) a \( V \)-generic class over \( P \), \( V \) and \( V[G] \) have the same cardinals and cofinalities, and \( V[G] \models \text{ZFC} + \text{GCH} \).

**Proof of Lemma 8.** Write \( G = G_\kappa * H \), where \( G_\kappa \) is \( V \)-generic over \( P_\kappa \), and \( H \) is a \( V[G_\kappa] \)-generic class over \( Q_\kappa \). We show \( V[G_\kappa][H] \models \text{ZFC} \), and by assuming for the time being that \( V[G_\kappa] \models \text{GCH} \) and has the same cardinals and cofinalities as \( V \), we show \( V[G_\kappa][H] \models \text{GCH} \) and has the same cardinals and cofinalities as \( V[G_\kappa] \) (and hence as \( V \)).

To do this, note that \( Q_\kappa \) is equivalent in \( V[G_\kappa] = V_1 \) to the Easton support iteration of \( \langle Q_\lambda^0 * Q_\lambda^1 : \lambda > \kappa \rangle \), so we assume without loss of generality that \( Q_\kappa \) is in fact this ordering. Note also that as we are assuming \( \kappa \) has no inaccessibles above it, \( Q_\kappa \) is in fact equivalent to the Easton support iteration of \( \langle Q_\lambda^0 * Q_\lambda^1 : \lambda > \kappa \rangle \) is a successor cardinal. We first show inductively that for any successor cardinal \( \delta^+ > \kappa \), forcing over \( V_1 \) with the iteration of \( \langle Q_\lambda^0 * Q_\lambda^1 : \delta < \lambda < \delta^+ \rangle \) is a successor cardinal) preserves cardinals, cofinalities, and GCH. If \( \delta \) is regular (meaning \( \delta \) is a successor cardinal since \( \kappa \) has no inaccessibles above it), then this iteration can be written as \( Q_{<\delta} \dot{Q}_{<\delta} \), where \( Q_{<\delta} \) is the iteration of \( \langle Q_\lambda^0 * Q_\lambda^1 : \kappa < \lambda < \delta \rangle \) is a successor cardinal). By induction, forcing over \( V_1 \) with \( Q_{<\delta} \) preserves cardinals, cofinalities, and GCH, so since forcing over \( V_1 \) with \( Q_{<\delta} \dot{Q}_{<\delta} \) will preserve GCH and the cardinals and cofinalities of \( V_1^{Q_{<\delta}} \), forcing over \( V_1 \) with \( Q_{<\delta} \dot{Q}_{<\delta} \) preserves cardinals, cofinalities, and GCH. If \( \delta \) is singular, let \( \gamma < \delta \) be a cardinal in \( V_1 \), and write the iteration of \( \langle Q_\lambda^0 * Q_\lambda^1 : \kappa < \lambda < \delta \rangle \) as \( Q_{<\gamma} * Q_{>\gamma} \), where \( Q_{<\gamma} \) is as above and \( Q_{>\gamma} \) is a term in \( V_1 \) for the rest of the iteration; if \( \gamma < \kappa \), then \( Q_{<\gamma} \) is trivial and \( Q_{>\gamma} \) is a term for the whole iteration. By induction, \( V_1^{Q_{<\gamma}} \models \langle \gamma \rangle \) is a cardinal, \( 2^\gamma = \gamma^+ \), and \( \text{cof}(\gamma) = \text{cof}^V(\gamma) \), so as \( V_1^{Q_{<\gamma}} \models \langle Q_{>\gamma} \rangle \) is \( \gamma \)-closed, \( V_1^{Q_{<\gamma} * Q_{>\gamma}} \models \langle \gamma \rangle \) is
a cardinal, \(2^\alpha = \gamma^+\) and \(\text{cof}(\gamma) = \text{cof}^{\aleph_1}(\gamma)^\omega\), i.e., GCH, cardinals, and cofinalities below \(\delta\) are preserved when forcing over \(V_1\) with \(Q_{\prec \gamma^+} \ast Q_{\geq \gamma^+}\). In addition, since the last sentence shows any \(f : \gamma \rightarrow \delta\) or \(f : \gamma \rightarrow \delta^+\), \(f \in V^{Q_{\prec \gamma^+} \ast Q_{\geq \gamma^+}}\), it is so that \(f \in V_{\prec \gamma^+}^{Q_{\prec \gamma^+}}\) for arbitrary \(\gamma < \delta\), the fact \(V_{\prec \gamma^+}^{Q_{\prec \gamma^+}}\) and \(V_1\) have the same cardinals and cofinalities, together with the fact \(V_{\prec \gamma^+}^{Q_{\prec \gamma^+} \ast Q_{\geq \gamma^+}} = \text{“}\delta\text{ is a singular limit of cardinals satisfying GCH”}\) yields that forcing over \(V_1\) with \(Q_{\prec \gamma^+} \ast Q_{\geq \gamma^+}\) preserves \(\delta\) is a singular cardinal of the same cofinality as in \(V_1\), \(2^\delta = \delta^+\), and \(\delta^+\) is a regular cardinal. Finally, as GCH in \(V_1\) tells us \(|Q_{\prec \gamma^+} \ast Q_{\geq \gamma^+}| = \delta^+\), forcing with \(Q_{\prec \gamma^+} \ast Q_{\geq \gamma^+}\) over \(V_1\) preserves cardinals and cofinalities \(\geq \delta^+\) and GCH \(\geq \delta^+\).

It is now easy to show \(V_2 = V[G_\kappa][H] = \text{ZFC + GCH}\) and has the same cardinals and cofinalities as \(V[G_\kappa] = V_1\). To show \(V_2 \models \text{GCH}\) and has the same cardinals and cofinalities as \(V_1\), let \(\gamma\) again be a cardinal in \(V_1\), and write \(Q_\kappa = Q_{\prec \gamma^+} \ast Q\), where \(Q\) is a term in \(V_1\) for the rest of \(Q_\kappa\). As before, \(V_{\prec \gamma^+}^{Q_{\prec \gamma^+}} = \text{“}\gamma = \gamma^+\) and \(\text{cof}(\gamma) = \text{cof}^{\aleph_1}(\gamma)^\omega\), so since \(V_{\prec \gamma^+}^{Q_{\prec \gamma^+}} = \text{“} Q\text{ is \(\gamma\)-closed”}, \ \(V_2 \models \text{“}\gamma = \gamma^+\) and \(\text{cof}(\gamma) = \text{cof}^{\aleph_1}(\gamma)^\omega\), i.e., by the arbitrariness of \(\gamma\), \(V_2 \models \text{GCH}\), and all cardinals of \(V_1\) are cardinals of the same cofinality in \(V_2\). Finally, as all functions \(f : \gamma \rightarrow \delta\), \(\delta \in V_1\) some ordinal, \(f \in V_2\) are so that \(f \in V_{\prec \gamma^+}^{Q_{\prec \gamma^+}}\) by the last sentence, it is the case \(V_2 \models \text{Power Set}\), and since \(V_2 \models \text{AC}\) and \(Q_\kappa\) is an Easton support iteration, by the usual arguments, the aforementioned fact implies \(V_2 \models \text{Replacement}\). Thus, \(V_2 \models \text{ZFC}\).

It remains to show that \(V[G_\kappa] \models \text{GCH}\) and has the same cardinals and cofinalities as \(V\). To do this, we first note that Easton support iterations of \(\delta\)-strategically closed partial orderings are \(\delta\)-strategically closed for any regular cardinal. The proof is via induction. If \(R_1\) is \(\delta\)-strategically closed and \(\models \text{“} R_2 \text{ is \(\delta\)-strategically closed”}\), then let \(p \in R_1\) be so that \(p \models \text{“} \hat{g} \text{ is a strategy for player II ensuring that the game which produces an increasing chain of elements of } R_2 \text{ of length } \delta \text{ can always be continued for } \alpha \leq \delta^\prime\). If \(II\) begins by picking \(r_0 = (p_0, q_0) \in R_1 \ast R_2\) so that \(p_0 \geq \delta\) has been chosen according to the strategy \(f\) for \(R_1\) and \(p_0 \models \text{“} q_0 \text{ has been chosen according to } \hat{g}'\), and at even stages \(\alpha + 2\) picks \(r_{\alpha + 2} = (p_{\alpha + 2}, q_{\alpha + 2})\) so that \(p_{\alpha + 2}\) has been chosen according to \(f\) and is so that \(p_{\alpha + 2} \models \text{“} \hat{q}_{\alpha + 2} \text{ has been chosen according to } \hat{g}'\) \text{ “} \), then at limit stages \(\lambda \leq \delta\), the chain \(r_0 = (p_0, q_0) \leq r_1 = (p_1, q_1) \leq \cdots \leq r_\alpha = (p_\alpha, q_\alpha) \leq \cdots (\alpha < \lambda)\) is so that \(II\) can find an upper bound \(p_\lambda\) for \(\langle p_\alpha : \alpha < \lambda \rangle\) using \(f\). By construction, \(p_\lambda \models \text{“} \hat{q}_\lambda : \alpha < \lambda \text{”}\) is so that at limit and even stages, II has played according to \(\hat{g}'\), so for some \(q_\lambda\), \(p_\lambda \models \text{“} \hat{q}_\lambda \text{ is an upper bound to } \langle q_\alpha : \alpha < \lambda \rangle \text{”}\), meaning the condition \(\langle p_\lambda, q_\lambda \rangle\) is as desired. These methods, together with the usual proof at limit stages (see [Ba, Theorem 2.5]) that the Easton support iteration of \(\delta\)-closed partial orderings is \(\delta\)-closed, yield that \(\delta\)-strategic closure is preserved at limit stages of all of our Easton support iterations of \(\delta\)-strategically closed partial orderings. In addition, the ideas of this paragraph will also show that Easton support iterations of \(< \delta^+\)-strategically closed partial orderings are \(< \delta^+\)-strategically closed for \(\delta\) any regular cardinal.

For \(\alpha < \kappa\) and \(P_{\alpha + 1} = P_\alpha \ast Q_\alpha\), since \(\lambda_\alpha < \delta_{\alpha + 1}\), the definition of \(Q_\alpha\) in \(V^{P_\alpha}\) implies \(V^{P_\alpha} \models \text{“} \#Q_\alpha < \delta_{\alpha + 1}\). This fact, together with Lemma 5 and the definition of \(Q_\alpha\) in \(V^{P_\alpha}\), now yields the proof that \(V^{P_{\alpha + 1}} \models \text{GCH}\) and has the same cardinals and cofinalities as \(V\) is virtually identical to the proof given in the first part of this lemma that \(V_2 \models \text{GCH}\) and has the same cardinals and cofinalities as \(V_1\), replacing
$\gamma$-closure with $\gamma$-strategic closure, which also implies that the forcing adds no new functions from $\gamma$ to the ground model.

If $\lambda$ is a limit ordinal so that $\lambda = \sup(\{\delta_\alpha : \alpha < \lambda\})$ is singular, then, again, the proof that $V^{P_\gamma} \models \text{GCH}$ and has the same cardinals and cofinalities as $V$ is virtually the same as the just referred to proof of the first part of this lemma for virtually identical reasons as in the previous sentence, keeping in mind that since $|P_\alpha| < \delta_\alpha$ inductively for $\alpha < \lambda$, $|P_\lambda| = \lambda^+$. If $\lambda \leq \kappa$ is a limit ordinal so that $\lambda = \lambda$, then for cardinals $\gamma \leq \lambda$, the proof that $V^{P_\gamma} \models \gamma$ is a cardinal and $\text{cof}(\gamma) = \text{cof}(V(\gamma))$ is once more as before, as is the proof that $V^{P_\gamma} \models \gamma^{++}$ for $\gamma < \lambda$. As again $|P_\alpha| < \delta_\alpha < \lambda$ for $\alpha < \lambda$, $|P_\lambda| = \lambda$, so $V^{P_\gamma} \models \gamma^{++}$ for $\gamma \geq \lambda$ a cardinal. Thus, $V[G_{\alpha}] \models \text{GCH}$ and has the same cardinals and cofinalities as $V$. This proves Lemma 8. \hfill $\square$

Lemma 9. If $\delta < \gamma$ and $V \models \gamma$ is $\gamma$ supercompact and $\gamma$ is regular, then for $G$ $\gamma$-generic over $P$, $V[G] \models \delta$ is $\gamma$ supercompact$^\ast$.

Proof of Lemma 9. Let $j : V \rightarrow M$ be an elementary embedding witnessing the $\gamma$ supercompactness of $\delta$ so that $M \models \delta$ is not $\gamma$ supercompact$^\ast$. For the $\alpha_0$ so that $\delta = \delta_{\alpha_0}$, let $P = P_{\alpha_0} \ast Q_{\alpha_0} \ast \hat{T}_{\alpha_0} \ast \hat{R}$, where $Q_{\alpha_0}$ is a term for the full support iteration of $(P_{\omega,\lambda}^0 \ast (P_{\omega,\lambda}^1[\hat{S}_\lambda]) \times P_{\omega,\lambda}^2[\hat{S}_\lambda]) : \delta^+ \leq \lambda \leq \gamma$ and $\lambda$ is regular), $\hat{T}_{\alpha_0}$ is a term for the rest of $Q_{\alpha_0}$, and $\hat{R}$ is a term for the rest of $P$. We show that $V^{P_{\alpha_0} \ast Q_{\alpha_0}} \models \delta$ is $\gamma$ supercompact$^\ast$. This will suffice, since $|V^{P_{\alpha_0} \ast Q_{\alpha_0} \ast \hat{T}_{\alpha_0} \ast \hat{R}}| \models \delta$ is $\gamma$-strategically closed$^\ast$, so as the regularity of $\gamma$ and GCH in $V^{P_{\alpha_0} \ast Q_{\alpha_0}}$ imply $V^{P_{\alpha_0} \ast Q_{\alpha_0}} \models \gamma$, if $V^{P_{\alpha_0} \ast Q_{\alpha_0}} \models \delta$ is $\gamma$ supercompact$^\ast$, then $V^{P_{\alpha_0} \ast Q_{\alpha_0} \ast \hat{T}_{\alpha_0} \ast \hat{R}} = V^P \models \delta$ is $\gamma$ supercompact via any ultrafilter $U \in V^{P_{\alpha_0} \ast Q_{\alpha_0}}$.

To this end, we first note we will actually show that for $G_{\alpha_0} \ast G'_{\alpha_0}$ the portion of $G$ $\gamma$-generic over $P_{\alpha_0} \ast Q'_{\alpha_0}$, the embedding $j$ extends to $k : V[G_{\alpha_0} \ast G'_{\alpha_0}] \rightarrow M[H]$ for some $H \subseteq j(P)$. As $j(\alpha) : \alpha < \gamma \in M$, this will be enough to allow the definition of the ultrafilter $x \in U$ iff $j(\alpha) : \alpha < \gamma \in k(x)$ to be given in $V[G_{\alpha_0} \ast G'_{\alpha_0}]$.

We construct $H$ in stages. In $M$, as $\delta = \delta_{\alpha_0}$ is the critical point of $j$, $j(P_{\alpha_0} \ast Q'_{\alpha_0}) = P_{\alpha_0} \ast \hat{R}'_{\alpha_0} \ast \hat{R}'''_{\alpha_0} \ast \hat{R}''''_{\alpha_0}$, where $\hat{R}'_{\alpha_0}$ will be a term for the full support iteration of $(P_{\omega,\lambda}^0 \ast (P_{\omega,\lambda}^1[\hat{S}_\lambda]) \times P_{\omega,\lambda}^2[\hat{S}_\lambda]) : \delta^+ \leq \lambda < \gamma$ and $\lambda$ is regular) $\ast (P_{\omega,\lambda}^0 \ast P_{\omega,\gamma}[\hat{S}_\gamma])$ (note that as $M \subseteq M$, GCH implies that $M \models \lambda$ is supercompact if $\lambda < \gamma$ is regular, so since $M \models \lambda$ is not $\gamma$ supercompact" $\hat{R}'_{\alpha_0}$ is indeed as just stated), $\hat{R}'''_{\alpha_0}$ will be a term for the rest of the portion of $j(P_{\alpha_0})$ defined below $j(\delta)$, and $\hat{R}''''_{\alpha_0}$ will be a term for $j(Q'_{\alpha_0})$. This will allow us to define $H$ as $H_{\alpha_0} \ast H'_{\alpha_0} \ast H''_{\alpha_0} \ast H'''_{\alpha_0}$. Factoring $G'_{\alpha_0}$ as $\langle G'_{\alpha_0} = (G'_{\omega,\lambda} \times G'_{\omega,\gamma}) : \delta^+ \leq \lambda \leq \gamma$ and $\lambda$ is regular$\rangle$, we let $H_{\alpha_0} \equiv G_{\alpha_0}$, and

$H'_{\alpha_0} = \langle G'_{\omega,\lambda} \ast (G'_{\omega,\lambda} \times G'_{\omega,\gamma}) : \delta^+ \leq \lambda < \gamma$ and $\lambda$ is regular$\rangle$ $\ast \langle G'_{\omega,\gamma} \ast G'_{\omega,\gamma} \rangle$.

Thus $H'_{\alpha_0}$ is the same as $G'_{\alpha_0}$, except, since $M \models \delta$ is not $\gamma$ supercompact", we omit the generic object $G'_{\omega,\gamma}$.

To construct $H'''_{\alpha_0}$, we first note that the definition of $P$ ensures $|P_{\alpha_0}| = \delta$ and, since $\delta$ is necessarily Mahlo, $P_{\alpha_0}$ is $\delta$-c.c. As $V[G_{\alpha_0}]$ and $M[G_{\alpha_0}]$ are both models
of GCH, the definition of $R'_0$ in $M[H_{\alpha_0}]$, Lemmas 4, 5, and 8, and the remark immediately following Lemma 5 then ensure that $M[H_{\alpha_0}] \models \text{“The portion of } R'_0 \text{ below } \gamma \text{ is } \gamma^+\text{-c.c. and the portion of } R'_0 \text{ at } \gamma \text{ is a } \gamma\text{-strategically closed partial ordering followed by a } \gamma^+\text{-c.c. partial ordering”}. \text{ Since } M' \subseteq M \text{ implies } (\gamma^+)^M = (\gamma^+)^M \text{ and } P_{\alpha_0} \text{ is } \delta\text{-c.c., Lemma 6.4 of [Ba] shows } V[G_{\alpha_0}] \text{ satisfies these facts as well. This means applying the argument of Lemma 6.4 of [Ba] twice, in concert with an application of the fact that a portion of } R'_0 \text{ at } \gamma \text{ is } \gamma\text{-strategically closed, shows } M[H_{\alpha_0} * H'_{\alpha_0}] = M[G_{\alpha_0} * H'_{\alpha_0}] \text{ is closed under } \gamma \text{ sequences with respect to } V[G_{\alpha_0} * H'_{\alpha_0}] \text{, i.e., if } f : \gamma \rightarrow M[H_{\alpha_0} * H'_{\alpha_0}], f \in V[G_{\alpha_0} * H'_{\alpha_0}], \text{ then } f \in M[H_{\alpha_0} * H'_{\alpha_0}] \text{. Therefore, as } M[H_{\alpha_0} * H'_{\alpha_0}] \models \text{“} R'_0 \text{ is both } \gamma\text{-strategically closed and } \langle \gamma^+ \text{-strategically closed”, these facts are true in } V[G_{\alpha_0} * H'_{\alpha_0}] \text{ as well.}}$

Observe now that GCH allows us to assume $\gamma^+ < j(\delta) < j(\delta^+) < \gamma^+$. Since $M[H_{\alpha_0} * H'_{\alpha_0}] \models \text{“} R'_0 = j(\delta) \text{ and } |P'(\alpha_0)| = j(\delta^+) \text{”} \text{ (this last fact follows from GCH in } M[H_{\alpha_0} * H'_{\alpha_0}])$, in $V[G_{\alpha_0} * H'_{\alpha_0}]$, we can let $D_\alpha : \alpha < \gamma^+$ be an enumeration of the dense open subsets of $R'_0$ present in $M[H_{\alpha_0} * H'_{\alpha_0}]$. The $\langle \gamma^+ \text{-strategically closure of } R'_0 \text{ in both } M[H_{\alpha_0} * H'_{\alpha_0}] \text{ and } V[G_{\alpha_0} * H'_{\alpha_0}] \text{ now allows us to meet all of these dense subsets as follows. Work in } V[G_{\alpha_0} * H'_{\alpha_0}] \text{. Player I picks } p_\alpha \in D_\alpha \text{ extending sup}(\{q_\beta : \beta < \alpha\}) \text{ (initially, } q_{-1} \text{ is the trivial condition), and player II responds by picking } q_\alpha \geq p_\alpha \text{ (so } q_\alpha \in D_\alpha \text{). By the } \langle \gamma^+ \text{-strategic closure of } R'_0 \text{ in } V[G_{\alpha_0} * H'_{\alpha_0}] \text{, player II has a winning strategy for this game, so } \langle q_\alpha : \alpha < \gamma^+ \rangle \text{ can be taken as an increasing sequence of conditions with } q_\alpha \in D_\alpha \text{ for } \alpha < \gamma^+. \text{ Clearly, } H''_{\alpha_0} = \{p \in R''_{\alpha_0} : 3\alpha < \gamma^+ \langle q_\alpha \geq p \rangle\} \text{ is our } M[H_{\alpha_0} * H'_{\alpha_0}]\text{-generic object over } R'_{\alpha_0} \text{ which has been constructed in } V[G_{\alpha_0} * H'_{\alpha_0}] \subseteq V[G_{\alpha_0} * G'_{\alpha_0}], \text{ so } H''_{\alpha_0} \in V[G_{\alpha_0} * G'_{\alpha_0}]. \text{ To construct } H'''_{\alpha_0} \text{, we note first that, as in our remarks in Lemma 8, since } \gamma \text{ must be below the least inaccessible } > \delta \text{ and } \gamma \text{ is regular, } \gamma = \sigma^+ \text{ for some } \sigma. \text{ This allows us to write in } V[G_{\alpha_0}] \text{ } Q''_{\alpha_0} = Q''_{\alpha_0} * Q'''_{\alpha_0}, \text{ where } Q''_{\alpha_0} \text{ is the full support iteration of } \langle P_{\omega,\lambda}^0 \star (P_{\omega,\lambda}^1[\lambda] \times P_{\omega,\lambda}^2[\lambda]) : \delta^+ \leq \lambda \leq \sigma \text{ and } \lambda \text{ is regular} \rangle \text{ and } Q'''_{\alpha_0} \text{ is a term for } P_{\omega,\gamma}^{0,\gamma} \star (P_{\omega,\gamma}^{1,\gamma} \times P_{\omega,\gamma}^{2,\gamma}[\lambda]). \text{ This factorization of } Q''_{\alpha_0} \text{ induces through } j \text{ in } M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}] \text{ a factorization of } R''_{\alpha_0} \text{ into } R^4_{\alpha_0} \star \tilde{R} = \langle \text{the full support iteration of } \langle P_{\omega,\lambda}^0 \star (P_{\omega,\lambda}^1[\lambda] \times P_{\omega,\lambda}^2[\lambda]) : j(\delta^+) \leq \lambda \leq j(\sigma) \text{ and } \lambda \text{ is regular} \rangle \star \langle P_{\omega,\gamma}^{0,\gamma} \star (P_{\omega,\gamma}^{1,\gamma} \times P_{\omega,\gamma}^{2,\gamma}[\lambda]) \rangle \rangle). \text{ Work now in } V[G_{\alpha_0} * H'_{\alpha_0}]. \text{ In } M[H_{\alpha_0} * H'_{\alpha_0}], \text{ as previously noted, } R''_{\alpha_0} \text{ is } \gamma\text{-strategically closed. Since } M[H_{\alpha_0} * H'_{\alpha_0}] \text{ has already been observed to be closed under } \gamma \text{ sequences with respect to } V[G_{\alpha_0} * H'_{\alpha_0}], \text{ and since any } \gamma \text{ sequence of elements of } M[H_{\alpha_0} * H'_{\alpha_0}] \text{ can be represented, in } M[H_{\alpha_0} * H'_{\alpha_0}], \text{ by a term which is actually a function } f : \gamma \rightarrow M[H_{\alpha_0} * H'_{\alpha_0}], \text{ } M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}] \text{ is closed under } \gamma \text{ sequences with respect to } V[G_{\alpha_0} * H'_{\alpha_0}], \text{ i.e., if } f : \gamma \rightarrow M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}], f \in V[G_{\alpha_0} * H'_{\alpha_0}], \text{ then } f \in M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]. \text{ Factor (in } V[G_{\alpha_0} * G'_{\alpha_0}]) \text{ } C'_{\alpha_0} \text{ as } C'_{\alpha_0} * C'''_{\alpha_0} \text{ with } C'''_{\alpha_0} = C_{\omega,\alpha} \star (G_{\omega,\lambda} \times G_{\omega,\lambda}), \text{ where } \delta^+ \leq \lambda \leq \sigma \text{ and } \lambda \text{ is regular} \text{ and } C'''_{\alpha_0} = C_{\omega,\gamma} \star (G_{\omega,\lambda} \times G_{\omega,\lambda}). \text{ The projection of } C'_{\alpha_0} \text{ onto } Q''_{\alpha_0} \text{ and } C'''_{\alpha_0} \text{ is the projection of } C'_{\alpha_0} \text{ onto } Q''_{\alpha_0}. \text{ By our definitions, } Q''_{\alpha_0} \in V[G_{\alpha_0}] \text{ and } C'''_{\alpha_0} \in V[G_{\alpha_0} * H'_{\alpha_0}]. \text{ Also, our construction to this point guarantees that in } V[G_{\alpha_0} * H'_{\alpha_0}], \text{ the embedding } j \text{ extends to } j^*: V[G_{\alpha_0}] \rightarrow M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]. \text{ Thus, as GCH in } V[G_{\alpha_0} * H'_{\alpha_0}] \text{ implies } V[G_{\alpha_0} * H'_{\alpha_0}] \models \text{“} Q''_{\alpha_0} = C'''_{\alpha_0} = \gamma \text{”, the last paragraph implies } \langle j^*(p) : p \in C'''_{\alpha_0} \rangle \subseteq M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}] \text{. Since } \{j^*(p) : p \in C'''_{\alpha_0} \} \subseteq R^4_{\alpha_0}, \text{ } M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}] \models \text{“} R^4_{\alpha_0}$
is equivalent to a $j^*(\delta) = j(\delta)$-directed closed partial ordering”, and $j(\delta) > \gamma$, $q = \sup\{j^*(p) : p \in G_{\alpha_0}\}$ can be taken as a condition in $R_{\alpha_0}^4$.

Note that GCH in $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$ implies $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}] = \langle R_{\alpha_0}^4 = j(\gamma)\rangle$, and by choice of $j : V \rightarrow M$, $V[G_{\alpha_0} * H_{\alpha_0}] = \langle j(\gamma)\rangle = \gamma^+$ and $|j(\gamma^+)| = \gamma^+$.

Hence, as the number of dense open subsets of $R_{\alpha_0}^4$ in $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$ is $\langle 2(\gamma) \rangle M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}] = \langle (j(\gamma)^+)^+M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}] \rangle$ which has cardinality $(\gamma^+)^{V = (\gamma^+)^+V[G_{\alpha_0} * H_{\alpha_0}]}$, we can let $(D_{\alpha_0} : \alpha < \gamma^+) \in V[G_{\alpha_0} * H_{\alpha_0}]$ enumerate all dense open subsets of $R_{\alpha_0}^4$ in $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$. The $\gamma$-closure of $R_{\alpha_0}^4$ in $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$ and hence in $V[G_{\alpha_0} * H_{\alpha_0}]$ now allows an $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$-generic object $H_{\alpha_0}$ over $R_{\alpha_0}^4$ containing $q$ to be constructed in the standard way in $V[G_{\alpha_0} * H_{\alpha_0}]$; namely let $q_0 \in D_0$ be so that $q_0 \geq q$, and at stage $\alpha < \gamma^+$, by the $\gamma$-closure of $R_{\alpha_0}^4$ in $V[G_{\alpha_0} * H_{\alpha_0}]$, let $q_\alpha \in D_\alpha$ be so that $q_\alpha \geq \sup(q_j : \beta < \alpha)$. As before, $H_{\alpha_0}^4 = \{ q \in R_{\alpha_0}^4 : 3\alpha < \gamma^+ [q_\alpha \geq p] \in V[G_{\alpha_0} * H_{\alpha_0}] \subseteq V[G_{\alpha_0} * G_{\alpha_0}']$ is clearly our desired generic object.

By the above construction, in $V[G_{\alpha_0} * G_{\alpha_0}']$, the embedding $j^* : V[G_{\alpha_0}] \rightarrow M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$ extends to an embedding $j^{**} : V[G_{\alpha_0} * G_{\alpha_0}'] \rightarrow M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$. We will be done once we have constructed in $V[G_{\alpha_0} * G_{\alpha_0}']$ the appropriate generic object for $R_{\alpha_0}^5 = P_{\omega,j(\gamma)}^0 \times (P_{\omega,j(\gamma)}^1[S_{j(\gamma)}]^4 \times P_{\omega,j(\gamma)}^2[S_{j(\gamma)}]) = (P_{\omega,j(\gamma)}^0 \times P_{\omega,j(\gamma)}^2[S_{j(\gamma)}]^4) \times P_{\omega,j(\gamma)}^2[S_{j(\gamma)}])$. To do this, first rewrite $G_{\alpha_0}^{\alpha_0} = (G_{\omega,\gamma}^0, G_{\omega,\gamma}^0)$. By the nature of the forcings, $G_{\omega,\gamma}^0 \times G_{\omega,\gamma}^0$ is $V[G_{\alpha_0} * G_{\alpha_0}']$-generic over a partial ordering which is $(\gamma, \infty)$-distributive. Thus, by a general fact about transference of genericities via elementary embeddings (folklore; see [C, Section 1.2, Fact 2, pp. 5-6]), since $j^{**} : V[G_{\alpha_0} * G_{\alpha_0}'] \rightarrow M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$ is so that every element of $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$ can be written $j^{**}(F)(a)$ with dom $(F)$ having cardinality $\gamma$, $j^{**}G_{\omega,\gamma}^0, G_{\omega,\gamma}^0$ generates an $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$-generic set $H_{\alpha_0}^5$.

It remains to construct $H_{\alpha_0}^{\alpha_0}$, our $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} \times H_{\alpha_0}^4 \times H_{\alpha_0}^5]$-generic object over $P_{\omega,j(\gamma)}[S_{j(\gamma)}]$. To do this, first note that $H_{\alpha_0}^4$ (which was constructed in $V[G_{\alpha_0} * H_{\alpha_0}^4]$) is $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$-generic over $R_{\alpha_0}^4$, a partial ordering which in $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$ is $j(\delta)$-closed. Since $j(\delta) > \gamma$ and $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$ is closed under $\gamma$ sequences with respect to $V[G_{\alpha_0} * H_{\alpha_0}]$, we can apply earlier reasoning to infer $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$ is closed under $\gamma$ sequences with respect to $V[G_{\alpha_0} * H_{\alpha_0}]$, i.e., if $j : \gamma \rightarrow M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H_{\alpha_0}]$, $f \in V[G_{\alpha_0} * H_{\alpha_0}]$, then $f \in M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$.

Choose in $V[G_{\alpha_0} * G_{\alpha_0}']$ an enumeration $(p_\alpha : \alpha < \gamma^+)$ of $G_{\omega,\gamma}^1$. Working now in $V[G_{\alpha_0} * G_{\alpha_0}']$, let $f$ be an isomorphism between (a dense subset of) $P_{\omega,j(\gamma)}[S_{j(\gamma)}]$ and $Q_{\gamma}^1$. This gives us a sequence $(f(p_\alpha) : \alpha < \gamma^+)$ of $\gamma^+$ many compatible elements of $Q_{\gamma}^1$. Letting $p_\alpha = f(p_\alpha)$, we may hence assume that $I = \langle p_\alpha : \alpha < \gamma^+ \rangle$ is an appropriately generic object for $Q_{\gamma}^1$. By Lemma 6, $V[G_{\alpha_0} * G_{\alpha_0}^0 * G_{\alpha_0}^1 * G_{\alpha_0}^2 \times G_{\omega,\gamma}^0 \times G_{\omega,\gamma}^0] = V[G_{\alpha_0} * G_{\alpha_0}']$ and $V[G_{\alpha_0} * G_{\alpha_0}^0 * G_{\alpha_0}^1 * G_{\alpha_0}^2 \times G_{\omega,\gamma}^0 * G_{\alpha_0}^1] = V[G_{\alpha_0} * H_{\alpha_0}]$ have the same $\gamma$ sequences of elements of $V[G_{\alpha_0} * G_{\alpha_0}']$ and hence of $V[G_{\alpha_0} * H_{\alpha_0}]$. Thus, any $\gamma$ sequence of elements of $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$ present in $V[G_{\alpha_0} * G_{\alpha_0}']$ is actually an element of $V[G_{\alpha_0} * H_{\alpha_0}]$ (so $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$ is really closed under $\gamma$ sequences with respect to $V[G_{\alpha_0} * G_{\alpha_0}']$).

For $\alpha \in (\gamma, \gamma^+]$ and $p \in Q_{\gamma}^1$, let $p(\alpha) = \langle (\rho, \sigma), \eta \rangle \in Q_{\gamma}^1 : \rho < \alpha \rangle$ and $I(\alpha) = \{ p(\alpha) : \alpha < \gamma^+ \}. Clearly V[G_{\alpha_0} * G_{\alpha_0}'] = \langle I(\alpha) = \gamma \rangle$ for all $\alpha \in (\gamma, \gamma^+]$". Thus, since
$Q^1_{j}(\gamma) \in M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0}]$ and $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0}] = \{Q^1_{j}(\gamma) \text{ is } j(\gamma)\text{-directed closed}\}$, the facts $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0}]$ is closed under $\gamma$ sequences with respect to $V[G_{\alpha_0} \ast G'_{\alpha_0}]$ and $I$ is compatible imply that $q_{\alpha} = \bigcup \{j^*(p) : p \in I(\alpha)\}$ for $\alpha \in (\gamma, \gamma^+)$. Further, if $\langle \rho, \sigma \rangle \in \text{dom}(q_{\alpha}) \setminus \text{dom}(\bigcup_{\beta < \alpha} q_{\beta}) = Q^1_{j}(\gamma)$ as $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0}]$ is closed under $\gamma$ sequences with respect to $V[G_{\alpha_0} \ast G'_{\alpha_0}]$, then $\sigma \in \bigcup j(\beta)$ if $\langle \rho, \sigma \rangle \in \text{dom}(q_{\alpha})$. Then $\beta$ be minimal so that $\sigma < j(\beta)$, and let $\rho$ and $\sigma$ be so that $\langle \rho, \sigma \rangle \in \text{dom}(q_{\alpha})$. It must thus be the case that for some $p \in I(\alpha)$, $\langle \rho, \sigma \rangle \in \text{dom}(j^*(p))$. Then by elementarity and the definitions of $I(\beta)$ and $I(\alpha)$, for $p(\beta) = q \in I(\beta)$, $j^*(q) = j^*(p)$, so $(\rho, \sigma) \in \text{dom}(j^*(p))$, it must be the case that $\langle \rho, \sigma \rangle \in \text{dom}(j^*(q))$. This means $\langle \rho, \sigma \rangle \in \text{dom}(q_{\beta})$, a contradiction.

We now define an $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0} \ast H^5_{\alpha_0}]$-generic object $H_{6,0}^g$ over $Q^1_{j}(\gamma)$ so that $p \in j^g(G_{\alpha_0} \ast G'_{\alpha_0})$ implies $j^g(p) \in H_{6,0}^g$. First, for $\beta \in (j(\gamma), j(\gamma^+))$, let $Q^1_{j}(\gamma) \in M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0}]$ be the forcing for adding $\beta$ many Cohen subsets to $j(\gamma)$, i.e., $Q^1_{j}(\gamma) = \{g : j(\beta) \times \beta \rightarrow \{0, 1\} : g \text{ is a function so that } |\text{dom}(g)| < j(\gamma)\}$, ordered by inclusion. Next, note that since $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0}] = \langle Q^1_{j}(\gamma) \text{ is } j(\gamma)^\text{-c.e. and } Q^1_{j}(\gamma) \text{ has } j(\gamma)^\text{ many maximal antichains}\rangle$. This means that if $A \in M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0}]$ is a maximal antichain of $Q^1_{j}(\gamma)$, then $A \subseteq Q^1_{j}(\gamma)$ for some $\beta \in (j(\gamma), j(\gamma^+))$. Also, since $V \subseteq V[G_{\alpha_0} \ast G'_{\alpha_0}] \subseteq V[G_{\alpha_0} \ast G''_{\alpha_0}] \subseteq V[G_{\alpha_0} \ast G'_{\alpha_0}]$ are all models of GCH containing the same cardinals and cofinalities, $V[G_{\alpha_0} \ast G'_{\alpha_0}] = \langle i(\beta) \text{ is } i(\gamma)^\text{, the preceding \text{ means we can let } \langle A_\alpha : \alpha < \gamma^+ \rangle \in V[G_{\alpha_0} \ast G'_{\alpha_0}] \text{ be an enumeration of the maximal antichains of } Q^1_{j}(\gamma) \text{ present in } M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0}] \rangle$.

Working in $V[G_{\alpha_0} \ast G'_{\alpha_0}]$, we define now an increasing sequence $\langle r_\alpha : \alpha \in (\gamma, \gamma^+) \rangle$ of elements of $Q^1_{j}(\gamma)$ so that $\forall \alpha < \gamma^+ r_\alpha \geq q_{\alpha}$ and $r_\alpha \in Q^1_{j}(\alpha)$ and so that $\forall A \in \langle A_\alpha : \alpha \in (\gamma, \gamma^+) \rangle \exists r_\beta \in A[r_\beta \geq r \forall r \in (\alpha, \alpha^+) \text{ such that } |\text{dom}(g)| < j(\gamma)\rangle$. Assuming we have such a sequence, $H_{6,0}^g = \{p \in Q^1_{j}(\gamma) : \exists r \in (\alpha, \alpha^+) \in (\alpha, \gamma^+) \geq r \geq p\}$ is our desired generic object. To define $r_\alpha : \alpha \in (\gamma, \gamma^+)$, if $\alpha$ is a limit, we let $r_\alpha = \bigcup_{\beta < \alpha} r_\beta$. By the facts $\langle q_\beta : \beta \in (\gamma, \gamma^+) \rangle$ is (strictly) increasing and $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0}]$ is closed under $\gamma$ sequences with respect to $V[G_{\alpha_0} \ast G'_{\alpha_0}]$, this definition is valid. Assuming now $r_\alpha$ has been defined and we wish to define $r_{\alpha+1}$, let $\langle r_\beta : \beta < \eta \leq \gamma \rangle$ be the subsequence of $A(\beta) : \beta < \alpha + 1$ containing each antichain $A$ so that $A \subseteq Q^1_{j}(\alpha+1)$. Since $q_{\alpha}, r_\alpha \in Q^1_{j}(\alpha), q_{\alpha+1} \in Q^1_{j}(\alpha+1)$, and $j(\alpha) < j(\alpha + 1)$, the condition $r_{\alpha+1} = r_\alpha \cup q_{\alpha+1}$ is well-defined, as by our earlier observations, any new elements of dom($q_{\alpha+1}$) won’t be present in either dom($q_{\alpha}$) or dom($r_\alpha$). We can thus, using the fact $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0}]$ is closed under $\gamma$ sequences with respect to $V[G_{\alpha_0} \ast G'_{\alpha_0}]$, define by induction an increasing sequence $\langle s_\beta : \beta < \eta \rangle$ so that $s_\beta \geq r_{\beta+1}$, $s_\beta = \bigcup_{\beta < \rho} s_\beta$ if $\rho$ is a limit, and $s_{\beta+1} \geq s_\beta$ is so that $s_{\beta+1}$ extends some element of $B_\beta$. The just mentioned closure fact implies $r_{\alpha+1} = \bigcup_{\beta \in \alpha} s_\beta$ is a well-defined condition.

In order to show $H_{6,0}^g$ is $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0} \ast H^5_{\alpha_0}]$-generic over $Q^1_{j}(\gamma)$, we must show that $\forall A \in \langle A_\alpha : \alpha \in (\gamma, \gamma^+) \rangle \exists \beta \in (\gamma, \gamma^+) \exists r \in A[r_\beta \geq r \forall r \in (\alpha, \gamma^+)]$. To do this, we
first note that \( j(\alpha) : \alpha < \gamma^+ \) is unbounded in \( j(\gamma^+) \). To see this, if \( \beta < j(\gamma^+) \) is an ordinal, then for some \( g : \gamma \rightarrow M \) representing \( \beta \), we can assume that for \( \lambda < \gamma \), 
\( g(\lambda) < \gamma^+ \). Thus, by the regularity of \( \gamma^+ \) in \( V \), \( \beta_0 = \bigcup_{\lambda < \gamma} g(\lambda) < \gamma^+ \), and \( j(\beta_0) > \beta \).

This means by our earlier remarks that if \( A \in \langle A_\alpha : \alpha < \gamma^+ \rangle, A = A_\rho \), then we can let \( \beta \in (\gamma, \gamma^+) \) be so that \( A \subseteq Q^{j(\beta)}_{\gamma} \). By construction, for \( \eta > \max(\beta, \rho) \), there is some \( r \in A \) so that \( r_\eta \geq r \). Finally, since any \( p \in Q^1_\gamma \) is so that for some \( \alpha \in (\gamma, \gamma^+) \), \( p = p|\alpha, H^\delta_{\alpha,0} \) is so that if \( p \in f''G_{\omega, \gamma} \), then \( j^{**}(p) \in H_{\alpha_0} \).

Note now that our earlier work ensures \( j^{**} \) extends to

\[
j^{***} : V[G_{\alpha_0} * G''_{\alpha_0} * G^0_{\omega, \gamma} * G^2_{\omega, \gamma}] \rightarrow M[H_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0} * H^5_{\alpha_0}].
\]

By Lemma 4, the isomorphism \( f \) is definable over \( V[G_{\alpha_0} * G''_{\alpha_0} * G^0_{\omega, \gamma} * G^2_{\omega, \gamma}] \). This means the notions \( j^{***}(f) \) and \( j^{***}(f^{-1}) \) make sense, so \( j^{***}(f) \) is a definable isomorphism over \( M[H_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0} * H^5_{\alpha_0}] \) between (a dense subset of) \( P^1_{\omega, j(\gamma)}[S_{j(\gamma)}] \) and \( Q^2_{\omega, j(\gamma)} \) and \( j^{***}(f^{-1}) \) is its inverse. If \( H^6_{\alpha_0} = \{ j^{***}(f^{-1})(p) : p \in H^6_{\alpha_0} \} \), then it is now easy to verify that \( H^6_{\alpha_0} \) is an \( M[H_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0} * H^5_{\alpha_0}] \)-generic object over (a dense subset of) \( P^1_{\omega, j(\gamma)}[S_{j(\gamma)}] \), so that \( p \in (\text{a dense subset of}) P^1_{\omega, j(\gamma)}[S_{j(\gamma)}] \) implies \( j^{***}(p) \in H^6_{\alpha_0} \). Therefore, for \( H'' = H^1_{\alpha_0} * H^5_{\alpha_0} * H^6_{\alpha_0} \) and \( H = H_{\alpha_0} * H''_{\alpha_0} * H^{\omega, \lambda} \), \( j : V \rightarrow M \) extends to \( k : V[G_{\alpha_0} * G'_{\alpha_0}] \rightarrow M[H] \), so \( V[G] \models \langle \delta, \gamma \rangle \) is supercompact if \( \gamma \) is regular. This proves Lemma 9. \( \square \) Lemma 9

**Lemma 10.** For \( \gamma \) regular, \( V[G] \models \langle \delta, \gamma \rangle \) is supercompact if and only if \( \delta \) is \( \gamma \)-strongly compact.

**Proof of Lemma 10.** Assume towards a contradiction that the lemma is false, and let \( \delta < \gamma \) be so that \( V[G] \models \langle \delta, \gamma \rangle \) is supercompact, \( \gamma \) is regular, and \( \gamma \) is the least such cardinal”. As before, let \( \delta = \delta_\alpha \), i.e., \( \delta \) is the \( \alpha \)-th inaccessible cardinal. If \( V \models \langle \delta_\alpha, \gamma \rangle \) is supercompact”, then Lemma 9 implies \( V[G] \models \langle \delta_\alpha, \gamma \rangle \) is supercompact”, so it must be the case that \( V \models \langle \delta_\alpha, \gamma \rangle \) is supercompact”. We therefore have \( \lambda_\alpha \leq \gamma \) for \( \lambda_\alpha \) the least regular cardinal so that \( V \models \langle \delta_\alpha, \lambda_\alpha \rangle \) is supercompact”.

In the manner of Lemma 9, write \( P = P_\alpha * Q_\alpha * Q'_\alpha \) where \( P_\alpha \) is the iteration through stage \( \alpha \), \( Q_\alpha \) is a term for the full support iteration of \( (P^0_{\omega, \lambda} + (P^1_{\omega, \lambda}[S_{\lambda}]) \times P^2_{\omega, \lambda}[S_{\lambda}]) : \delta^+ \leq \lambda < \lambda_\alpha \) for \( \lambda \) is regular) \( \times (P^0_{\omega, \lambda_\alpha} + P^2_{\omega, \lambda_\alpha}[S_{\lambda_\alpha}]), \) and \( Q'_\alpha \) is a term for the rest of \( P \). By our previous results, \( V^{P_\alpha, Q_\alpha} \models \langle \delta_\alpha \rangle \) is \( \alpha \)-strongly compact”, and \( V^{P_\alpha, Q_\alpha} \) is \( \alpha \)-strategically closed” (where \( \delta_\alpha \) is the least inaccessible > \( \delta_\alpha \)). It must thus be the case that \( V^{P_\alpha, Q_\alpha} \models \langle \delta_\alpha \rangle \) is \( \alpha \)-strongly compact”, so of course, as \( \lambda_\alpha \leq \gamma \), \( V[G] \models \langle \delta_\alpha \rangle \) is \( \gamma \)-strongly compact”. This proves Lemma 10. \( \square \) Lemma 10

**Lemma 11.** For \( \gamma \) regular, \( V[G] \models \langle \delta, \gamma \rangle \) is supercompact if and only if \( V \models \langle \delta, \gamma \rangle \) is supercompact”.

**Proof of Lemma 11.** By Lemma 9, if \( V \models \langle \delta, \gamma \rangle \) is supercompact and \( \gamma \) is regular”, then \( V[G] \models \langle \delta, \gamma \rangle \) is supercompact”. If \( V[G] \models \langle \delta, \gamma \rangle \) is supercompact and \( \gamma \) is regular” but \( V \models \langle \delta, \gamma \rangle \) is not \( \gamma \)-supercompact”, then as in Lemma 10, for the \( \alpha \) so that \( \delta = \delta_\alpha \), \( \lambda_\alpha \leq \gamma \) for \( \lambda_\alpha \) the least regular cardinal so that \( V \models \langle \delta_\alpha, \lambda_\alpha \rangle \) is supercompact”. The proof of Lemma 10 then immediately yields that \( V[G] \models \langle \delta_\alpha \rangle \) is \( \alpha \)-strongly compact”. This proves Lemma 11. \( \square \) Lemma 11
The proof of Lemma 11 completes the proof of our Theorem in the case when \( \kappa \) is the unique supercompact cardinal in the universe and has no inaccessibles above it. This guarantees the Theorem to hold non-trivially. \( \square \) Theorem

3. The General Case

We will now prove our Theorem under the assumption that there may be more than one supercompact cardinal in the universe (including a proper class of supercompact cardinals) and that the large cardinal structure above any given supercompact can be rather complicated, including possibly many inaccessibles, measurables, etc. Before defining the forcing conditions, a few intuitive remarks are in order. We will proceed using the same general paradigm as in the last section, namely iterating the forcings of Section 1 using Easton supports so as to destroy those “bad” instances of strong compactness which can be destroyed and so as to resurrect and preserve all instances of supercompactness. For each inaccessible \( \delta_i \), a certain coding ordinal \( \theta_i < \delta_i \) will be chosen when possible which we will use to define \( P^0_{\theta_i, \lambda} \), \( P^1_{\theta_i, \lambda}[S_{\theta_i, \lambda}] \), and \( P^2_{\theta_i, \lambda}[S_{\theta_i, \lambda}] \), where \( S_{\theta_i, \lambda} \) is the non-reflecting stationary set of ordinals of cofinality \( \theta_i \) added to \( \lambda^+ \) by \( P^0_{\theta_i, \lambda} \). We will need to have different values of \( \theta_i \), instead of having \( \theta_i = \omega \) as in the last section, so as to destroy the \( \lambda \) strong compactness of some \( \delta \) and yet preserve the \( \lambda \) supercompactness of a \( \delta' \neq \delta \) when necessary. When \( \theta_i \) can’t be defined, we won’t necessarily be able to destroy the \( \lambda \) strong compactness of \( \delta_i \), although we will be able to preserve the \( \lambda \) supercompactness of \( \delta_i \) if appropriate. This will happen when instances of the results of \( \text{[Me]} \) and \( \text{[A]} \) occur, i.e., when there are certain limits of supercompactness.

Getting specific, let \( \langle \delta_i : i \in \text{Ord} \rangle \) enumerate the inaccessibles of \( V \models \text{GCH} \), and let \( \lambda_i > \delta_i \) be the least regular cardinal so that \( V \models \delta_i \text{ is } \lambda_i \text{ supercompact} \) if such a \( \lambda_i \) exists. If no such \( \lambda_i \) exists, i.e., if \( \delta_i \) is supercompact, then let \( \lambda_i = \Theta \), where we think of \( \Theta \) as some giant “ordinal” larger than any \( \alpha \in \text{Ord} \). If possible, choose \( \theta_i < \delta_i \) as the least regular cardinal so that \( \theta_i < \delta_j < \delta_i \) implies \( \lambda_j < \delta_i \) (whenever \( j < i \)). Note that \( \theta_i \) is undefined for \( \delta_i \) if \( \delta_i \) is a limit of cardinals which are \( < \delta_i \) supercompact, because for \( j < i \), if \( \delta_j < \delta_i \) supercompact, then \( \lambda_j < \delta_i \).

We define now a class Easton support iteration \( P = \langle \langle P_\alpha, Q_\alpha : \alpha \in \text{Ord} \rangle \rangle \) as follows: 1. \( P_0 \) is trivial. 2. Assuming \( P_\alpha \) has been defined, \( P_{\alpha+1} = P_\alpha * Q_\alpha \), where \( Q_\alpha \) is a term for the trivial partial ordering unless \( \alpha \) is regular and for some inaccessible \( \delta = \delta_i < \alpha \) with \( \delta_i \) defined, either \( \delta_i \) is \( \alpha \) supercompact or \( \alpha = \lambda_i \). Under these circumstances \( Q_\alpha \) is a term for

\[
\prod_{\{i < \alpha : \delta_i < \alpha \text{ supercompact}\}} \prod_{\{i < \alpha : \delta_i < \alpha \text{ supercompact}\}} P^1_{\delta_i, \alpha}[S_{\delta_i, \alpha}] \times \prod_{\{i < \alpha : \lambda_i = \delta_i \}} P^0_{\delta_i, \alpha} \times \prod_{\{i < \alpha : \alpha = \lambda_i \}} P^1_{\delta_i, \alpha}[S_{\delta_i, \alpha}] = (P^0_\alpha * P^1_\alpha) \times (P^2_\alpha * P^3_\alpha),
\]

with the proviso that elements of \( P^0_\alpha \) and \( P^2_\alpha \) will have full support, and elements of \( P^1_\alpha \) and \( P^3_\alpha \) will have support \( < \alpha \). Note that unless \( |\{i < \alpha : \delta_i < \alpha \text{ supercompact}\}| = \alpha \), the elements of \( P^i_\alpha \) will have full support for \( i = 0, 1, 2, 3 \).

The following lemma is the natural analogue to Lemma 8.

Lemma 12. For \( G \) a \( V \)-generic class over \( P \), \( V \) and \( V[G] \) have the same cardinals and cofinalities, and \( V[G] \models \text{ZFC + GCH} \).

Proof of Lemma 12. We show inductively that for any \( \alpha \), \( V \) and \( V^{P_\alpha} \) have the same cardinals and cofinalities, and \( V^{P_\alpha} \models \text{GCH} \). This will suffice to show \( V[G] \models \text{ZFC + GCH} \).
GCH and has the same cardinals and cofinalities as \( V \), since if \( \dot{R} \) is a term so that \( P_\alpha*\dot{R} = P_\alpha \) then \( \| P_\alpha \| < \alpha \). The iteration \( \dot{R} \) is \(<\alpha\)-strategically closed”, meaning \( V^{P_\alpha}\times\dot{R} \) and \( V^{P_\alpha} \) have the same cardinals and cofinalities \( \leq \alpha \) and GCH holds in both of these models for cardinals \(<\alpha\).

Assume now \( V \) and \( V^{P_\alpha} \) have the same cardinals and cofinalities, and \( V^{P_\alpha} = GCH \). We show \( V \) and \( V^{P_\alpha+1} = V^{P_\alpha}*Q_\alpha \) have the same cardinals and cofinalities, and \( V^{P_\alpha+1} = GCH \). If \( Q_\alpha \) is a term for the trivial partial ordering, this is clearly the case, so we assume \( \dot{Q}_\alpha \) is not a term for the trivial partial ordering. Let \( \dot{Q}_\alpha' \) be a term for \( (\dot{P}_\alpha^0 \times \dot{P}_\alpha^1) \times (\prod_{\{i<\alpha: \delta_i = \alpha \supseteq \lambda_i \}} (\dot{P}_\delta \times [\dot{S}_{\delta,\alpha}] \times \dot{P}_\delta^3)) \) where as earlier, the elements of \( \dot{P}_\alpha^0 \) and \( \dot{P}_\alpha^1 \) will have full support, and the elements of \( \dot{P}_\alpha^1 \) and \( \dot{P}_\alpha^3 \) will have support \(<\alpha\). We are now able to rewrite \( \dot{Q}_\alpha \) as \( \prod_{\{i<\alpha: \delta_i = \supseteq \alpha \}} (\prod_{\{i<\alpha: \delta_i = \lambda_i \}} \dot{P}_\delta \times [\dot{S}_{\delta,\alpha}] \times \dot{P}_\delta^3) \) and \( \dot{Q}_\alpha \) will have full support, and the elements of \( \dot{P}_\alpha^6 \) will have support \(<\alpha\). By Lemma 4, in \( V^{P_\alpha} \), each \( P_\delta \) is isomorphic to \((\dot{P}_\delta \times [\dot{S}_{\delta,\alpha}] \times \dot{P}_\delta^3)\) is equivalent to \( Q_\alpha \). We therefore have \( \dot{Q}_\alpha \) is equivalent to \( \prod_{\{i<\alpha: \delta_i = \lambda_i \}} \gamma \) where \( \gamma = |\{i < \alpha: \delta_i \text{ is } \alpha \text{ supercompact or } \alpha = \lambda_i \}| \), (\gamma is a cardinal in both \( V \) and \( V^{P_\alpha} \) by induction), i.e., the full support product of \( \gamma \) copies of \( Q_\alpha \) followed by the \(<\alpha\) support product of \( \gamma \) copies of \( Q_\alpha \). Since \( \gamma \leq \alpha \), \( \prod_{\beta < \gamma} Q_\alpha \) is isomorphic to the usual ordering for adding \( \gamma \) many Cohen subsets to \( \alpha^+ \) using conditions of support \(<\alpha^+\), and since \( \prod_{\beta < \gamma} Q_\alpha \) is composed of elements having support \(<\alpha\), \( \prod_{\beta < \gamma} Q_\alpha \) is isomorphic to a single partial ordering for adding \( \alpha^+ \) many Cohen subsets to \( \alpha \) using conditions of support \(<\alpha\).

Hence, \( V^{P_{\alpha+1}}=V^{P_\alpha}*Q_\alpha \) and \( V^{P_\alpha} \) have the same cardinals and cofinalities, and \( V^{P_\alpha}*Q_\alpha \) \( = \) GCH, so \( V^{P_{\alpha+1}}=Q_\alpha \) and \( V \) have the same cardinals and cofinalities. And, for \( G_\alpha \) the projection of \( G \) onto \( P_\alpha \), if \( H \) is \( V[G_\alpha]=\text{generic over } Q_\alpha \), for any \( i < \alpha \) so that \( \alpha = \lambda_i \), we can omit the portion of \( H \) generic over \( P_\delta \) and \( \dot{S}_{\delta,\alpha} \) and thus obtain a \( V[G_\alpha]=\text{generic object } H' \) for \( Q_\alpha \). Since \( V \subseteq V[G_\alpha][H'] \subseteq V[H] \), as in Lemma 5, it must therefore be the case that \( V \), \( V^{P_{\alpha+1}}=V^{P_\alpha+1} \), and \( V^{P_{\alpha+1}}=Q_\alpha \) all have the same cardinals and cofinalities and satisfy GCH.

To complete the proof of Lemma 12, if now \( \alpha \) is a limit ordinal, the proof that \( V \) and \( V^{P_\alpha} \) have the same cardinals and cofinalities and \( V^{P_\alpha} \) is GCH is the same as the proof given in the last paragraph of Lemma 8, since the iteration still has enough strategic closure and can easily be seen by GCH to be so that for any \( \beta < \alpha \), \( |P_\beta| \leq \alpha \). And, since for any \( \alpha \), \( \| P_\alpha \| < \alpha \)-strategically closed”, all functions \( f: \gamma \rightarrow \beta \) for \( \gamma < \alpha \) and \( \beta \) any ordinal in \( V[G] \) are so that \( f \in V^{P_\alpha} \). Thus, since \( P \) is an Easton support iteration, as in Lemma 8, \( V[G] \) satisfies Power Set and Replacement. This proves Lemma 12.

\( \square \) Lemma 12

We remark that if we rewrite \( \dot{Q}_\alpha \) as \( (\dot{P}_\alpha^0 \times \dot{P}_\alpha^1) \times (\dot{P}_\alpha^2 \times \dot{P}_\alpha^3) \), then the ideas used in the proof of Lemma 12 combined with an argument analogous to the one in the remark following the proof of Lemma 5 show \( \| P_\alpha \|=\text{\textquoteleft\textquoteleft}P_\alpha^0 \times \dot{P}_\alpha^1 \times \dot{P}_\alpha^3 \text{\textquoteright\textquoteright} \text{is } \alpha^+\text{-c.c.\textquoteright\textquoteright} \). Also, by their definitions, \( \| P_\alpha \|=\dot{P}_\alpha^0 \times \dot{P}_\alpha^1 \text{is } \alpha^+\text{-strategically closed} \). These
observations will be used in the proof of the following lemma, which is the natural analogue to Lemma 9.

**Lemma 13.** If \( \delta < \gamma \) and \( V \models \text{“} \delta \text{ is } \gamma \text{ supercompact and } \gamma \text{ is regular} \), then for \( G \) \( V\)-generic over \( P \), \( V[G] \models \text{“} \delta \text{ is } \gamma \text{ supercompact} \).

**Proof of Lemma 13.** We mimic the proof of Lemma 9. Let \( j : V \to M \) be an elementary embedding witnessing the \( \gamma \text{-supercompactness of } \delta \) so that \( M \models \text{“} \delta \text{ is not } \gamma \text{-supercompact} \”, \) and let \( \alpha_0 \) be so that \( \delta = \delta_{\alpha_0} \).

Let \( P = P_\delta \ast \dot{Q}_\delta^j \ast \dot{R} \), where \( P_\delta \) is the iteration through stage \( \delta \), \( \dot{Q}_\delta^j \) is a term for the iteration \( \langle \langle P_\alpha / P_\delta, \dot{Q}_\alpha \rangle : \alpha \leq \gamma \rangle \), and \( \dot{R} \) is a term for the rest of \( P \). As before, since \( \lbrack P_\delta / \dot{Q}_\delta^j \rbrack \text{ “} \dot{R} \text{ is } \gamma \text{-strategically closed} \text{”} \), the regularity of \( \gamma \) and \( \text{GCH in } V_{P_\delta \ast \dot{Q}_\delta^j} \text{ mean} \) it suffices to show \( V_{P_\delta \ast \dot{Q}_\delta^j} \models \text{“} \delta \text{ is } \gamma \text{-supercompact} \”).

We will again show that \( j : V \to M \) extends to \( k : V[G_\delta \ast G_\delta^j] \to M[H] \) for some \( H \subseteq j(P) \). In \( M \), \( j(P_\delta \ast \dot{Q}_\delta^j) = P_\delta \ast \dot{R}_\delta^j \ast \dot{R}_\delta^{\alpha_0} \), where \( \dot{R}_\delta^j \) will be a term for the iteration (as defined in \( M_{P_\delta} \langle \langle P_\alpha / P_\delta, \dot{Q}_\alpha \rangle : \alpha \leq \gamma \rangle \), \( \dot{R}_\delta^j \) will be a term for the iteration (as defined in \( M_{P_\delta \ast \dot{R}_\delta^j} \) \( \langle \langle P_\alpha / P_\gamma, \dot{Q}_\alpha \rangle : \gamma + 1 \leq \alpha < j(\delta) \rangle \), and \( \dot{R}_\delta^{\alpha_0} \) will be a term for the iteration (as defined in \( M_{P_\delta \ast \dot{R}_\delta^j \ast \dot{R}_\delta^{\alpha_0}} \) \( \langle \langle P_\alpha / P_\gamma, \dot{Q}_\alpha \rangle : j(\delta) \leq \alpha < j(\gamma) \rangle \)). By the facts that \( \text{GCH holds in both } V \text{ and } M \), \( M^\gamma \subseteq M \), and \( M \models \text{“} \delta \text{ is } \gamma \text{-supercompact } \text{ but } \delta \text{ is not } \gamma \text{-supercompact} \text{”} \), \( \dot{R}_\delta^j \) will actually be a term for the iteration \( \langle \langle P_\alpha / P_\delta, \dot{Q}_\alpha \rangle \rangle \; : \; \delta \leq \alpha < \gamma \rangle \langle \langle P_0^j \ast P_\gamma^j \rangle \times \langle \langle P_2^j \ast P_\delta^j \rangle \), where the term for the iteration \( \langle \langle P_\alpha / P_\delta, \dot{Q}_\alpha \rangle : \delta \leq \alpha < \gamma \rangle \rangle \) is the same as in \( V \), any term of the form \( (\bar{P}_{\theta_i} \ast \gamma, \dot{P}_{\theta_i} \ast \dot{\gamma}, \dot{\gamma}, \theta_{\delta}, \gamma) \ast \langle \langle \dot{\theta}_{\delta}, \dot{\gamma} \rangle \rangle \) appearing in \( \dot{R}_\delta^j \) (more specifically, \( \dot{P}_{\delta}^j \) or \( \dot{P}_{\delta}^{\alpha_0} \)) is identical to one appearing in \( \dot{Q}_\delta^j \), and if \( \dot{P}_{\theta_i} \ast \gamma, \dot{P}_{\theta_i} \ast \dot{\gamma} \ast \theta_{\delta}, \gamma \rangle \) appears in \( \dot{R}_\delta^j \) (more specifically, \( \dot{P}_{\gamma}^j \ast \dot{P}_{\delta}^j \)), then either it appears as an identical term in \( \dot{Q}_\delta^j \), or (as is the case, e.g., when \( i = \alpha_0 \) and \( \theta_i \) is defined) it appears as the term \( (\bar{P}_{\theta_i} \ast \gamma, \dot{P}_{\theta_i} \ast \dot{\gamma}, \theta_{\delta}, \gamma) \ast \dot{\theta}_{\delta}, \dot{\gamma}, \theta_{\delta}, \gamma \rangle \) in \( \dot{Q}_\delta^j \). This allows us to define \( H_\delta = G_\delta \), where \( G_\delta \) is the projection of \( G \text{-} V \)-generic over \( P_\delta \), and \( H_\delta = K \ast K' \text{, where } K \text{ is the projection of } G \text{ onto } \langle \langle P_\alpha / P_\delta, \dot{Q}_\alpha \rangle : \delta \leq \alpha < \gamma \rangle \text{ and } K' \text{ is the projection of } G \text{ onto } \langle \langle P_0^j \ast P_\gamma^j \rangle \times \langle \langle P_2^j \ast P_\delta^j \rangle \rangle \text{ as defined in } M \).

To construct the next portion of the generic object \( H''_\delta \), note that as in Lemma 9, the definition of \( P_\delta \) ensures \( |P_\delta| = \delta \) and \( P_\delta \) is \( \delta \)-c.c. Thus, as before, \( \text{GCH in } V[G_\delta] \) and \( M[G_\delta] \), the definition of \( R^j_\delta \), the fact \( M^\gamma \subseteq M \), and some applications of Lemma 6.4 of [Ba] allow us to conclude that \( M[H_\delta \ast H''_\delta] = M[G_\delta] \ast H''_\delta \) is closed under \( \gamma \)-sequences with respect to \( V[G_\delta, H''_\delta] \). Thus, any partial ordering which is \( \prec \gamma - \text{strategically closed in } M[H_\delta \ast H''_\delta] \) is actually \( \prec \gamma^+ - \text{strategically closed in } V[G_\delta \ast H''_\delta] \).

Observe now that if \( \langle T_\alpha \rangle \ast \alpha < \eta \) is so that each \( T_\alpha \) is a \( \rho^+ \)-strategically closed for some cardinal \( \rho \), then \( \prod_{\alpha < \eta} T_\alpha \) is also \( \prec \rho^+ - \text{strategically closed} \), for if \( \langle f_\alpha \rangle \ast \alpha < \eta \) is so that each \( f_\alpha \) is a winning strategy for player II for \( T_\alpha \), then \( \prod_{\alpha < \eta} f_\alpha \), i.e., pick the \( \alpha \)-th coordinate according to \( f_\alpha \), is a winning strategy for player II for \( \prod_{\alpha < \eta} T_\alpha \).

This observation easily implies \( \lbrack P_\delta \ast \dot{R}_\delta^j \rbrack \text{ “} \dot{R}_\delta^j \text{ is } \prec \gamma^+ - \text{strategically closed} \text{“} \) in either \( V[G_\delta \ast H''_\delta] \) or \( M[H_\delta \ast H''_\delta] \). The definition of the iteration \( R''_\delta \) then allows us, as in Lemma 9, to construct in \( V[G_\delta \ast H''_\delta] \subseteq V[G_\delta \ast G_\delta^j] \) an \( M[H_\delta \ast H''_\delta] \)-generic object.
$H^\gamma_5$ over $R^\gamma_5$. As in Lemma 9, $M[H_5 \ast H^\gamma_5 \ast H^\gamma_5]$ is closed under $\gamma$ sequences with respect to $V[G_5 \ast H^\gamma_5]$. Write $\hat{R}_5^\gamma$ as $\hat{R}_5^\gamma \ast \hat{R}_5^\gamma$, where $\hat{R}_5^\gamma$ is a term for the iteration $\langle \langle P_\alpha / P_\delta, \hat{Q}_\alpha \rangle : j(\delta) \leq \alpha < j(\gamma) \rangle$ and $\hat{R}_5^\gamma$ is a term for $\hat{Q}_{j(\gamma)}$. Also, write in $V$ $Q_5^\delta = Q_5^\delta \ast \hat{Q}_5^\gamma$, where $\hat{Q}_5^\gamma$ is a term for the iteration $\langle \langle P_\alpha / P_\delta, \hat{Q}_\alpha \rangle : \delta \leq \alpha < \gamma \rangle$ and $\hat{Q}_5^\gamma$ is a term for $\hat{Q}_\gamma$, and let $G_5^\gamma = G_5^\gamma \ast G_5^\gamma$ be the corresponding factorization of $G_5$. For any non-trivial term $\hat{Q}_\alpha = (P_0^\alpha \ast P_1^\alpha) \times (P_2^\alpha \ast P_3^\alpha)$ appearing in $\hat{R}_5^\gamma$, Lemma 4 and the fact that elements of $P_0^\alpha$ will have full support and elements of $P_4^\alpha$ will have support $< \alpha$ imply that in $M$, for $T = P_5 \times \hat{R}_5^\gamma \ast \hat{R}_5^\gamma \ast \langle \langle P_\delta / P_3(\delta), \hat{Q}_\delta \rangle : j(\delta) \leq \beta < \alpha \rangle$, $\parallel P_{\theta_i,\alpha}^0 \parallel_\theta$ “(A dense subset of) $\hat{R}_5^\gamma$ is $\gamma^+$-directed closed”. Further, if $\alpha \in \{j(\delta), j(\gamma)\}$ is so that for some $i, \alpha = \lambda_i$, then it must be the case that $j(\delta) < \delta_i$, for if $\delta_i \leq j(\delta)$, then by a theorem of Magidor [Ma2], since $M = \{ \delta_i < j(\delta) \}$ supercompact and $j(\delta)$ is $j(\gamma)$ supercompact, $M = \{ \delta_i \}$ is $j(\gamma)$ supercompact”, a contradiction to the fact $M = \{ \alpha = \lambda_i < j(\gamma) \}$. Hence, by the definition of $\theta_i$, it must be the case that $j(\delta) \leq \theta_i$, i.e., since $j(\delta) > \gamma$, $\theta_i > \gamma$. This means $\parallel P_{\theta_i,\alpha}^0 \parallel_\theta$ and $P_{\theta_i,\alpha}^0 [\hat{S}_\theta, \alpha]$ are $\gamma^+$-directed closed”, so as elements of $P_0^\alpha$ will have full support and elements of $P_4^\alpha$ will have support $< \alpha$, $\parallel P_4^\alpha \parallel_\theta$ “(A dense subset of) $\hat{R}_5^\gamma$ is $\gamma^+$-directed closed”. Therefore, using the extension of $j, j^*$ : $V[G_5] \to M[H_5 \ast H^\gamma_5 \ast H^\gamma_5]$ which we have produced in $V[G_5 \ast H_5]$, the fact that GCH in $M[H_5 \ast H^\gamma_5 \ast H^\gamma_5]$ implies $M[H_5 \ast H^\gamma_5 \ast H^\gamma_5] = \{ j(\gamma) \} = j(\gamma)$ and $2^\gamma = j(\gamma)^+$, $V[G_5 \ast H^\gamma_5] \models \{ j(\gamma^+) \} = \gamma^+$, and the closure properties of $M[H_5 \ast H^\gamma_5 \ast H^\gamma_5]$, we can produce in $V[G_5 \ast H_5^\gamma]$ as in Lemma 9 an upper bound $q$ for $\{ j^*(p) : p \in G_5^\gamma \}$ and an $M[H_5 \ast H^\gamma_5 \ast H^\gamma_5]$-generic object $H_5^\gamma$ for $\theta$ such that $q \in H_5^\gamma$. Again, as in Lemma 9, $M[H_5 \ast H^\gamma_5 \ast H^\gamma_5 \ast H^\gamma_5]$ is closed under $\gamma$-sequences with respect to $V[G_5 \ast H_5]$. Therefore, by the remarks after the proof of Lemma 12 and the proof of Lemma 6, $M[H_5 \ast H^\gamma_5 \ast H^\gamma_5 \ast H^\gamma_5]$ is closed under $\gamma$-sequences with respect to $V[G_5 \ast G_5^\gamma]$.

Rewrite $\hat{R}_5^\gamma$ as

\[
\left( \prod_{\{ i < j(\gamma) : \delta_i \text{ is } j(\gamma) \text{ supercompact} \}} (P_{\theta_i,j(\gamma)}^0 \ast P_{\theta_i,j(\gamma)}^2 \ast \hat{S}_{\theta_i,j(\gamma)}) \right) \times \left( \prod_{\{ i < j(\gamma) : j(\delta_i) = \lambda_i \}} P_{\theta_i,j(\gamma)}^0 \right) \]

\[
\ast \left( \prod_{\{ i < j(\gamma) : \delta_i \text{ is } j(\gamma) \text{ supercompact or } j(\delta_i) = \lambda_i \}} P_{\theta_i,j(\gamma)}^1 \right) \]

\[
= \hat{R}_5^\gamma \ast \hat{R}_5^\gamma,
\]

where all elements of $\hat{R}_5^\gamma$ will have full support, and all elements of $\hat{R}_5^\gamma$ will have support $< j(\gamma)$. By our earlier observation that products of (appropriately) strategically closed partial orderings retain the same amount of strategic closure, it is clearly the case that $Q_\gamma^\gamma$, the portion of $Q_\gamma$ corresponding to $R_5^\gamma$, i.e., $Q_\gamma^\gamma = \prod_{\{ i < j(\gamma) : j(\delta_i) = \gamma \text{ supercompact} \}} (P_{\theta_i,j(\gamma)}^0 \ast P_{\theta_i,j(\gamma)}^2 \ast \hat{S}_{\theta_i,j(\gamma)}) \times \prod_{\{ i < j(\gamma) : j(\delta_i) = \lambda_i \}} P_{\theta_i,j(\gamma)}^0$, is $\gamma$-strategically closed and therefore is $(\gamma, \infty)$-distributive. Hence, as we again have that in $V[G_5 \ast G_5^\gamma]$, $j^*$ extends to $j^{**} : V[G_5 \ast G_5^\gamma] \to M[H_5 \ast H^\gamma_5 \ast H^\gamma_5 \ast H^\gamma_5]$, we can
use $j^{**}$ as in the proof of Lemma 9 to transfer $G_\delta^4$, the projection of $G_\delta^{**}$ onto $Q_\gamma^*$, via the general transference principle given in [C], Section 1.2, Fact 2, pp. 5-6, to an $M[H_\delta^* * H_\delta^* * H_\delta^* * H_\delta^*]-generic$ object $H_\delta^5$ over $R_\delta^6$.

By its construction, since $p \in G_\delta^4$ implies $j^{**}(p) \in H_\delta^5$, $j^{**}$ extends in $V[G_\delta * G_\delta^4]$ to $j^{***}: V[G_\delta * G_\delta^{**} * G_\delta^4] \rightarrow M[H_\delta^* * H_\delta^* * H_\delta^{**} * H_\delta^4]$. And, since $R_\delta^6$ is $\gamma$-strategically closed, $M[H_\delta^* * H_\delta^* * H_\delta^{**} * H_\delta^4]$ and $M[H_\delta^* * H_\delta^* * H_\delta^{**} * H_\delta^4]$ contain the same $\gamma$ sequences of elements of $M[H_\delta^* * H_\delta^* * H_\delta^{**} * H_\delta^4]$ with respect to $V[G_\delta * G_\delta^4]$. As any $\gamma$ sequence of elements of $M[H_\delta^* * H_\delta^* * H_\delta^{**} * H_\delta^4]$ can be represented, in $M[H_\delta^* * H_\delta^* * H_\delta^{**} * H_\delta^4]$, by a term which is actually a function $f: \gamma \rightarrow M[H_\delta^* * H_\delta^* * H_\delta^{**} * H_\delta^4]$, and as $M[H_\delta^* * H_\delta^* * H_\delta^{**} * H_\delta^4]$ is closed under $\gamma$ sequences with respect to $V[G_\delta * G_\delta^4]$, $M[H_\delta^* * H_\delta^* * H_\delta^{**} * H_\delta^4]$ is closed under $\gamma$ sequences with respect to $V[G_\delta * G_\delta^4]$.

It remains to construct the $M[H_\delta^* * H_\delta^* * H_\delta^{**} * H_\delta^4]-generic$ object $H_\delta^5$ over $R_\delta^6$. To do this, take $Q_\gamma^*$ to be the portion of $Q_\gamma$ corresponding to $R_\delta^6$, i.e., $Q_\gamma^*$ is the $< \gamma$ support product $\prod_{\{i < \gamma: i \text{ is } \gamma\text{-supercompact or } i = \lambda_i\}} P_{\theta_i, \gamma}[S_{\theta_i, \gamma}]$, with $G_\delta^5$ the projection of $G_\delta^{**}$ onto $Q_\gamma^*$. Next, for the purpose of the remainder of the proof of this lemma, if $p \in R_\delta^6$ and $i < j(\gamma)$ is an ordinal, say that $i \in \text{support}(p)$ iff for some non-trivial component $\bar{p}$ of $p$, $\bar{p} \in P_{\theta_i, j(\gamma)}^0$. Analogously, it is clear what $i \in \text{support}(p)$ for $p \in R_\delta^7$ means. Now, let $A = \{i < j(\gamma): \text{ for some } p \in j^{***}G_\delta^4, i \in \text{support}(p)\}$, and let $B = \{i < j(\gamma): \text{ for some } q \in R_\delta^6, i \in \text{support}(q) \text{ but } i \notin \text{support}(p) \text{ for any } p \in j^{***}G_\delta^4\}$. Write $A = A_0 \cup A_1$, where $A_0 = \{i \in A: j(\gamma) = \lambda_i\}$ and $A_1 = \{i \in A: j(\gamma) \neq \lambda_i\}$. Note that since $H_\delta^5 = \{q \in R_\delta^6: \exists p \in j^{***}G_\delta^4[q \leq p]\}, A_0, A_1, B \in M[H_\delta^* * H_\delta^* * H_\delta^{**} * H_\delta^4].$

If $i \in A_1$, then by the genericity of $H_\delta^5$, $P_{\theta_i, j(\gamma)}[S_{\theta_i, j(\gamma)}]$ contains a dense sub-ordering $P_i^*$ given by Lemma 4 which is isomorphic to $Q_{j(\gamma)}^1$. Hence, we can infer that the $(< j(\gamma) \text{ support}) \text{ product } \prod_{i \in A_1} P_i^*$ is dense in the $(< j(\gamma) \text{ support}) \text{ product } \prod_{i \in A_1} P_{\theta_i, j(\gamma)}[S_{\theta_i, j(\gamma)}]$. Therefore, our remarks in the last paragraph imply $\prod_{i \in A_0} P_{\theta_i, j(\gamma)}[S_{\theta_i, j(\gamma)}] \times \prod_{i \in A_1} P_i^*$ is $\gamma^+\text{-directed closed, the fact that } M[H_\delta^* * H_\delta^* * H_\delta^{**} * H_\delta^4] \text{ is closed under } \gamma \text{ sequences with respect to } V[G_\delta * G_\delta^4] \text{ means that we can in essence ignore }$
each sequence $\bar{Z}$ as above and apply the arguments used in Lemma 9 to construct the generic object for $Q_{\lambda_0}(\gamma)$, to construct an $M[H_\delta \ast H_\delta^0 \ast H_\delta^1 \ast H_\delta^2 \ast H_\delta^3 \ast H_\delta^4 \ast H_\delta^5 \ast H_\delta^6 \ast H_\delta^7 \ast H_\delta^8 \ast H_\delta^9 \ast H_\delta^{10}]$-generic object $H_\delta^{6,0}$ for $\prod_{i \in \Lambda_0} \prod_{i \in \Lambda_0} P_i^{\lambda_i} \ast [S_{\lambda_i}] \times \prod_{i \in \Lambda_1} P_i^{\lambda_i} \ast [S_{\lambda_i}]$ as before, since $\prod_{i \in \Lambda_0} \prod_{i \in \Lambda_0} P_i^{\lambda_i} \ast [S_{\lambda_i}] \times \prod_{i \in \Lambda_1} P_i^{\lambda_i}$ is $\gamma$-directed closed, $M[H_\delta \ast H_\delta^0 \ast H_\delta^1 \ast H_\delta^2 \ast H_\delta^3 \ast H_\delta^4 \ast H_\delta^5 \ast H_\delta^6 \ast H_\delta^{6,0}]$ is closed under $\gamma$ sequences with respect to $V[G_\delta \ast G_\delta']$.

By our remarks following the proof of Lemma 12 and the ideas used in the remark following the proof of Lemma 5, $\prod_{i \in \mathcal{B}} P_i^{\lambda_i}/\mathcal{B} \ast [S_{\lambda_i}]$ is $j(\gamma^+)\ast$-c.c. in $M[H_\delta \ast H_\delta^0 \ast H_\delta^1 \ast H_\delta^2 \ast H_\delta^3 \ast H_\delta^4 \ast H_\delta^5 \ast H_\delta^6 \ast H_\delta^7 \ast H_\delta^8 \ast H_\delta^9 \ast H_\delta^{10}]$. So since $\prod_{i \in \mathcal{B}} P_i^{\lambda_i}/\mathcal{B} \ast [S_{\lambda_i}]$ has cardinality $j(\gamma^+)$ in $M[H_\delta \ast H_\delta^0 \ast H_\delta^1 \ast H_\delta^2 \ast H_\delta^3 \ast H_\delta^4 \ast H_\delta^5 \ast H_\delta^6 \ast H_\delta^7 \ast H_\delta^8 \ast H_\delta^9 \ast H_\delta^{10}]$, we can thus as in Lemma 9 let $\langle \lambda_0 : \alpha < \gamma^+ \rangle$ enumerate in $V[G_\delta \ast G_\delta']$ the maximal antichains of $\prod_{i \in \mathcal{B}} P_i^{\lambda_i}/\mathcal{B} \ast [S_{\lambda_i}]$ with respect to $M[H_\delta \ast H_\delta^0 \ast H_\delta^1 \ast H_\delta^2 \ast H_\delta^3 \ast H_\delta^4 \ast H_\delta^5 \ast H_\delta^6 \ast H_\delta^7 \ast H_\delta^8 \ast H_\delta^9 \ast H_\delta^{10}]$, and we can once more mimic the construction in Lemma 9 of $H_\delta^0$ to produce in $V[G_\delta \ast G_\delta']$ an $M[H_\delta \ast H_\delta^0 \ast H_\delta^1 \ast H_\delta^2 \ast H_\delta^3 \ast H_\delta^4 \ast H_\delta^5 \ast H_\delta^6 \ast H_\delta^7 \ast H_\delta^8 \ast H_\delta^9 \ast H_\delta^{10}]$-generic object $H_\delta^{6,1}$ over $\prod_{i \in \mathcal{B}} P_i^{\lambda_i}/\mathcal{B} \ast [S_{\lambda_i}]$. If we now let $H_\delta^6 = H_\delta^{6,0} \ast H_\delta^{6,1}$ and $H = H_\delta^1 \ast H_\delta^2 \ast H_\delta^3 \ast H_\delta^4 \ast H_\delta^5 \ast H_\delta^6 \ast H_\delta^7 \ast H_\delta^8 \ast H_\delta^9 \ast H_\delta^{10}$, then our construction guarantees $j : V \to M$ extends to $k : V[G_\delta \ast G_\delta'] \to M[H_\delta^6]$, so $V[G_\delta^6] \models \"\delta \ast \gamma \ast \text{supercompact}\"$. This proves Lemma 13.

We remark that the proof of Lemma 13 will work whether or not $\theta_{\lambda_0}$ is defined.

We prove now the natural analogue of Lemma 10.

**Lemma 14.** For $\gamma$ regular, $V[G_\delta^6] \models \"\delta \ast \gamma \ast \text{strongly compact if} \delta \ast \gamma \ast \text{supercompact, except possibly if for the i so that} \delta = \delta_i, \theta_i \ast \text{is undefined}\"$.

**Proof of Lemma 14.** As in Lemma 10, we assume towards a contradiction that the lemma is false, and let $\delta = \delta_{\lambda_0} < \gamma$ be so that $V[G_\delta] \models \"\delta \ast \gamma \ast \text{supercompact,} \theta_{\lambda_0} \ast \text{is defined,} \gamma \ast \text{is regular, and} \gamma \ast \text{is the least such cardinal}\"$. Since Lemma 13 implies that if $V \models \"\delta \ast \gamma \ast \text{supercompact}\"$, then $V[G_\delta] \models \"\delta \ast \gamma \ast \text{supercompact}\"$, as in Lemma 10, it must be the case that $\lambda_{\lambda_0} \leq \gamma$.

Write $P = P_{\lambda_0} \ast Q_{\lambda_0} \ast R$, where $P_{\lambda_0}$ is the forcing through stage $\lambda_{\lambda_0}$, $Q_{\lambda_0}$ is a term for the forcing at stage $\lambda_{\lambda_0}$, and $R$ is a term for the rest of the forcing. In $V[P_{\lambda_0}]$, since $V \models \"\delta = \delta_{\lambda_0} \ast \text{isn’t} \lambda_{\lambda_0} \ast \text{supercompact}\"$, we can write $Q_{\lambda_0}$ as $T_0 \times T_1$, where $T_1$ is $P_{\lambda_0}^{\lambda_0} \ast P_{\lambda_0}^{\lambda_0} \ast [S_{\lambda_0}] \times \lambda_{\lambda_0}$, and $T_0$ is the rest of $Q_{\lambda_0}$. Since $V[P_{\lambda_0}] \models \"T_0 \times P_{\lambda_0}^{\lambda_0} \ast \text{is} \ast \lambda_{\lambda_0} \ast \text{strategically closed}\"$ (and hence adds no new bounded subsets of $\lambda_{\lambda_0}$ when forcing over $V[P_{\lambda_0}]$), the arguments of Lemma 3 apply in $V[P_{\lambda_0} \ast (T_0 \times P_{\lambda_0}^{\lambda_0} \ast \lambda_{\lambda_0})]$ to show $V[P_{\lambda_0} \ast (T_0 \times P_{\lambda_0}^{\lambda_0} \ast \lambda_{\lambda_0}) \ast P_{\lambda_0}^{\lambda_0} \ast [S_{\lambda_0}] \times \lambda_{\lambda_0}] \models V[P_{\lambda_0}] \ast Q_{\lambda_0} \ast \lambda_{\lambda_0} \ast \text{isn’t} \lambda_{\lambda_0} \ast \text{strongly compact since} \lambda_{\lambda_0} \ast \text{doesn’t carry a} \lambda_{\lambda_0} \ast \text{additive uniform ultrafilter}\"$.

It remains to show that $V[P_{\lambda_0}] \ast Q_{\lambda_0} \ast R \ast V \models \"\delta_{\lambda_0} \ast \text{isn’t} \lambda_{\lambda_0} \ast \text{strongly compact}\"$. If this weren’t the case, then let $U$ be a term in $V[P_{\lambda_0}] \ast Q_{\lambda_0} \ast R$ so that $\bar{R} \ast \bar{U}$ is a $\delta_{\lambda_0}$-additive uniform ultrafilter over $\lambda_{\lambda_0}$.” Since $\bar{P}_{\lambda_0} \ast Q_{\lambda_0} \ast R \ast \bar{R}$ is $< \lambda_{\lambda_0} \ast \text{strategically closed}$ and $V[P_{\lambda_0}] \ast Q_{\lambda_0} \ast G \ast \text{GCH}$, we can let $\langle \lambda_0 : \alpha < \lambda_{\lambda_0} \rangle$ be in $V[P_{\lambda_0}] \ast Q_{\lambda_0}$ a listing of all of the subsets of $\lambda_{\lambda_0}$, as in the construction of $H_{\lambda_0}^\beta$ in Lemma 9, we can let
\[ r_{\alpha} : \alpha < \lambda_i^+ \] be an increasing sequence of elements of \( R \) so that \( r_{\alpha} \models \text{“} x_{\alpha} \in \hat{U} \text{”} \).

If now in \( V^{P_{\lambda_0}} \otimes \hat{Q}_{\lambda_0} \) we define \( U' \) by \( x_{\alpha} \in U' \Leftrightarrow r_{\alpha} \models \text{“} x_{\alpha} \in \hat{U} \text{”} \), then it is routine to check that \( U' \) is a \( \delta_{\text{in}} \)-additive uniform ultrafilter over \( \lambda_0 \) in \( V^{P_{\lambda_0}} \otimes \hat{Q}_{\lambda_0} \), which contradicts that there is no such ultrafilter in \( V^{P_{\lambda_0}} \otimes \hat{Q}_{\lambda_0} \). Thus, \( V' = \text{“} \delta_{\text{in}} \text{ isn’t } \lambda_0 \text{ strongly compact} \text{”} \), a contradiction to \( V[G] \models \text{“} \delta \text{ is } \gamma \text{ strongly compact} \text{”} \). This proves Lemma 14.

Note that the analogue to Lemma 11 holds if \( \delta = \delta_i \) and \( \theta_i \) is defined, i.e., for \( \gamma \) regular, \( V[G] \models \text{“} \delta \text{ is } \gamma \text{ supercompact} \text{”} \) if \( V \models \text{“} \delta \text{ is } \gamma \text{ supercompact} \text{”} \) and \( \delta = \delta_i \) and \( \theta_i \) is defined. The proof uses Lemmas 13 and 14 and is exactly the same as the proof of Lemma 11.

Lemmas 12–14 complete the proof of our Theorem in the general case.

\[ \square \text{ Theorem} \]

4. CONCLUDING REMARKS

In conclusion, we would like to mention that it is possible to use generalizations of the methods of this paper to answer some further questions concerning the possible relationships amongst strongly compact, supercompact, and measurable cardinals. In particular, it is possible to show, using generalizations of the methods of this paper, that the result of [Me] which states that the least measurable cardinal \( \kappa \) which is the limit of strongly compact or supercompact cardinals is not \( 2^\kappa \) supercompact is best possible. Specifically, if \( V \models \text{“} \text{ZFC + GCH + } \kappa \text{ is the least supercompact limit of supercompact cardinals } + \lambda > \kappa^+ \text{ is a regular cardinal which either is inaccessible or is the successor of a cardinal of cofinality } > \kappa + h : \kappa \rightarrow \kappa \text{ is a function so that for some elementary embedding } j : V \rightarrow M \text{ witnessing the } < \lambda \text{ supercompactness of } \kappa, j(h)(\kappa) = \lambda', \text{ then there is some generic extension } V[G] \models \text{“} \text{ZFC + For every cardinal } \delta < \kappa \text{ which is an inaccessible limit of supercompact cardinals and every cardinal } \gamma \in [\delta, h(\delta)), 2^\gamma = h(\delta) + \text{ For every cardinal } \gamma \in [\kappa, \lambda), 2^\gamma = \lambda + \kappa \text{ is } < \lambda \text{ supercompact + } \kappa \text{ is the least measurable limit of supercompact cardinals} \text{”} \) \].

It is also possible to show using generalizations of the methods of this paper that if \( V \models \text{“} \text{ZFC + GCH + } \kappa < \lambda \text{ are such that } \kappa < \lambda \text{ supercompact, } \lambda > \kappa^+ \text{ is a regular cardinal which either is inaccessible or is the successor of a cardinal of cofinality } > \kappa + h : \kappa \rightarrow \kappa \text{ is a function so that for some elementary embedding } j : V \rightarrow M \text{ witnessing the } < \lambda \text{ supercompactness of } \kappa, j(h)(\kappa) = \lambda', \text{ then there is some cardinal and cofinality preserving generic extension } V[G] \models \text{“} \text{ZFC + For every inaccessible } \delta < \kappa \text{ and every cardinal } \gamma \in [\delta, h(\delta)), 2^\gamma = h(\delta) + \text{ For every cardinal } \gamma \in [\kappa, \lambda), 2^\gamma = \lambda + \kappa \text{ is } < \lambda \text{ supercompact + } \kappa \text{ is the least measurable cardinal} \text{”} \) \]. This generalizes a result of Woodin (see [CW]), who showed, in response to a question posed to him by the first author, that it was possible to start from a model for \( \text{“} \text{ZFC + GCH + } \kappa < \lambda \text{ are such that } \kappa \text{ is } \lambda^+ \text{ supercompact and } \lambda \text{ is regular} \text{”} \) and use Radin forcing to produce a model for \( \text{“} \text{ZFC + } 2^\kappa = \lambda + \kappa \text{ is } \delta \text{ supercompact for all regular } \delta < \lambda + \kappa \text{ is the least measurable cardinal} \text{”} \). In addition, it is possible to iterate the forcing used in the construction of the above model to show, for instance, that if \( V \models \text{“} \text{ZFC + GCH + There is a proper class of cardinals } \kappa \text{ so that } \kappa \text{ is } \kappa^+ \text{ supercompact} \text{”} \), then there is some cardinal and cofinality preserving generic extension \( V[G] \models \text{“} \text{ZFC + } 2^\kappa = \kappa^{++} \text{ if } \kappa \text{ is inaccessible + There is a proper class of measurable cardinals + } \forall \kappa[\kappa \text{ is measurable iff } \kappa \text{ is } \kappa^+ \text{ strongly} \]
compact iff $\kappa$ is $\kappa^+$ supercompact] + No cardinal $\kappa$ is $\kappa^{++}$ strongly compact”. In this result, there is nothing special about $\kappa^+$, and each $\kappa$ can be $\lambda$ supercompact for $\lambda = \kappa^+$, $\lambda = \kappa^{++}$, or $\lambda$ essentially any “reasonable” value below $2^\kappa$. The proof of these results will appear in [AS].

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