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CONFLUENCE OF CYCLES FOR HYPERGEOMETRIC FUNCTIONS ON $Z_{2,n+1}$

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Abstract. The hypergeometric function of general type, which is a generalization of the classical confluent hypergeometric functions, admits an integral representation derived from a character of a linear abelian group. For the hypergeometric function on the space of $2 \times (n+1)$ matrices, a basis of cycles for the integral is constructed by a limit process, which is called a process of confluence. The determinant of the period matrix is explicitly evaluated to show the independence of the cycles.

Introduction

The hypergeometric function of type λ is introduced in [KHT1]. It contains the classical confluent hypergeometric functions—Kummer, Bessel, Hermite and Airy functions—and their generalizations in several variables as specializations, and is expected to be a substantial object in the special function theory.

For the hypergeometric function of type λ, we have an integral representation whose kernel is given by the character of a linear abelian group. Then several properties of the hypergeometric function are described in terms of linear abelian groups. However, without specifying domains of integration (cycles), we could study only formal properties. For the regular singular case, the cycles are studied by Aomoto [A2], [A3] and Kita [Kt], and there a topological theory is established for such cycles [IK1], [IK2]. For the confluent case, there is no systematic study of cycles.

In this paper we construct cycles for the hypergeometric function of general type with 1-dimensional integral representation. In [KHT2] we have defined the confluence of linear abelian groups, which govern the hypergeometric functions, and then obtained the confluence of cocycles. We shall show in §2 that we can define the confluence of cycles so as to be compatible with the confluence of cocycles. Then by step by step confluence starting from the twisted cycles for the regular singular case owing to Kita, we construct cycles for the confluent case (Theorem 1.2.4).

Our construction makes it possible to evaluate the determinant of the period matrix associated with the hypergeometric function. We give the explicit form of the determinant in Theorem 3.1.3, and this shows the independence of the cycles

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we have constructed. As a corollary we see that the rank of the hypergeometric system of type λ on $Z_{2,n+1}$ is $n-1$ (§3.4).

The confluence of cycles together with the confluence of cocycles defines the confluence of functions. Thus our result will make way for an analysis of fine structures of the hypergeometric function. For cycles for multi-dimensional integral representations, here we cite the works by Kimura [Km2], [Km3].

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Notation. $e(\alpha) := \exp(2\pi\sqrt{-1}\alpha)$ for $\alpha \in \mathbb{C}$.

1. Integral representation of the hypergeometric function and the main theorem

1.1. Hypergeometric function of type λ. To fix the notation, we recall the definition of the hypergeometric function of type λ.

Let n, r be integers such that $1 \le r < n$. Set

$$Z_{r+1,n+1} := \left\{ z = (z_{ij})_{\substack{0\le i\le r\\ 0\le j\le n}} \in \mathrm{M}(r+1, n+1; \mathbb{C}) \ ; \ \mathrm{rank}\ z = r+1 \right\}.$$

Let

$$\lambda = (1+\lambda_0, 1+\lambda_1, \dots, 1+\lambda_\ell)$$

be a composition of $n+1$; i.e.

$$\begin{aligned} &1+\lambda_0, 1+\lambda_1, \dots, 1+\lambda_\ell \in \mathbb{N}, \\ &(1+\lambda_0) + (1+\lambda_1) + \cdots + (1+\lambda_\ell) = n+1. \end{aligned}$$

Set

$$H_\lambda := J(1+\lambda_0) \times J(1+\lambda_1) \times \cdots \times J(1+\lambda_\ell) \subset \mathrm{GL}(n+1, \mathbb{C}),$$

where $J(m)$ denotes the Jordan group of size m in $\mathrm{GL}(m, \mathbb{C})$. Then every element h of H_λ is written as follows:

$$h = \bigoplus_{k=0}^{\ell} h^{(k)}, \quad h^{(k)} = \begin{pmatrix} h_0^{(k)} & h_1^{(k)} & \cdots & h_{\lambda_k}^{(k)} \\ & \ddots & \ddots & \vdots \\ & & \ddots & h_1^{(k)} \\ & & & h_0^{(k)} \end{pmatrix} \quad (0 \le k \le \ell). \tag{1.1.1}$$

We define the biholomorphic mapping $\iota_\lambda : H_\lambda \longrightarrow \prod_{k=1}^{\ell} (\mathbb{C}^\times \times \mathbb{C}^{\lambda_k})$ by

$$\iota_\lambda(h) = (\ \dots, (h_0^{(k)}, h_1^{(k)}, \dots, h_{\lambda_k}^{(k)}), \dots\ ).$$

According to the composition λ, we sometimes write $z \in Z_{r+1,n+1}$ and $\alpha \in \mathbb{C}^{n+1}$ as follows:

$$\begin{cases} z = (z_{ij}) =: (z^{(0)}, z^{(1)}, \dots, z^{(\ell)}), \\ \qquad z^{(k)} = (z_0^{(k)}, \dots, z_{\lambda_k}^{(k)}) \in \mathrm{M}(r+1, 1+\lambda_k) \quad (0 \le k \le \ell), \\ \qquad z_j^{(k)} = \begin{pmatrix} z_{0j}^{(k)} \\ \vdots \\ z_{rj}^{(k)} \end{pmatrix} \quad (0 \le k \le \ell,\ 0 \le j \le \lambda_k), \end{cases} \tag{1.1.2}$$

$$\alpha = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} =: \begin{pmatrix} \alpha^{(0)} \\ \alpha^{(1)} \\ \vdots \\ \alpha^{(\ell)} \end{pmatrix}, \quad \alpha^{(k)} = \begin{pmatrix} \alpha_0^{(k)} \\ \alpha_1^{(k)} \\ \vdots \\ \alpha_{\lambda_k}^{(k)} \end{pmatrix} \in \mathbb{C}^{\lambda_k+1} \quad (0 \le k \le \ell). \tag{1.1.3}$$

Definition 1.1.1. We define the polynomial $\theta_k(x_1, \dots, x_k)$ of k variables $x_1, \dots, x_k$ $(k = 1, 2, \dots)$ by the following generating function:

$$\log(1 + x_1 T + x_2 T^2 + \cdots + x_k T^k + \cdots) = \sum_{k=1}^{\infty} \theta_k(x_1, \dots, x_k) T^k. \tag{1.1.4}$$

It follows that

$$\theta_k(x_1, \dots, x_k) = \sum_{j=1}^{k} \frac{(-1)^{j-1}}{j} \sum_{\substack{i_1+\cdots+i_j=k \\ i_1,\dots,i_j \ge 1}} x_{i_1} \cdots x_{i_j}. \tag{1.1.5}$$

The character χ_λ of the universal covering group $\tilde{H}_\lambda$ is parametrized by $\alpha \in \mathbb{C}^{n+1}$:

(1.1.6)

$$\chi_\lambda(h) = \chi_\lambda(h, \alpha) = \prod_{k=0}^{\ell} (h_0^{(k)})^{\alpha_0^{(k)}} \exp\left[\sum_{j=1}^{\lambda_k} \alpha_j^{(k)} \theta_j\left(\frac{h_1^{(k)}}{h_0^{(k)}}, \dots, \frac{h_j^{(k)}}{h_0^{(k)}}\right)\right], \quad h \in \tilde{H}_\lambda,$$

where we write h and α as in (1.1.1) and (1.1.3). $\mathrm{GL}(r+1, \mathbb{C})$ (resp. H_λ) acts on $Z_{r+1,n+1}$ by left (resp. right) multiplication of matrices:

$$\mathrm{GL}(r+1, \mathbb{C}) \times Z_{r+1,n+1} \times H_\lambda \longrightarrow Z_{r+1,n+1}, \quad (g, z, h) \mapsto gzh.$$

Definition 1.1.2. Take $\alpha \in \mathbb{C}^{n+1}$. We assume

$$\sum_{k=0}^{\ell} \alpha_0^{(k)} = -(r+1). \tag{1.1.7}$$

The *hypergeometric function of type* λ *on* $Z_{r+1,n+1}$ is a (multi-valued) function $\Phi_\lambda(z)$ on $Z_{r+1,n+1} \setminus \{\text{divisors}\}$ satisfying

$$\Phi_\lambda(gz) = (\det g)^{-1} \Phi_\lambda(z), \quad \forall g \in \mathrm{GL}(r+1, \mathbb{C}), \tag{1.1.8}$$

$$\Phi_\lambda(zh) = \Phi_\lambda(z) \chi_\lambda(h, \alpha), \quad \forall h \in H_\lambda, \tag{1.1.9}$$

$$\left(\frac{\partial^2}{\partial z_{ip} \partial z_{jq}} - \frac{\partial^2}{\partial z_{iq} \partial z_{jp}}\right) \Phi_\lambda(z) = 0, \quad 0 \le i, j \le r,\ 0 \le p, q \le n. \tag{1.1.10}$$

To indicate the parameter α, we write $\Phi_\lambda(z) = \Phi_\lambda(z, \alpha)$. The system of the infinitesimal version of (1.1.8) and (1.1.9) together with (1.1.10) is called the *hypergeometric system of type* λ *on* $Z_{r+1,n+1}$.

We want to find a systematic way to find an r-chain Δ in $\mathbb{P}^r(\mathbb{C})$ so that

$$\Phi_\lambda(z, \alpha) = \int_\Delta \chi_\lambda(\iota_\lambda^{-1}(tz), \alpha) \omega, \tag{1.1.11}$$

where

$$t = (t_0, t_1, \dots, t_r) \in \mathbb{C}^{r+1},$$
$$\omega = \sum_{i=0}^{r} (-1)^i t_i dt_0 \wedge \cdots \wedge dt_{i-1} \wedge dt_{i+1} \wedge \cdots \wedge dt_r,$$

is convergent, and is a solution of the system (1.1.8)-(1.1.10). In this paper we only work when $r = 1$.

1.2. Integral representation in the affine space $(r = 1)$**.** In the following of this paper we set $r = 1$. We normalize $z \in Z_{2,n+1}$ as

$$\begin{pmatrix} z_{00} \\ z_{10} \end{pmatrix} = \begin{pmatrix} z_{00}^{(0)} \\ z_{10}^{(0)} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{1.2.1}$$

Then the integral representation (1.1.11) is reduced to an integral on the affine space $\mathbb{C}$. Setting $u = t_1/t_0$, from (1.1.6), (1.1.7), (1.1.11) and (1.2.1) we obtain

$$\Phi_\lambda(z, \alpha) = \int_\Delta U_\lambda(u; z, \alpha) du, \tag{1.2.2}$$

where

(1.2.3)

$$U_\lambda(u; z, \alpha) = \exp\left[\sum_{j=1}^{\lambda_0} \alpha_j^{(0)} \theta_j\big(z_{01}^{(0)} + u z_{11}^{(0)}, \dots, z_{0j}^{(0)} + u z_{1j}^{(0)}\big)\right]$$
$$\times \prod_{k=1}^{\ell} (z_{00}^{(k)} + u z_{10}^{(k)})^{\alpha_0^{(k)}} \exp\left[\sum_{j=1}^{\lambda_k} \alpha_j^{(k)} \theta_j\left(\frac{z_{01}^{(k)} + u z_{11}^{(k)}}{z_{00}^{(k)} + u z_{10}^{(k)}}, \dots, \frac{z_{0j}^{(k)} + u z_{1j}^{(k)}}{z_{00}^{(k)} + u z_{10}^{(k)}}\right)\right].$$

The chain Δ is a sum of paths in $\mathbb{C}$, whose construction is the theme of this paper. On the chain Δ we require that the integral (1.2.2) converges, and that the integration by parts works so that (1.2.2) satisfies (1.1.8), (1.1.9) and (1.1.10). Thus there comes the notion of twisted homology groups with supports.

Set $I_\lambda := \{\, k \in \{0, 1, \dots, \ell\} \,;\, \lambda_k > 0 \,\}$. Take $z \in Z_{2,n+1}$ satisfying (1.2.1) and

$$\begin{cases} \det \begin{pmatrix} z_{00}^{(i)} & z_{00}^{(j)} \\ z_{10}^{(i)} & z_{10}^{(j)} \end{pmatrix} \neq 0 \quad (0 \le i, j \le \ell,\ i \neq j), \\ \det \begin{pmatrix} z_{00}^{(i)} & z_{01}^{(i)} \\ z_{10}^{(i)} & z_{11}^{(i)} \end{pmatrix} \neq 0 \quad (i \in I_\lambda), \end{cases} \tag{1.2.4}$$

take $\alpha \in \mathbb{C}^{n+1}$ satisfying (1.1.7) and

$$\begin{cases} \alpha_0^{(k)} \notin \mathbb{Z} \quad (1 \le k \le \ell), \\ \alpha_{\lambda_k}^{(k)} \neq 0 \quad (k \in I_\lambda), \end{cases} \tag{1.2.5}$$

and fix them. We set

$$\zeta^{(k)} := -\frac{z_{00}^{(k)}}{z_{10}^{(k)}} \in \mathbb{P}^1(\mathbb{C}) \quad (0 \le k \le \ell). \tag{1.2.6}$$

Note that $\zeta^{(0)} = \infty$ by (1.2.1). Set $X := \mathbb{P}^1(\mathbb{C}) \setminus \{\zeta^{(0)}, \zeta^{(1)}, \dots, \zeta^{(\ell)}\}$. Let

$$\nabla := d + d \log U_\lambda(u; z, \alpha) \wedge \tag{1.2.7}$$

be the connection associated with (1.2.2). Let $\mathcal{S}$ be the complex local system of the local solutions of $\nabla\varphi = 0$, and let $\check{\mathcal{S}}$ denote the dual of $\mathcal{S}$. Note that U_λ is a section of $\check{\mathcal{S}}$.

For a p-simplex Δ in X and a branch $U_{\lambda\Delta}$ of U_λ on Δ, we denote by $\Delta \otimes U_{\lambda\Delta}$ the pair $(\Delta, U_{\lambda\Delta})$. We define a family of supports Ψ by

$$\Psi := \{A \subset X \text{ closed } ;\ \Re \log U_\lambda(u; z, \alpha)|_A \to -\infty \text{ as } u \to \zeta^{(k)}\ (\forall k \in I_\lambda)\}.$$

Set

$$C_p^{\ell f}({}^\Psi X, \check{\mathcal{S}}) := \left\{ \sum c_\Delta \Delta \otimes U_{\lambda\Delta}\ ;\ c_\Delta \in \mathbb{C}, \{\Delta\} : \text{locally finite}, \bigcup \text{supp}\Delta \in \Psi \right\},$$

the group of locally finite twisted p-chains in X with supports in Ψ. The boundary operator

$$\partial : C_p^{\ell f}({}^\Psi X, \check{\mathcal{S}}) \to C_{p-1}^{\ell f}({}^\Psi X, \check{\mathcal{S}})$$

is defined by

$$\partial(\Delta \otimes U_{\lambda\Delta}) = \sum_{j=0}^{p} (-1)^j \langle 01 \cdots \hat{j} \cdots p\rangle \otimes U_{\lambda_{\langle 01\cdots\hat{j}\cdots p\rangle}},$$

where $\Delta = \langle 01 \cdots p\rangle$ is a p-simplex, and $U_{\lambda_{\langle 01\cdots\hat{j}\cdots p\rangle}}$ is the branch on $\langle 01 \cdots \hat{j} \cdots p\rangle$ determined by the branch $U_{\lambda\Delta}$ on Δ. Then we see that $\partial \circ \partial = 0$. Set

$$Z_p^{\ell f}({}^\Psi X, \check{\mathcal{S}}) := \ker(\partial : C_p^{\ell f}({}^\Psi X, \check{\mathcal{S}}) \to C_{p-1}^{\ell f}({}^\Psi X, \check{\mathcal{S}})),$$
$$B_p^{\ell f}({}^\Psi X, \check{\mathcal{S}}) := \partial C_{p+1}^{\ell f}({}^\Psi X, \check{\mathcal{S}}).$$

Define

$$H_p^{\ell f}({}^\Psi X, \check{\mathcal{S}}) := Z_p^{\ell f}({}^\Psi X, \check{\mathcal{S}})/B_p^{\ell f}({}^\Psi X, \check{\mathcal{S}}),$$

and we call it the p-th *twisted homology group of locally finite chains in X with supports in Ψ* (cf. [P1] and [P2]).

Now we are going to give elements of $H_1^{\ell f}({}^\Psi X, \check{\mathcal{S}})$ to state the main theorem. First we look at the condition defining the family of supports Ψ. If $\lambda_0 > 0$, for $j = 0, 1, \ldots, \lambda_0 - 1$ we set

$$\Theta_j^{(0)} := \left\{ \vartheta \in \mathbb{R}\ ;\ \frac{4j+1}{2\lambda_0}\pi - \frac{\psi^{(0)}}{\lambda_0} < \vartheta < \frac{4j+3}{2\lambda_0}\pi - \frac{\psi^{(0)}}{\lambda_0} \right\},$$

where $\psi^{(0)} := \arg \alpha_{\lambda_0}^{(0)} + (\lambda_0 - 1)\pi + \lambda_0 \arg z_{11}^{(0)}$. For $k \in I_\lambda$, $k \geq 1$, we set

$$\Theta_j^{(k)} := \left\{ \vartheta \in \mathbb{R}\ ;\ \frac{4j+1}{2\lambda_k}\pi + \frac{\psi^{(k)}}{\lambda_k} < \vartheta < \frac{4j+3}{2\lambda_k}\pi + \frac{\psi^{(k)}}{\lambda_k} \right\}$$

for $j = 0, 1, \ldots, \lambda_k - 1$, where

$$\psi^{(k)} := \arg \alpha_{\lambda_k}^{(k)} + (\lambda_k - 1)\pi + \lambda_k \arg((z_{10}^{(k)} z_{01}^{(k)} - z_{00}^{(k)} z_{11}^{(k)})/z_{10}^{(k)2}).$$

By using (1.2.3) and Definition 1.1.1, we obtain

Lemma 1.2.1. (i) *Suppose $\lambda_0 > 0$. If $\arg u$ is constant and belongs to $\bigcup_{j=0}^{\lambda_0-1} \Theta_j^{(0)}$, then*

$$\Re \log U_\lambda(u; z, \alpha) \to -\infty \quad \textit{as } u \to \zeta^{(0)} = \infty.$$

(ii) *Suppose* $\lambda_k > 0$ *for some* $k \geq 1$. *If* $\arg u$ *is constant and belongs to* $\bigcup_{j=0}^{\lambda_k - 1} \Theta_j^{(k)}$, *then*

$$\Re \log U_\lambda(u; z, \alpha) \to -\infty \quad as\ u \to \zeta^{(k)}.$$

Next we give topological chains. For $r \in \mathbb{R}_{>0}$, $\vartheta, \underline{\vartheta}, \overline{\vartheta} \in \mathbb{R}$, $\zeta, \eta \in \mathbb{P}^1(\mathbb{C})$ we define paths $\ell_r(\zeta; \vartheta)$, $\overline{\ell}_r(\zeta; \vartheta)$, $c_r(\zeta; \underline{\vartheta}, \overline{\vartheta})$ and $b(\zeta, \eta)$ in X by

$$\begin{cases} \ell_r(\zeta; \vartheta)(t) = (1-t)\zeta + t(\zeta + r\exp(\sqrt{-1}\vartheta)), \quad t \in (0,1], \quad \text{if } \zeta \in \mathbb{C}, \\ \ell_r(\infty; \vartheta)(t) = t^{-1} r \exp(\sqrt{-1}\vartheta), \quad t \in (0,1], \end{cases}$$

$$\begin{cases} \overline{\ell}_r(\zeta; \vartheta)(t) = t\zeta + (1-t)(\zeta + r\exp(\sqrt{-1}\vartheta), \quad t \in [0,1), \quad \text{if } \zeta \in \mathbb{C}, \\ \overline{\ell}_r(\infty; \vartheta)(t) = (1-t)^{-1} r \exp(\sqrt{-1}\vartheta), \quad t \in [0,1), \end{cases}$$

$$\begin{cases} c_r(\zeta; \underline{\vartheta}, \overline{\vartheta})(t) = \zeta + r\exp\sqrt{-1}((1-t)\underline{\vartheta} + t\overline{\vartheta}), \quad t \in [0,1], \quad \text{if } \zeta \in \mathbb{C}, \\ c_r(\infty; \underline{\vartheta}, \overline{\vartheta})(t) = r\exp\sqrt{-1}((1-t)\underline{\vartheta} + t\overline{\vartheta}), \quad t \in [0,1], \end{cases}$$

$$b(\zeta, \eta)(t) = (1-t)\zeta + t\eta, \quad t \in [0,1], \quad \text{if } \zeta, \eta \in \mathbb{C}.$$

Using these notations, we shall define locally finite chains $\Delta_j^{\lambda,(k)}(z, \alpha)$ ($k = 0, 1, \ldots, \ell$; $j = 0, 1, \ldots, \lambda_k$). For the sake of simplicity, we assume that $\zeta^{(1)}, \ldots, \zeta^{(\ell)}$ are arranged as

$$\arg(\zeta^{(j)} - \zeta^{(k)}) \approx \begin{cases} 0, & \text{if } j > k, \\ \pi, & \text{if } j < k, \end{cases} \quad j, k = 1, \ldots, \ell. \tag{1.2.8}$$

Definition 1.2.2. Take $R > 0$ so large that $|\zeta^{(k)}| < R/2$ ($k = 1, \ldots, \ell$), and take $r > 0$ so small that $|\zeta^{(j)} - \zeta^{(k)}| > 2r$ ($j, k = 1, \ldots, \ell,\ j \neq k$). For every $k \in I_\lambda$, fix a permutation σ_k of λ_k letters, and choose

$$\vartheta_j^{(k)} \in \Theta_{\sigma_k(j)}^{(k)}$$

for $j = 0, 1, \ldots, \lambda_k - 1$.
(I) When $0 \in I_\lambda$ (i.e. $\lambda_0 > 0$),
(i) for $j = 1, \ldots, \lambda_0 - 1$, we define

$$\Delta_j^{\lambda,(0)}(z, \alpha) := \ell_R(\zeta^{(0)}; \vartheta_{j-1}^{(0)}) + c_R(\zeta^{(0)}; \vartheta_{j-1}^{(0)}, \vartheta_j^{(0)}) + \overline{\ell}_R(\zeta^{(0)}; \vartheta_j^{(0)});$$

(ii) if $1 \notin I_\lambda$ (i.e. $\lambda_1 = 0$), we define

$$\begin{aligned} \Delta_{\lambda_0}^{\lambda,(0)}(z, \alpha) := {} & \ell_R(\zeta^{(0)}; \vartheta_{\lambda_0 - 1}^{(0)}) + c_R(\zeta^{(0)}; \vartheta_{\lambda_0 - 1}^{(0)}, \pi) + b(Re^{\pi\sqrt{-1}}, \zeta^{(1)} + re^{\pi\sqrt{-1}}) \\ & + \frac{1}{e(\alpha_0^{(1)}) - 1} c_r(\zeta^{(1)}; \pi, 3\pi); \end{aligned}$$

(iii) if $1 \in I_\lambda$ (i.e. $\lambda_1 > 0$), we define

$$\begin{aligned} \Delta_{\lambda_0}^{\lambda,(0)}(z, \alpha) := {} & \ell_R(\zeta^{(0)}; \vartheta_{\lambda_0 - 1}^{(0)}) + c_R(\zeta^{(0)}; \vartheta_{\lambda_0 - 1}^{(0)}, \pi) + b(Re^{\pi\sqrt{-1}}, \zeta^{(1)} + re^{\pi\sqrt{-1}}) \\ & + c_r(\zeta^{(1)}; \pi, \vartheta_0^{(1)}) + \overline{\ell}_r(\zeta^{(1)}; \vartheta_0^{(1)}). \end{aligned}$$

(II) When $k \geq 1$, $k \notin I_\lambda$ (i.e. $\lambda_k = 0$),

(i) if $k+1 \notin I_\lambda$ (i.e. $\lambda_{k+1}=0$), we define

$$\Delta_0^{\lambda,(k)}(z,\alpha) := \frac{1}{e(\alpha_0^{(k)})-1} c_r(\zeta^{(k)};0,2\pi) + b(\zeta^{(k)}+r,\zeta^{(k+1)}+re^{\pi\sqrt{-1}}) + \frac{-1}{e(\alpha_0^{(k+1)})-1} c_r(\zeta^{(k+1)};\pi,3\pi);$$

(ii) if $k \in I_\lambda$ (i.e. $\lambda_k > 0$), we define

$$\Delta_0^{\lambda,(k)}(z,\alpha) := \frac{1}{e(\alpha_0^{(k)})-1} c_r(\zeta^{(k)};0,2\pi) + b(\zeta^{(k)}+r,\zeta^{(k+1)}+re^{\pi\sqrt{-1}}) + c_r(\zeta^{(k+1)};\pi,\vartheta_0^{(k+1)}) + \overline{\ell}_r(\zeta^{(k+1)};\vartheta_0^{(k+1)}).$$

(III) When $k \geq 1$, $k \in I_\lambda$ (i.e. $\lambda_k > 0$),
(i) we define

$$\Delta_0^{\lambda,(k)}(z,\alpha) := \ell_r(\zeta^{(k)};\vartheta_0^{(k)}) + c_r(\zeta^{(k)};\vartheta_0^{(k)},\vartheta_0^{(k)}+2\pi) + \overline{\ell}_r(\zeta^{(k)};\vartheta_0^{(k)}+2\pi);$$

(ii) for $j=1,\dots,\lambda_k-1$, we define

$$\Delta_j^{\lambda,(k)}(z,\alpha) := \ell_r(\zeta^{(k)};\vartheta_{j-1}^{(k)}) + c_r(\zeta^{(k)};\vartheta_{j-1}^{(k)},\vartheta_j^{(k)}) + \overline{\ell}_r(\zeta^{(k)};\vartheta_j^{(k)});$$

(iii) if $k+1 \notin I_\lambda$ (i.e. $\lambda_{k+1}=0$), we define

$$\Delta_{\lambda_k}^{\lambda,(k)}(z,\alpha) := \ell_r(\zeta^{(k)};\vartheta_{\lambda_k-1}^{(k)}) + c_r(\zeta^{(k)};\vartheta_{\lambda_k-1}^{(k)},0) + b(\zeta^{(k)}+r,\zeta^{(k+1)}+re^{\pi\sqrt{-1}}) + \frac{-1}{e(\alpha_0^{(k+1)})-1} c_r(\zeta^{(k+1)};\pi,3\pi);$$

(iv) if $k+1 \in I_\lambda$ (i.e. $\lambda_{k+1} > 0$), we define

$$\Delta_{\lambda_k}^{\lambda,(k)}(z,\alpha) := \ell_r(\zeta^{(k)};\vartheta_{\lambda_k-1}^{(k)}) + c_r(\zeta^{(k)};\vartheta_{\lambda_k-1}^{(k)},0) + b(\zeta^{(k)}+r,\zeta^{(k+1)}+re^{\pi\sqrt{-1}}) + c_r(\zeta^{(k+1)};\pi,\vartheta_0^{(k+1)}) + \overline{\ell}_r(\zeta^{(k+1)};\vartheta_0^{(k+1)}).$$

We use $\Delta_j^{\lambda,(k)}$ for $\Delta_j^{\lambda,(k)}(z,\alpha)$ if there is no possibility of confusion. The locally finite chains $\Delta_j^{\lambda,(k)}$ are illustrated in Figure 1.

With each locally finite chain $\Delta_j^{\lambda,(k)}$ we associate a branch of the multi-valued function $U_\lambda(u;z,\alpha)$ in such a manner that, C and C' being component paths of $\Delta_j^{\lambda,(k)}$, if the beginning points of C and C' are coincide, or if the beginning point of C coincides only with the end point of C', then the branches at that common point conincide, and the branch at any point in C is determined by the analytic continuation of the branch at the beginning point along C. Hence the branch at every point on $\Delta_j^{\lambda,(k)}$ is uniquely determined by a branch at a point on $\Delta_j^{\lambda,(k)}$. We denote by $\Delta_j^{\lambda,(k)} \otimes U_\lambda$ the locally finite twisted chain thus determined. By a similar argument in [Kt, II] together with Lemma 1.2.1, we obtain

Proposition 1.2.3. *The locally finite twisted chains* $\Delta_j^{\lambda,(k)}(z,\alpha)\otimes U_\lambda$ $(k=0,1,\dots,\ell;\ j=0,1,\dots,\lambda_k)$ *are twisted cycles: namely,*

$$\Delta_j^{\lambda,(k)}(z,\alpha)\otimes U_\lambda \in Z_1^{\ell f}({}^\Psi X,\check{\mathcal{S}}).$$

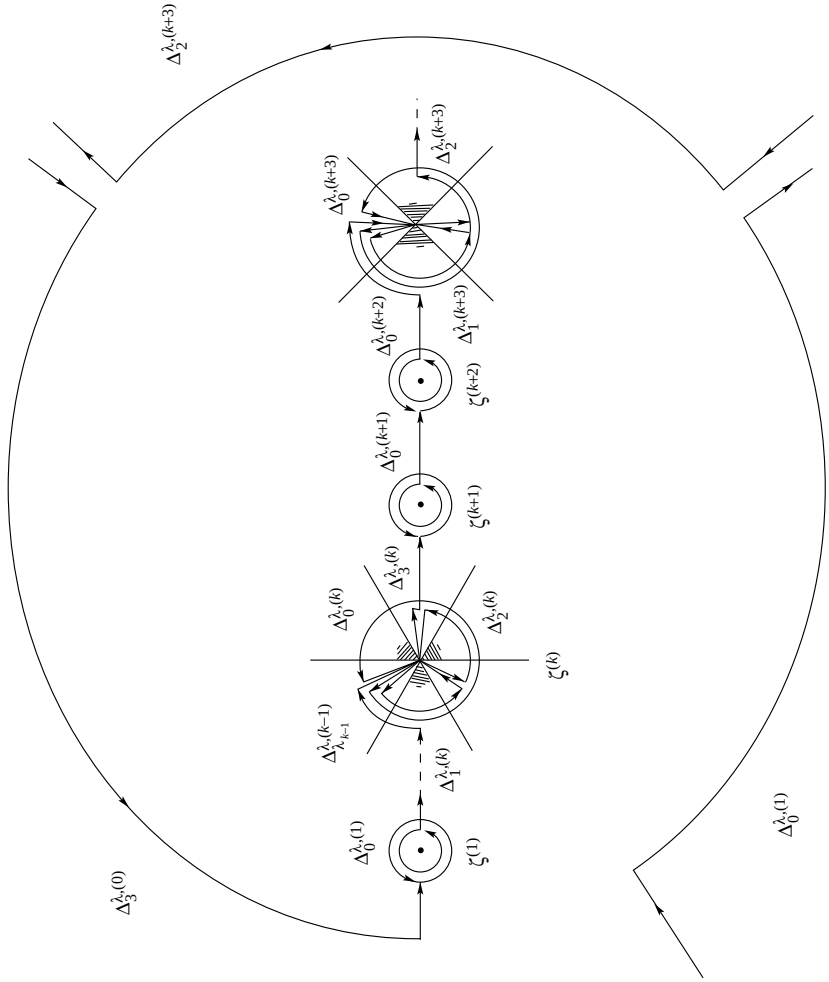

FIGURE 1. Cycles $\Delta_j^{\lambda,(k)}$ for $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_\ell)$ with $\lambda_0 = 3$, $\lambda_1 = 0, \ldots, \lambda_k = 3$, $\lambda_{k+1} = \lambda_{k+2} = 0$, $\lambda_{k+3} = 2, \ldots$

We denote the homology class of the locally finite twisted cycle $\Delta_j^{\lambda,(k)}(z,\alpha)\otimes U_\lambda$ by $[\Delta_j^{\lambda,(k)}(z,\alpha)\otimes U_\lambda]$. Now we state our main theorem.

Theorem 1.2.4. *Assume* (1.2.4) *and* (1.2.5). *Then the set of* $(n-1)$*-elements*

$$\begin{aligned}&\{[\Delta_j^{\lambda,(0)}(z,\alpha)\otimes U_\lambda]\ ;\ j=1,\dots,\lambda_0\}\\ &\qquad\cup\bigcup_{k=1}^{\ell-1}\{[\Delta_j^{\lambda,(k)}(z,\alpha)\otimes U_\lambda]\ ;\ j=0,1,\dots,\lambda_k\}\\ &\qquad\cup\{[\Delta_j^{\lambda,(\ell)}(z,\alpha)\otimes U_\lambda]\ ;\ j=0,1,\dots,\lambda_\ell-1\}\end{aligned}$$

makes a basis of $H_1^{\ell f}({}^\Psi X,\check{\mathcal{S}})$. (*When* $\lambda_0=0$ (*resp.* $\lambda_\ell=0$), *the first* (*resp. the last*) *set of the above union is empty.*)

The theorem is a special case of Theorem 2 in [Kt, II] when $I_\lambda=\emptyset$ (see also [A2], [A3]). For the confluent case ($I_\lambda\neq\emptyset$), we construct the twisted cycles $\Delta_j^{\lambda,(k)}(z,\alpha)\otimes U_\lambda$ by confluence in §2. In §3 we evaluate the determinant of the period matrix to prove Theorem 1.2.4.

1.3. Confluence of cocycles. Let a composition λ, $z\in Z_{2,n+1}$ and $\alpha\in\mathbb{C}^{n+1}$ be fixed as in §1.2. We denote by Ω^p the rational p-forms in u with poles at $\{\zeta^{(0)},\zeta^{(1)},\dots,\zeta^{(\ell)}\}$ $(p=0,1)$; i.e.

$$\begin{aligned}\Omega^0&=\mathbb{C}\left[u,\frac{1}{u-\zeta^{(1)}},\dots,\frac{1}{u-\zeta^{(\ell)}}\right],\\ \Omega^1&=\Omega^0du.\end{aligned}$$

Then we obtain the rational de Rham complex

$$(\Omega^\bullet,\nabla):0\longrightarrow\Omega^0\xrightarrow{\nabla}\Omega^1\longrightarrow 0,$$

where ∇ is the connection defined by (1.2.7). We set

$$\varphi_\lambda^k(z,\alpha)=U_\lambda(z,\alpha)u^{k-1}du$$

for $k=1,2,\dots,n-1$. By $[\varphi_\lambda^k(z,\alpha)]$ we denote the cohomology class defined by $\varphi_\lambda^k(z,\alpha)$.

Theorem 1.3.1 (Kimura [Km1]). (i) $H^0(\Omega^\bullet,\nabla)=0$.
(ii) $\{[\varphi_\lambda^k(z,\alpha)];k=1,2,\dots,n-1\}$ *is a basis of* $H^1(\Omega^\bullet,\nabla)$.

Now we proceed to a confluence of cocycles.

In [KHT2] we gave a manner of confluence of linear abelian groups H_λ, which induces a confluence of characters of the groups, and hence a confluence of cocycles associated with the integral representations of the hypergeometric functions of type λ. Here we recall a manner of confluence which will be used later.

For $m\in\mathbb{N}_0$, we consider the two compositions

$$\begin{aligned}\lambda&=(1+\lambda_0,\dots,1+\lambda_{i-1},\overset{\overset{i}{\smile}}{1+m},1,1,\dots,1),\\ \mu&=(1+\lambda_0,\dots,1+\lambda_{i-1},1+(m+1),1,\dots,1)\end{aligned}$$

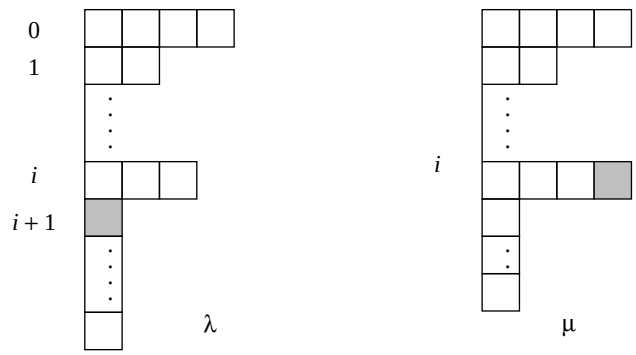

FIGURE 2

of $n+1$. In this case μ is said to be *simply adjacent* to λ, and we write as $\lambda \Rightarrow \mu$. The diagrams associated with the compositions λ and μ are given in Figure 2. We define a holomorphic mapping $g : \mathbb{C} \setminus \{0\} \to \mathrm{GL}(n+1, \mathbb{C})$ by

$$g(\varepsilon) = \left(\begin{array}{c|ccccc|c} I_i & & & & & & \\ \hline & 1 & & & & 1 & \\ & & 1 & & & \varepsilon & \\ & & & \ddots & & \vdots & \\ & & & & 1 & \varepsilon^m & \\ & & & & & \varepsilon^{m+1} & \\ \hline & & & & & & I_{n-i-m-1} \end{array}\right), \quad \varepsilon \in \mathbb{C} \setminus \{0\}. \tag{1.3.1}$$

For $z \in Z_{2,n+1}$ and for $\alpha \in \mathbb{C}^{n+1}$, set

$$z(\varepsilon) := z\, g(\varepsilon), \quad \alpha(\varepsilon) := g(\varepsilon)^{-1}\alpha. \tag{1.3.2}$$

Then we have

Proposition 1.3.2.

$$\lim_{\varepsilon \to 0} U_\lambda(u; z(\varepsilon), \alpha(\varepsilon)) = U_\mu(u; z, \alpha).$$

A similar relationship holds for the cocycles $\varphi^k_\lambda(z, \alpha)$.

Corollary 1.3.3.

$$\lim_{\varepsilon \to 0} \varphi^k_\lambda(z(\varepsilon), \alpha(\varepsilon)) = \varphi^k_\mu(z, \alpha),$$

for $k = 1, 2, \ldots, n-1$.

2. Construction of cycles by means of confluence

In this section we construct cycles for the integral representation (1.2.2) of the hypergeometric function of type λ. For every composition λ, there is a chain

$$\epsilon = \lambda^{(0)} \Rightarrow \lambda^{(1)} \Rightarrow \cdots \Rightarrow \lambda^{(N)} = \lambda \tag{2.0.1}$$

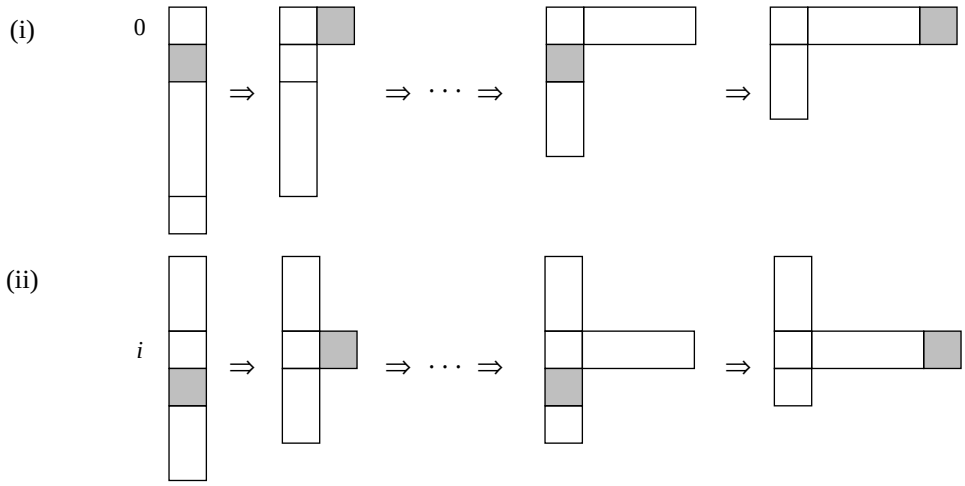

FIGURE 3

of compositions such that $\lambda^{(j)}$ is simply adjacent to $\lambda^{(j-1)}$ for $j = 1, 2, \ldots, N$, where

$$\epsilon = (1, 1, \ldots, 1),$$

which we call the *basic composition*. As we have noted after Theorem 1.2.4, it is already known that the theorem holds when $\lambda = \epsilon$, and we have a basis of cycles for the hypergeometric function of type ϵ. Hence, to obtain cycles for a general composition, it suffices to construct cycles for μ from those for λ by means of confluence when μ is simply adjacent to λ. Thus we study the confluence of cycles for two particular cases (i) and (ii) in Figure 3, and then we will obtain the cycles for a general composition. In the following we fix the following notation:

$$\begin{aligned} z &= (z_{ij})_{\substack{0\le i\le 1\\ 0\le j\le n}} = \begin{pmatrix} 1 & z_{01} & \cdots & z_{0n} \\ 0 & z_{11} & \cdots & z_{1n} \end{pmatrix}, \\ \zeta_j &= -\frac{z_{0j}}{z_{1j}}, \quad j = 1, \ldots, n, \\ \alpha &= \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}. \end{aligned} \tag{2.0.2}$$

2.1. Case (i). In this subsection we restrict ourselves to the composition

$$[m] := (1 + m, 1, 1, \ldots, 1)$$

of $n + 1$, where m is a non-negative integer. Here we change some notation in §1.2 according to $\lambda = [m]$. When $m > 0$, $\alpha_m \neq 0$ and $z_{11} \neq 0$, we set

(2.1.1)
$$\Theta_j^{[m]}(z,\alpha) := \left\{ \vartheta \in \mathbb{R}\ ;\ \frac{4j+1}{2m}\pi - \frac{\psi_m(z,\alpha)}{m} < \vartheta < \frac{4j+3}{2m}\pi - \frac{\psi_m(z,\alpha)}{m} \right\}$$
for $j = 0, 1, \dots, m-1$, where

(2.1.2)
$$\psi_m(z,\alpha) := \arg \alpha_m + (m-1)\pi + m \arg z_{11}.$$

Definition 2.1.1. For $j = 1, 2, \dots, n-1$, we define locally finite chains $\Delta_j^{[m]}(z,\alpha)$ as follows. Fix a permutation σ of m letters.
(I) When $m > 0$, for $j = 1, \dots, m-1$ we define
$$\begin{aligned}\Delta_j^{[m]}(z,\alpha) &:= \Delta_j^{[m],(0)}(z,\alpha)\\ &= \ell_R(\infty;\vartheta_{j-1}^{[m]}) + c_R(\infty;\vartheta_{j-1}^{[m]},\vartheta_j^{[m]}) + \overline{\ell}_R(\infty;\vartheta_j^{[m]}),\end{aligned}$$
and for $j = m$ we define
$$\begin{aligned}\Delta_m^{[m]}(z,\alpha) &:= \Delta_m^{[m],(0)}(z,\alpha)\\ &= \ell_R(\infty;\vartheta_{m-1}^{[m]}) + c_R(\infty;\vartheta_{m-1}^{[m]},\pi) + b(Re^{\pi\sqrt{-1}},\zeta_{m+1} + re^{\pi\sqrt{-1}})\\ &\quad + \frac{1}{e(\alpha_{m+1})-1} c_r(\zeta_{m+1};\pi,3\pi),\end{aligned}$$
where $\vartheta_j^{[m]} \in \Theta_{\sigma(j)}^{[m]}(z,\alpha)$.
(II) For $j = m+1, \dots, n-1$, we define
$$\begin{aligned}\Delta_j^{[m]}(z,\alpha) &:= \Delta_0^{[m],(j-m)}(z,\alpha)\\ &= \frac{1}{e(\alpha_j)-1} c_r(\zeta_j;0,2\pi) + b(\zeta_j + r, \zeta_{j+1} + re^{\pi\sqrt{-1}})\\ &\quad + \frac{-1}{e(\alpha_{j+1})-1} c_r(\zeta_{j+1};\pi,3\pi).\end{aligned}$$

We study the confluence $[m] \to [m+1]$.

We assume that

(2.1.3)
$$\det\begin{pmatrix}1 & z_{01}\\ 0 & z_{11}\end{pmatrix} \neq 0, \quad \det\begin{pmatrix}z_{0i} & z_{0j}\\ z_{1i} & z_{1j}\end{pmatrix} \neq 0 \quad (i,j = 0, m+2, m+3, \dots, n,\ i \neq j),$$
and

(2.1.4)
$$\alpha_0 + \sum_{j=m+2}^{n} \alpha_j = -2, \quad \alpha_{m+1} \neq 0, \quad \alpha_j \notin \mathbb{Z} \quad (j = 0, m+2, m+3, \dots, n).$$

Set
$$g(\varepsilon) = \begin{pmatrix} 1 & & & & 1 & \\ & 1 & & & \varepsilon & \\ & & \ddots & & \vdots & \\ & & & 1 & \varepsilon^m & \\ & & & & \varepsilon^{m+1} & \\ & & & & & I_{n-m-1}\end{pmatrix},$$
and set

(2.1.5)

$$\begin{aligned} z(\varepsilon) &:= zg(\varepsilon) \\ &= \begin{pmatrix} 1 & z_{01} & \cdots & z_{0m} & 1+\varepsilon z_{01}+\cdots+\varepsilon^{m+1}z_{0\,m+1} & z_{0\,m+2} & \cdots & z_{0n} \\ 0 & z_{11} & \cdots & z_{1m} & \varepsilon z_{11}+\cdots+\varepsilon^{m+1}z_{1\,m+1} & z_{1\,m+2} & \cdots & z_{1n} \end{pmatrix}, \\ \alpha(\varepsilon) &:= g(\varepsilon)^{-1}\alpha = \begin{pmatrix} \alpha_0-\varepsilon^{-(m+1)}\alpha_{m+1} \\ \alpha_1-\varepsilon^{-m}\alpha_{m+1} \\ \vdots \\ \alpha_m-\varepsilon^{-1}\alpha_{m+1} \\ \varepsilon^{-(m+1)}\alpha_{m+1} \\ \alpha_{m+2} \\ \vdots \\ \alpha_n \end{pmatrix}. \end{aligned}$$

We are going to examine the variation of $\Delta_j^{[m]}(z(\varepsilon),\alpha(\varepsilon))$ as $\varepsilon\to 0$.

For $j\geq m+2$, from Definition 2.1.1 and (2.1.5) it follows that $\Delta_j^{[m]}(z(\varepsilon),\alpha(\varepsilon))$ is independent of ε, and we have

$$\Delta_j^{[m]}(z(\varepsilon),\alpha(\varepsilon)) = \Delta_j^{[m]}(z,\alpha) = \Delta_j^{[m+1]}(z,\alpha) \quad (j\geq m+2). \tag{2.1.6}$$

First we assume $m=0$. For $j=1$, we have

$$\begin{aligned} \Delta_1^{[0]}(z(\varepsilon),\alpha(\varepsilon)) = {} & \frac{1}{e(\varepsilon^{-1}\alpha_1)-1}c_r(\zeta_1(\varepsilon);0,2\pi)+b(\zeta_1(\varepsilon)+r,\zeta_2+re^{\pi\sqrt{-1}}) \\ & + \frac{-1}{e(\alpha_2)-1}c_r(\zeta_2;\pi,3\pi), \end{aligned}$$

where

$$\zeta_1(\varepsilon) := -\frac{1+\varepsilon z_{01}}{\varepsilon z_{11}},$$

which tends to ∞ as $\varepsilon\to 0$. Take $\vartheta\in\mathbb{R}$ satisfying

$$-\frac{\pi}{2} < \arg\alpha_1-\vartheta < \frac{\pi}{2} \pmod{2\pi}. \tag{2.1.7}$$

Lemma 2.1.2. *Suppose that*

$$\arg\varepsilon = \vartheta + O(|\varepsilon|) \quad \textit{as } |\varepsilon|\to 0. \tag{2.1.8}$$

Then we have:
(i) *There is a* $\vartheta_0^{[1]}\in\Theta_0^{[1]}(z,\alpha)$ *such that* $\arg(\zeta_1(\varepsilon))\sim\vartheta_0^{[1]}$ *as* $\varepsilon\to 0$.
(ii) $\dfrac{1}{e(\varepsilon^{-1}\alpha_1)-1}\displaystyle\int_{c_r(\zeta_1(\varepsilon);0,2\pi)} U_{[0]}(u;z(\varepsilon),\alpha(\varepsilon))du\to 0$ *as* $\varepsilon\to 0$.
(iii) $\displaystyle\int_{b(\zeta_1(\varepsilon)+r,\zeta_2+re^{\pi\sqrt{-1}})} U_{[0]}(u;z(\varepsilon),\alpha(\varepsilon))du$ *converges as* $\varepsilon\to 0$.

Proof. (i) We have

$$\begin{aligned} \arg(\zeta_1(\varepsilon)) &= \arg\left(-\frac{1+\varepsilon z_{01}}{\varepsilon z_{11}}\right) = \pi+\arg(1+\varepsilon z_{01})-\arg(\varepsilon z_{11}) \\ &\sim \pi-\arg\varepsilon-\arg z_{11}, \end{aligned}$$

and the last member of the above belongs to $\Theta_0^{[1]}(z,\alpha)$ by virtue of (2.1.7) and (2.1.8).

(ii) We note that

$$U_{[0]}(u;z(\varepsilon),\alpha(\varepsilon)) = (1+\varepsilon(z_{01}+uz_{11}))^{\varepsilon^{-1}\alpha_1}\prod_{j=2}^{n}(z_{0j}+uz_{1j})^{\alpha_j}. \tag{2.1.9}$$

Then by using (2.1.7) and (2.1.8) we have

$$\int_{c_r(\zeta_1(\varepsilon);0,2\pi)} U_{[0]}(u;z(\varepsilon),\alpha(\varepsilon))du \to 0 \tag{2.1.10}$$

as $\varepsilon \to 0$. On the other hand, as a function in ε, $1/(e(\varepsilon^{-1}\alpha_1)-1)$ has a pole of order 1 when $\varepsilon^{-1}\alpha_1 \in \mathbb{Z}$. By (2.1.7) and (2.1.8), $\varepsilon^{-1}\alpha_1 \in \mathbb{Z}$ implies $\varepsilon^{-1}\alpha_1 \in \mathbb{N}$, and in this case the integral of (2.1.9) over $c_r(\zeta_1(\varepsilon);0,2\pi)$ becomes 0. Thus the poles are removable, and then we can modify the estimate of (2.1.10) to obtain the assertion (ii).

The assertion (iii) is shown by noting the assertion (i). □

Using Lemma 2.1.2, we obtain

Proposition 2.1.3. *Suppose* (2.1.8) *with* (2.1.7). *Define*

$$\vartheta_0^{[1]} := \pi - \vartheta - \arg z_{11}.$$

Then we have $\vartheta_0^{[1]} \in \Theta_0^{[1]}(z,\alpha)$, *and*

$$\int_{\Delta_1^{[0]}(z(\varepsilon),\alpha(\varepsilon))} U_{[0]}(u;z(\varepsilon),\alpha(\varepsilon))du \to \int_{\Delta_1^{[1]}(z,\alpha)} U_{[1]}(u;z,\alpha)du$$

as $\varepsilon \to 0$.

We sum up Proposition 2.1.3, (2.1.6) and Proposition 1.3.2 into

$$(\Delta_j^{[0]} \otimes U_{[0]})(z(\varepsilon),\alpha(\varepsilon)) \to (\Delta_j^{[1]} \otimes U_{[1]})(z,\alpha)$$

as $\varepsilon \to 0$ with (2.1.8), $j = 1,\dots,n-1$.

Next we assume $m \geq 1$. For simplicity, we assume that the permutation σ which appears in the definition of $\Delta_j^{[m]}(z,\alpha)$ is the identity; namely we assume

$$\vartheta_j^{[m]} \in \Theta_j^{[m]}(z,\alpha) \quad (j = 0,1,\dots,m-1).$$

We set

$$\zeta_{m+1}(\varepsilon) := -\frac{1+\varepsilon z_{01}+\cdots+\varepsilon^{m+1}z_{0\,m+1}}{\varepsilon z_{11}+\cdots+\varepsilon^{m+1}z_{1\,m+1}},$$

which tends to ∞ as $\varepsilon \to 0$. Take $\vartheta \in \mathbb{R}$ satisfying

$$-\frac{\pi}{2} < \arg\alpha_{m+1} - (m+1)\vartheta < \frac{\pi}{2} \pmod{2\pi}. \tag{2.1.11}$$

Then we may assume that

$$\frac{\arg\alpha_{m+1}}{m+1} - \frac{4(j-m)+1}{2(m+1)}\pi < \vartheta < \frac{\arg\alpha_{m+1}}{m+1} - \frac{4(j-m)-1}{2(m+1)}\pi \tag{2.1.12}$$

for some $j \in \{0,1,\dots,m\}$. Suppose that

$$\arg\varepsilon = \vartheta + O(|\varepsilon|) \quad \textit{as } |\varepsilon| \to 0. \tag{2.1.13}$$

Then in a similar way as Lemma 2.1.2, (i), we have

$$\arg(\zeta_{m+1}(\varepsilon)) \sim \pi - \vartheta - \arg z_{11} \in \Theta_j^{[m+1]}(z,\alpha). \tag{2.1.14}$$

Lemma 2.1.4. (i) *Suppose* (2.1.13) *with* (2.1.12). *For any* $i \in \{0,1,\dots,m-1\}$, *there is a* $k \in \{0,1,\dots,m\} \setminus \{j\}$ *such that*

$$\Theta_i^{[m]}(z(\varepsilon),\alpha(\varepsilon)) \cap \Theta_k^{[m+1]}(z,\alpha) \neq \emptyset.$$

(ii) *Suppose* (2.1.13) *with* (2.1.12). *For any* $i \in \{0,1,\dots,m-1\}$, *we have*

$$\pi - \vartheta - \arg z_{11} \notin \Theta_i^{[m]}(z(\varepsilon),\alpha(\varepsilon))$$

Proof. (i) By (2.1.2), (2.1.5) and (2.1.13), we have

$$\begin{aligned}\psi_m(z(\varepsilon),\alpha(\varepsilon)) &= \arg(\alpha_m - \varepsilon^{-1}\alpha_{m+1}) + (m-1)\pi + m\arg z_{11}\\ &\sim \arg\alpha_{m+1} + m\pi + m\arg z_{11} - \vartheta.\end{aligned} \tag{2.1.15}$$

Putting (2.1.12) and (2.1.15) into (2.1.1), we obtain

$$\tilde{\Theta}_i^{[m]}(z,\alpha) \subset \Theta_i^{[m]}(z(\varepsilon),\alpha(\varepsilon)), \tag{2.1.16}$$

where

$$\begin{aligned}\tilde{\Theta}_i^{[m]}(z,\alpha) := \Bigg(&\frac{-2m^2+3m+4mi+4i-4j+2}{2m(m+1)}\pi - \arg z_{11} - \frac{\arg\alpha_{m+1}}{m+1},\\ &\frac{-2m^2+5m+4mi+4i-4j+2}{2m(m+1)}\pi - \arg z_{11} - \frac{\arg\alpha_{m+1}}{m+1}\Bigg)\end{aligned}$$

(an open interval). We shall show that, for any $i \in \{0,1,\dots,m-1\}$, there is one and only one $k \in \mathbb{Z}$ such that

$$\tilde{\Theta}_i^{[m]}(z,\alpha) \cap \Theta_k^{[m+1]}(z,\alpha) \neq \emptyset. \tag{2.1.17}$$

Since both of the lengths of the open intervals $\tilde{\Theta}_i^{[m]}(z,\alpha)$ and $\Theta_k^{[m+1]}(z,\alpha)$ are $\pi/(m+1)$, (2.1.17) is equivalent to

$$\begin{aligned}\Bigg|&\left(\frac{-2m^2+3m+4mi+4i-4j+2}{2m(m+1)}\pi - \arg z_{11} - \frac{\arg\alpha_{m+1}}{m+1}\right)\\ &-\left(\frac{4k+1}{2(m+1)}\pi - \frac{\arg\alpha_{m+1} + m\pi + (m+1)\arg z_{11}}{m+1}\right)\Bigg| < \frac{\pi}{m+1}.\end{aligned}$$

Then we have

$$\frac{2mi+2i-2j+1}{2m} < k < \frac{2m+2mi+2i-2j+1}{2m}. \tag{2.1.18}$$

The difference between the right- and left-hand sides of (2.1.18) is 1, and the left hand side does not belong to $\mathbb{Z}$ since the numerator is an odd integer while the denominator is even. Hence (2.1.18) has one and only one integer solution k. In our context we can regard $\Theta_k^{[m+1]}(z,\alpha)$ as a subset of $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, and so k is determined as an element of $\mathbb{Z}/(m+1)\mathbb{Z}$. Thus we may assume that $k \in \{0,1,\dots,m\}$.

Now for each of m integers i there corresponds only one integer k in $m+1$ integers $0,1,\dots,m$. Then there is one integer $k \in \{0,1,\dots,m\}$ which has no associate. We shall show that the integer k is j; i.e.

$$\tilde{\Theta}_i^{[m]}(z,\alpha) \cap \Theta_j^{[m+1]}(z,\alpha) = \emptyset.$$

Put $k = j$ in (2.1.18) to obtain

$$j - 1 + \frac{1}{2(m+1)} < i < j - \frac{1}{2(m+1)},$$

which has no integer solution i.

(ii) From (2.1.1) and (2.1.15) it follows that

$$\begin{aligned}\Theta_i^{[m]}(z(\varepsilon), \alpha(\varepsilon)) \approx \Bigg(&\frac{4i+1}{2m}\pi - \frac{\arg \alpha_{m+1}}{m} - \pi - \arg z_{11} + \frac{\vartheta}{m}, \\ &\frac{4i+3}{2m}\pi - \frac{\arg \alpha_{m+1}}{m} - \pi - \arg z_{11} + \frac{\vartheta}{m}\Bigg).\end{aligned} \tag{2.1.19}$$

Assume that

$$\pi - \vartheta - \arg z_{11} \in \Theta_i^{[m]}(z(\varepsilon), \alpha(\varepsilon)) \tag{2.1.20}$$

would hold for some i. Then from (2.1.19) and (2.1.20) it should follow that

$$\frac{\arg \alpha_{m+1}}{m+1} - \frac{4(i-m)+3}{2(m+1)}\pi < \vartheta < \frac{\arg \alpha_{m+1}}{m+1} - \frac{4(i-m)+1}{2(m+1)}\pi,$$

which is never compatible with (2.1.12). □

For $i = 0, 1, \ldots, m-1$, we denote by $k(i)$ the integer in $\{0, 1, \ldots, m\} \setminus \{j\}$ which is congruent modulo $m+1$ with the unique integer k satisfying (2.1.18). By checking the proof of Lemma 2.1.4 in detail, we see that

$$k(j + i \bmod m) \equiv j + i + 1 \mod (m+1)$$

for $i = 0, 1, \ldots, m-1$.

Proposition 2.1.5. *Suppose* (2.1.13) *with* (2.1.12). *If necessary, for* $i = 0, 1, \ldots, m-1$ *we retake* $\vartheta_i^{[m]} \in \Theta_i^{[m]}(z(\varepsilon), \alpha(\varepsilon))$ *so that*

$$\vartheta_i^{[m]} \in \tilde{\Theta}_i^{[m]}(z, \alpha) \cap \Theta_{k(i)}^{[m+1]}(z, \alpha). \tag{2.1.21}$$

(*This modification causes only homologous change on the cycles* $\Delta_i^{[m]}(z(\varepsilon), \alpha(\varepsilon))$.) *Define* $\vartheta_i^{[m+1]}$ $(i = 0, 1, \ldots, m)$ *by*

$$\begin{cases} \vartheta_i^{[m+1]} := \vartheta_i^{[m]} \quad (i = 0, 1, \ldots, m-1), \\ \vartheta_m^{[m+1]} := \pi - \vartheta - \arg z_{11}, \end{cases} \tag{2.1.22}$$

and define a permutation σ *of* $m+1$ *letters by*

$$\sigma(i) = k(i) \quad (i = 0, 1, \ldots, m-1), \quad \sigma(m) = j. \tag{2.1.23}$$

Then we have $\vartheta_i^{[m+1]} \in \Theta_{\sigma(i)}^{[m+1]}(z, \alpha)$ *for* $i = 0, 1, \ldots, m$, *and*

$$(\Delta_i^{[m]} \otimes U_{[m]})(z(\varepsilon), \alpha(\varepsilon)) \to (\Delta_i^{[m+1]} \otimes U_{[m+1]})(z, \alpha)$$

for $i = 0, 1, \ldots, m$, *as* $\varepsilon \to 0$ *with* (2.1.13).

Proof. $\vartheta_i^{[m+1]} \in \Theta_{\sigma(i)}^{[m+1]}(z, \alpha)$ follows from (2.1.14), (2.1.21), (2.1.22) and (2.1.23). For $i = 1, \ldots, m-1$, $\Delta_i^{[m]}(z(\varepsilon), \alpha(\varepsilon))$ is independent of ε, and coincides with $\Delta_i^{[m+1]}(z, \alpha)$. On $\Delta_m^{[m]}(z(\varepsilon), \alpha(\varepsilon))$ and $\Delta_{m+1}^{[m]}(z(\varepsilon), \alpha(\varepsilon))$, the assertion follows from a similar statement to Lemma 2.1.2, (ii) and (iii), which holds by virtue of (2.1.11). □

2.2. Case (ii). First we note that we use some notation in §2.1 for another meaning.

We are concerned with the composition

$$[m] := (1, \dots, 1, \overset{\overset{i}{\smile}}{1+m}, 1, 1, \dots, 1)$$

of $n+1$, where m is a non-negative integer. When $m > 0$, $\alpha_{i+m} \neq 0$, $z_{1i} \neq 0$ and $z_{1i}z_{0\,i+1} - z_{0i}z_{1\,i+1} \neq 0$, we set

(2.2.1)
$$\Theta_j^{[m]}(z,\alpha) := \left\{ \vartheta \in \mathbb{R} \ ; \ \frac{4j+1}{2m}\pi + \frac{\psi_m(z,\alpha)}{m} < \vartheta < \frac{4j+3}{2m}\pi + \frac{\psi_m(z,\alpha)}{m} \right\}$$

for $j = 0, 1, \dots, m-1$, where

$$\psi_m(z,\alpha) := \arg \alpha_{i+m} + (m-1)\pi + m \arg\left(\frac{z_{1i}z_{0\,i+1} - z_{0i}z_{1\,i+1}}{z_{1i}^2} \right). \tag{2.2.2}$$

Definition 2.2.1. For $j = 1, 2, \dots, n-1$, we define locally finite chains $\Delta_j^{[m]}(z,\alpha)$ as follows. Fix a permutation σ of m letters.
(I) For $j = 1, 2, \dots, i-2, i+m+1, i+m+2, \dots, n-1$, we define

$$\begin{aligned}
\Delta_j^{[m]}(z,\alpha) &:= \begin{Bmatrix} \Delta_0^{[m],(j)}(z,\alpha) & \text{if} & j \leq i-2 \\ \Delta_0^{[m],(j-m)}(z,\alpha) & \text{if} & j \geq i+m+1 \end{Bmatrix} \\
&= \frac{1}{e(\alpha_j)-1} c_r(\zeta_j; 0, 2\pi) + b(\zeta_j + r, \zeta_{j+1} + re^{\pi\sqrt{-1}}) \\
&\quad + \frac{-1}{e(\alpha_{j+1})-1} c_r(\zeta_{j+1}; \pi, 3\pi).
\end{aligned}$$

(II) When $m = 0$, also for $j = i-1, i$ we define

$$\Delta_j^{[m]}(z,\alpha) := \Delta_0^{[m],(j)}(z,\alpha).$$

(III) When $m > 0$, we define

$$\begin{aligned}
\Delta_{i-1}^{[m]}(z,\alpha) &:= \Delta_0^{[m],(i-1)}(z,\alpha) \\
&= \frac{1}{e(\alpha_{i-1})-1} c_r(\zeta_{i-1}; 0, 2\pi) + b(\zeta_{i-1} + r, \zeta_i + re^{\pi\sqrt{-1}}) \\
&\quad + c_r(\zeta_i; \pi, \vartheta_0^{[m]}) + \overline{\ell}_r(\zeta_i; \vartheta_0^{[m]}),
\end{aligned}$$

$$\begin{aligned}
\Delta_i^{[m]}(z,\alpha) &: = \Delta_0^{[m],(i)}(z,\alpha) \\
&= \ell_r(\zeta_i; \vartheta_0^{[m]}) + c_r(\zeta_i; \vartheta_0^{[m]}, \vartheta_0^{[m]} + 2\pi) + \overline{\ell}_r(\zeta_i; \vartheta_0^{[m]} + 2\pi),
\end{aligned}$$

$$\begin{aligned}
\Delta_j^{[m]}(z,\alpha) &: = \Delta_{j-i}^{[m],(i)}(z,\alpha) \\
&= \ell_r(\zeta_i; \vartheta_{j-i-1}^{[m]}) + c_r(\zeta_i; \vartheta_{j-i-1}^{[m]}, \vartheta_{j-i}^{[m]}) + \overline{\ell}_r(\zeta_i; \vartheta_{j-i}^{[m]})
\end{aligned}$$

for $j = i+1, \dots, i+m-1$, and

$$\begin{aligned}\Delta_{i+m}^{[m]}(z,\alpha) &:= \Delta_m^{[m],(i)}(z,\alpha)\\ &= \ell_r(\zeta_i;\vartheta_{m-1}^{[m]}) + c_r(\zeta_i;\vartheta_{m-1}^{[m]},0) + b(\zeta_i + r, \zeta_{i+m+1} + re^{\pi\sqrt{-1}})\\ &\quad + \frac{-1}{e(\alpha_{i+m+1})-1} c_r(\zeta_{i+m+1};\pi,3\pi),\end{aligned}$$

where $\vartheta_j^{[m]} \in \Theta_{\sigma(j)}^{[m]}$.

We assume that

$$\det\begin{pmatrix} z_{0i} & z_{0\,i+1} \\ z_{1i} & z_{1\,i+1}\end{pmatrix} \neq 0, \quad \det\begin{pmatrix} z_{0k} & z_{0j} \\ z_{1k} & z_{1j}\end{pmatrix} \neq 0 \qquad (k,j = 0,1,\dots,i,i+m+2,i+m+3,\dots,n,\ k\neq j), \tag{2.2.3}$$

and

$$\sum_{j=0}^{i}\alpha_j + \sum_{j=i+m+2}^{n}\alpha_j = -2, \quad \alpha_{i+m+1}\neq 0, \qquad \alpha_j \notin \mathbb{Z} \quad (j = 0,1,\dots,i,i+m+2,i+m+3,\dots,n). \tag{2.2.4}$$

Define $g(\varepsilon)$ by (1.3.1), and define $z(\varepsilon)$, $\alpha(\varepsilon)$ by (1.3.2). Then we can study the confluence $[m] \to [m+1]$ as $\varepsilon \to 0$.

For $j = 1,2,\dots,i-2,i+m+2,i+m+3,\dots,n-1$, we see by the definition that $\Delta_j^{[m]}(z(\varepsilon),\alpha(\varepsilon))$ is independent of ε, and we have

$$\Delta_j^{[m]}(z(\varepsilon),\alpha(\varepsilon)) = \Delta_j^{[m]}(z,\alpha) = \Delta_j^{[m+1]}(z,\alpha) \quad (j \le i-2,\ j \ge i+m+2). \tag{2.2.5}$$

First we study the confluence $[0] \to [1]$ (i.e. $m = 0$). Owing to (2.2.5) we study the confluence of $\Delta_j^{[m]}(z(\varepsilon),\alpha(\varepsilon))$ only for $j = i-1$, i and $i+1$. Set

$$\zeta_{i+1}(\varepsilon) := -\frac{z_{0i} + \varepsilon z_{0\,i+1}}{z_{1i} + \varepsilon z_{1\,i+1}},$$

which tends to ζ_i as $\varepsilon \to 0$. Since, in Definition 2.2.1, the particular value r has no effect on the integral of $U_{[m]}(u;z,\alpha)$ over $\Delta_j^{[m]}(z,\alpha)$, we may have

$$\begin{aligned}\Delta_{i-1}^{[0]}(z(\varepsilon),\alpha(\varepsilon)) = &\ \frac{1}{e(\alpha_{i-1})-1} c_r(\zeta_{i-1};0,2\pi)\\ &+ b(\zeta_{i-1}+r, \zeta_i + r(\varepsilon)e^{\pi\sqrt{-1}}) + \frac{-1}{e(\alpha_i - \varepsilon^{-1}\alpha_{i+1})-1} c_{r(\varepsilon)}(\zeta_i;\pi,3\pi),\end{aligned}$$

$$\begin{aligned}\Delta_{i}^{[0]}(z(\varepsilon),\alpha(\varepsilon)) = &\ \frac{1}{e(\alpha_i - \varepsilon^{-1}\alpha_{i+1})-1} c_{r(\varepsilon)}(\zeta_i;0,2\pi)\\ &+ b(\zeta_i + r(\varepsilon), \zeta_{i+1}(\varepsilon) + r(\varepsilon)e^{\pi\sqrt{-1}})\\ &+ \frac{-1}{e(\varepsilon^{-1}\alpha_{i+1})-1} c_{r(\varepsilon)}(\zeta_{i+1}(\varepsilon);\pi,3\pi),\end{aligned}$$

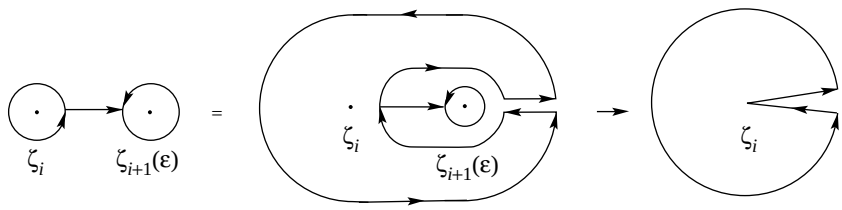

FIGURE 4

$$\begin{aligned}\Delta^{[0]}_{i+1}(z(\varepsilon),\alpha(\varepsilon)) = {}& \frac{1}{e(\varepsilon^{-1}\alpha_{i+1})-1}c_{r(\varepsilon)}(\zeta_{i+1}(\varepsilon);0,2\pi)\\ &+ b(\zeta_{i+1}(\varepsilon)+r(\varepsilon),\zeta_{i+2}+re^{\pi\sqrt{-1}})\\ &+ \frac{-1}{e(\varepsilon^{-1}\alpha_{i+1})-1}c_r(\zeta_{i+2};\pi,3\pi),\end{aligned}$$

where

$$0 < r(\varepsilon) < \frac{1}{2}|\zeta_i - \zeta_{i+1}(\varepsilon)| \quad \text{as} \quad \varepsilon \to 0. \tag{2.2.6}$$

Take $\vartheta \in \mathbb{R}$ satisfying

$$-\frac{\pi}{2} < \arg\alpha_{i+1} - \vartheta < \frac{\pi}{2} \pmod{2\pi}, \tag{2.2.7}$$

and suppose that

$$\arg\varepsilon = \vartheta + O(|\varepsilon|) \quad \text{as } |\varepsilon| \to 0. \tag{2.2.8}$$

Define $\vartheta^{[1]}_0$ by

$$\vartheta^{[1]}_0 = \vartheta + \arg\left(\frac{z_{1i}z_{0\,i+1} - z_{0i}z_{1\,i+1}}{z^2_{1i}}\right) - \pi, \tag{2.2.9}$$

which belongs to $\Theta^{[1]}_0$ by virtue of (2.2.7) and (2.2.8). Then, in a similar manner as Proposition 2.1.3 together with (2.2.6), we have

$$(\Delta^{[0]}_{i+1} \otimes U_{[0]})(z(\varepsilon),\alpha(\varepsilon)) \to (\Delta^{[1]}_{i+1} \otimes U_{[1]})(z,\alpha).$$

Similarly we modify the estimate as in Lemma 2.1.2, (ii), (iii) by using (2.2.6) to obtain

$$(e(\alpha_i - \varepsilon^{-1}\alpha_{i+1}) - 1)\cdot(\Delta^{[0]}_i \otimes U_{[0]})(z(\varepsilon),\alpha(\varepsilon)) \to (\Delta^{[1]}_i \otimes U_{[1]})(z,\alpha).$$

The variation of the chain $\Delta^{[0]}_i(z(\varepsilon),\alpha(\varepsilon))$ is illustrated in Figure 4.

For the study of $\Delta^{[0]}_{i-1}(z(\varepsilon),\alpha(\varepsilon))$, without loss of generality we fix the branch of $U_{[0]}(u;z,\alpha)$ over $\Delta^{[0]}_j(z,\alpha)$ as follows: Noting (1.2.8), we assume that

$$\begin{aligned}\arg(u-\zeta_k) &\approx 0 \quad \text{for} \quad k = 1,\dots,j,\\ \arg(u-\zeta_k) &\approx \pi \quad \text{for} \quad k = j+1,\dots,n\end{aligned} \tag{2.2.10}$$

when $u \in b(\zeta_j + r, \zeta_{j+1} + re^{\pi\sqrt{-1}})$, $j = 1,\dots,n-1$. As explained after Definition 1.2.2, (2.2.9) determines the branches. When ε is involved in the twisted chains, we understand the branches as the continuations of those determined as above,

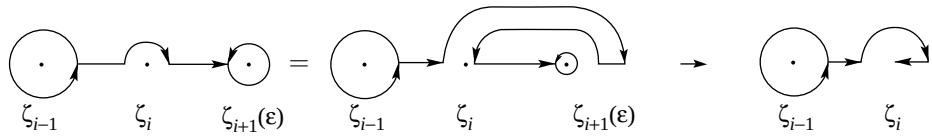

FIGURE 5

where we have assumed that $\arg\varepsilon$ is so taken that $\arg(\zeta_{i+1}(\varepsilon)-\zeta_i)\approx 0$ when $|\varepsilon|$ is sufficiently large. Then we obtain

$$\begin{aligned}
&(\Delta_{i-1}^{[0]}\otimes U_{[0]}+\Delta_i^{[0]}\otimes U_{[0]})(z(\varepsilon),\alpha(\varepsilon))\\
&=\Bigg[\frac{1}{e(\alpha_{i-1})-1}c_r(\zeta_{i-1};0,2\pi)+b(\zeta_{i-1}+r,\zeta_i+r(\varepsilon)e^{\pi\sqrt{-1}})\\
&\quad+c_{r(\varepsilon)}(\zeta_i;\pi,0)+b(\zeta_i+r(\varepsilon),\zeta_{i+1}(\varepsilon)+r(\varepsilon)e^{\pi\sqrt{-1}})\\
&\quad+\frac{-1}{e(\varepsilon^{-1}\alpha_{i+1})-1}c_{r(\varepsilon)}(\zeta_{i+1}(\varepsilon);\pi,3\pi)\Bigg]\otimes U_{[0]}(u;z(\varepsilon),\alpha(\varepsilon)).
\end{aligned}\tag{2.2.11}$$

In Figure 5 we illustrate the variation of the chain in the right-hand side of (2.2.11). Thus the right-hand side of (2.2.11) converges to $(\Delta_{i-1}^{[1]}\otimes U_{[1]})(z,\alpha)$.

Summing up the above, we have obtained the following result.

Proposition 2.2.2. *Suppose* (2.2.8) *with* (2.2.7). *We fix the branch of* $U_{[0]}(u;z,\alpha)$ *by* (2.2.10). *Define* $\vartheta_0^{[1]}$ *by* (2.2.9). *Then we have*

$$(\Delta_{i-1}^{[0]}\otimes U_{[0]}+\Delta_i^{[0]}\otimes U_{[0]})(z(\varepsilon),\alpha(\varepsilon))\to(\Delta_{i-1}^{[1]}\otimes U_{[1]})(z,\alpha),$$
$$(e(\alpha_i-\varepsilon^{-1}\alpha_{i+1})-1)\cdot(\Delta_i^{[0]}\otimes U_{[0]})(z(\varepsilon),\alpha(\varepsilon))\to(\Delta_i^{[1]}\otimes U_{[1]})(z,\alpha),$$

and

$$(\Delta_{i+1}^{[0]}\otimes U_{[0]})(z(\varepsilon),\alpha(\varepsilon))\to(\Delta_{i+1}^{[1]}\otimes U_{[1]})(z,\alpha),$$

as $\varepsilon\to 0$.

Next we assume $m\geq 1$. Analogous argument to Proposition 2.1.5 yields the following result.

Proposition 2.2.3. *Take* $\vartheta\in\mathbb{R}$ *satisfying*

$$-\frac{\pi}{2}<\arg\alpha_{i+m+1}-(m+1)\vartheta<\frac{\pi}{2}\pmod{2\pi}.\tag{2.2.12}$$

If necessary, for $j=0,1,\dots,m-1$ *we retake* $\vartheta_j^{[m]}\in\Theta_j^{[m]}(z(\varepsilon),\alpha(\varepsilon))$, *which causes only homologous change on the cycles* $\Delta_j^{[m]}(z(\varepsilon),\alpha(\varepsilon))$. *Suppose that*

$$\arg\varepsilon=\vartheta+O(|\varepsilon|)\quad\text{as } |\varepsilon|\to 0.\tag{2.2.13}$$

Then, by appropriately defining a permutation σ *of* $m+1$ *letters, for* $j=i-1,i,\dots,i+m,i+m+1$ *we have*

$$(\Delta_j^{[m]}\otimes U_{[m]})(z(\varepsilon),\alpha(\varepsilon))\to(\Delta_j^{[m+1]}\otimes U_{[m+1]})(z,\alpha)$$

as $\varepsilon\to 0$.

3. Determinant of the period matrix

3.1. For general composition λ. In §1.2, Definition 1.2.2, we have given $n-1$ elements $\Delta_j^{\lambda,(i)}(z,\alpha)$ of the homology group $H_1^{\ell f}({}^{\Psi}X,\check{\mathcal{S}})$, and in §1.3, Theorem 1.3.1, we have given $n-1$ generators $\varphi_\lambda^k(z,\alpha)$ of the cohomology group $H^1(\Omega^\bullet,\nabla)$. Between these elements we define the pairing by the integral:

$$\langle \Delta_j^{\lambda,(i)}(z,\alpha), \varphi_\lambda^k(z,\alpha)\rangle := \int_{\Delta_j^{\lambda,(i)}(z,\alpha)} \varphi_\lambda^k(z,\alpha).$$

In this section we shall show that this pairing gives a perfect pairing between $H_1^{\ell f}({}^{\Psi}X,\check{\mathcal{S}})$ and $H^1(\Omega^\bullet,\nabla)$.

Definition 3.1.1. We call the $(n-1)\times(n-1)$ matrix

$$\big(\langle \Delta_j^{\lambda,(i)}(z,\alpha), \varphi_\lambda^k(z,\alpha)\rangle\big)_{\substack{i,j\\ 1\le k\le n-1}}$$

the *period matrix* associated with the hypergeometric function of type λ. We denote the determinant of this matrix by $D_\lambda(z,\alpha)$:

$$D_\lambda(z,\alpha) := \det\big(\langle \Delta_j^{\lambda,(i)}(z,\alpha), \varphi_\lambda^k(z,\alpha)\rangle\big)_{\substack{i,j\\ 1\le k\le n-1}}.$$

When λ is the basic composition $\epsilon=(1,1,\dots,1)$, the determinant $D_\epsilon(z,\alpha)$ is evaluated by Varchenko [V] and Terasoma [T2]:

Theorem 3.1.2 ([V], [T2]). *Suppose that $\zeta_1,\dots,\zeta_n$ and $\alpha_1,\dots,\alpha_n$ are real numbers, and that they satisfy $\zeta_1<\cdots<\zeta_n$ and $\alpha_i>-1$ $(1\le i\le n)$. Then we have*

$$\begin{aligned} D_\epsilon(z,\alpha) = &\prod_{k=1}^n {z_{1k}}^{(n-1)\alpha_k}\cdot \frac{\prod_{k=1}^n \Gamma(\alpha_k+1)}{\Gamma(\sum_{k=1}^n \alpha_k+n)}\\ &\times \prod_{k=1}^n\Big\{\prod_{j=1}^{k-1}(\zeta_j-\zeta_k)^{\alpha_k}\cdot \prod_{j=k+1}^n(\zeta_j-\zeta_k)^{\alpha_k+1}\Big\}. \end{aligned}$$

Using this result, by step by step confluences, we evaluate the determinant $D_\lambda(z,\alpha)$ for a general composition λ.

To state our result, here we define several functions. First we recall that the function $\theta_k(x)=\theta_k(x_1,\dots,x_k)$ is defined in Definition 1.1.1. For $k=0,1,2,\dots$, the polynomial $\psi_k(x)=\psi_k(x_1,\dots,x_k)$ is defined by

$$\frac{1}{1+x_1T+x_2T^2+\cdots} =: \sum_{k=0}^\infty \psi_k(x)T^k. \tag{3.1.1}$$

For $k=1,2,\dots,$ and for $j=0,1,\dots,k$ (or $k-1$), the polynomials $\xi_{k,j}(x)=\xi_{k,j}(x_1,\dots,x_k)$ and $\eta_{k,j}(x)=\eta_{k,j}(x_2,\dots,x_k)$ are defined by

$$\theta_k\left(x_1\frac{1-T^2}{1-T},\dots,x_k\frac{1-T^{k+1}}{1-T}\right) =: \sum_{j=0}^k \xi_{k,j}(x_1,\dots,x_k)T^j, \tag{3.1.2}$$

$$\theta_k\left(0,-x_2T,\dots,-x_kT\cdot\frac{1-T^{k-1}}{1-T}\right) =: \sum_{j=0}^{k-1} \eta_{k,j}(x_2,\dots,x_k)T^j. \tag{3.1.3}$$

Let

$$y = \begin{pmatrix} y_{00} & y_{01} & \cdots & y_{0j} & \cdots \\ y_{10} & y_{11} & \cdots & y_{1j} & \cdots \end{pmatrix}$$

be a set of indeterminates. We define

$$p_k(y) := \sum_{j=0}^{k-1} \frac{y_{00}y_{1\,k-j} - y_{0\,k-j}y_{10}}{y_{00}y_{11} - y_{01}y_{10}} \psi_j\left(\frac{y_{11}}{y_{10}}, \dots, \frac{y_{1j}}{y_{10}}\right) \quad (k = 1, 2, \dots), \tag{3.1.4}$$

where $\psi_j(x)$ is given by (3.1.1). Noting that $p_1(y) = 1$, we set

$$p_+(y) = (p_2(y), p_3(y), \dots). \tag{3.1.5}$$

Also we define

$$q_k(y) := \sum_{j=0}^{k} y_{0\,k-j}\psi_j\left(\frac{y_{12}}{y_{11}}, \dots, \frac{y_{1\,j+1}}{y_{11}}\right) \quad (k = 2, 3, \dots) \tag{3.1.6}$$

by the help of (3.1.1), and we set

$$q_+(y) = (q_2(y), q_3(y), \dots). \tag{3.1.7}$$

For another indeterminate ζ, we set

$$c(y; \zeta) := -\frac{y_{00}y_{11} - y_{01}y_{10}}{y_{10}(y_{00} + y_{10}\zeta)}. \tag{3.1.8}$$

Finally, we set

$$\begin{aligned} I'_\lambda &:= \{k \in \{1, \dots, \ell\}\,;\, \lambda_k > 0\} = I_\lambda \setminus \{0\}, \\ J'_\lambda &:= \{k \in \{1, \dots, \ell\}\,;\, \lambda_k = 0\}. \end{aligned} \tag{3.1.9}$$

Then we see that $\{0, 1, \dots, \ell\}$ is a disjoint union of I'_λ, J'_λ and $\{0\}$.

Theorem 3.1.3. *On $z \in Z_{2,n+1}$ which is written as* (1.3.1)*, we assume* (1.2.1) *and* (1.2.4)*. On $\alpha \in \mathbb{C}^{n+1}$ which is written as* (1.1.3)*, we assume* (1.1.7) *and* (1.2.5)*. Further we assume*

$$\begin{aligned} &\arg z_{10}^{(k)} \approx 0 \quad (1 \le k \le \ell), \\ &\arg \alpha_0^{(k)} \approx 0 \quad (1 \le k \le \ell), \\ &\arg \alpha_{\lambda_k}^{(k)} \approx 0 \quad (k \in I_\lambda), \\ &\arg(\zeta^{(k)} - \zeta^{(j)}) \approx \begin{cases} 0, & k > j, \\ \pi, & k < j, \end{cases} \end{aligned} \tag{3.1.10}$$

where $\zeta^{(k)}$ *is defined by* (1.2.6). *Then we have*

$$\begin{aligned}
D_\lambda(z,\alpha) = \prod_{k\in J'_\lambda} \Bigg\{ & \prod_{\substack{j\in J'_\lambda\\ j<k}} [(j),(k)]^{\alpha_0^{(k)}} \cdot \prod_{\substack{j\in I'_\lambda\\ j<k}} [(j),(k)]^{(\lambda_j+1)\alpha_0^{(k)}} \\
& \times \prod_{\substack{j\in J'_\lambda\\ j>k}} [(j),(k)]^{\alpha_0^{(k)}+1} \cdot \prod_{\substack{j\in I'_\lambda\\ j>k}} [(j),(k)]^{(\lambda_j+1)(\alpha_0^{(k)}+1)} \Bigg\} \\
\times \prod_{k\in I'_\lambda} \Bigg\{ & \prod_{\substack{j\in J'_\lambda\\ j<k}} [(j),(k)]^{\alpha_0^{(k)}} \cdot \prod_{\substack{j\in I'_\lambda\\ j<k}} [(j),(k)]^{(\lambda_j+1)\alpha_0^{(k)}} \\
& \times \prod_{\substack{j\in J'_\lambda\\ j>k}} [(j),(k)]^{\alpha_0^{(k)}+\lambda_k+1} \cdot \prod_{\substack{j\in I'_\lambda\\ j>k}} [(j),(k)]^{(\lambda_j+1)(\alpha_0^{(k)}+\lambda_k+1)} \Bigg\} \\
\times \prod_{i=0}^{\ell} & g_\lambda^{(i)}(z,\alpha),
\end{aligned}$$

where

$$\begin{aligned}
[(j),(k)] &:= \zeta^{(j)} - \zeta^{(k)}, \\
g_\lambda^{(0)}(z,\alpha) &:= \frac{1}{\Gamma(\sum_{i=1}^{\ell} \alpha_0^{(i)} + n)}
\end{aligned}$$

if $\lambda_0 = 0$,

$$\begin{aligned}
& g_\lambda^{(0)}(z,\alpha) \\
& \quad := (2\pi)^{(\lambda_0-1)/2} \cdot (\alpha_{\lambda_0}^{(0)})^{-n+(\lambda_0+1)/2-\sum_{i=1}^{\ell}\alpha_0^{(i)}} \cdot (z_{11}^{(0)})^{\lambda_0(-n+(\lambda_0+1)/2-\sum_{i=1}^{\ell}\alpha_0^{(i)})} \\
& \qquad \times \exp\Bigg[\Bigg(-(\lambda_0-1)n + \frac{\lambda_0(\lambda_0+1)}{4} - \frac{1}{2} + \sum_{i=1}^{\ell} \alpha_0^{(i)} \Bigg)\pi\sqrt{-1} \\
& \qquad\quad + \Bigg(\sum_{j\in J'_\lambda} + \sum_{j\in I'_\lambda} (\lambda_j+1) \Bigg) \sum_{k=1}^{\lambda_0} \alpha_k^{(0)} \theta_k(z_{01}^{(0)} + z_{11}^{(0)}\zeta^{(j)}, \dots, z_{0k}^{(0)} + z_{1k}^{(0)}\zeta^{(j)}) \\
& \qquad\quad + (\lambda_0-1) \sum_{k=1}^{\lambda_0} \alpha_k^{(0)} \theta_k\Bigg(\frac{z_{12}^{(0)}}{z_{11}^{(0)}}, \dots, \frac{z_{1\,k+1}^{(0)}}{z_{11}^{(0)}} \Bigg) - \sum_{k=1}^{\lambda_0} \alpha_k^{(0)} \sum_{j=1}^{k-1} \eta_{k,j}(q_+(z^{(0)})) \\
& \qquad\quad + \sum_{j=1}^{\lambda_0-1} \frac{(-1)^{j-1}}{j(j+1)} (\alpha_{\lambda_0}^{(0)})^{-j} \sum_{\substack{m_1+\cdots+m_{j+1}=\lambda_0\\ m_1,\dots,m_{j+1}\geq 1}} \alpha_{\lambda_0-m_1}^{(0)} \cdots \alpha_{\lambda_0-m_{j+1}}^{(0)} \Bigg]
\end{aligned}$$

if $\lambda_0 > 0$, *and for* $i = 1, \dots, \ell$,

$$g_\lambda^{(i)}(z,\alpha) := (z_{10}^{(i)})^{(n-1)\alpha_0^{(i)}} \cdot \Gamma(\alpha_0^{(i)} + 1)$$

if $\lambda_i = 0$,

$$
\begin{aligned}
&g_\lambda^{(i)}(z,\alpha)\\
&:= (2\pi)^{(\lambda_i+1)/2}\cdot(\alpha_{\lambda_i}^{(i)})^{\alpha_0^{(i)}+(\lambda_i+1)/2}\cdot(z_{10}^{(i)})^{(n-2\lambda_i-1)\alpha_0^{(i)}-\lambda_i(\lambda_i+1)}\\
&\quad\times(z_{00}^{(i)}z_{11}^{(i)}-z_{01}^{(i)}z_{10}^{(i)})^{\lambda_i\alpha_0^{(i)}+\lambda_i(\lambda_i+1)/2}\\
&\quad\times\exp\Bigg[\left(1+\frac{\lambda_i(\lambda_i+1)}{4}+\lambda_i\alpha_0^{(i)}\right)\pi\sqrt{-1}+(n-1)\sum_{k=1}^{\lambda_i}\alpha_k^{(i)}\theta_k\left(\frac{z_{11}^{(i)}}{z_{10}^{(i)}},\dots,\frac{z_{1k}^{(i)}}{z_{10}^{(i)}}\right)\\
&\qquad+\left(\sum_{j\in J'_\lambda}+\sum_{j\in I'_\lambda}(\lambda_j+1)\right)\\
&\qquad\times\sum_{k=1}^{\lambda_i}\alpha_k^{(i)}\theta_k(c(z^{(i)};\zeta^{(j)})p_1(z^{(i)}),\dots,c(z^{(i)};\zeta^{(j)})p_k(z^{(i)}))\\
&\qquad+\lambda_i\sum_{k=1}^{\lambda_i}\alpha_k^{(i)}\theta_k(p_+(z^{(i)}))-\sum_{k=1}^{\lambda_i}\alpha_k^{(i)}\sum_{j=1}^{k}\xi_{k,j}(p_+(z^{(i)}))\\
&\qquad+\sum_{j=1}^{\lambda_i-1}\frac{(-1)^{j-1}}{j(j+1)}(\alpha_{\lambda_i}^{(i)})^{-j}\sum_{\substack{m_1+\cdots+m_{j+1}=\lambda_i\\ m_1,\dots,m_{j+1}\geq 1}}\alpha_{\lambda_i-m_1}^{(i)}\cdots\alpha_{\lambda_i-m_{j+1}}^{(i)}\Bigg]
\end{aligned}
$$

if $\lambda_i > 0$.

By taking a chain (2.0.1) of compositions into consideration, the proof of Theorem 3.1.3 is reduced to ones for two particular cases (i) and (ii) in Figure 3 (§2). In §3.2 and §3.3 we prove the theorem for the cases (i) and (ii), respectively. Theorem 3.1.3 itself is proved at the end of §3.3.

3.2. For the case (i). Similarly as in §2.1, we restrict ourselves to the composition

$$[m] = (1+m,1,1,\dots,1)$$

of $n+1$. We restate Theorem 3.1.3 for the composition $[m]$. Since the theorem is reduced to Theorem 3.1.2 when $m=0$, we assume $m>0$.

Proposition 3.2.1. *Assumptions on z and α are as in Theorem 3.1.3. Assume that $m>0$. Then we have*

$$
\begin{aligned}
D_{[m]}(z,\alpha) = {}&(2\pi)^{(m-1)/2}\cdot{\alpha_m}^{-n+(m+1)/2-\sum_{j=m+1}^n\alpha_j}\cdot{z_{11}}^{m(-n+(m+1)/2-\sum_{j=m+1}^n\alpha_j)}\\
&\times\prod_{j=m+1}^n{z_{1j}}^{(n-1)\alpha_j}\cdot\prod_{j=m+1}^n\Gamma(\alpha_j+1)\\
&\times\prod_{k=m+1}^n\left\{\prod_{j=m+1}^{k-1}(\zeta_j-\zeta_k)^{\alpha_k}\cdot\prod_{j=k+1}^n(\zeta_j-\zeta_k)^{\alpha_k+1}\right\}\\
&\times\exp\Bigg[\left(-(m-1)n+\frac{m(m+1)}{4}-\frac{1}{2}+\sum_{j=m+1}^n\alpha_j\right)\pi\sqrt{-1}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=m+1}^{n}\sum_{k=1}^{m}\alpha_k\theta_k(z_{01}+z_{11}\zeta_j,\dots,z_{0k}+z_{1k}\zeta_j)\\
&+(m-1)\sum_{k=1}^{m}\alpha_k\theta_k\left(\frac{z_{12}}{z_{11}},\dots,\frac{z_{1\,k+1}}{z_{11}}\right)\\
&-\sum_{k=1}^{m}\alpha_k\sum_{j=1}^{k-1}\eta_{k,j}(q_2(z),\dots,q_k(z))\\
&+\sum_{j=1}^{m-1}\frac{(-1)^{j-1}}{j(j+1)}{\alpha_m}^{-j}\sum_{\substack{m_1+\cdots+m_{j+1}=m\\ m_1,\dots,m_{j+1}\geq 1}}\alpha_{m-m_1}\cdots\alpha_{m-m_{j+1}}\Bigg].
\end{aligned}
$$

Proof. We prove the theorem by induction on m.

First we shall obtain $D_{[1]}(z,\alpha)$ from $D_{[0]}(z,\alpha)$ by the confluence. Define $z(\varepsilon)$ and $\alpha(\varepsilon)$ by (2.1.5) with $m=0$. Assume (2.1.8) with (2.1.7). Then, owing to Propositions 1.3.2 and 2.1.3, we obtain

$$D_{[1]}(z,\alpha)=\lim_{\varepsilon\to 0}D_{[1]}(z(\varepsilon),\alpha(\varepsilon)).$$

In order to fix branches of power functions so we can use the result of Theorem 3.1.2, we assume further

$$\arg\varepsilon\approx 0 \tag{3.2.1}$$

when $|\varepsilon|$ is *sufficiently large*. We set

$$\zeta_1(\varepsilon)=-\frac{1+\varepsilon z_{01}}{\varepsilon z_{11}}.$$

Then it follows from Theorem 3.1.2 that

(3.2.2)

$$
\begin{aligned}
D_{[0]}(z(\varepsilon),\alpha(\varepsilon))=&\,(\varepsilon z_{11})^{(n-1)\varepsilon^{-1}\alpha_1}\prod_{k=2}^{n}{z_{1k}}^{(n-1)\alpha_k}\\
&\times\frac{\Gamma(\varepsilon^{-1}\alpha_1+1)\prod_{k=2}^{n}\Gamma(\alpha_k+1)}{\Gamma(\varepsilon^{-1}\alpha_1+\sum_{k=2}^{n}\alpha_k+n)}\\
&\times\prod_{j=2}^{n}(\zeta_j-\zeta_1(\varepsilon))^{\varepsilon^{-1}\alpha_1+1}\\
&\times\prod_{k=2}^{n}\left\{(\zeta_1(\varepsilon)-\zeta_k)^{\alpha_k}\prod_{j=2}^{k-1}(\zeta_j-\zeta_k)^{\alpha_k}\prod_{j=k+1}^{n}(\zeta_j-\zeta_k)^{\alpha_k+1}\right\}.
\end{aligned}
$$

By (2.1.7) and (2.1.8) we have

$$-\frac{\pi}{2} < \arg(\varepsilon^{-1}\alpha_1) < \frac{\pi}{2}$$

for sufficiently small ε, and then we can use Stirling's formula to obtain

(3.2.3)

$$\begin{aligned}
&\frac{\Gamma(\varepsilon^{-1}\alpha_1+1)}{\Gamma(\varepsilon^{-1}\alpha_1+\sum_{k=2}^n \alpha_k+n)} \\
&\sim \frac{\sqrt{2\pi}(\varepsilon^{-1}\alpha_1)^{\varepsilon^{-1}\alpha_1+1/2}e^{-\varepsilon^{-1}\alpha_1}}{\sqrt{2\pi}(\varepsilon^{-1}\alpha_1+\sum_{k=2}^n \alpha_k+n-1)^{\varepsilon^{-1}\alpha_1+\sum_{k=2}^n \alpha_k+n-1/2}e^{-(\varepsilon^{-1}\alpha_1+\sum_{k=2}^n \alpha_k+n-1)}} \\
&= e^{n-1+\sum_{k=2}^n \alpha_k}\left(1+\frac{\sum_{k=2}^n \alpha_k+n-1}{\varepsilon^{-1}\alpha_1}\right)^{-\varepsilon^{-1}\alpha_1-1/2} \\
&\quad \times \varepsilon^{n-1+\sum_{k=2}^n \alpha_k}\left(\alpha_1+\varepsilon\left(\sum_{k=2}^n \alpha_k+n-1\right)\right)^{-(\sum_{k=2}^n \alpha_k+n-1)}.
\end{aligned}$$

Noting that

$$\zeta_j-\zeta_1(\varepsilon)=\varepsilon^{-1}{z_{11}}^{-1}(1+\varepsilon(z_{01}+z_{11}\zeta_j)),$$

we obtain

$$\begin{aligned}
&\prod_{j=2}^n(\zeta_j-\zeta_1(\varepsilon))^{\varepsilon^{-1}\alpha_1+1} \\
&= (\varepsilon z_{11})^{-(n-1)\varepsilon^{-1}\alpha_1}\cdot\varepsilon^{-(n-1)}\cdot {z_{11}}^{-(n-1)}\prod_{j=2}^n(1+\varepsilon(z_{01}+z_{11}\zeta_j))^{\varepsilon^{-1}\alpha_1+1}
\end{aligned} \tag{3.2.4}$$

and

$$\begin{aligned}
&\prod_{k=2}^n(\zeta_1(\varepsilon)-\zeta_k)^{\alpha_k} \\
&= \prod_{k=2}^n\{(\zeta_k-\zeta_1(\varepsilon))e^{\pi\sqrt{-1}}\}^{\alpha_k} \\
&= \varepsilon^{-\sum_{k=2}^n \alpha_k}\cdot {z_{11}}^{-\sum_{k=2}^n \alpha_k}\cdot e^{\pi\sqrt{-1}\sum_{k=2}^n \alpha_k}\prod_{k=2}^n(1+\varepsilon(z_{01}+z_{11}\zeta_k))^{\alpha_k}.
\end{aligned} \tag{3.2.5}$$

Put (3.2.3), (3.2.4) and (3.2.5) into (3.2.2), and let ε tend to 0. Then we have

$$\begin{aligned}
&D_{[0]}(z(\varepsilon),\alpha(\varepsilon)) \\
&\sim \prod_{k=2}^n {z_{1k}}^{(n-1)\alpha_k}\cdot\prod_{k=2}^n\Gamma(\alpha_k+1)\cdot\prod_{k=2}^n\left\{\prod_{j=2}^{k-1}(\zeta_j-\zeta_k)^{\alpha_k}\prod_{j=k+1}^n(\zeta_j-\zeta_k)^{\alpha_k+1}\right\} \\
&\times {\alpha_1}^{-(\sum_{k=2}^n \alpha_k+n-1)}\cdot {z_{11}}^{-(\sum_{k=2}^n \alpha_k+n-1)} \\
&\times \exp\left[\pi\sqrt{-1}\sum_{k=2}^n \alpha_k+\sum_{j=2}^n(\varepsilon^{-1}\alpha_1+1)\log(1+\varepsilon(z_{01}+z_{11}\zeta_j))\right].
\end{aligned}$$

Now the theorem for $m = 1$ follows by noting

$$(\varepsilon^{-1}\alpha_1 + 1)\log(1 + \varepsilon(z_{01} + z_{11}\zeta_j)) \to \alpha_1\theta_1(z_{01} + z_{11}\zeta_j).$$

Next we compute the confluence of $D_{[m]}(z, \alpha)$ to obtain $D_{[m+1]}(z, \alpha)$. Define $z(\varepsilon)$ and $\alpha(\varepsilon)$ by (2.1.5). Assume (2.1.13) with (2.1.11). Then it follows from Propositions 1.3.2 and 2.1.5 that

$$D_{[m+1]}(z, \alpha) = \lim_{\varepsilon\to 0} D_{[m]}(z(\varepsilon), \alpha(\varepsilon)). \tag{3.2.6}$$

Further we assume (3.2.1) and

$$\arg(-\varepsilon^k) \approx \pi \quad (k = 1, 2, \dots) \tag{3.2.7}$$

when $|\varepsilon|$ is *sufficiently large*. We set

$$\zeta_{m+1}(\varepsilon) = -\frac{1 + \sum_{k=1}^{m+1} \varepsilon^k z_{0k}}{\sum_{k=1}^{m+1} \varepsilon^k z_{1k}}.$$

Further we set

$$\begin{aligned} Q^k_{[m]}(z) &:= (m-1)\theta_k\left(\frac{z_{12}}{z_{11}}, \dots, \frac{z_{1\,k+1}}{z_{11}}\right) - \sum_{j=1}^{k-1} \eta_{k,j}(q_2(z), \dots, q_k(z)), \\ A^j_{[m]}(\alpha) &:= \frac{(-1)^{j-1}}{j(j+1)} \sum_{\substack{m_1+\cdots+m_{j+1}=m \\ m_1,\dots,m_{j+1}\geq 1}} \alpha_{m-m_1}\cdots\alpha_{m-m_{j+1}}. \end{aligned} \tag{3.2.8}$$

By (1.1.4), (3.1.1), (3.1.3) and (3.1.6), we see that $Q^k_{[m]}(z)$ depends only on $z_{01}, \dots, z_{0k},\ z_{11}, \dots, z_{1k}, z_{1\,k+1}$. Moreover by using Sublemma 3.2.3 which will be given later, we see that $Q^m_{[m]}(z)$ does not depned on $z_{1\,m+1}$. Hence we have

$$Q^k_{[m]}(z(\varepsilon)) = Q^k_{[m]}(z) \quad (k = 1, 2, \dots, m). \tag{3.2.9}$$

Now, by noting (3.2.9), we obtain

$$\begin{aligned} &D_{[m]}(z(\varepsilon), \alpha(\varepsilon)) \\ &= (2\pi)^{(m-1)/2}(\alpha_m - \varepsilon^{-1}\alpha_{m+1})^{-n+(m+1)/2-\varepsilon^{-m-1}\alpha_{m+1}-\sum_{j=m+2}^n \alpha_j} \\ &\quad \times {z_{11}}^{m(-n+(m+1)/2-\varepsilon^{-m-1}\alpha_{m+1}-\sum_{j=m+2}^n \alpha_j)} \\ &\quad \times \left(\sum_{k=1}^{m+1} \varepsilon^k z_{1k}\right)^{(n-1)\varepsilon^{-m-1}\alpha_{m+1}} \cdot \prod_{j=m+2}^{n} {z_{1j}}^{(n-1)\alpha_j} \end{aligned} \tag{3.2.10}$$

$$
\begin{aligned}
&\times \Gamma(\varepsilon^{-m-1}\alpha_{m+1}+1)\cdot \prod_{j=m+2}^{n}\Gamma(\alpha_j+1)\\
&\times \prod_{j=m+2}^{n}(\zeta_j-\zeta_{m+1}(\varepsilon))^{\varepsilon^{-m-1}\alpha_{m+1}+1}\\
&\times \prod_{k=m+2}^{n}\left\{(\zeta_{m+1}(\varepsilon)-\zeta_k)^{\alpha_k}\cdot \prod_{j=m+2}^{k-1}(\zeta_j-\zeta_k)^{\alpha_k}\cdot \prod_{j=k+1}^{n}(\zeta_j-\zeta_k)^{\alpha_k+1}\right\}\\
&\times \exp\Bigg[\left(-(m-1)+\frac{m(m+1)}{4}-\frac{1}{2}+\varepsilon^{-m-1}\alpha_{m+1}+\sum_{j=m+2}^{n}\alpha_j\right)\pi\sqrt{-1}\\
&\qquad+\sum_{k=1}^{m}(\alpha_k-\varepsilon^{-m-1+k}\alpha_{m+1})\theta_k(z_{01}+z_{11}\zeta_{m+1}(\varepsilon),\dots,z_{0k}+z_{1k}\zeta_{m+1}(\varepsilon))\\
&\qquad+\sum_{j=m+2}^{n}\sum_{k=1}^{m}(\alpha_k-\varepsilon^{-m-1+k}\alpha_{m+1})\theta_k(z_{01}+z_{11}\zeta_j,\dots,z_{0k}+z_{1k}\zeta_j)\\
&\qquad+\sum_{k=1}^{m}(\alpha_k-\varepsilon^{-m-1+k}\alpha_{m+1})Q^k_{[m]}(z)+\sum_{j=1}^{m-1}\frac{A^j_{[m]}(\alpha(\varepsilon))}{(\alpha_m-\varepsilon^{-1}\alpha_{m+1})^j}\Bigg].
\end{aligned}
$$

Owing to (2.1.11) and (2.1.13), we have

$$
-\frac{\pi}{2}<\arg(\varepsilon^{-m-1}\alpha_{m+1})<\frac{\pi}{2}
$$

for sufficiently small ε, and then we can apply Stirling's formula to obtain

$$
\Gamma(\varepsilon^{-m-1}\alpha_{m+1}+1)\sim\sqrt{2\pi}(\varepsilon^{-m-1}\alpha_{m+1})^{\varepsilon^{-m-1}\alpha_{m+1}+1/2}e^{-\varepsilon^{-m-1}\alpha_{m+1}}. \tag{3.2.11}
$$

By noting (3.1.10), (3.2.1) and (3.2.7), we have

(3.2.12)

$$
\begin{aligned}
\alpha_m-\varepsilon^{-1}\alpha_{m+1}&=(\varepsilon^{-1}\alpha_{m+1}-\alpha_m)e^{\pi\sqrt{-1}}\\
&=\varepsilon^{-1}\alpha_{m+1}\left(1-\varepsilon\frac{\alpha_m}{\alpha_{m+1}}\right)e^{\pi\sqrt{-1}},\\
\zeta_j-\zeta_{m+1}(\varepsilon)&=(\varepsilon z_{11})^{-1}\cdot\left(1+\sum_{k=1}^{m+1}\varepsilon^k(z_{0k}+z_{1k}\zeta_j)\right)\cdot\left(\sum_{k=0}^{m}\varepsilon^k\frac{z_{1\,k+1}}{z_{11}}\right)^{-1},\\
\zeta_{m+1}(\varepsilon)-\zeta_k&=(\zeta_k-\zeta_{m+1}(\varepsilon))e^{\pi\sqrt{-1}}\\
&=(\varepsilon z_{11})^{-1}(1+O(\varepsilon))\cdot e^{\pi\sqrt{-1}}.
\end{aligned}
$$

Putting (3.2.11) and (3.2.12) into (3.2.10), we obtain

$$
\begin{aligned}
&D_{[m]}(z(\varepsilon),\alpha(\varepsilon))\\
&\sim C(z,\alpha)\times\left(1-\varepsilon\frac{\alpha_m}{\alpha_{m+1}}\right)^{-\varepsilon^{-m-1}\alpha_{m+1}}\cdot\left(\sum_{k=0}^{m}\varepsilon^k\frac{z_{1\,k+1}}{z_{11}}\right)^{m\varepsilon^{-m-1}\alpha_{m+1}}\\
&\quad\times\prod_{j=m+2}^{n}\left(1+\sum_{k=1}^{m+1}\varepsilon^k(z_{0k}+z_{1k}\zeta_j)\right)^{\varepsilon^{-m-1}\alpha_{m+1}}
\end{aligned}
\tag{3.2.13}
$$

$$\begin{aligned}
&\times \exp\Bigg[-\varepsilon^{-m-1}\alpha_{m+1} \\
&\quad + \sum_{k=1}^{m}(\alpha_k - \varepsilon^{-m-1}\alpha_{m+1})\theta_k(z_{01} + z_{11}\zeta_{m+1}(\varepsilon), \dots, z_{0k} + z_{1k}\zeta_{m+1}(\varepsilon)) \\
&\quad - \varepsilon^{-m-1}\alpha_{m+1}\sum_{j=m+2}^{n}\sum_{k=1}^{m}\varepsilon^k\theta_k(z_{01} + z_{11}\zeta_j, \dots, z_{0k} + z_{1k}\zeta_j) \\
&\quad + \sum_{k=1}^{m}(\alpha_k - \varepsilon^{-m-1+k}\alpha_{m+1})Q^k_{[m]}(z) + \sum_{j=1}^{m-1}\frac{A^j_{[m]}(\alpha(\varepsilon))}{(\alpha_m - \varepsilon^{-1}\alpha_{m+1})^j}\Bigg],
\end{aligned}$$

where we have set

$$\begin{aligned}
C(z,\alpha) = {} & (2\pi)^{m/2}\cdot {\alpha_{m+1}}^{-n+(m+2)/2-\sum_{j=m+2}^{n}\alpha_j} \\
& \cdot {z_{11}}^{(m+1)(-n+(m+2)/2-\sum_{j=m+2}^{n}\alpha_j)} \\
& \times \prod_{j=m+2}^{n}{z_{1j}}^{(n-1)\alpha_j}\cdot\prod_{j=m+2}^{n}\Gamma(\alpha_j+1) \\
& \times \prod_{k=m+2}^{n}\left\{\prod_{j=m+2}^{k-1}(\zeta_j-\zeta_k)^{\alpha_k}\cdot\prod_{j=k+1}^{n}(\zeta_j-\zeta_k)^{\alpha_k+1}\right\} \\
& \times \exp\Bigg[\left(-mn + \frac{(m+1)(m+2)}{4} - \frac{1}{2} + \sum_{j=m+2}^{n}\alpha_j\right)\pi\sqrt{-1} \\
& \qquad + \sum_{j=m+2}^{n}\sum_{k=1}^{m}\alpha_k\theta_k(z_{01} + z_{11}\zeta_j, \dots, z_{0k} + z_{1k}\zeta_j)\Bigg].
\end{aligned} \tag{3.2.14}$$

We are going to compute the right-hand side of (3.2.13). Setting $w_{kj} := z_{0k} + z_{1k}\zeta_j$, by (1.1.4) we have

(3.2.15)

$$\begin{aligned}
&\prod_{j=m+2}^{n}\left(1 + \sum_{k=1}^{m+1}\varepsilon^k w_{kj}\right)^{\varepsilon^{-m-1}\alpha_{m+1}} \\
&\times \exp\left[-\varepsilon^{-m-1}\alpha_{m+1}\sum_{j=m+2}^{n}\sum_{k=1}^{m}\varepsilon^k\theta_k(w_{1j}, \dots, w_{kj})\right] \\
&= \exp\left[-\varepsilon^{-m-1}\alpha_{m+1}\sum_{j=m+2}^{n}\left\{\log\left(1 + \sum_{k=1}^{m+1}\varepsilon^k w_{kj}\right) - \sum_{k=1}^{m}\varepsilon^k\theta_k(w_{1j}, \dots, w_{kj})\right\}\right] \\
&\to \exp\left[\sum_{j=m+2}^{n}\alpha_{m+1}\theta_{m+1}(w_{1j}, \dots, w_{kj})\right].
\end{aligned}$$

By Lemma 3.2.2 which will be shown later, we have

$$\begin{aligned}
&\theta_k(z_{01} + z_{11}\zeta_{m+1}(\varepsilon), \dots, z_{0k} + z_{1k}\zeta_{m+1}(\varepsilon)) \\
&\qquad = -\frac{1}{k}\varepsilon^{-k} + \bar{\theta}_k + \sum_{j=1}^{\infty}\varepsilon^j\eta_{k+j,k}(q_2, \dots, q_{k+j}),
\end{aligned} \tag{3.2.16}$$

where we have set $q_k := q_k(z)$ (see (3.1.6)) and

$$\bar{\theta}_k := \theta_k(z_{12}/z_{11}, \dots, z_{1\,k+1}/z_{11}).$$

Using (3.2.8) and (3.2.16), we have

(3.2.17)

$$\begin{aligned}
&\left(\sum_{k=0}^{m} \varepsilon^k \frac{z_{1\,k+1}}{z_{11}}\right)^{m\varepsilon^{-m-1}\alpha_{m+1}} \\
&\quad \times \exp\Bigg[\sum_{k=1}^{m}(\alpha_k - \varepsilon^{-m-1+k}\alpha_{m+1})\theta_k(z_{01} + z_{11}\zeta_{m+1}(\varepsilon), \dots, z_{0k} + z_{1k}\zeta_{m+1}(\varepsilon)) \\
&\qquad + \sum_{k=1}^{m}(\alpha_k - \varepsilon^{-m-1+k}\alpha_{m+1})Q^k_{[m]}(z)\Bigg] \\
&= \exp\Bigg[m\varepsilon^{-m-1}\alpha_{m+1}\log\left(1 + \sum_{k=1}^{m}\varepsilon^k \frac{z_{1\,k+1}}{z_{11}}\right) \\
&\qquad + \sum_{k=1}^{m}(\alpha_k - \varepsilon^{-m-1+k}\alpha_{m+1})\left(-\frac{\varepsilon^{-k}}{k} + \bar{\theta}_k + \sum_{j=1}^{\infty}\varepsilon^j \eta_{k+j,k}(q_+)\right) \\
&\qquad + \sum_{k=1}^{m}(\alpha_k - \varepsilon^{-m-1+k}\alpha_{m+1})Q^k_{[m]}(z)\Bigg] \\
&= \exp\Bigg[\varepsilon^{-m-1}\alpha_{m+1}\Bigg\{\sum_{k=1}^{m}\varepsilon^k\left((m-1)\bar{\theta}_k - \sum_{j=1}^{k-1}\eta_{k,j}(q_+) - Q^k_{[m]}(z)\right) \\
&\qquad\qquad + \varepsilon^{m+1}\left(m\bar{\theta}_k - \sum_{j=1}^{m}\eta_{m+1,j}(q_+)\right) + O(\varepsilon^{m+2})\Bigg\} \\
&\qquad - \sum_{k=1}^{m}\frac{\alpha_k}{k}\varepsilon^{-k} + \varepsilon^{-m-1}\alpha_{m+1}\sum_{k=1}^{m}\frac{1}{k} + \sum_{k=1}^{m}\alpha_k\big(\bar{\theta}_k + Q^k_{[m]}(z)\big) + O(\varepsilon)\Bigg] \\
&= \exp\left[\sum_{k=1}^{m+1}\alpha_k Q^k_{[m+1]}(z) - \sum_{k=1}^{m}\frac{\alpha_k}{k}\varepsilon^{-k} + \varepsilon^{-m-1}\alpha_{m+1}\sum_{k=1}^{m}\frac{1}{k} + O(\varepsilon)\right].
\end{aligned}$$

By Lemma 3.2.4 which will also be shown later, we have

$$\begin{aligned}
\sum_{j=1}^{m-1}\frac{A^j_{[m]}(\alpha(\varepsilon))}{(\alpha_m - \varepsilon^{-1}\alpha_{m+1})^j} &= \varepsilon^{-m-1}\alpha_{m+1}\left(-\sum_{k=1}^{m}\frac{1}{k} + 1\right) - \sum_{k=1}^{m+1}\frac{\varepsilon^{-m-1+k}}{k}\cdot\frac{{\alpha_m}^k}{{\alpha_{m+1}}^{k-1}} \\
&\quad + \sum_{k=1}^{m}\varepsilon^{-k}\frac{\alpha_k}{k} + \sum_{j=1}^{m}\frac{A^j_{[m+1]}(\alpha)}{{\alpha_{m+1}}^j} + O(\varepsilon).
\end{aligned}$$

Hence we have

(3.2.18)

$$\left(1-\varepsilon\frac{\alpha_m}{\alpha_{m+1}}\right)^{-\varepsilon^{-m-1}\alpha_{m+1}}\exp\Bigg[-\varepsilon^{-m-1}\alpha_{m+1}+\sum_{j=1}^{m-1}\frac{A^j_{[m]}(\alpha(\varepsilon))}{(\alpha_m-\varepsilon^{-1}\alpha_{m+1})^j} -\sum_{k=1}^{m}\frac{\alpha_k}{k}\varepsilon^{-k}+\varepsilon^{-m-1}\alpha_{m+1}\sum_{k=1}^{m}\frac{1}{k}\Bigg]$$
$$\to\ \exp\left[\sum_{j=1}^{m}\frac{A^j_{[m+1]}(\alpha)}{{\alpha_{m+1}}^j}\right].$$

Combining (3.2.6), (3.2.13), (3.2.14), (3.2.15), (3.2.17) and (3.2.18), we see that the theorem for $m+1$ holds, which completes the induction, and hence the proof. □

Lemma 3.2.2.

$$\begin{aligned}&\theta_k(z_{01}+z_{11}\zeta_{m+1}(\varepsilon),\dots,z_{0k}+z_{1k}\zeta_{m+1}(\varepsilon))\\ &\quad= -\frac{1}{k}\varepsilon^{-k}+\theta_k\left(\frac{z_{12}}{z_{11}},\dots,\frac{z_{1\,k+1}}{z_{11}}\right)+\sum_{j=1}^{\infty}\varepsilon^j\eta_{k+j,k}(q_2,\dots,q_{k+j}).\end{aligned}$$

Proof. Set

$$\begin{aligned}g(\varepsilon)&=\sum_{k=0}^{\infty}\varepsilon^k z_{0k},\\ h(\varepsilon)&=\sum_{k=0}^{\infty}\varepsilon^k\frac{z_{1\,k+1}}{z_{11}},\\ f(\varepsilon)&=\frac{g(\varepsilon)}{h(\varepsilon)}.\end{aligned}$$

Then we have

$$f(\varepsilon)=\sum_{k=0}^{\infty}\varepsilon^k q_k,$$

where

$$q_k=\sum_{j=0}^{k}z_{0\,k-j}\psi_j\left(\frac{z_{12}}{z_{11}},\dots,\frac{z_{1\,j+1}}{z_{11}}\right)=q_k(z)$$

(see (3.1.6)). By considering modulo $O(\varepsilon^{m+2})$, we have

$$\zeta_{m+1}(\varepsilon)=-(\varepsilon z_{11})^{-1}\cdot\frac{g(\varepsilon)}{h(\varepsilon)}=-(\varepsilon z_{11})^{-1}f(\varepsilon).$$

Consider the generating function

$$F(T)=\sum_{k=1}^{\infty}(\varepsilon T)^k\theta_k(z_{01}+z_{11}\zeta_{m+1}(\varepsilon),\dots,z_{0k}+z_{1k}\zeta_{m+1}(\varepsilon)). \tag{3.2.19}$$

By the help of (1.1.4) we obtain

(3.2.20)

$$\begin{aligned}
F(T) &= \log(1 + (z_{01} + z_{11}\zeta_{m+1}(\varepsilon))\varepsilon T + \cdots + (z_{0k} + z_{1k}\zeta_{m+1}(\varepsilon))(\varepsilon T)^k + \cdots) \\
&= \log(g(\varepsilon T) - f(\varepsilon)\cdot T \cdot h(\varepsilon T)) \\
&= \log(f(\varepsilon T) - Tf(\varepsilon)) + \log h(\varepsilon T) \\
&= \log(1-T) + \log\left(1 - q_2 T\varepsilon^2 - \cdots - q_k T\frac{1-T^{k-1}}{1-T}\varepsilon^k - \cdots \right) + \log h(\varepsilon T) \\
&= -\sum_{k=1}^{\infty}\frac{T^k}{k} + \sum_{k=1}^{\infty}\varepsilon^k\theta_k\left(0, -q_2T, \ldots, -q_kT\frac{1-T^{k-1}}{1-T}\right) \\
&\quad + \sum_{k=1}^{\infty}(\varepsilon T)^k\theta_k\left(\frac{z_{12}}{z_{11}}, \ldots, \frac{z_{1\,k+1}}{z_{11}}\right).
\end{aligned}$$

Sublemma 3.2.3.

$$\theta_k\left(0, -q_2T, \ldots, -q_kT\frac{1-T^{k-1}}{1-T}\right)$$

is a polynomial in T of degree $k-1$ with the leading coefficient $-q_k$.

Proof. By (1.1.5) we have

$$\begin{aligned}
&\theta_k\left(0, -q_2T, \ldots, -q_kT\frac{1-T^{k-1}}{1-T}\right) \\
&= -\sum_{j=1}^{k}\frac{1}{j}T^j \sum_{i_1+\cdots+i_j=k}(q_{i_1}(1+T+\cdots+T^{i_1-2}))\cdots(q_{i_j}(1+T+\cdots+T^{i_j-2})),
\end{aligned}$$

from which the sublemma follows. □

In general, for a power series $G(T)$ in T, we denote by $[G(T)]_k$ the coefficient of $G(T)$ in T^k. Then, noting Sublemma 3.2.3, from (3.2.19) and (3.2.20) we obtain

$$\begin{aligned}
&\theta_k(z_{01} + z_{11}\zeta_{m+1}(\varepsilon), \ldots, z_{0k} + z_{1k}\zeta_{m+1}(\varepsilon)) \\
&\quad = \varepsilon^{-k}[F(T)]_k \\
&\quad = -\frac{1}{k}\varepsilon^{-k} + \theta_k\left(\frac{z_{12}}{z_{11}}, \ldots, \frac{z_{1\,k+1}}{z_{11}}\right) + \sum_{j=1}^{\infty}\varepsilon^j\eta_{k+j,k}(q_2, \ldots, q_{k+j}).
\end{aligned}$$

□

Lemma 3.2.4.

$$\begin{aligned}
\sum_{j=1}^{m-1}\frac{A^j_{[m]}(\alpha(\varepsilon))}{(\alpha_m - \varepsilon^{-1}\alpha_{m+1})^j} &= \varepsilon^{-m-1}\alpha_{m+1}\left(-\sum_{k=1}^{m}\frac{1}{k} + 1\right) - \sum_{k=1}^{m+1}\frac{\varepsilon^{-m-1+k}}{k}\cdot\frac{{\alpha_m}^k}{{\alpha_{m+1}}^{k-1}} \\
&\quad + \sum_{k=1}^{m}\varepsilon^{-k}\frac{\alpha_k}{k} + \sum_{j=1}^{m}\frac{A^j_{[m+1]}(\alpha)}{{\alpha_{m+1}}^j} + O(\varepsilon).
\end{aligned}$$

Proof. By (3.2.8) and (2.1.5) we have

$$\begin{aligned}
&\sum_{j=1}^{m-1} \frac{A^j_{[m]}(\alpha(\varepsilon))}{(\alpha_m - \varepsilon^{-1}\alpha_{m+1})^j} \\
&= \varepsilon^{-m-1}\alpha_{m+1} \sum_{j=1}^{m-1} \frac{(-1)^{j-1}}{j(j+1)} \left(1 - \varepsilon \frac{\alpha_m}{\alpha_{m+1}}\right)^{-j} \\
&\qquad \times \sum_{m_1+\cdots+m_{j+1}=m} \left(1 - \varepsilon^{m_1+1}\frac{\alpha_{m-m_1}}{\alpha_{m+1}}\right) \cdots \left(1 - \varepsilon^{m_{j+1}+1}\frac{\alpha_{m-m_{j+1}}}{\alpha_{m+1}}\right).
\end{aligned}$$

Set

$$B(T) = \left(1 - \varepsilon^2 \frac{\alpha_{m-1}}{\alpha_{m+1}}\right)T + \left(1 - \varepsilon^3 \frac{\alpha_{m-2}}{\alpha_{m+1}}\right)T^2 + \cdots + \left(1 - \varepsilon^{k+1} \frac{\alpha_{m-k}}{\alpha_{m+1}}\right)T^k + \cdots ;$$

then we have

$$\begin{aligned}
&\sum_{k=1}^{\infty} T^k \sum_{m_1+\cdots+m_{j+1}=k} \left(1 - \varepsilon^{m_1+1}\frac{\alpha_{m-m_1}}{\alpha_{m+1}}\right) \cdots \left(1 - \varepsilon^{m_{j+1}+1}\frac{\alpha_{m-m_{j+1}}}{\alpha_{m+1}}\right) \\
&= B(T)^{j+1}.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&\sum_{j=1}^{m-1} \frac{A^j_{[m]}(\alpha(\varepsilon))}{(\alpha_m - \varepsilon^{-1}\alpha_{m+1})^j} \\
&= \varepsilon^{-m-1}\alpha_{m+1} \left[\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j(j+1)} \left(1 - \varepsilon \frac{\alpha_m}{\alpha_{m+1}}\right)^{-j} B(T)^{j+1}\right]_m ,
\end{aligned} \tag{3.2.21}$$

where we have noticed that $B(T) = O(T)$. Now set

$$F(T) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j(j+1)} \left(1 - \varepsilon \frac{\alpha_m}{\alpha_{m+1}}\right)^{-j} B(T)^{j+1}. \tag{3.2.22}$$

Then we have

$$\begin{aligned}
F(T) &= \left(1 - \varepsilon \frac{\alpha_m}{\alpha_{m+1}} + B(T)\right) \log\left(1 - \varepsilon \frac{\alpha_m}{\alpha_{m+1}}\right) \\
&\quad - \left(1 - \varepsilon \frac{\alpha_m}{\alpha_{m+1}} + B(T)\right) \log\left(1 - \varepsilon \frac{\alpha_m}{\alpha_{m+1}} + B(T)\right) + B(T).
\end{aligned}$$

Noting that

$$1 - \varepsilon \frac{\alpha_m}{\alpha_{m+1}} + B(T) = \frac{1}{1-T}\left(1 + (T-1)\varepsilon \sum_{k=0}^{\infty} \frac{\alpha_{m-k}}{\alpha_{m+1}} (\varepsilon T)^k\right),$$

we have

$$\begin{aligned}
&\log\left(1 - \varepsilon \frac{\alpha_m}{\alpha_{m+1}} + B(T)\right) \\
&= \sum_{k=1}^{\infty} \frac{T^k}{k} + \sum_{k=1}^{\infty} \varepsilon^k \theta_k \left((T-1)\frac{\alpha_m}{\alpha_{m+1}}, (T-1)T\frac{\alpha_{m-1}}{\alpha_{m+1}}, \ldots, (T-1)T^{k-1}\frac{\alpha_{m-k+1}}{\alpha_{m+1}}\right).
\end{aligned}$$

Hence we obtain

(3.2.23)

$$
\begin{aligned}
F(T) = & -\left\{\frac{1}{1-T} - \varepsilon \sum_{k=0}^{\infty} \frac{\alpha_{m-k}}{\alpha_{m+1}} (\varepsilon T)^k\right\} \\
& \times \left\{\sum_{k=1}^{\infty} \frac{1}{k} \varepsilon^k \left(\frac{\alpha_m}{\alpha_{m+1}}\right)^k + \sum_{k=1}^{\infty} \frac{T^k}{k}\right. \\
& \left. + \sum_{k=1}^{\infty} \varepsilon^k \theta_k \left((T-1)\frac{\alpha_m}{\alpha_{m+1}}, (T-1)T\frac{\alpha_{m-1}}{\alpha_{m+1}}, \ldots, (T-1)T^{k-1}\frac{\alpha_{m-k+1}}{\alpha_{m+1}}\right)\right\} \\
& + \sum_{k=1}^{\infty} T^k - \sum_{k=2}^{\infty} \varepsilon^k \frac{\alpha_{m-k+1}}{\alpha_{m+1}} T^{k-1}.
\end{aligned}
$$

Sublemma 3.2.5.

(i) $\theta_k\left((T-1)\frac{\alpha_m}{\alpha_{m+1}}, (T-1)T\frac{\alpha_{m-1}}{\alpha_{m+1}}, \ldots, (T-1)T^{k-1}\frac{\alpha_{m-k+1}}{\alpha_{m+1}}\right)$ *is a polynomial in T of degree at most k.*

(ii) $\frac{1}{1-T}\theta_k\left((T-1)\frac{\alpha_m}{\alpha_{m+1}}, (T-1)T\frac{\alpha_{m-1}}{\alpha_{m+1}}, \ldots, (T-1)T^{k-1}\frac{\alpha_{m-k+1}}{\alpha_{m+1}}\right)$ *is a polynomial in T of degree at most $k-1$.*

This sublemma is also shown by using (1.1.5).

Regarding $F(T)$ as a power series in ε, we write as

$$F(T) = \sum_{k=0}^{\infty} \varepsilon^k f_k(T).$$

From (3.2.23) it follows that

(3.2.24)

$$
\begin{aligned}
f_0(T) &= -\sum_{k=0}^{\infty} T^k \cdot \sum_{k=1}^{\infty} \frac{T^k}{k} + \sum_{k=1}^{\infty} T^k, \\
f_1(T) &= \frac{\alpha_m}{\alpha_{m+1}}\left(-\sum_{k=0}^{\infty} T^k + 1 + \sum_{k=1}^{\infty} \frac{T^k}{k}\right), \\
f_k(T) &= -\frac{1}{1-T} \cdot \frac{1}{k} \cdot \left(\frac{\alpha_m}{\alpha_{m+1}}\right)^k + \sum_{\substack{i+j=k \\ i,j\geq 1}} \frac{\alpha_{m-i+1}}{\alpha_{m+1}} T^{i-1} \cdot \frac{1}{j}\left(\frac{\alpha_m}{\alpha_{m+1}}\right)^j \\
& - \frac{1}{1-T}\theta_k\left((T-1)\frac{\alpha_m}{\alpha_{m+1}}, \ldots, (T-1)T^{k-1}\frac{\alpha_{m-k+1}}{\alpha_{m+1}}\right) \\
& + \frac{\alpha_{m-k+1}}{\alpha_{m+1}} T^{k-1} \sum_{j=1}^{\infty} \frac{T^j}{j} \\
& + \sum_{\substack{i+j=k \\ i,j\geq 1}} \frac{\alpha_{m-i+1}}{\alpha_{m+1}} T^{i-1} \cdot \theta_k\left((T-1)\frac{\alpha_m}{\alpha_{m+1}}, \ldots, (T-1)T^{k-1}\frac{\alpha_{m-k+1}}{\alpha_{m+1}}\right) \\
& - \frac{\alpha_{m-k+1}}{\alpha_{m+1}} T^{k-1} \qquad (k \geq 2).
\end{aligned}
$$

Then, by noting Sublemma 3.2.5, it follows from (3.2.24) that

(3.2.25)
$$[f_k(T)]_m = -\frac{1}{k}\left(\frac{\alpha_m}{\alpha_{m+1}}\right)^k + \frac{1}{m-k+1}\cdot\frac{\alpha_{m-k+1}}{\alpha_{m+1}} \quad (1 \le k \le m),$$

$$[f_{m+1}(T)]_m = -\frac{1}{m+1}\left(\frac{\alpha_m}{\alpha_{m+1}}\right)^{m+1} + \frac{1}{\alpha_{m+1}}\sum_{j=1}^{m}\frac{(-1)^{j-1}}{j(j+1)}{\alpha_{m+1}}^{-j}\sum_{\substack{m_1+\cdots+m_{j+1}=m\\ m_1,\ldots,m_{j+1}\ge 1}}\alpha_{m+1-m_1}\cdots\alpha_{m+1-m_{j+1}}.$$

Thus we have

(3.2.26)
$$\begin{aligned}[F(T)]_m &= \sum_{k=0}^{\infty}\varepsilon^k[f_k(T)]_m\\ &= -\sum_{k=1}^{m}\frac{1}{k} + 1 + \sum_{k=1}^{m}\varepsilon^k\left(-\frac{1}{k}\left(\frac{\alpha_m}{\alpha_{m+1}}\right)^k + \frac{1}{m-k+1}\cdot\frac{\alpha_{m-k+1}}{\alpha_{m+1}}\right)\\ [0.5pc]\ &+ \varepsilon^{m+1}\left(-\frac{1}{m+1}\left(\frac{\alpha_m}{\alpha_{m+1}}\right)^{m+1} + \frac{1}{\alpha_{m+1}}\sum_{j=1}^{m}\frac{A^j_{[m+1]}(\alpha)}{{\alpha_{m+1}}^j}\right)\\ &+ O(\varepsilon^{m+2}).\end{aligned}$$

The lemma follows from (3.2.21), (3.2.22) and (3.2.26). □

Remark 3.2.6. In [T3] Terasoma has evaluated the determinant $D_{[m]}(z,\alpha)$ in a different manner, and has obtained another expression of the determinant.

3.3. For the case (ii). In this subsection we are concerned with the composition

$$[m] := (1,\ldots,1,\overset{\overset{i}{\smile}}{1+m},1,\ldots,1)$$

as in §2.2. We assume $m > 0$.

Proposition 3.3.1. *Assumptions on z and α are as in Theorem* 3.1.3. *Assume that $m > 0$. Then we have*

$$
\begin{aligned}
D_{[m]}(z,\alpha) = \ & (2\pi)^{(m+1)/2} \cdot {\alpha_{i+m}}^{\alpha_i+(m+1)/2} \cdot {z_{1i}}^{(n-1-2m)\alpha_i - m(m+1)} \\
& \times (z_{0i}z_{1\,i+1} - z_{1i}z_{0\,i+1})^{m\alpha_i + m(m+1)/2} \prod_{k\neq i,\dots,i+m} {z_{1k}}^{(n-1)\alpha_k} \\
& \times \frac{\prod_{k\neq i,\dots,i+m}\Gamma(\alpha_k+1)}{\Gamma(\sum_{k\neq i+1,\dots,i+m}\alpha_k + n)} \\
& \times \prod_{k=1}^{i-1}\left\{\prod_{j=1}^{k-1}(\zeta_j-\zeta_k)^{\alpha_k}\cdot(\zeta_i-\zeta_k)^{m(\alpha_k+1)}\cdot \prod_{\substack{j=k+1\\ j\neq i+1,\dots,i+m}}^{n}(\zeta_j-\zeta_k)^{\alpha_k+1}\right\} \\
& \times \prod_{j=1}^{i-1}(\zeta_j-\zeta_i)^{\alpha_i}\cdot \prod_{j=i+m+1}^{n}(\zeta_j-\zeta_i)^{\alpha_i+m+1} \\
& \times \prod_{k=i+m+1}^{n}\left\{\prod_{\substack{j=1\\ j\neq i+1,\dots,i+m}}^{k-1}(\zeta_j-\zeta_k)^{\alpha_k}\cdot(\zeta_i-\zeta_k)^{m\alpha_k}\cdot\prod_{j=k+1}^{n}(\zeta_j-\zeta_k)^{\alpha_k+1}\right\} \\
& \times \exp\Bigg[\left(1+\frac{m(m+1)}{4}+m\alpha_i\right)\pi\sqrt{-1} \\
& \quad + (n-1)\sum_{k=1}^{m}\alpha_{i+k}\theta_k\left(\frac{z_{1\,i+1}}{z_{1i}},\dots,\frac{z_{1\,i+k}}{z_{1i}}\right) \\
& \quad + \sum_{j\neq i,\dots,i+m}\sum_{k=1}^{m}\alpha_{i+k}\theta_k\big(c(z^{(i)};\zeta_j)p_1(z^{(i)}),\dots,c(z^{(i)};\zeta_j)p_k(z^{(i)})\big) \\
& \quad + m\sum_{k=1}^{m}\alpha_{i+k}\theta_k(p_+(z^{(i)})) - \sum_{k=1}^{m}\alpha_{i+k}\sum_{j=1}^{k}\xi_{k,j}(p_+(z^{(i)})) \\
& \quad + \sum_{j=1}^{m-1}\frac{(-1)^{j-1}}{j(j+1)}{\alpha_{i+m}}^{-j}\sum_{\substack{m_1+\cdots+m_{j+1}=m\\ m_1,\dots,m_{j+1}\geq 1}}\alpha_{i+m-m_1}\cdots\alpha_{i+m-m_{j+1}}\Bigg].
\end{aligned}
$$

Proof. This proposition is shown in a similar manner as in the proof of Proposition 3.2.1 except for the confluence $[0]\to[1]$.

Define $z(\varepsilon)$ and $\alpha(\varepsilon)$ by (1.3.2) with $m=0$. Assume (2.2.8) with (2.2.7). Then, from Propositions 1.3.2 and 2.2.2, it follows that

$$
D_{[1]}(z,\alpha) = \lim_{\varepsilon\to 0}(e(\alpha_i-\varepsilon^{-1}\alpha_{i+1})-1)D_{[0]}(z(\varepsilon),\alpha(\varepsilon)).
$$

By noting that

$$
\begin{aligned}
\Gamma(\alpha_i-\varepsilon^{-1}\alpha_{i+1}+1) &= \frac{-\pi}{\Gamma(\varepsilon^{-1}\alpha_{i+1}-\alpha_i)\sin\pi(\alpha_i-\varepsilon^{-1}\alpha_{i+1})}, \\
e(\alpha_i-\varepsilon^{-1}\alpha_{i+1})-1 &= 2\sqrt{-1}\exp((\alpha_i-\varepsilon^{-1}\alpha_{i+1})\pi\sqrt{-1})\sin\pi(\alpha_i-\varepsilon^{-1}\alpha_{i+1}),
\end{aligned}
$$

we can modify the proof of Proposition 3.2.1 to obtain the proof. Details are omitted here. □

Proof of Theorem 3.1.3. Consider the composition

$$\lambda(i;m) := (1+\lambda_0,\dots,1+\lambda_{i-1},\overset{\overset{i}{\smile}}{1+m},1,\dots,1)$$

of $n+1$. When $\lambda = \lambda(0;m)$, the theorem is just Proposition 3.2.1. Suppose that $i>0$. Define $z(\varepsilon)$ and $\alpha(\varepsilon)$ by (1.3.2), and assume (2.2.13) with (2.2.12). Then by Propositions 1.3.2, 2.1.3 and 2.1.5, we obtain

$$\begin{aligned} D_{\lambda(i;1)}(z,\alpha) &= \lim_{\varepsilon\to 0}(e(\alpha_0^{(i)} - \varepsilon^{-1}\alpha_0^{(i+1)}) - 1)D_{\lambda(i;0)}(z(\varepsilon),\alpha(\varepsilon)),\\ D_{\lambda(i;m+1)}(z,\alpha) &= \lim_{\varepsilon\to 0} D_{\lambda(i;m)}(z(\varepsilon),\alpha(\varepsilon)) \quad (m>0). \end{aligned}$$

Thus the theorem is proved by induction similarly as in Proposition 3.3.1. Details are omitted here. □

3.4. Proof of the main theorem.

Proof of Theorem 1.2.4. First we note that, by Proposition 6 in [KHT1], the dimension of $H_1^{\ell f}({}^{\Psi}X,\check{\mathcal{S}})$ is at most $n-1$. As we have noted in §3.1, there is a pairing between $H_1^{\ell f}({}^{\Psi}X,\check{\mathcal{S}})$ and $H^1(\Omega^\bullet,\Delta)$, the latter of which has dimension $n-1$ by Theorem 1.3.1, (ii). Theorem 3.1.3 shows that, on the assumption (1.2.4) and (1.2.5), this pairing is perfect. Then it follows that the $(n-1)$ elements

$$\{\,[\Delta_j^{\lambda,(k)}(z,\alpha)\otimes U_\lambda]\,\}_{k,j}$$

of $H_1^{\ell f}({}^{\Psi}X,\check{\mathcal{S}})$ are linearly independent. This completes the proof. □

In the above we have also proved the following fact.

Corollary 3.4.1. *Assume* (1.2.4) *and* (1.2.5). *Then the rank of the hypergeometric system of type* λ *on* $Z_{2,n+1}$ *is* $n-1$.

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