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THE IMAGE OF THE BP THOM MAP FOR EILENBERG–MAC LANE SPACES

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Abstract. Fundamental classes in BP cohomology of Eilenberg–Mac Lane spaces are defined. The image of the Thom map from BP cohomology to mod-p cohomology is determined for arbitrary Eilenberg–Mac Lane spaces. This image is a polynomial subalgebra generated by infinitely many elements obtained by applying a maximum number of Milnor primitives to the fundamental class in mod-p cohomology. This subalgebra in mod p cohomology is invariant under the action of the Steenrod algebra, and it is annihilated by all Milnor primitives. We also show that BP cohomology determines Morava K cohomology for Eilenberg–Mac Lane spaces.

1. Introduction and Summary of Results

Let $\mathbb{Z}_p = \mathbb{Z}/p$ be the ring of mod-p integers, and let $\mathbb{Z}_{(p)} \subset Q$ be the ring of integers localized at a prime p, even or odd. Let BP be the Brown-Peterson spectrum at the prime p. The BP spectrum was originally constructed in [BP]. Later, Quillen [Q], [A2] gave a very nice construction using formal group laws. For a topological space X, let BP^*X denote (unreduced) BP cohomology of X. Its coefficient ring is a polynomial algebra $BP^*(\mathrm{pt}) = \mathbb{Z}_{(p)}[v_1, v_2, \dots, v_n, \dots]$ with $|v_n| = -2(p^n - 1)$. Let X be an infinite dimensional CW complex. Its BP cohomology ring $BP^*(X)$ has the skeletal filtration topology given in (3-17) in §3. We set $v_0 = p$. Let $H\mathbb{Z}_p$ denote the mod-p Eilenberg–Mac Lane spectrum, and we use the notation $H\mathbb{Z}_p^*(X)$ to denote the usual (unreduced) mod p cohomology of the space X.

As is well-known, Eilenberg–Mac Lane spaces are fundamental building blocks of topological spaces from the homotopical point of view. Thus, the BP cohomology of Eilenberg–Mac Lane spaces should shed some light on the general (unstable) structure of BP cohomology rings of topological spaces. In this paper, we produce infinite families of nontrivial elements of fundamental importance in the BP cohomology of Eilenberg–Mac Lane spaces. Our main result (see Theorem A below) states that the image of the BP Thom map

$$\rho_* : BP^*\big(K(\pi, q)\big) \to H\mathbb{Z}_p^*\big(K(\pi, q)\big), \qquad q \geq 1, \tag{1-1}$$

is a polynomial subalgebra generated by the images of the above BP cohomology elements, unless $K(\pi, q)$ has a factor $K(\mathbb{Z}, 1) = S^1$. If there is a $K(\mathbb{Z}, 1)$ factor, there is an exterior algebra tensor factor for the image. The Thom map $\rho : BP \to H\mathbb{Z}_p$

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is a ring spectra map and its behavior on the coefficient ring is given by $\rho_*(v_n) = 0$ for all $n \geq 0$.

To describe our result more precisely, let $\mathcal{A}(p)^*$ be the mod-p Steenrod algebra and let $Q_n \in \mathcal{A}(p)^*$ be the nth Milnor primitive for any $n \geq 0$. Its degree is $|Q_n| = 2p^n - 1$. The Q_n generate exterior subalgebras of $\mathcal{A}(p)^*$ for any prime p. For more details on Steenrod algebras, see §2. For any sequence $S = (s_1, s_2, \ldots, s_n)$ of n strictly increasing positive integers, let $Q_S = Q_{s_n} \cdots Q_{s_2} Q_{s_1}$ be a product of Milnor primitives. For the Eilenberg–Mac Lane space $K(\mathbb{Z}_{(p)}, n+2)$ for the group $\mathbb{Z}_{(p)}$ in degree $n+2$, let $\tau_{n+2} \in H\mathbb{Z}_p^{n+2}\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ be the fundamental cohomology class. For the mod-p^j Eilenberg–Mac Lane space $K(\mathbb{Z}/p^j, n+1)$ in degree $n+1$ for a positive integer j, let $\iota_{n+1} \in H\mathbb{Z}_p^{n+1}\big(K(\mathbb{Z}/p^j, n+1)\big)$ be the fundamental class.

Theorem A [Theorem 7-2]. *Let p be a prime, even or odd, and let $n \geq 1$.*

(I) *The image of the Thom map*

$$\rho_* : BP^*\big(K(\mathbb{Z}_{(p)}, n+2)\big) \to H\mathbb{Z}_p^*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$$

is an $\mathcal{A}(p)^*$-invariant polynomial subalgebra with infinitely many generators:

$$\operatorname{Im} \rho_* = \mathbb{Z}_p[Q_{s_n} \cdots Q_{s_1} \tau_{n+2} \mid 0 < s_1 < \cdots < s_n]. \tag{1-2}$$

This subalgebra is annihilated by all Milnor primitives, and any polynomial generator $Q_S \tau_{n+2}$ is obtained by applying a Steenrod reduced power operation to the element $Q_n \cdots Q_1 \tau_{n+2}$.

(II) *The image of the Thom map*

$$\rho_* : BP^*\big(K(\mathbb{Z}/p^j, n+1)\big) \to H\mathbb{Z}_p^*\big(K(\mathbb{Z}/p^j, n+1)\big)$$

is an $\mathcal{A}(p)^*$-invariant polynomial subalgebra with infinitely many generators:

$$\operatorname{Im} \rho_* = \mathbb{Z}_p[Q_{s_n} \cdots Q_{s_1} \delta_j \iota_{n+1} \mid 0 < s_1 < \cdots < s_n], \tag{1-3}$$

where δ_j is the jth Bockstein operator. This subalgebra is annihilated by all Milnor primitives, and any polynomial generator $Q_S \delta_j \iota_{n+1}$ is obtained from $Q_n \cdots Q_1 \delta_j \iota_{n+1}$ by applying a Steenrod reduced power operation.

A brief description of the mod-p cohomology ring of Eilenberg–Mac Lane spaces in terms of the Milnor basis is given in §2. For more details, see [T]. From this description, we can observe that on the fundamental class τ_{n+2}, any product of n distinct Milnor primitives (different from Q_0) can act nontrivially, but all products of $n+1$ Milnor primitives act trivially on τ_{n+2}. A similar fact holds for ι_{n+1}. We remark that for cohomology classes other than the fundamental classes, these facts may not hold. For example, as was pointed out by the referee, we have $Q_2 Q_1 Q_0 \iota_2 = 0$ in $H\mathbb{Z}_2^*\big(K(\mathbb{Z}/2, 2)\big)$, but $Q_2 Q_1 Q_0[\iota_2 Q_0 \iota_2 Q_1 \iota_2]$ is nonzero [JW3].

The above Theorem A can be generalized to arbitrary Eilenberg–Mac Lane spaces as follows. Let G be a finitely generated positively graded $\mathbb{Z}_{(p)}$-module, say, $G = \bigoplus_{0<q\in\mathbb{Z}} G_q$. The corresponding generalized (connected) Eilenberg–Mac Lane space $K(G)$ is defined by $K(G) = \prod_q K(G_q, q)$. Then the image of the BP Thom map $\rho_* : BP^*\big(K(G)\big) \to H\mathbb{Z}_p^*\big(K(G)\big)$ is a tensor product of polynomial algebras of the form in (1-2) or in (1-3), if $K(G)$ does not have a factor $K(\mathbb{Z}, 1)$. For details, see Theorem 7-3.

In the subalgebra given in (1-2), the lowest degree element is $Q_n \cdots Q_1 \tau_{n+2}$ of degree $2(1 + p + \cdots + p^n)$. This element is the image of an element $\vartheta \in$

$BP^{2(1+p+\cdots+p^n)}\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ defined by composition of the following maps:

$$\vartheta : K(\mathbb{Z}_{(p)}, n+2) = \underline{BP}\langle 0\rangle_{n+2} \xrightarrow{\Delta_1} \underline{BP}\langle 1\rangle_{2p+n+1} \xrightarrow{\Delta_2} \cdots \cdots \xrightarrow{\Delta_n} \underline{BP}\langle n\rangle_{2(1+p+\cdots+p^n)} \xrightarrow{\theta\langle n\rangle} \underline{BP}_{2(1+p+\cdots+p^n)}. \tag{1-4}$$

Here, Δ_i's are maps realizing the connecting homomorphisms in Sullivan exact sequences, and the last map $\theta\langle n\rangle$ is an inclusion map provided by Wilson's Splitting Theorem (see Theorem 3-1). This element ϑ is very fundamental in the BP cohomology of Eilenberg–Mac Lane spaces, and we call ϑ the *BP fundamental class* of the Eilenberg–Mac Lane space $K(\mathbb{Z}_{(p)}, n+2)$ [Definition 3-2]. By applying certain BP operations on ϑ, we can produce nontrivial elements ϑ_S with the property $\rho_*(\vartheta_S) = Q_S\tau_{n+2}$ for each sequence S of n strictly increasing positive integers [Theorem 3-5]. We also have corresponding statements for mod-p^j Eilenberg–Mac Lane spaces [Theorem 3-6].

As a byproduct of our proof of Theorem A, we also show that for the Eilenberg–Mac Lane spaces, BP cohomology "determines" Morava K theory in the following sense.

Theorem B [Theorem 6-7]. *Let $K(G)$ be the generalized Eilenberg–Mac Lane space associated to a positively graded finitely generated $\mathbb{Z}_{(p)}$-module $G = \bigoplus_{q=1}^{r} G_q$. Then for any positive integer $m \geq r$, the following canonical map is surjective*:

$$v_m^{-1} BP^*\big(K(G)\big) \to K(m)^*\big(K(G)\big). \tag{1-5}$$

Furthermore, the canonical map $BP^*\big(K(G)\big) \to K(m)^*\big(K(G)\big)$ before localization is surjective in degrees $* < 2(p^m - 1)$.

Actually, this is a key fact used in the proof of Theorem A. Since $K(m)$ theory is periodic with the period $2(p^m - 1)$, the second statement in Theorem B means that all $K(m)^*$-algebra generators for $K(m)^*\big(K(G)\big)$ come from BP cohomology.

The organization of this paper is as follows. In §2, we present a description of the mod-p cohomology of Eilenberg–Mac Lane spaces in terms of Milnor basis elements of the mod-p Steenrod algebra rather than in terms of admissible monomials in Steenrod reduced power operations and Bockstein operators, which is the traditional description. In our description, the admissibility condition is automatically satisfied by Milnor elements and the excess condition assumes a very simple form. All the details for this section can be found in [T]. In §3, we produce an infinite family of nontrivial elements in the BP cohomology of Eilenberg–Mac Lane spaces and determine their images in the corresponding mod-p cohomology. These image elements generate very interesting polynomial subalgebras in the mod-p cohomology rings of Eilenberg–Mac Lane spaces. These subalgebras are invariant under the action of the Steenrod algebra. In §4, we describe the Kronecker pairing in *connective* Morava K theory of Eilenberg–Mac Lane spaces. The cases for mod-p^j Eilenberg–Mac Lane spaces turn out to be a lot more complicated than the cases for mod-p or integral Eilenberg–Mac Lane spaces. These results on Kronecker pairings are used to identify images of certain BP cohomology elements in Morava K theory in §5. The results in §4 are obtained using information on Kronecker pairings in mod-p cohomology. The reason for working with connective Morava K theory is that there is no canonical spectra map from non-connective Morava K theories to $H\mathbb{Z}_p$, but there are canonical maps from connective Morava K theories to mod-p theory. In §5, we describe the algebra structure of the Morava K cohomology of

Eilenberg–Mac Lane spaces. Ravenel–Wilson [RW2] computed the Hopf ring structure of Morava K homology of Eilenberg–Mac Lane spaces. We dualize their result. In the mod-p^j case, we explicitly calculate the exponent of algebra generators of the cohomology rings. In §6, we consider the canonical map from BP cohomology to Morava K cohomology of Eilenberg–Mac Lane spaces and we prove Theorem B. In §7, we prove Theorem A, our main result. The key idea of the proof of Theorem A is the diagram (7-3).

The results in this paper originated in the author's Master thesis at the University of Tokyo in 1983. Some of the results were subsequently applied to finite H-spaces in [Y].

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After this paper was finished, Steve Wilson pointed out to the author that some of the results described in §2 are contained in [K]. [JW3] also contains some related formulae. We thank the referee for numerous comments.

2. Mod-p Cohomology Rings of Eilenberg–Mac Lane Spaces and Milnor Basis Elements

In the literature, the mod-p cohomology ring of an Eilenberg–Mac Lane space is traditionally described in terms of admissible monomials in Steenrod reduced power operations and Bockstein operators satisfying the so-called excess condition. In this section, we give an alternate and much simpler description of the mod-p cohomology of Eilenberg–Mac Lane spaces. Complete details can be found in [T]. This description is given in terms of the Milnor basis for the mod-p Steenrod algebra [M1].

Let p be a prime, even or odd. Let $\mathcal{A}(p)^*$ be the mod-p Steenrod algebra. This is a Hopf algebra, and let $\mathcal{A}(p)_*$ be the dual Hopf algebra whose structure is determined in [M1] as follows:

$$\begin{aligned} &\mathcal{A}(2)_* = \mathbb{Z}_2[\zeta_1, \zeta_2, \dots, \zeta_r, \dots], \\ &\mathcal{A}(p)_* = \Lambda_{\mathbb{Z}_p}(\tau_0, \tau_1, \dots, \tau_r, \dots) \otimes \mathbb{Z}_p[\xi_1, \xi_2, \dots, \xi_r, \dots], \quad p \text{ an odd prime,} \end{aligned} \tag{2-1}$$

where degrees of generators are given by $|\zeta_i| = 2^i - 1$, $|\tau_r| = 2p^r - 1$, $|\xi_r| = 2(p^r - 1)$. Here, $\mathbb{Z}_p[\ \cdot\ ]$ and $\Lambda_{\mathbb{Z}_p}(\ \cdot\ )$ denote a polynomial algebra and an exterior algebra, respectively, generated by the elements indicated over the finite field $\mathbb{Z}_p$.

Let p be an odd prime for a moment. Let $R = (r_1, r_2, \dots)$ range over all sequences of non-negative integers which are almost all zero, and let $E = (\varepsilon_0, \varepsilon_1, \dots)$ range over all sequences of zeroes and ones which are almost all zero. We put $\tau(E)\xi(R) = \tau_0^{\varepsilon_0}\tau_1^{\varepsilon_1}\tau_2^{\varepsilon_2}\cdots\xi_1^{r_1}\xi_2^{r_2}\cdots \in \mathcal{A}(p)^*$. Then the collection of elements $\{\tau(E)\xi(R)\}_{E,R}$ forms a $\mathbb{Z}_p$-basis of $\mathcal{A}(p)_*$. Let $Q^E\mathcal{P}^R$ be the element in $\mathcal{A}(p)^*$ dual to $\tau(E)\xi(R)$. Let Q_n be the element dual to τ_n for $n \geq 0$. These operators Q_n are primitive and generate an exterior subalgebra of $\mathcal{A}(p)^*$. We also have $Q^E = \pm Q_0^{\varepsilon_0}Q_1^{\varepsilon_1}\cdots$. The mod-$p$ Steenrod algebra $\mathcal{A}(p)^*$ is spanned by the set of elements $\{Q^E\mathcal{P}^R\}_{E,R}$ as a $\mathbb{Z}_p$-vector space.

Let $\Delta_j = (0, \dots, 0, 1, 0, \dots)$ be the sequence with 1 at the jth entry and 0 everywhere else for $j \geq 1$. Let Δ_0 denote the zero sequence. For example, the above sequence R can be written as $R = \sum_{j\geq 1} r_j\Delta_j$. The element of the form

$\mathcal{P}^R Q_k$ can be expressed in terms of the above basis by the following formula:

(2-2)
$$\mathcal{P}^R Q_k = Q_k \mathcal{P}^R + Q_{k+1}\mathcal{P}^{R-p^k\Delta_1} + Q_{k+2}\mathcal{P}^{R-p^k\Delta_2} + \cdots + Q_{k+j}\mathcal{P}^{R-p^k\Delta_j} + \cdots .$$

Here on the right hand side, if the sequence $R - p^k\Delta_j$ contains a negative integer, then the corresponding operator is regarded as zero.

Next, let $p = 2$. For any sequence $K = (k_1, k_2, \cdots)$ of non-negative integers which are almost all zero, let $\zeta^K = \zeta_1^{k_1}\zeta_2^{k_2}\cdots \in \mathcal{A}(2)_*$ and let Sq^K be the element in $\mathcal{A}(2)^*$ dual to ζ^K. We let $Q_i = \mathrm{Sq}^{\Delta_{i+1}}$ for $i \geq 0$ and $\mathcal{P}^R = \mathrm{Sq}^{2R}$. Then the elements Q_n are primitive, and they generate an exterior subalgebra satisfying $Q_iQ_j = Q_jQ_i$ and $Q_i^2 = 0$, as in the odd prime case. Also elements $Q^E\mathcal{P}^R = \mathrm{Sq}^{E+2R}$ span the mod-2 Steenrod algebra $\mathcal{A}(2)^*$ over $\mathbb{Z}_2$, where $E + 2R = (\varepsilon_0 + 2r_1, \varepsilon_1 + 2r_2, \dots)$.

The difference between the odd prime case and the even prime case is that in the odd prime case, Steenrod reduced powers $\mathcal{P}^R$ form a subalgebra, whereas in the mod 2 case they do not. Even so, the use of the notation $Q^E\mathcal{P}^R$ allows us to treat all the primes simultaneously. The formula (2-2) holds for the mod-2 case also [M2]. We remark that the multiplication rule for $\mathcal{P}^{R_1}\mathcal{P}^{R_2}$ is the same as the multiplication rule for $\mathrm{Sq}^{R_1}\mathrm{Sq}^{R_2}$ for any sequence of non-negative integers R_1 and R_2 [M1].

For sequences $E = (\varepsilon_0, \varepsilon_1, \dots)$ and $R = (r_1, r_2, \cdots)$ as above, we let $|E| = \sum_j \varepsilon_j$ and $|R| = \sum_j r_j$. Let t denote a shifting operator to the left on sequences: it is given by $t(E) = (\varepsilon_1, \varepsilon_2, \dots)$ and $t(R) = (r_2, r_3, \dots)$ for the above E and R. From E and R, we can form a new sequence $I(E, R) = (\varepsilon_0, r_1, \varepsilon_1, r_2, \cdots)$.

The following notation is used. For a graded $\mathbb{Z}_p$-vector space $V = V^+ \oplus V^-$, where V^+ is the even degree part and V^- is the odd degree part, let $F[V] = \Lambda_{\mathbb{Z}_p}(V^-) \otimes \mathbb{Z}_p[V^+]$ be the free graded algebra generated by V.

Theorem 2-1 (Mod-p cohomology rings for Eilenberg–Mac Lane spaces [T])**.** *Let p be an odd prime and $n \geq 0$.*

(I) *The action of the Milnor element $Q^E\mathcal{P}^R \in \mathcal{A}(p)^*$ on the fundamental class $\iota_{n+1} \in H\mathbb{Z}_p^{n+1}\big(K(\mathbb{Z}/p^j, n+1)\big)$ and the structure of the mod-p cohomology ring are described as follows*:

$$Q^E\mathcal{P}^R\iota_{n+1} = 0 \quad \textit{if} \ \ |E| + 2|R| > n+1. \tag{2-3}$$

$$Q^E\mathcal{P}^R\iota_{n+1} = \big(Q^{t(E)}\mathcal{P}^{t(R)}\iota_{n+1}\big)^p \quad \textit{if} \ \ |E| + 2|R| = n+1 \ \textit{and} \ \varepsilon_0 = 0. \tag{2-4}$$

$$H\mathbb{Z}_p^*\big(K(\mathbb{Z}/p^j, n+1)\big) = \mathbb{Z}_p\big[Q^E\mathcal{P}^R\iota_{n+1} \ \big|\ |E| + 2|R| < n+1+\varepsilon_0.\big]. \tag{2-5}$$

Here when Q_0 directly acts on ι_{n+1}, it should be regarded as $\delta_j\iota_{n+1}$, where δ_j is the jth Bockstein map $\delta_j : K(\mathbb{Z}/p^j, n+1) \to K(\mathbb{Z}_{(p)}, n+2) \to K(\mathbb{Z}/p, n+2)$.

(II) *Let $\tau_{n+2} \in H\mathbb{Z}_p^{n+2}\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ be the fundamental class. Then the action of $\mathcal{A}(p)^*$ on τ_{n+2} and the structure of the cohomology ring are described as follows*:

$$Q^E\mathcal{P}^R\tau_{n+2} = 0 \quad \textit{if} \ \ |E| + 2|R| > n+2. \tag{2-6}$$

$$Q^E\mathcal{P}^R\tau_{n+2} = \big(Q^{t(E)}\mathcal{P}^{t(R)}\tau_{n+2}\big)^p \quad \textit{if} \ \ |E| + 2|R| = n+2 \ \textit{and} \ \varepsilon_0 = 0. \tag{2-7}$$

(2-8)
$$H\mathbb{Z}_p^*\big(K(\mathbb{Z}_{(p)}, n+2)\big) = \mathbb{Z}_p\left[Q^E\mathcal{P}^R\tau_{n+2} \ \middle|\ \begin{array}{l} |E| + 2|R| < n+2+\varepsilon_0 \ \textit{and} \ I(E,R) \\ \textit{ends with an entry from } R. \end{array}\right].$$

In (2-8), when $|E| = n$, from the requirement on $I(E, R)$ we must have $|R| = 1$, $\varepsilon_0 = 1$, and the corresponding operator is of the form $\pm Q_{s_{n-1}} \cdots Q_{s_1} Q_0 \mathcal{P}^{\Delta_{s_n}} \tau_{n+2}$ with $0 < s_1 < \cdots < s_{n-1} < s_n$. But by (2-2), we have $\mathcal{P}^{\Delta_{s_n}} Q_0 = Q_0 \mathcal{P}^{\Delta_{s_n}} + Q_{s_n}$. So the above indecomposable polynomial generator is actually equal to $\pm Q_{s_n} \cdots Q_{s_1} \tau_{n+2} \neq 0$.

Theorem 2-2 (Mod-2 cohomology rings for Eilenberg–Mac Lane spaces [T])**.** *Let $p = 2$ and $n \geq 0$.*

(I) *Let $\iota_{n+1} \in H\mathbb{Z}_2^{n+1}\big(K(\mathbb{Z}/2^j, n+1)\big)$ be the fundamental class. Assume $j = 1$ when $n = 0$. Then the action of the mod-2 Steenrod algebra $\mathcal{A}(2)^*$ on ι_{n+1} and the structure of the mod-2 cohomology ring are described as follows*:

$$\mathrm{Sq}^K \iota_{n+1} = 0 \quad \textit{if} \ \ |K| > n+1. \tag{2-9}$$

$$\mathrm{Sq}^K \iota_{n+1} = \big(\mathrm{Sq}^{t(K)} \iota_{n+1}\big)^2 \quad \textit{if} \ \ |K| = n+1. \tag{2-10}$$

$$\begin{aligned} H\mathbb{Z}_2^*\big(K(\mathbb{Z}/2^j, n+1)\big) &= \mathbb{Z}_2\big[\mathrm{Sq}^K \iota_{n+1} \ \big| \ |K| < n+1 \ \big] \\ &= \mathbb{Z}_2\big[Q^E \mathcal{P}^R \iota_{n+1} \ \big| \ |E| + 2|R| < n+1 \ \big]. \end{aligned} \tag{2-11}$$

Here when Q_0 is directly applied to ι_{n+1}, it should be regarded as $\delta_j \iota_{n+1}$, where $\delta_j : K(\mathbb{Z}/2^j, n+1) \to K(\mathbb{Z}_{(2)}, n+2) \to K(\mathbb{Z}/2, n+2)$ is the jth Bockstein map.

(II) *Let $\tau_{n+2} \in H\mathbb{Z}_2^{n+2}\big(K(\mathbb{Z}_{(2)}, n+2)\big)$ be the fundamental class. Then the action of $\mathcal{A}(2)^*$ on τ_{n+2} and the structure of the mod-2 cohomology ring are given as follows*:

$$\mathrm{Sq}^K \tau_{n+2} = 0 \quad \textit{if} \ \ |K| > n+2. \tag{2-12}$$

$$\mathrm{Sq}^K \tau_{n+2} = \big(\mathrm{Sq}^{t(K)} \tau_{n+2}\big)^2 \quad \textit{if} \ \ |K| = n+2. \tag{2-13}$$

(2-14)

$$\begin{aligned} H\mathbb{Z}_2^*\big(K(\mathbb{Z}_{(2)}, n+2)\big) &= \mathbb{Z}_2\left[\mathrm{Sq}^K \tau_{n+2} \ \middle| \ \begin{array}{l} |K| < n+2 \ \textit{with} \ r_k > 1 \ \textit{if} \\ K = (r_1, \ldots, r_k, 0, 0, \ldots) \end{array} \right] \\ &= \mathbb{Z}_2\left[Q^E \mathcal{P}^R \tau_{n+2} \ \middle| \ \begin{array}{l} |E| + 2|R| < n+2 \ \textit{and} \ I(E,R) \\ \textit{ends with an entry from } R \end{array} \right]. \end{aligned}$$

Remarks. (i) In (I) of Theorem 2-2, when $j > 1$ and $n = 0$, the cohomology ring is not a polynomial algebra. It is given by

$$H\mathbb{Z}_2^*\big(K(\mathbb{Z}/2^j, 1)\big) = \mathbb{Z}_2[\delta_j \iota_1] \otimes \Lambda_{\mathbb{Z}_2}(\iota_1). \tag{2-15}$$

(ii) In (2-14), when $|E| = n$, $Q^E \tau_{n+2}$ is not a generator of the mod-2 cohomology ring, unlike the odd prime case. It turns out that it is a square of a certain polynomial generator. See (3-11) below.

The point of Theorems 2-1 and Theorem 2-2 is the way the excess condition simplifies to a condition on the size of $|E|+2|R|$. The admissibility condition, which was necessary for the traditional description in terms of monomials of Steenrod reduced power operations and Bockstein operators, is automatically taken care of for Milnor elements. The main ingredient of the proof of the above theorems in [T] is a certain decomposition formula for Milnor's Steenrod reduced powers.

3. Fundamental BP Cohomology Classes for Eilenberg–Mac Lane Spaces

In this section, we produce infinite families of nontrivial BP cohomology classes for Eilenberg–Mac Lane spaces. These elements are produced through the action of BP operations on "fundamental BP cohomology classes." We also identify their images into mod-p cohomology under the BP Thom map.

Let p be a fixed prime, even or odd. Let BP be the Brown-Peterson spectrum at the prime p. For each integer $n \geq -1$, let $BP\langle n\rangle$ be a BP-module spectrum whose homotopy groups are given by $BP\langle n\rangle_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_n]$. We have $BP\langle -1\rangle = H\mathbb{Z}_p$ and $BP\langle 0\rangle = H\mathbb{Z}_{(p)}$. These BP-module spectra fit into the following BP tower of BP-module maps for $m \geq 0$ [W]:

$$BP \to \cdots \to BP\langle m+1\rangle \xrightarrow{\rho\langle m+1,m\rangle} BP\langle m\rangle \to BP\langle -1\rangle = H\mathbb{Z}_p. \tag{3-1}$$

For convenience, let $BP \xrightarrow{\rho_{\langle m\rangle}} BP\langle m\rangle \xrightarrow{\rho^{\langle m\rangle}} H\mathbb{Z}_p$ be partial Thom maps. Let $\underline{BP}\langle m\rangle_i$ be the ith space in the Ω-spectrum for $BP\langle m\rangle$ theory. For each nonnegative integer m, we have the following diagram which is homotopy commutative up to a nonzero element in $\mathbb{Z}_p$ [JW1]:

$$\begin{array}{ccc} \underline{BP}\langle m-1\rangle_i & \xrightarrow{\Delta_m} & \underline{BP}\langle m\rangle_{i+2p^m-1} \\ \Big\downarrow{\scriptstyle \rho^{\langle m-1\rangle}} & & \Big\downarrow{\scriptstyle \rho^{\langle m\rangle}} \\ K(\mathbb{Z}_p, i) & \xrightarrow{Q_m} & K(\mathbb{Z}_p, i+2p^m-1), \end{array} \tag{3-2}$$

where Δ_m is an infinite loop map realizing the connecting map in the cofibre sequence $\Sigma^{2(p^m-1)}BP\langle m\rangle \xrightarrow{v_m} BP\langle m\rangle \to BP\langle m-1\rangle \xrightarrow{\Delta_m} \Sigma^{2p^m-1}BP\langle m\rangle$, and Q_m is the mth Milnor primitive in the Steenrod algebra. The infinite loop spaces $\underline{BP}\langle m\rangle_*$ are important because they are building blocks of the spaces in the Ω-spectrum of BP theory in the following way.

Theorem 3-1 (Splitting Theorem) [W]**.** *For $k \leq 2(1+p+\cdots+p^m)$, the kth space of the Ω-spectrum for BP theory decomposes into a product as follows*:

$$\underline{BP}_k \cong \underline{BP}\langle m\rangle_k \times \prod_{j\geq m+1} \underline{BP}\langle j\rangle_{k+2(p^j-1)}. \tag{3-3}$$

When $k < 2(1+p+\cdots+p^m)$, the above decomposition is as H-spaces.

This Wilson's Splitting Theorem provides us with a canonical ready-made BP cohomology class $\theta\langle m\rangle \in BP^k(\underline{BP}\langle m\rangle_k)$ for $k \leq 2(1+p+\cdots+p^m)$ given by

$$\theta\langle m\rangle : \underline{BP}\langle m\rangle_k \xrightarrow{\mathrm{Id}\times(\prod_j *)} \underline{BP}\langle m\rangle_k \times \prod_{j\geq m+1} \underline{BP}\langle j\rangle_{k+2(p^j-1)} \cong \underline{BP}_k, \tag{3-4}$$

where $*$ means a constant map to the base point. In a sense, these classes $\theta\langle m\rangle$ are BP analogues of the mod-p fundamental cohomology classes of Eilenberg–Mac Lane spaces. Note that $\rho \circ \theta\langle m\rangle = \rho^{\langle m\rangle} : \underline{BP}\langle m\rangle_k \to K(\mathbb{Z}_p, k)$. Consider the following composition of the connecting maps:

(3-5)

$$K(\mathbb{Z}_{(p)}, n+2) = \underline{BP}\langle 0\rangle_{n+2} \xrightarrow{\Delta_1} \underline{BP}\langle 1\rangle_{2p+n+1} \xrightarrow{\Delta_2} \cdots \xrightarrow{\Delta_n} \underline{BP}\langle n\rangle_{2(1+p+\cdots+p^n)}.$$

Let $\delta_j : K(\mathbb{Z}/p^j, n+1) \to K(\mathbb{Z}_{(p)}, n+2)$ be the Bockstein map for any positive integer j. When $j = 1$, we have $\delta_1 = \Delta_0$.

Definition 3-2. (Fundamental BP cohomology classes for Eilenberg–Mac Lane spaces) For any non-negative integer $n \geq 0$, the pull-back of the BP fundamental class $\theta\langle n\rangle$ for the space $\underline{BP}\langle n\rangle_{2(1+p+\cdots+p^n)}$ by the map $\Delta_n \circ \cdots \circ \Delta_1$ is denoted by $\vartheta = \vartheta_{n+2}$ and is called the BP *fundamental class* for $K(\mathbb{Z}_{(p)}, n+2)$:

$$\begin{aligned}\vartheta = (\Delta_n \circ \cdots \circ \Delta_1)^*\theta\langle n\rangle = \theta\langle n\rangle \circ \Delta_n \circ \cdots \circ \Delta_1 \\ \in BP^{2(1+p+\cdots+p^n)}\big(K(\mathbb{Z}_{(p)}, n+2)\big).\end{aligned} \tag{3-6}$$

The pull-back $\delta_j^*\vartheta \in BP^{2(1+p+\cdots+p^n)}\big(K(\mathbb{Z}/p^j, n+1)\big)$ of ϑ by the Bockstein map δ_j, denoted by $\vartheta^{(j)} = \vartheta^{(j)}_{n+1}$, is called the BP *fundamental class* for $K(\mathbb{Z}/p^j, n+1)$.

These BP fundamental classes for Eilenberg–Mac Lane spaces are nontrivial.

Proposition 3-3. *Let $\rho_* : BP^*\big(K(\mathbb{Z}_{(p)}, n+2)\big) \to H\mathbb{Z}_p^*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ be the Thom map. Then the image of the BP fundamental class ϑ_{n+2} under ρ_* is nontrivial, and it is given by*

$$\rho_*(\vartheta_{n+2}) = Q_nQ_{n-1}\cdots Q_1\tau_{n+2} \neq 0 \in H\mathbb{Z}_p^{2(1+p+\cdots+p^n)}\big(K(\mathbb{Z}_{(p)}, n+2)\big), \tag{3-7}$$

where $\tau_{n+2} \in H\mathbb{Z}_p^{n+2}\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ is the mod-p cohomology fundamental class. Similarly, under the Thom map $\rho_* : BP^*\big(K(\mathbb{Z}/p^j, n+1)\big) \to H\mathbb{Z}_p^*\big(K(\mathbb{Z}/p^j, n+1)\big)$, the image of the BP fundamental class $\vartheta^{(j)}_{n+1}$ for $K(\mathbb{Z}/p^j, n+1)$ is given by

$$\rho_*\big(\vartheta^{(j)}_{n+1}\big) = Q_nQ_{n-1}\cdots Q_1\delta_j\iota_{n+1} \neq 0 \in H\mathbb{Z}_p^{2(1+p+\cdots+p^n)}\big(K(\mathbb{Z}/p^j, n+1)\big), \tag{3-8}$$

where $\iota_{n+1} \in H\mathbb{Z}_p^{n+1}\big(K(\mathbb{Z}/p^j, n+1)\big)$ is the mod-p cohomology fundamental class.

Proof. Using the identity $\rho \circ \theta\langle n\rangle = \rho^{\langle n\rangle}$ and the diagram (3-2), we see that the cohomology class $\rho_*(\vartheta_{n+2})$ is represented by a map $\rho \circ \theta\langle n\rangle \circ \Delta_n \circ \cdots \circ \Delta_1 = Q_n \circ \cdots \circ Q_1 \circ \rho^{\langle 0\rangle}$, up to multiplication by a nonzero element $\lambda \in \mathbb{Z}_p$. Here,

$$\rho^{\langle 0\rangle} : K(\mathbb{Z}_{(p)}, n+2) = \underline{BP}\langle 0\rangle_{n+2} \to \underline{BP}\langle -1\rangle_{n+2} = K(\mathbb{Z}/p, n+2)$$

is a mod-p cohomology fundamental class. Letting the nonzero multiplicative factor λ be absorbed in this fundamental class, we have $\rho_*(\vartheta_{n+2}) = Q_n\cdots Q_1\tau_{n+2}$, where $\tau_{n+2} = \lambda\rho^{\langle 0\rangle}$. Similarly,

$$\rho_*\big(\vartheta^{(j)}\big) = \delta_j^*\rho_*(\vartheta) = \delta_j^*(Q_n\cdots Q_1\tau_{n+2}) = Q_n\cdots Q_1\delta_j^*(\tau_{n+2}) = Q_n\cdots Q_1\delta_j\iota_{n+1}.$$

The elements $Q_n\cdots Q_1\tau_{n+2}$ and $Q_n\cdots Q_1\delta_j\iota_{n+1}$ are nonzero polynomial generator in the mod p cohomology ring from the description given in Theorems 2-1 when p is an odd prime, and they are squares of polynomial generators when $p = 2$ by Theorem 2-2. (See also the remark right after Lemma 3-4 below.) This completes the proof of Proposition 3-3. □

For the description of our next results, we need some notation. Let $\mathcal{S}_n^+$ be the set of all sequences of n strictly increasing positive integers. Namely,

$$\mathcal{S}_n^+ = \{(s_1, s_2, \ldots, s_n) \in \mathbb{Z}^n \mid 0 < s_1 < s_2 < \cdots < s_n\}. \tag{3-9}$$

To any such sequence $S = (s_1, s_2, \ldots, s_n)$, we put $Q_S = Q_{s_n}Q_{s_{n-1}}\cdots Q_{s_1}$. We also associate a sequence $R(S) = \sum_{j=1}^n p^j\Delta_{s_j-j}$ to each such sequence $S \in \mathcal{S}_n^+$, and the corresponding Steenrod reduced power operation is denoted by $\mathcal{P}^{R(S)}$. We quote another result from [T].

Lemma 3-4 [T]**.** *Let $n \geq 1$. For any sequence $S \in S_n^+$, we have*

(3-10)
$$\mathcal{P}^{R(S)} Q_n \cdots Q_1 \tau_{n+2} = Q_S \tau_{n+2} \neq 0 \in H\mathbb{Z}_p^{2(1+p^{s_1}+\cdots+p^{s_n})}\big(K(\mathbb{Z}_{(p)}, n+2)\big),$$
$$\mathcal{P}^{R(S)} Q_n \cdots Q_1 \delta_j \iota_{n+1} = Q_S \delta_j \iota_{n+1} \neq 0 \in H\mathbb{Z}_p^{2(1+p^{s_1}+\cdots+p^{s_n})}\big(K(\mathbb{Z}/p^j, n+1)\big).$$

Remark. From the paragraph right after Theorem 2-1, we know that when p is an odd prime, $Q_S \tau_{n+2}$ is indecomposable and is a polynomial generator. When $p = 2$, the formula (2-13) implies that

$$Q_S \tau_{n+2} = \big(Q_{s_1-1} \cdots Q_{s_{n-1}-1} \mathcal{P}^{\Delta_{s_n-1}} \tau_{n+2}\big)^2, \tag{3-11}$$

where the element inside the parentheses on the right hand side is a polynomial generator of the mod-2 cohomology ring of $K(\mathbb{Z}_{(2)}, n+2)$ described in (2-14). A similar statement holds for $Q_S \delta_j \iota_{n+1}$.

The algebras of stable cohomology operations $BP^*(BP)$ and $H\mathbb{Z}_p^*(H\mathbb{Z}_p)$ have Hopf algebra conjugation maps

$$c : BP^*(BP) \to BP^*(BP), \qquad c : H\mathbb{Z}_p^*(H\mathbb{Z}_p) \to H\mathbb{Z}_p^*(H\mathbb{Z}_p).$$

These maps are anti-automorphisms. By a result of Zahler [Z, Lemma 3-7], we have the following commutative diagram for any BP operation r_R:

$$\begin{array}{ccc} BP^*(X) & \xrightarrow{c(r_R)} & BP^*(X) \\ {\scriptstyle \rho_*}\big\downarrow & & \big\downarrow{\scriptstyle \rho_*} \\ H\mathbb{Z}_p^*(X) & \xrightarrow{\mathcal{P}^R} & H\mathbb{Z}_p^*(X). \end{array} \tag{3-12}$$

Theorem 3-5. *Let p be a fixed prime, even or odd. For each strictly increasing sequence of n positive integers $S \in \mathcal{S}_n^+$, let $\vartheta_S = c(r_{R(S)})\vartheta \in BP^*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$. Then $|\vartheta_S| = 2(1+p^{s_1}+\cdots+p^{s_n})$ and the Thom map $\rho_* : BP^*\big(K(\mathbb{Z}_{(p)}, n+2)\big) \to H\mathbb{Z}_p^*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ is such that*

$$\rho_*\big(\vartheta_S\big) = Q_S \tau_{n+2} \neq 0 \in H\mathbb{Z}_p^*\big(K(\mathbb{Z}_{(p)}, n+2)\big). \tag{3-13}$$

Consequently, the image of the Thom map contains a polynomial subalgebra generated by the above elements:

$$\operatorname{Im} \rho_* \supset \mathbb{Z}_p[Q_S \tau_{n+2} \mid S \in \mathcal{S}_n^+]. \tag{3-14}$$

Proof. By (3-12) and Lemma 3-4, we have $\rho_*\big(\vartheta_S\big) = \rho_*\big(c(r_{R(S)})\vartheta\big) = \mathcal{P}^{R(S)}\rho_*(\vartheta) = \mathcal{P}^{R(S)} Q_n \cdots Q_1 \tau_{n+2} = Q_S \tau_{n+2}$ for any $S \in S_n^+$. This shows (3-13). By Theorems 2-1 and 2-2, these elements generate a polynomial subalgebra of $H\mathbb{Z}_p^*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ for any prime p, even or odd. From this, (3-14) follows. □

We remark that there are many elements in $BP^*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ which map to $Q_S \tau_{n+2}$ under the Thom map. The above choice of elements ϑ_S is one possible choice. Pulling back the above result by the Bockstein map $\delta_j : K(\mathbb{Z}/p^j, n+1) \to K(\mathbb{Z}_{(p)}, n+2)$, we obtain the following result.

Theorem 3-6. *For each $S \in \mathcal{S}_n^+$, let $\vartheta_S^{(j)} = c\big(r_{R(S)}\big)\vartheta^{(j)} \in BP^*\big(K(\mathbb{Z}/p^j, n+1)\big)$. Then its degree is given by $|\vartheta_S^{(j)}| = 2(1+p^{s_1}+\cdots+p^{s_n})$ and, under the Thom map $\rho_* : BP^*\big(K(\mathbb{Z}/p^j, n+1)\big) \to H\mathbb{Z}_p^*\big(K(\mathbb{Z}/p^j, n+1)\big)$, we have*

$$\rho_*\big(\vartheta_S^{(j)}\big) = Q_S \delta_j \iota_{n+1} \neq 0 \in H\mathbb{Z}_p^*\big(K(\mathbb{Z}/p^j, n+1)\big). \tag{3-15}$$

The image of the BP Thom map contains a polynomial subalgebra generated by the above elements:

$$\operatorname{Im}\rho_* \supset \mathbb{Z}_p[Q_S\delta_j\iota_{n+1} \mid S \in \mathcal{S}_n^+]. \tag{3-16}$$

In Theorem 7-2, we will show that the inclusion relations (3-14) and (3-16) are actually equalities.

Due to the fact described in the remark right after Lemma 3-4, when $p = 2$, all elements in the polynomial subalgebras in the right hand sides of (3-14) and (3-16) are squares. Namely, when $p = 2$, we have

$$\mathbb{Z}_2[Q_S\tau_{n+2} \mid S \in \mathcal{S}_n^+] \subset \big[H\mathbb{Z}_2^*\big(K(\mathbb{Z}_{(2)}, n+2)\big)\big]^2,$$
$$\mathbb{Z}_2[Q_S\delta_j\iota_{n+1} \mid S \in \mathcal{S}_n^+] \subset \big[H\mathbb{Z}_2^*\big(K(\mathbb{Z}/2^j, n+1)\big)\big]^2.$$

In generalized cohomology theories, we often have to consider infinite sums of elements. To discuss their convergence, we introduce a topology in cohomology groups. For generalized cohomology theories for CW complexes, the topology coming from the skeletal filtration is convenient. Let $\mathbf{E}$ be a spectrum and let X be a CW complex. For any non-negative integer $r \geq 0$, let

$$F^{(r)}\big(\mathbf{E}^*(X)\big) = \operatorname{Ker}\big(\iota_{r-1}^* : \mathbf{E}^*(X) \to \mathbf{E}^*(X^{(r-1)})\big), \tag{3-17}$$

where $X^{(r-1)}$ is the $r-1$ skeleton of X, and $\iota_{r-1} : X^{(r-1)} \to X$ is the inclusion map. This defines a decreasing filtration $\mathbf{E}^*(X) = F^{(0)} \supset F^{(1)} \supset \cdots \supset F^{(\ell)} \supset \cdots \supset \bigcap_{\ell\geq 0} F^{(\ell)} = F^{(\infty)}$. Here $F^{(\infty)}$ is the group of elements of infinite filtration, or phantom elements. The phantom elements may be present in BP cohomology. This skeletal filtration has a multiplicative property: $F^{(r)} \cdot F^{(s)} \subset F^{(r+s)}$ for any $r, s \geq 0$. This can be seen easily using the fact that the restriction of the diagonal map $\Delta : X \to X \times X$ to $X^{(r+s-1)}$ is homotopic to a map into $X^{(r-1)} \times X \cup X \times X^{(s-1)}$.

Now we examine the filtration of our elements ϑ_S in the BP cohomology ring of Eilenberg–Mac Lane spaces.

Lemma 3-7. *Let X be a CW complex. Any element $x \in BP^r(X)$ of degree r is in the filtration r, that is, $x \in F^{(r)}\big(BP^*(X)\big)$. In particular, for each $S = (s_1, \ldots, s_n) \in \mathcal{S}_n^+$, the element $\vartheta_S \in BP^{2(1+p^{s_1}+\cdots+p^{s_n})}\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ is such that*

$$\vartheta_S \in F^{(2(1+p^{s_1}+\cdots+p^{s_n}))}\big(BP^*K(\mathbb{Z}_{(p)}, n+2)\big). \tag{3-18}$$

Proof. Let $\underline{BP}_r$ be the rth space in the Ω-spectrum for BP theory. It is $(r-1)$-connected. So the map from the $r-1$ skeleton of X to $\underline{BP}_r$ is always null homotopic, and we have $[X^{(r-1)}, \underline{BP}_r] = 0$. Thus for any $x \in BP^r(X) = [X, \underline{BP}_r]$ its restriction to the $r-1$ skeleton vanishes. Hence $x \in F^{(r)}\big(BP^*(X)\big)$. (3-18) is an immediate consequence of this. □

4. Kronecker Pairings for Connective Morava K Theory of Eilenberg–Mac Lane Spaces

In this section, we study the images of BP cohomology classes ϑ_S and $\vartheta_S^{(j)}$ in connective Morava K cohomology theories in terms of Kronecker pairings in connective Morava K theories. Relevant elements in $k(m)$ homology theories are produced by $\circ$-products. See Propositions 4-5 and 4-7 for precise statements.

First we prove one elementary lemma which relates Kronecker pairings in various generalized (co)homology theories. Let $\mathbf{E}$ be a ring spectrum with multiplication

map $\mu : \mathbf{E} \wedge \mathbf{E} \to \mathbf{E}$. Also, let Σ^r be the sphere spectrum suspended r times. Let $\mathbf{X}$ be a spectrum. For a cohomology class $\alpha \in \mathbf{E}^r(\mathbf{X})$ represented by a spectra map $\alpha : \mathbf{X} \to \Sigma^r \mathbf{E}$ and a homology class $h \in \mathbf{E}_s(\mathbf{X})$ represented by a spectra map $h : \Sigma^s \to \mathbf{E} \wedge \mathbf{X}$, their Kronecker pairing $\langle \alpha, h \rangle_{\mathbf{E}} \in \mathbf{E}_{s-r} = \mathbf{E}^{r-s}$ in $\mathbf{E}$ theory is defined by the following composition of spectra maps:

$$\langle \alpha, h \rangle_E : \Sigma^s \xrightarrow{h} \mathbf{E} \wedge \mathbf{X} \xrightarrow{1 \wedge \alpha} \mathbf{E} \wedge \Sigma^r \mathbf{E} \xrightarrow{\Sigma^r \mu} \Sigma^r \mathbf{E}. \tag{4-1}$$

Lemma 4-1. *Let $\eta : \mathbf{E} \to \mathbf{G}$ be a ring map between ring spectra. Then for any elements $\alpha \in \mathbf{E}^r(\mathbf{X})$ and $h \in \mathbf{E}_s(\mathbf{X})$, we have*

$$\langle \eta_*(\alpha), \eta_*(h) \rangle_{\mathbf{G}} = \eta_*\big(\langle \alpha, h \rangle_{\mathbf{E}}\big) \in \mathbf{G}_{s-r} = \mathbf{G}^{r-s}. \tag{4-2}$$

Proof. For $\alpha : \mathbf{X} \to \Sigma^r \mathbf{E}$ and $h : \Sigma^s \to \mathbf{E} \wedge \mathbf{X}$, elements $\eta_*(\alpha)$ and $\eta_*(h)$ are represented by spectra maps $\eta_*(\alpha) : \mathbf{X} \xrightarrow{\alpha} \Sigma^r \mathbf{E} \xrightarrow{\Sigma^r \eta} \Sigma^r \mathbf{G}$ and $\eta_*(h) : \Sigma^s \xrightarrow{h} \mathbf{E} \wedge \mathbf{X} \xrightarrow{\eta \wedge 1} \mathbf{G} \wedge \mathbf{X}$. Now consider the following commutative diagram:

$$\begin{array}{ccccccc}
\Sigma^r & \xrightarrow{h} & \mathbf{E} \wedge \mathbf{X} & \xrightarrow{1 \wedge \alpha} & \mathbf{E} \wedge \Sigma^r \mathbf{E} & \xrightarrow{\Sigma^r \mu} & \Sigma^r \mathbf{E} \\
\| & & \downarrow{\scriptstyle \eta \wedge 1} & & \downarrow{\scriptstyle \eta \wedge \Sigma^r \eta} & & \downarrow{\scriptstyle \Sigma^r \eta} \\
\Sigma^r & \xrightarrow{\eta_*(h)} & \mathbf{G} \wedge \mathbf{X} & \xrightarrow{1 \wedge \eta_*(\alpha)} & \mathbf{G} \wedge \Sigma^r \mathbf{G} & \xrightarrow{\Sigma^r \mu} & \Sigma^r \mathbf{G}.
\end{array}$$

Here, μ denotes multiplication maps in both ring spectra $\mathbf{E}$ and $\mathbf{G}$. The composition of the upper horizontal arrows is $\langle \alpha, h \rangle_{\mathbf{E}} \in \mathbf{E}^{r-s}$ and the composition in the lower horizontal arrows is $\langle \eta_*(\alpha), \eta_*(h) \rangle_{\mathbf{G}} \in \mathbf{G}^{r-s}$. Thus the commutativity of the above diagram proves (4-2). $\square$

The Kronecker pairing we are interested in is the one for the connective Morava K theory spectrum $k(m)$, that is, $\langle\ ,\ \rangle_{k(m)} : k(m)^*(X) \otimes k(m)_*(X) \to k(m)^*$ for $m \geq 1$. The coefficient ring of this theory is given by $k(m)^* = \mathbb{Z}_p[v_m]$, where $|v_m| = -2(p^m - 1)$ in cohomology theory and $|v_m| = 2(p^m - 1)$ in homology theory. If $x = x^{k(m)} \in k(m)^2(\mathbb{CP}^\infty)$ is a complex orientation, we have $k(m)^*(\mathbb{CP}^\infty) = k(m)^*[[x]]$. The formal group law for $k(m)$ theory has height m with the property $[p]_{k(m)}(x) = v_m x^{p^m}$. For each non-negative integer i, let $\beta_i \in k(m)_{2i}(\mathbb{CP}^\infty)$ be the homology element dual to x^i with respect to the Kronecker pairing. Then it follows that $k(m)_*(\mathbb{CP}^\infty) = \bigoplus_{i \geq 0} k(m)_* \beta_i$. It is known that as an algebra, $k(m)$ homology of $\mathbb{CP}^\infty$ is generated by elements $\beta_{(i)} = \beta_{p^i}$ for $i \geq 0$ with only the relations given by

$$\beta^p_{(m+i-1)} = v_m^{p^i} \beta_{(i)} \quad \text{for all } i \geq 0. \tag{4-3}$$

Here, $\beta_{(i)} = 0$ when $i < 0$. See [RW1], [RW2] for more details. For our purpose, we need to calculate the $k(m)$ homology of $K(\mathbb{Z}/p^j, 1)$. The Gysin sequence for the fibration $K(\mathbb{Z}/p^j, 1) \xrightarrow{\delta_j} \mathbb{CP}^\infty \xrightarrow{p^j} \mathbb{CP}^\infty$ is of the following form since $k(m)_*(\mathbb{CP}^\infty)$ is even dimensional:

$$\begin{aligned} 0 \to k(m)_{2i}\big(K(\mathbb{Z}/p^j, 1)\big) \xrightarrow{\delta_{j*}} k(m)_{2i}\big(\mathbb{CP}^\infty\big) \xrightarrow{\Phi = \cap [p^j](x)} k(m)_{2i-2}\big(\mathbb{CP}^\infty\big) \\ \xrightarrow{\partial} k(m)_{2i-1}\big(K(\mathbb{Z}/p^j, 1)\big) \to 0. \end{aligned} \tag{4-4}$$

Here $[p^j]_{k(m)}(x)$ is the $k(m)$ theory Euler class of the fibration and is given by

$$[p^j]_{k(m)}(x) = \underbrace{[p]\circ[p]\circ\cdots\circ[p]}_{j}(x) = v_m^{(1+p^m+\cdots+p^{m(j-1)})}x^{p^{jm}} = v_m^{\frac{p^{jm}-1}{p^m-1}}x^{p^{jm}}. \tag{4-5}$$

From the Gysin sequence (4-4), $k(m)$ homology of $K(\mathbb{Z}/p^j,1)$ is calculated as follows.

Proposition 4-2. (I) *As a $k(m)_*$-module, the additive structure of the $k(m)$ homology of $K(\mathbb{Z}/p^j,1)$ is given by*

$$k(m)_*\big(K(\mathbb{Z}/p^j,1)\big) = \bigoplus_{i=0}^{p^{jm}-1} k(m)_*\cdot a_i \oplus \bigoplus_{i\geq 0} k(m)_*/\big(v_m^{\frac{p^{jm}-1}{p^m-1}}\big)\cdot c_i, \tag{4-6}$$

where $a_i \in k(m)_{2i}\big(K(\mathbb{Z}/p^j,1)\big)$ and $c_i \in k(m)_{2i+1}\big(K(\mathbb{Z}/p^j,1)\big)$ are uniquely determined by $\delta_{j*}(a_i) = \beta_i$ and $\partial\beta_i = c_i$ in the Gysin sequence (4-4). *The element a_i is dual to $(x^{(j)})^i$, where $x^{(j)} = \delta_j^*(x) \in k(m)^2\big(K(\mathbb{Z}/p^j,1)\big)$.*

(II) *As a $k(m)_*$-algebra, the even dimensional subalgebra $k(m)_{2*}\big(K(\mathbb{Z}/p^j,1)\big)$ is generated by elements $a_{(i)} = a_{p^i}$ for $0 \leq i < mj$ with relations*

$$a_{(m+i-1)}^p = v_m^{p^i}a_{(i)}, \quad \text{where} \quad a_{(i)} = 0 \quad \text{when} \quad i < 0. \tag{4-7}$$

Products of odd degree elements vanish, namely, $c_k \cdot c_\ell = 0$ for any $k,\ell \geq 0$.

(III) *Let $x = x^H = \rho_*\big(x^{(j)}\big) \in H\mathbb{Z}_p^2\big(K(\mathbb{Z}/p^j,1)\big)$ be the complex orientation induced from $k(m)$ theory. Then the Kronecker pairing in $H\mathbb{Z}_p$ theory is given by $\langle x^k, \rho_*\big(a_i\big)\rangle_{H\mathbb{Z}_p} = \delta_{ki}$ for any $0 \leq i < p^{mj}$. In particular,*

$$\langle Q_k\iota_1, \rho_*\big(a_{(i)}\big)\rangle_{H\mathbb{Z}_p} = \delta_{ki} \quad \text{for all} \quad 0 \leq i,k < jm. \tag{4-8}$$

where $\iota_1 \in H\mathbb{Z}_p^1\big(K(\mathbb{Z}/p^j,1)\big)$ is the fundamental class such that $\delta_j\iota_1 = x$.

Proof. First we examine the behaviour of the map Φ in the Gysin sequence (4-4). Since $\beta_i \in k(m)_{2i}(\mathbb{CP}^\infty)$ is dual to $x^i \in k(m)^{2i}(\mathbb{CP}^\infty)$, we have $\beta_k \cap x^\ell = \beta_{k-\ell}$, where $\beta_{k-\ell} = 0$ when $k < \ell$. Hence Φ is given by

$$\Phi\big(\beta_{p^{jm}+i}\big) = \beta_{p^{jm}+i} \cap [p^j]_{k(m)}(x) = \beta_{p^{jm}+i} \cap \big(v_m^{\frac{p^{jm}-1}{p^m-1}}x^{p^{jm}}\big) = v_m^{\frac{p^{jm}-1}{p^m-1}}\beta_i$$

for $i \geq 0$, and $\Phi(\beta_k) = 0$ for $k < p^{jm}$. Thus, the kernel of Φ is a $k(m)_*$ free module generated by β_k with $0 \leq k < p^{jm}$. Let $a_i \in k(m)_{2i}\big(K(\mathbb{Z}/p^j,1)\big)$ be the unique element such that $\delta_{j*}\big(a_i\big) = \beta_i$ for $0 \leq i < p^{jm}$. Also let $c_i = \partial\beta_i \in k(m)_{2i+1}\big(K(\mathbb{Z}/p^j,1)\big)$. From the Gysin sequence, the $k(m)_*$ additive structure of $k(m)$ homology of $K(\mathbb{Z}/p^j,1)$ is given by (4-6). Since β_i is dual to x^i, it follows that a_i is dual to $\big(x^{(j)}\big)^i$, where $x^{(j)} = \delta_j^*(x) \in k(m)^2\big(K(\mathbb{Z}/p^j,1)\big)$.

Since δ_j is an H space map, δ_{j*} is a Hopf algebra monomorphism. Pulling back the relation (4-3), we get the relation (4-7). Since all the odd degree elements are v_m-torsion and all the even degree elements are v_m-torsion free, the product of two odd degree elements must vanish. This proves (II).

Since a_i is dual to $(x^{(j)})^i$, from Lemma 4-1 we see that $\rho_*(a_i)$ is dual to $(x^H)^i = \rho_*\big((x^{(j)})^i\big)$. Since $Q_k\iota_1 = \mathcal{P}^{p^{k-1}}\cdots\mathcal{P}^p\mathcal{P}\delta_j\iota_1 = x^{p^k}$ for $k \geq 0$, (4-8) follows. $\square$

Now we let $j = 1$ and we consider the mod-p Eilenberg–Mac Lane space. For $i \geq 0$, let $a_i \in H\mathbb{Z}_{p\,2i}\big(K(\mathbb{Z}/p,1)\big)$ be the element dual to $(x^{(1)})^i$. Then the mod-p

homology ring contains elements $a_{(i)} = a_{p^i}$ for all $i \geq 0$. The cup product of $n+1$ cohomology elements of degree 1 in mod-p theory is induced by the map

$$\mu : K(\mathbb{Z}/p,1) \times K(\mathbb{Z}/p,1) \times \cdots \times K(\mathbb{Z}/p,1) \to K(\mathbb{Z}/p,n+1), \tag{4-9}$$

which factors through a smash product. For any sequence of strictly increasing positive integers $S = (s_1, s_2, \ldots, s_n) \in \mathcal{S}_n^+$, the image of $a_{(0)} \otimes a_{(s_1)} \otimes \cdots \otimes a_{(s_n)}$ by the induced map μ_* in mod-p theory is denoted by $a(S) = a_{(0)} \circ a_{(s_1)} \circ \cdots \circ a_{(s_n)} \in H\mathbb{Z}_{p_*}\big(K(\mathbb{Z}/p,n+1)\big)$. The degree of this element is $2(1+p^{s_1}+\cdots+p^{s_n})$. The basic property of these elements is the following.

Lemma 4-3. *The Kronecker pairing in $H\mathbb{Z}_p$ theory for $K(\mathbb{Z}/p,n+1)$ is such that*

$$\langle Q_S \delta \iota_{n+1}, a(S')\rangle_{H\mathbb{Z}_p} = \delta_{SS'} \quad \textit{for any} \quad S, S' \in \mathcal{S}_n^+, \tag{4-10}$$

where $\iota_{n+1} \in H\mathbb{Z}_p^{n+1}\big(K(\mathbb{Z}/p,n+1)\big)$ is the fundamental class and δ is the Bockstein map.

Proof. Let $S = (s_1, s_2, \ldots, s_n)$ and $S' = (s_1', s_2', \ldots, s_n')$ be sequences in $\mathcal{S}_n^+$. Since $\mu^*(\iota_{n+1}) = \iota_1 \otimes \iota_1 \otimes \cdots \otimes \iota_1$ with $n+1$ tensor factors, we have

$$\begin{aligned}\langle Q_S \delta \iota_{n+1}, a(S')\rangle &= \langle Q_S \delta(\iota_1 \otimes \cdots \otimes \iota_1)\,,\, a_{(0)} \otimes a_{(s_1')} \otimes \cdots \otimes a_{(s_n')}\rangle \\ &= \sum_{\sigma \in \mathfrak{S}_n} \pm \langle \delta\iota_1, a_{(0)}\rangle \langle Q_{s_{\sigma(1)}}\iota_1, a_{(s_1')}\rangle \cdots \langle Q_{s_{\sigma(n)}}\iota_1, a_{(s_n')}\rangle.\end{aligned}$$

Since both S and S' are strictly increasing sequences and $\langle Q_k \iota_1, a_{(\ell)}\rangle = \delta_{k\ell}$ for $k, \ell \geq 0$, the nontrivial contribution in the above summation can only come from the term with $\sigma = 1$, and in this case, the pairing gives $\delta_{SS'}$. This proves (4-10). □

Now we consider $\circ$-products of elements in $k(m)_*\big(K(\mathbb{Z}/p,1)\big)$. Since only those $a_{(i)}$ with $0 \leq i < m$ are nontrivial in $k(m)_*\big(K(\mathbb{Z}/p,1)\big)$, we consider the following subset of $\mathcal{S}_n^+$ for any integer m such that $0 \leq n < m$:

$$\mathcal{S}_n^+(m) = \{(s_1, s_2, \ldots, s_n) \in \mathbb{Z}_+^n \mid 0 < s_1 < s_2 < \cdots < s_n < m\} \subset \mathcal{S}_n^+. \tag{4-11}$$

When $n = 0$, we set $\mathcal{S}_0^+(m) = \emptyset$. Applying $k(m)$ homology to (4-9), we obtain $(n+1)$-fold $\circ$-products. For any $S = (s_1, s_2, \ldots, s_n) \in \mathcal{S}_n^+(m)$, let $a(S) = a_{(0)} \circ a_{(s_1)} \circ a_{(s_2)} \circ \cdots \circ a_{(s_n)}$ be an element in $k(m)$ homology of $K(\mathbb{Z}/p,n+1)$ of degree $d(S) = 2(1+p^{s_1}+\cdots+p^{s_n})$. Whether we are considering elements $a(S)$ in $H\mathbb{Z}_p$ theory or in $k(m)$ theory should be clear from the context. The following observation will be useful later.

Lemma 4-4. *Let p be any prime. Let $0 \leq n < m$. Then for any $S, S' \in \mathcal{S}_n^+(m)$, we have $|d(S) - d(S')| < 2(p^m - 1) - 1$.*

Proof. When S ranges over all $S \in \mathcal{S}_n^+(m)$, the largest possible $d(S)$ occurs when $S = S_{\max} = (m-n, m-n+1, \ldots, m-1)$, and the smallest possible $d(S)$ occurs when $S = S_{\min} = (1, 2, \ldots, n)$. The difference of the corresponding degrees is

$$d(S_{\max}) - d(S_{\min}) = 2\frac{p^m - 1}{p-1} - 2(p + p^2 + \cdots + p^{m-n-1}) - 2\frac{p^{n+1}-1}{p-1}.$$

Since the third term on the right hand side is at least 2, this is strictly less than $2(p^m-1)/(p-1) - 1$, which is less than or equal to $2(p^m-1)-1$ for any prime p, even or odd. This completes the proof. □

From Theorem 3-6, for any $S \in \mathcal{S}_n^+$ we have nontrivial BP cohomology elements $\vartheta_S \in BP^*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ and $\vartheta_S^{(1)} \in BP^*\big(K(\mathbb{Z}/p, n+1)\big)$. Let their images under the map $\rho_{(m)} : BP \to k(m)$ be also denoted by $\vartheta_S = \vartheta_S^{k(m)} \in k(m)^*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ and $\vartheta_S^{(1)} \in k(m)^*\big(K(\mathbb{Z}/p, n+1)\big)$. All of these elements are nontrivial for any $S \in \mathcal{S}_n^+$ because their images under the map $\rho^{(m)} : k(m) \to H\mathbb{Z}_p$ into mod-p cohomology are nontrivial by Theorem 3-6. Let $b(S) = \delta_*\big(a(S)\big) \in k(m)_*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ for $S \in \mathcal{S}_n^+(m)$, where $\delta : K(\mathbb{Z}/p, n+1) \to K(\mathbb{Z}_{(p)}, n+2)$ is the Bockstein map.

Using Lemma 4-3, we deduce some properties of $k(m)$ Kronecker pairings for mod-p and integral Eilenberg–Mac Lane spaces. The case for mod-p^j Eilenberg–Mac Lane spaces is more complicated and it is treated in Proposition 4-7 below.

Proposition 4-5. (I) *Let $m \geq 1$. The Kronecker pairing in $k(m)$ theory of the mod-p Eilenberg–Mac Lane space*

$$\langle\ ,\ \rangle_{k(m)} : k(m)^*\big(K(\mathbb{Z}/p, n+1)\big) \otimes k(m)_*\big(K(\mathbb{Z}/p, n+1)\big) \to k(m)_*$$

satisfies the following identities:

$$\langle \vartheta_S^{(1)}, a(S')\rangle_{k(m)} = \delta_{SS'} \in k(m)_*, \tag{4-12}$$

for any $S, S' \in \mathcal{S}_n^+(m)$. In particular, $a(S) \neq 0 \in k(m)_*\big(K(\mathbb{Z}/p, n+1)\big)$ for any $S \in \mathcal{S}_n^+(m)$.

(II) *The Kronecker pairing in $k(m)$ theory of the $\mathbb{Z}_{(p)}$ Eilenberg–Mac Lane space $\langle\ ,\ \rangle_{k(m)} : k(m)^*\big(K(\mathbb{Z}_{(p)}, n+2)\big) \otimes k(m)_*\big(K(\mathbb{Z}_{(p)}, n+2)\big) \to k(m)_*$ is such that*

$$\langle \vartheta_S, b(S')\rangle_{k(m)} = \delta_{SS'} \in k(m)_*, \tag{4-13}$$

for any $S, S' \in \mathcal{S}_n^+(m)$. In particular, $b(S) \neq 0 \in k(m)_*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ for any $S \in \mathcal{S}_n^+(m)$.

Proof. We apply Lemma 4-1 to the $k(m)$ theory Thom map $\rho : k(m) \to H\mathbb{Z}_p$. For any $S, S' \in \mathcal{S}_n^+(m)$, we have

$$\begin{aligned}\rho_*\big(\langle \vartheta_S^{(1)}, a(S')\rangle_{k(m)}\big) &= \big\langle \rho_*\big(\vartheta_S^{(1)}\big), \rho_*\big(a(S')\big)\big\rangle_{H\mathbb{Z}_p} \\ &= \big\langle Q_S\delta\iota_{n+1}, a_{(0)} \circ a_{(s_1')} \circ \cdots \circ a_{(s_n')}\big\rangle_{H\mathbb{Z}_p} \\ &= \delta_{SS'}.\end{aligned}$$

The second equality above is due to (3-15) for the first entry, and commutativity of the Thom map and the $\circ$-products for the second entry. The third equality is due to (4-10) in Lemma 4-3. Since the kernel of $\rho_* : k(m)_* \to H\mathbb{Z}_{p*} = \mathbb{Z}/p$ is the ideal (v_m) generated by v_m, we have $\langle \vartheta_S^{(1)}, a(S')\rangle_{k(m)} \in \delta_{SS'} + (v_m)$. However, the degree of $\langle \vartheta_S^{(1)}, a(S')\rangle_{k(m)} \in k(m)_*$ is $d(S) - d(S')$, which is strictly less than $|v_m| = 2(p^m - 1)$ by Lemma 4-4. Thus we must have (4-12) for degree reasons.

For the $\mathbb{Z}_{(p)}$ Eilenberg–Mac Lane space, we have

$$\langle \vartheta_S, b(S')\rangle_{k(m)} = \langle \vartheta_S, \delta_*\big(a(S')\big)\rangle_{k(m)} = \langle \delta^*(\vartheta_S), a(S')\rangle_{k(m)} = \langle \vartheta_S^{(1)}, a(S')\rangle_{k(m)} = \delta_{SS'},$$

by our previous calculation. This proves (4-13). □

In [RW2], Morava K theories of Eilenberg–Mac Lane spaces are calculated. For our purpose, we need to reprove some of their results in the context of *connective* Morava K theory. However, most of the statements in [RW2] also hold in the connective $k(m)$ theory, because we will be dealing with v_m-torsion free elements.

Let $\mathbb{Z}/p^j \to \mathbb{Z}/p^{j+1}$ be the inclusion map induced by multiplication by p, and $\mathbb{Z}/p^{j+1} \to \mathbb{Z}/p^j$ be the projection map. Let α, β be induced maps:

$$K(\mathbb{Z}/p^j, n+1) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} K(\mathbb{Z}/p^{j+1}, n+1). \tag{4-14}$$

These maps fit into the following homotopy commutative diagram:

$$\begin{array}{ccccc} K(\mathbb{Z}_{(p)}, n+2) & = & K(\mathbb{Z}_{(p)}, n+2) & \xrightarrow{p} & K(\mathbb{Z}_{(p)}, n+2) \\ \uparrow{\scriptstyle \delta_j} & & \uparrow{\scriptstyle \delta_{j+1}} & & \uparrow{\scriptstyle \delta_j} \\ K(\mathbb{Z}/p^j, n+1) & \xrightarrow{\alpha} & K(\mathbb{Z}/p^{j+1}, n+1) & \xrightarrow{\beta} & K(\mathbb{Z}/p^j, n+1). \end{array} \tag{4-15}$$

From this diagram, and the definition of elements $a_i \in k(m)_{2i}\big(K(\mathbb{Z}/p^j, 1)\big)$ in Proposition 4-2, we have

$$\alpha_*(a_i) = a_i \quad \text{for} \quad 0 \le i < p^{jm}, \qquad \beta_*\big(a_{(m+i)}\big) = v_m^{p^i} a_{(i)} \quad \text{for} \quad i < mj. \tag{4-16}$$

These are the same as Lemma 5-7 in [RW2] for the nonconnective Morava K theory of Eilenberg–Mac lane spaces. The reason for the validity of these formulae is that it is not necessary to invert v_m in their proof. Lemma 4-6 below corresponds to Proposition 11.4 in [RW2], in which v_m is set equal to 1 in cyclically graded Morava K theory. In $k(m)$ theory, we have to count the power of v_m. Let $\Delta = \sum_{i=1}^n \Delta_i = (1, 1, \dots, 1)$ be a sequence of 1's repeated n times. For the sake of completeness, we give the proof of the next lemma, demonstrating that it is not necessary to invert v_m for the proof.

Lemma 4-6. *For any sequence $S = (s_1, s_2, \dots, s_n) \in \mathcal{S}_n^+(m)$ and any $j \ge 2$, the iterated map $\alpha_*^{j-1} : k(m)_*\big(K(\mathbb{Z}/p, n+1)\big) \to k(m)_*\big(K(\mathbb{Z}/p^j, n+1)\big)$ is such that*

$$v_m^{(p^{s_1}+p^{s_2}+\cdots+p^{s_n})(1+p^m+\cdots+p^{m(j-2)})} \alpha_*^{j-1}\big(a(S)\big) = a\big(S + m(j-1)\Delta\big). \tag{4-17}$$

Proof. The proof is essentially the same as the proof of Proposition 11.4 in [RW2], and is obtained by using the following homotopy commutative diagram:

$$\begin{array}{ccc} K(\mathbb{Z}/p^j, 1) \times K(\mathbb{Z}/p^{j+1}, n) & \xrightarrow{\alpha \times 1} & K(\mathbb{Z}/p^{j+1}, 1) \times K(\mathbb{Z}/p^{j+1}, n) \\ \downarrow{\scriptstyle 1\times\beta} & & \downarrow{\scriptstyle \mu} \\ K(\mathbb{Z}/p^j, 1) \times K(\mathbb{Z}/p^j, n) & & K(\mathbb{Z}/p^{j+1}, n+1) \\ \downarrow{\scriptstyle \mu} & & \| \\ K(\mathbb{Z}/p^j, n+1) & \xrightarrow{\alpha} & K(\mathbb{Z}/p^{j+1}, n+1). \end{array} \tag{4-18}$$

We apply $k(m)$ homology to the above diagram. Let $j \ge 1$. In the top left corner, consider the following pair of elements: $a_{(0)} \in k(m)_2\big(K(\mathbb{Z}/p^j, 1)\big)$ and $a_{(s_1+mj)} \circ \cdots \circ a_{(s_n+mj)} \in k(m)_*\big(K(\mathbb{Z}/p^{j+1}, n)\big)$ for $S = (s_1, \dots, s_n) \in \mathcal{S}_n^+(m)$.

Using the first formula in (4-16), if we go to the right top corner and then to the right bottom corner, the above pair of elements becomes

$$a_{(0)} \circ a_{(s_1+mj)} \circ \cdots \circ a_{(s_n+mj)} = a(S + mj\Delta) \in k(m)_*\big(K(\mathbb{Z}/p^{j+1}, n+1)\big).$$

On the other hand, at the left bottom corner, the above pair of elements becomes

$$\begin{aligned} &a_{(0)} \circ \beta_*\big(a_{(s_1+mj)} \circ \cdots \circ a_{(s_n+mj)}\big) \\ &\quad = a_{(0)} \circ \big(v_m^{p^{s_1+m(j-1)}} a_{(s_1+m(j-1))} \circ \cdots \circ v_m^{p^{s_n+m(j-1)}} a_{(s_n+m(j-1))}\big) \\ &\quad = v_m^{(p^{s_1}+\cdots+p^{s_n})p^{m(j-1)}} a(S+m(j-1)\Delta). \end{aligned}$$

Here, we used the second formula in (4-16) and the fact that β_* preserves $\circ$-products because β is a ring map. Applying α_* to the above element and using commutativity of the diagram (4-18), we finally obtain the identity

$$v_m^{(p^{s_1}+\cdots+p^{s_n})p^{m(j-1)}} \alpha_*\big(a(S+m(j-1)\Delta)\big) = a(S+mj\Delta).$$

Repeatedly using this identity, we obtain (4-17). □

In the mod-p homology Gysin sequence for the fibration $K(\mathbb{Z}/p^j,1) \xrightarrow{\delta_j} \mathbb{CP}^\infty \xrightarrow{p^j} \mathbb{CP}^\infty$, the Euler class is zero. So we see that ${\delta_j}_*$ is an isomorphism in even degrees in mod-p homology. It follows that $\beta_* = 0$ on the even degree homology groups, due to commutativity of the right square in (4-15) with $n = 1$. One can also show that β_* is an isomorphism in odd degrees using the fact that the connecting homomorphism in the Gysin sequence for mod-p homology of $K(\mathbb{Z}/p^j,1)$ is an isomorphism in odd degree groups. If we apply mod-p homology theory to the diagram (4-18) with $n = 1$, then a diagram chasing using the fact that $\beta_* = 0$ in even degrees shows that

$$a_k \circ a_\ell = 0 \quad \text{in} \quad H\mathbb{Z}_p^*\big(K(\mathbb{Z}/p^j,2)\big) \quad \text{for} \quad j \geq 2, \tag{4-19}$$

for any $k, \ell \geq 1$. Thus, all $\circ$-products in $k(m)_*\big(K(\mathbb{Z}/p^j,n+1)\big)$ map to zero in mod-p homology when $j \geq 2$. However, some $\circ$-product elements are nonzero in $k(m)$ theory of $K(\mathbb{Z}/p^j,n+1)$, as shown in the next proposition.

Proposition 4-7. *Let $\langle\ ,\ \rangle_{k(m)} \colon k(m)^*\big(K(\mathbb{Z}/p^j,n+1)\big) \otimes k(m)_*\big(K(\mathbb{Z}/p^j,n+1)\big) \to k(m)_*$ be the Kronecker pairing in $k(m)$ theory of the mod-p^j Eilenberg–Mac Lane space. Then for any $S, S' \in \mathcal{S}_n^+(m)$, we have*

$$\big\langle \vartheta_S^{(j)}, a\big(S'+m(j-1)\Delta\big)\big\rangle_{k(m)} = \delta_{SS'} \cdot v_m^{(p^{s_1}+p^{s_2}+\cdots+p^{s_n})(1+p^m+\cdots+p^{m(j-2)})}, \tag{4-20}$$

where $S = (s_1, s_2, \ldots, s_n)$. In particular, elements $a\big(S'+m(j-1)\Delta\big)$ are nontrivial in $k(m)_*\big(K(\mathbb{Z}/p^j,n+1)\big)$ for any $S \in \mathcal{S}_n^+(m)$.

Proof. Since $\delta_{j+1} \circ \alpha = \delta_j$ for all $j \geq 1$ by (4-15), we have $\delta_j \circ \alpha^{j-1} = \delta_1 = \delta$. Thus, ${\delta_j}_* \circ \alpha_*^{j-1}\big(a(S')\big) = \delta_*\big(a(S')\big) = b(S')$. This implies that the $k(m)$ Kronecker product is such that

$$\begin{aligned} \big\langle \vartheta_S^{(j)}, \alpha_*^{j-1}\big(a(S')\big)\big\rangle &= \big\langle \delta_j^*(\vartheta_S), \alpha_*^{j-1}\big(a(S')\big)\big\rangle = \big\langle \vartheta_S, {\delta_j}_* \circ \alpha_*^{j-1}\big(a(S')\big)\big\rangle \\ &= \langle \vartheta_S, b(S')\rangle = \delta_{SS'}. \end{aligned}$$

The last identity is due to (4-13). Now, by Lemma 4-6,

$$\begin{aligned} \big\langle \vartheta_S^{(j)}, a\big(S'+m(j-1)\Delta\big)\big\rangle &= v_m^{(p^{s_1}+p^{s_2}+\cdots+p^{s_n})(1+p^m+\cdots+p^{m(j-2)})} \big\langle \vartheta_S^{(j)}, \alpha_*^{j-1}\big(a(S')\big)\big\rangle \\ &= \delta_{SS'} \cdot v_m^{(p^{s_1}+p^{s_2}+\cdots+p^{s_n})(1+p^m+\cdots+p^{m(j-2)})}. \end{aligned}$$

This proves (4-20). □

Although we do not know the precise structure of the connective Morava K theory of Eilenberg–Mac Lane spaces, we do know Morava K theory of these spaces thanks to [RW2]. It turns out that to determine the image of the Thom map from BP cohomology of Eilenberg–Mac Lane spaces to mod-p cohomology, the knowledge of Morava K theory $K(m)^*(\,\cdot\,)$ of these spaces for all m is necessary and sufficient.

5. Morava K Cohomology Rings of Eilenberg–Mac Lane Spaces

In [RW2], the Hopf algebra structure of the Morava K homology of Eilenberg–Mac Lane spaces was determined. In this section, we dualize their calculation and explicitly describe the algebra structure of Morava K *cohomology* of Eilenberg–Mac Lane spaces. Our calculation in this section is primarily directed towards the determination of the precise exponent of algebra generators of $K(m)$ cohomology of mod-p^j Eilenberg–Mac Lane spaces (see Proposition 5-3 below). For this, we only need the additive structure of $K(m)$ homology and the behaviour of Verschiebung maps on them from the description in [RW2]. For the complete description of the Hopf algebra structure of $K(m)_*\big(K(\pi,n)\big)$ for a finitely generated abelian group π, see [RW2] for the odd prime case, and see the Appendix of [JW4] for $p=2$.

First, we briefly describe $K(m)_*\big(K(\mathbb{Z}/p^j,n+1)\big)$ for $m\geq n+1$. Let

(5-1)
$$\mathcal{S}^{(j)}_{n+1}(m)=\{(s_0+m\ell;s_1,s_2,\dots,s_n)\mid 0\leq \ell<j,\ 0\leq s_0<s_1<\cdots<s_n<m\}$$

be a set of sequences of $n+1$ non-negative integers. For each $S\in\mathcal{S}^{(j)}_{n+1}(m)$, let

$$a_S=a_{(s_0+m\ell;s_1,s_2,\dots,s_n)}=a_{(s_0+m\ell)}\circ a_{(s_1+m(j-1))}\circ\cdots\circ a_{(s_n+m(j-1))} \tag{5-2}$$

be an element in $K(m)_*\big(K(\mathbb{Z}/p^j,n+1)\big)$. Note that in the expression in terms of the $\circ$-product, the index s_i is replaced by $s_i+m(j-1)$ for $1\leq i\leq n$. In [RW2], it is shown that these elements are nonzero and that a $K(m)_*$-module basis of the $K(m)$ homology is given by a set of elements $\big\{\prod_{S\in\mathcal{S}^{(j)}_{n+1}(m)}a_S^{j_S}\big\}_j$, where the exponent j_S is such that $0\leq j_S<p$ for all S and all but finitely many exponents j_S are zero. The Verschiebung map $V:K(m)^*(\,\cdot\,)\to K(m)^*(\,\cdot\,)$, which is the dual of the pth power map in $K(m)$ homology, is described as follows (cf. [RW2, Theorem 11.1]):

(5-3)
$$\begin{gathered}V\big(a_{(0;s_1,s_2,\dots,s_n)}\big)=0,\\ V\big(a_{(s_0+m\ell;s_1,s_2,\dots,s_n)}\big)=a_{(s_0-1+m\ell;s_1-1,\dots,s_n-1)}\quad\text{if } s_0>0 \text{ and } 0\leq\ell\leq j-1,\\ V\big(a_{(m\ell;s_1,s_2,\dots,s_n)}\big)=(-1)^n a_{(s_1-1+m(\ell-1);s_2-1,\dots,s_n-1,m-1)}\quad\text{if } 1\leq\ell\leq j-1.\end{gathered}$$

For any $S\in\mathcal{S}^{(j)}_{n+1}(m)$, let y_S be the $K(m)$ cohomology element dual to a_S with respect to the above additive $K(m)_*$-module basis for $K(m)_*\big(K(\mathbb{Z}/p^j,n+1)\big)$. Dualizing (5-3), we can describe the effect of the pth power map on elements y_S.

Lemma 5-1. (i) *Let $0\leq n<m$. For any $(s_1,s_2,\dots,s_n)\in\mathcal{S}^+_n(m)$, the $k(m)$ homology element $a_{(0;s_1,\dots,s_n)}$ is primitive and its dual $K(m)$ cohomology element $y_{(0;s_1,\dots,s_n)}$ is indecomposable.*

(ii) *pth powers of elements* $y_S \in K(m)^*\big(K(\mathbb{Z}/p^j, n+1)\big)$ *for* $S \in \mathcal{S}^{(j)}_{n+1}(m)$ *are given as follows*:

(5-4)
$$y^p_{(s_0+m(j-1);s_1,\dots,s_{n-1},m-1)} = 0,$$
$$y^p_{(s_0+m\ell;s_1,\dots,s_n)} = y_{(s_0+1+m\ell;s_1+1,\dots,s_n+1)} \quad \text{if } s_n < m-1 \text{ and } 0 \le \ell \le j-1,$$
$$y^p_{(s_0+m(\ell-1);s_1,\dots,s_{n-1},m-1)} = (-1)^n y_{(m\ell;s_0+1,s_1+1,\dots,s_{n-1}+1)} \quad \text{if } 1 \le \ell \le j-1.$$

Any element of the form y_S *for* $S \in \mathcal{S}^{(j)}_{n+1}(m)$ *is a* p^r*th power of an element of the form* $y_{(0;s_1,s_2,\dots,s_n)}$ *for some sequence* $0 < s_1 < \cdots < s_n$ *and for some non-negative integer* r.

Proof. For (i), we prove it by induction on n. Since the coalgebra structure of $K(m)_*\big(K(\mathbb{Z}/p^j, 1)\big)$ is given by $\psi(a_k) = \sum_{i+j=k} a_i \otimes a_j$, it follows that $a_{(0)} = a_1$ is primitive. Now for a given n such that $0 \le n < m-1$, suppose that $a_{(0;s_1,\dots,s_n)}$ is primitive for any sequence $(s_1, s_2, \dots, s_n) \in \mathcal{S}^+_n(m)$. Since the coalgebra map and the $\circ$-product are compatible, we then have
$$\psi(a_{(0;s_1,\dots,s_n,s_{n+1})}) = \big(a_{(0;s_1,\dots,s_n)} \otimes 1_{n+1} + 1_{n+1} \otimes a_{(0;s_1,\dots,s_n)}\big) \circ \sum_{r+s=p^{s_{n+1}}} a_r \otimes a_s.$$
Since $1_{n+1} \circ a_r = \delta_{r0} 1_{n+2}$, the only nontrivial term in the above right hand side is $a_{(0;s_1,\dots,s_n)} \circ a_{(s_{n+1})} \otimes 1_{n+1} + 1_{n+1} \otimes a_{(0;s_1,\dots,s_n)} \circ a_{(s_{n+1})}$. This shows that the element $a_{(0;s_1,\dots,s_n,s_{n+1})}$ is primitive. This completes the inductive step, and we have shown that $a_{(0;J)}$ is primitive for any $J \in \mathcal{S}^+_n(m)$ with $0 \le n < m$. Thus it follows that the dual cohomology elements $y_{(0;J)}$ for any $J \in \mathcal{S}^+_n$ are indecomposable. Note that due to the first identity in (5-3), the elements $y_{(0;s_1,\dots,s_n)}$ are not pth powers of any other elements.

For (ii), we first observe that elements of the form $a_{(s_0+m(j-1);s_1,\dots,s_{n-1},m-1)}$ for some $0 \le s_0 < \cdots < s_{n-1} < m-1$ are not in the image of the Verschiebung map described in (5-3). Consequently, pth powers of their dual elements are 0. This proves the first identity of (5-4). The other two are obtained by dualizing the second and the third identity in (5-3).

If we apply the Verschiebung operator V to a_S for $S \in \mathcal{S}^{(j)}_{n+1}(m)$ repeatedly, it will eventually vanish by (5-3). Suppose that $V^{r+1}(a_S) = 0$ for some $r \ge 0$ and that $V^r(a_S) \ne 0$. Then, $V^r(a_S)$ must be of the form $a_{(0;s'_1,\dots,s'_n)}$ for some $0 < s'_1 < \cdots < s'_n < m$. Dualizing this, we see that $y^{p^r}_{(0;s'_1,\dots,s'_n)} = y_S$. This proves the last statement. □

Repeated application of (5-4) yields the following formulae.

Lemma 5-2. *Let* $0 \le \ell \le j-1$ *and* $S = (s_1, \dots, s_n) \in \mathcal{S}^+_n(m)$. *We let* $s_0 = 0$. *Then in the Morava* K *cohomology* $K(m)^*\big(K(\mathbb{Z}/p^j, n+1)\big)$, *we have the following identity for* $0 \le r \le n$:

(5-5)
$$y^{p^{m-s_{n-r}}}_{(\ell m;S)} = (-1)^{(r+1)n} y_{((r+1+\ell)m;S'_r)},$$

where the sequence $S'_r \in \mathcal{S}^+_n(m)$ *is given by*
$$S'_r = \big(s_{n-r+1} - s_{n-r}, s_{n-r+2} - s_{n-r}, \dots, s_n - s_{n-r}, \\ m - s_{n-r}, s_1 + (m - s_{n-r}), \dots, s_{n-r-1} + (m - s_{n-r})\big).$$

Here, the right hand side of (5-5) *is zero if* $r+1+\ell \geq j$. *Furthermore, we have*

$$y_{(0;s_1,\dots,s_n)}^{p^{km}} = y_{(k(n+1)m;s_1,\dots,s_n)}. \tag{5-6}$$

Proof. (5-5) is obtained by repeated use of (5-4). For example, repeatedly applying the second formula in (5-4) $m-s_n-1$ times with $s_0=0$, we obtain

$$y_{(m\ell;S)}^{p^{m-s_n-1}} = y_{(m-s_n-1+m\ell;s_1+m-s_n-1,\dots,s_{n-1}+m-s_n-1,m-1)}.$$

We raise both sides to the pth power. Then the third formula in (5-4) gives

$$y_{(m\ell;S)}^{p^{m-s_n}} = (-1)^n y_{(m(\ell+1);m-s_n,s_1+(m-s_n),\dots,s_{n-1}+(m-s_n))}.$$

This proves (5-5) for the case $r=0$. By repeating the above procedure r times more, we obtain (5-5).

When $r=n$, (5-5) becomes $y_{(\ell m;S)}^{p^m} = (-1)^{n(n+1)} y_{((n+1+\ell)m;S)}$. Here note that $n(n+1)$ is even. Applying this formula k times with $\ell=0$, we obtain (5-6). □

Let $S=(s_1,\dots,s_n)\in\mathcal{S}_n^+(m)$, that is, $0<s_1<\cdots<s_n<m$. We now calculate the exponent of the element $y_{(0;S)}$ in the cohomology ring $K(m)^*\big(K(\mathbb{Z}/p^j,n+1)\big)$ and describe its ring structure.

Proposition 5-3. *Let* $0\leq n<m$. *For any* $j>0$, *the* $K(m)$ *cohomology ring of the Eilenberg–Mac Lane space* $K(\mathbb{Z}/p^j,n+1)$ *is a tensor product of finitely many truncated* $K(m)^*$*-polynomial algebras given as follows*:

$$K(m)^*\big(K(\mathbb{Z}/p^j,n+1)\big) = K(m)^*\big[y_{(0;S)}\bigm| S\in\mathcal{S}_{n+1}^+(m)\big]\big/\big(y_{(0;S)}^{p^{h(S)}}\big), \tag{5-7}$$

where $h(S)=km+(m-s_{n+1-r})$ *if* k *and* r *are integers determined by the equation* $j=k(n+1)+r$ *with* $1\leq r\leq n+1$. *We set* $s_0=0$. *The degree of the cohomology element* $y_{(0;S)}$ *is* $2(1+p^{s_1}+\cdots+p^{s_n})$.

Proof. First we calculate the exponent of $y_{(0;S)}$. Let k,r be integers determined by $j=k(n+1)+r$ with $1\leq r\leq n+1$ for given j,n. Then from (5-6), for any $S\in\mathcal{S}_{n+1}^+(m)$, we have $y_{(0;S)}^{p^{km}} = y_{(k(n+1)m;S)}$. Raising both sides to the power $p^{m-s_{n-r+2}}$ and applying (5-5), we have

$$y_{(0;S)}^{p^{km+(m-s_{n-r+2})}} = (-1)^{(r-1)n} y_{((k(n+1)+r-1)m;S'_{r-2})},$$

where the sequence $S'_{r-2}\in\mathcal{S}_n^+(m)$ is given by

$$\begin{aligned} S'_{r-2} = \big(&s_{n-r+3}-s_{n-r+2},\dots,s_n-s_{n-r+2},m-s_{n-r+2},\\ &s_1+(m-s_{n-r+2}),\dots,s_{n-r+1}+(m-s_{n-r+2})\big). \end{aligned}$$

Note that by our choice of integers k,r, the first subscript of the element y_* on the right hand side is $m\big(k(n+1)+r-1\big)=m(j-1)$. Furthermore, raising both sides of this identity to the power p to the $(m-1)-(s_{n-r+1}+m-s_{n-r+2}) = s_{n-r+2}-s_{n-r+1}-1$-st, and using the second formula in (5-4), we have

$$y_{(0;S)}^{p^{km+(m-s_{n-r+1})-1}} = (-1)^{(r-1)n} y_{(s_{n-r+2}-s_{n-r+1}-1+(j-1)m;S'')},$$

where the sequence $S''\in\mathcal{S}_n^+(m)$ is given by

$$\begin{aligned} S'' = (&s_{n-r+3}-s_{n-r+1}-1,\dots,s_n-s_{n-r+1}-1,(m-s_{n-r+1})-1,\\ &s_1+(m-s_{n-r+1})-1,\dots,s_{n-r}+(m-s_{n-r+1})-1,m-1) \end{aligned}$$

with $m-1$ as the last entry. Since $(s_{n-r+2}-s_{n-r+1}-1+(j-1)m; S'')$ is a sequence in $\mathcal{S}^{(j)}_{n+1}(m)$ defined in (5-1), the corresponding $K(m)$ cohomology element is nonzero because it pairs nontrivially with a homology element $a_{(s_{n-r+2}-s_{n-r+1}-1+(j-1)m;S'')}$. So we have $y_{(0;S)}^{p^{km+(m-s_{n-r+1})-1}} \neq 0$. However, the first formula in (5-4) implies that the pth power of this element is zero: $y_{(0;S)}^{p^{km+(m-s_{n-r+1})}} = 0$. Thus, the exponent of the cohomology element $y_{(0;S)}$ for $S \in \mathcal{S}^+_n(m)$ is precisely $p^{h(S)}$, where $h(S) = km+(m-s_{n-r+1})$. Here k, r are determined by $j = k(n+1)+r$ with $1 \leq r \leq n+1$. This shows that the right hand side of (5-7) is contained in the left hand side.

Now, $K(m)^*$-module generators of the left hand side of (5-7) are the dual elements of the $K(m)_*$-module basis of $K(m)_*\big(K(\mathbb{Z}/p^j, n+1)\big)$ which is given by $\big\{\prod_{S\in\mathcal{S}^{(j)}_{n+1}(m)} a_S^{j_S}\big\}_j$, where j runs over the set of all functions $j : \mathcal{S}^{(j)}_{n+1}(m) \to \{0, 1, \ldots, p-1\}$. For any $S \in \mathcal{S}^{(j)}_{n+1}(m)$ the nontrivial $K(m)$ cohomology element y_S is dual to a_S. Hence the set $\big\{\prod_{S\in\mathcal{S}^{(j)}_{n+1}(m)} y_S^{j_S}\big\}_j$ is a $K(m)^*$-module basis for $K(m)^*\big(K(\mathbb{Z}/p^j, n+1)\big)$, where $0 \leq j_S < p$. Since y_S is a p^rth power of an element $y_{(0;S')}$ for some $S' \in \mathcal{S}^+_n(m)$ and for some non-negative integer r by Lemma 5-1, the left hand side in (5-7) is contained in the right hand side. Thus nontrivial monomials of $y_{(0;S)}$'s form a $K(m)^*$-module basis for $K(m)^*\big(K(\mathbb{Z}/p^j, n+1)\big)$. This completes the proof of (5-7). □

We say a CW complex is locally finite if there are finitely many cells in each dimension. Although BP cohomology can have elements of infinite filtration, such elements do not exist in $K(m)$ cohomology groups.

Lemma 5-4. *Let X be a locally finite CW complex. Then with respect to the skeletal filtration, the topological space $K(m)^*(X)$ is complete Hausdorff.*

Proof. We examine the following Milnor exact sequence [M3]:

$$0 \to \varprojlim{}_r{}^1 K(m)^{t-1}\big(X^{(r-1)}\big) \to K(m)^t(X) \to \varprojlim{}_r K(m)^t\big(X^{(r-1)}\big) \to 0.$$

By assumption, the $r-1$ skeleton $X^{(r-1)}$ is a finite complex for each $r \geq 0$. Since $K(m)^* = \mathbb{Z}_p[v_m, v_m^{-1}]$, the group $K(m)^{t-1}\big(X^{(r-1)}\big)$ is a finite group for all $r \geq 0$. Hence the $\varprojlim^1$ term vanishes and there are no elements of infinite filtration in $K(m)^t(X)$ for any t. Thus the skeletal filtration topology in $K(m)^*(X)$ is complete and Hausdorff for a locally finite CW complex. □

Next, we describe the algebra structure of $K(m)^*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ for $m > n$. For any positive integer j, let $\delta_j : K(\mathbb{Z}/p^j, n+1) \to K(\mathbb{Z}_{(p)}, n+2)$ be the Bockstein map. For any $S \in \mathcal{S}^+_n(m)$, let

(5-8)
$$b(S) = \delta_{j*}\big(v_m^{-(p^{s_1}+\cdots+p^{s_n})(1+p^m+\cdots+p^{m(j-2)})} a_{(0;S)}\big) \in K(m)_*\big(K(\mathbb{Z}_{(p)}, n+2)\big),$$

where δ_{j*} is the induced map on $K(m)$ homology. This definition is independent of the choice of j by Lemma 4-6. Let

(5-9)
$$\mathcal{S}^{(\infty)}_{n+1}(m) = \{(s_0 + m\ell; s_1, \ldots, s_n) \in \mathbb{Z}^{n+1} \mid 0 \leq s_0 < s_1 < \cdots < s_n < m,\ \ell \geq 0\}.$$

In [RW2], a nontrivial element $b_J \in K(m)_*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ is defined for each $J \in \mathcal{S}_{n+1}^{(\infty)}(m)$, and it is shown that as a $K(m)_*$-module basis of $K(m)_*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ we may take the set $\big\{\prod_{J\in\mathcal{S}_{n+1}^{(\infty)}(m)} b_J^{i_J}\big\}_i$, where $0 \le i_J < p$ for all J. In this notation, the element $b(S)$ in (5-8) is equal to $b_{(0;S)}$. Let $x_S \in K(m)^*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ be the element dual to $b(S) \in K(m)_*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ for $S \in \mathcal{S}_n^+(m)$, with respect to such a basis. Note that the degree of x_S is $2(1+p^{s_1}+\cdots+p^{s_n})$ if $S = (s_1, \ldots, s_n)$.

The following theorem is stated in [RW2].

Proposition 5-5 [[RW2], Theorem 12.4]**.** *The $K(m)$ cohomology of Eilenberg–Mac Lane space $K(\mathbb{Z}_{(p)}, n+2)$ for $0 \le n < m$ is a power series ring given by*

$$K(m)^*\big(K(\mathbb{Z}_{(p)}, n+2)\big) = K(m)^*[[x_S \mid S \in \mathcal{S}_n^+(m)]], \tag{5-10}$$

where $|x_S| = 2(1+p^{s_1}+\cdots+p^{s_n})$ if $S = (s_1, \ldots, s_n) \in \mathcal{S}_n^+(m)$.

To compare the two sets of cohomology elements $\{\delta_j^*(x_S)\}$ and $\{y_{(0;S)}\}$ in the cohomology ring $K(m)^*\big(K(\mathbb{Z}/p^j, n+1)\big)$, we modify elements in the latter set as follows. For $S \in \mathcal{S}_n^+(m)$, we put

$$y_S^{(j)} = v_m^{(p^{s_1}+\cdots+p^{s_n})(1+p^m+\cdots+p^{m(j-2)})} y_{(0;S)}. \tag{5-11}$$

Note that the degree of the $K(m)$ cohomology element $y_S^{(j)}$ is $2(1+p^{s_1}+\cdots+p^{s_n})$.

Lemma 5-6. *Let $0 \le n < m$, $j > 0$, and let δ_j be the Bockstein map as above. Then for any $S \in \mathcal{S}_n^+(m)$, in $K(m)^*\big(K(\mathbb{Z}/p^j, n+1)\big)$, we have*

$$\delta_j^*(x_S) = y_S^{(j)} + (\textit{decomposables in } y_{S'}^{(j)}\textit{'s with coefficients in } \mathbb{Z}_p[v_m]). \tag{5-12}$$

Proof. For any $S, S' \in \mathcal{S}_n^+(m)$, from the definition of x_S, we have

$$\begin{aligned}\delta_{SS'} = \langle x_S, b(S')\rangle_{K(m)} &= \big\langle x_S, \delta_{j*}\big(v_m^{-(p^{s_1}+\cdots+p^{s_n})(1+p^m+\cdots+p^{m(j-2)})} a_{(0;S')}\big)\big\rangle_{K(m)} \\ &= v_m^{-(p^{s_1}+\cdots+p^{s_n})(1+p^m+\cdots+p^{m(j-2)})} \langle \delta_j^*(x_S), a_{(0;S')}\rangle_{K(m)},\end{aligned}$$

which means $\langle \delta_j^*(x_S), a_{(0;S')}\rangle_{K(m)} = \delta_{SS'} \cdot v_m^{(p^{s_1}+\cdots+p^{s_n})(1+p^m+\cdots+p^{m(j-2)})}$. On the other hand, since $y_{(0;S)}$ is the cohomology element dual to $a_{(0;S)}$ for any $S \in \mathcal{S}_n^+(m)$, we have $\langle \delta_j^*(x_S) - v_m^{(p^{s_1}+\cdots+p^{s_n})(1+p^m+\cdots+p^{m(j-2)})} y_{(0;S)}, a_{(0;S')}\rangle_{K(m)} = 0$. From this we have $\delta_j^*(x_S) = y_S^{(j)} +$(decomposables in $y_{S'}^{(j)}$'s with coefficients in $K(m)^*$). Since $|x_S| = 2(1+p^{s_1}+\cdots+p^{s_n}) < 2(p^m-1)$ and $|y_{S'}^{(j)}| = 2(1+p^{s'_1}+\cdots+p^{s'_n}) < 2(p^m-1)$, all the coefficients of monomials of $y_{S'}^{(j)}$'s must lie in $\mathbb{Z}_p[v_m] \subset K(m)^*$ for degree reasons. This completes the proof of (5-12). □

The above relation (5-12) is used in the proof of Proposition 6-6 below.

6. BP Cohomology Determines Morava K Theory for Eilenberg–Mac Lane Spaces

Let $n \ge 0$. Recall from §3 that for each sequence $S = (s_1, \ldots, s_n) \in \mathcal{S}_n^+$, we have a nontrivial element

$$\vartheta_S^{BP} \in BP^*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$$

of degree $|\vartheta_S| = 2(1+p^{s_1}+\cdots+p^{s_n})$. For any $S \in \mathcal{S}_n^+$ and for any $m > s_n$, we denote its image in $k(m)^*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ and in $K(m)^*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ by $\vartheta_S^{k(m)}$ and $\vartheta_S^{K(m)}$, respectively. If there is no danger of confusion, we also use the notation ϑ_S without superscripts for these elements. We know that the element $\vartheta_S^{k(m)}$ is nontrivial in $k(m)$ cohomology by Proposition 4-5 (II). The next lemma shows that $\vartheta_S^{K(m)}$ is also nontrivial in $K(m)$ cohomology.

Lemma 6-1. *Let $0 \le n < m$. The Kronecker pairing in $k(m)$ theory $\langle\ ,\ \rangle_{K(m)}$: $K(m)^*\big(K(\mathbb{Z}_{(p)}, n+2)\big) \otimes K(m)_*\big(K(\mathbb{Z}_{(p)}, n+2)\big) \to K(m)_*$ is such that*

$$\langle \vartheta_S^{K(m)}, b(S')\rangle_{K(m)} = \delta_{SS'}, \tag{6-1}$$

for any $S, S' \in \mathcal{S}_n^+(m)$. In particular, $\vartheta_S^{K(m)} \neq 0$ for any $S \in \mathcal{S}_n^+(m)$.

Proof. The Kronecker pairing of the above type of elements in connective Morava K theory for Eilenberg–Mac Lane spaces is described in (4-13). We then apply Lemma 4-1 to the localization map $\ell(m) : k(m) \to K(m)$ to obtain (6-1). □

Recall from Lemma 3-7 that for any $S \in \mathcal{S}_n^+$, the corresponding element ϑ_S^{BP} belongs to the filtration $F^{(2(1+p^{s_1}+\cdots+p^{s_n}))}\big(BP^*K(\mathbb{Z}_{(p)}, n+2)\big)$. The argument we used there cannot be applied to the element $\vartheta_S^{K(m)} \in K(m)^*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$, because $K(m)$ is not a connective spectrum. However, we have the following lemma.

Lemma 6-2. *Let $\eta : \mathbf{E} \to \mathbf{G}$ be a spectra map. Let X be a CW complex. Then the induced cohomology map $\eta_* : \mathbf{E}^*(X) \to \mathbf{G}^*(X)$ preserves the skeletal filtrations*:

$$\eta_*\big(F^{(r)}(\mathbf{E}^*X)\big) \subset F^{(r)}(\mathbf{G}^*X) \qquad \textit{for any} \quad r \ge 0. \tag{6-2}$$

Proof. This is straightforward from the following diagram:

$$\begin{array}{ccc} \mathbf{E}^*(X) & \xrightarrow{\eta_*} & \mathbf{G}^*(X) \\ \downarrow & & \downarrow \\ \mathbf{E}^*\big(X^{(r-1)}\big) & \xrightarrow{\eta_*} & \mathbf{G}^*\big(X^{(r-1)}\big). \end{array}$$

Here, vertical maps are restrictions to $r-1$ skeletons. □

Applying this lemma to the canonical map $\eta : BP \to K(m)$ and using Lemma 3-7, we obtain the following result.

Corollary 6-3. *Let $0 \le n < m$ and let $S = (s_1, \ldots, s_n) \in \mathcal{S}_n^+(m)$. Then, for any positive integer j, in $K(m)$ cohomology theories, we have*

$$\begin{aligned} \vartheta_S^{K(m)} &\in F^{(2(1+p^{s_1}+\cdots+p^{s_n}))}\big(K(m)^*K(\mathbb{Z}_{(p)}, n+2)\big), \\ \vartheta_S^{(j)} &\in F^{(2(1+p^{s_1}+\cdots+p^{s_n}))}\big(K(m)^*K(\mathbb{Z}/p^j, n+1)\big). \end{aligned} \tag{6-3}$$

Next we identify elements $\vartheta_S^{K(m)}$ in the cohomology ring $K(m)^*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$, which is a power series ring generated by x_S's as described in Proposition 5-5.

Lemma 6-4. *For each $S \in \mathcal{S}_n^+(m)$, the element $\vartheta_S^{K(m)} \in K(m)^*\big(K(\mathbb{Z}_{(p)}, n+2)\big) = K(m)^*[[x_S \mid S \in \mathcal{S}_n^+(m)]]$ is of the form*

$$\vartheta_S^{K(m)} = x_S + \begin{pmatrix} \textit{decomposables in } x_{S'}\textit{'s for } S' \in \mathcal{S}_n^+(m) \\ \textit{with coefficients in } \mathbb{Z}_p[v_m] \end{pmatrix}. \tag{6-4}$$

Furthermore, the Morava K theory is generated by the elements $\vartheta_S^{K(m)}$:

$$K(m)^*\big(K(\mathbb{Z}_{(p)}, n+2)\big) = K(m)^*[[\vartheta_S^{K(m)} \mid S \in \mathcal{S}_n^+(m)]], \tag{6-5}$$

Proof. Since x_S is dual to $b(S)$ by definition, the Kronecker product in $k(m)$ theory is such that $\langle x_S, b(S')\rangle_{K(m)} = \delta_{SS'}$ for any $S, S' \in \mathcal{S}_n^+(m)$. Thus, together with (6-1), we have $\langle \vartheta_S^{K(m)} - x_S, b(S')\rangle_{K(m)} = 0$ for any $S, S' \in \mathcal{S}_n^+(m)$. This shows that $\vartheta_S^{K(m)}$ is of the form $x_S +$(decomposables in $x_{S'}$'s with coefficients in $K(m)^*$). Since $|\vartheta_S^{K(m)}| < 2(p^m - 1) = -|v_m|$, the coefficients of decomposable elements in $x_{S'}$'s must lie in the subring $\mathbb{Z}_p[v_m]$. This proves (6-4).

Using (6-4), we can formally express x_S as a power series in $\vartheta_{S'}^{K(m)}$'s with coefficients in $\mathbb{Z}_p[v_m]$. Since the cohomology ring $K(m)^*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ is complete Hausdorff in each degree with respect to the filtration topology by Lemma 5-4, and the filtration of $\vartheta_{S'}^{K(m)}$ is $2(1 + p^{s'_1} + \cdots + p^{s'_n})$, the above power series in $\vartheta_{S'}^{K(m)}$'s converges to x_S in the cohomology group. Thus x_S's can be replaced by $\vartheta_S^{K(m)}$'s, and we have (6-5). This completes the proof. □

From Lemma 6-4, the following proposition follows.

Proposition 6-5. *Let $0 \le n < m$. The canonical map $BP^*\big(K(\mathbb{Z}_{(p)}, n+2)\big) \to K(m)^*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ is surjective in degrees $* < 2(p^m - 1)$. All the $K(m)^*$-algebra generators of $K(m)^*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ are in this degree range and they come from $BP^*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$. Consequently, the following canonical map is surjective*:

$$v_m^{-1} BP^*\big(K(\mathbb{Z}_{(p)}, n+2)\big) \longrightarrow K(m)^*\big(K(\mathbb{Z}_{(p)}, n+2)\big). \tag{6-6}$$

Proof. First note that for any $S \in \mathcal{S}_n^+(m)$, $\vartheta_S^{BP} \in BP^*\big(K(\mathbb{Z}(p), n+2)\big)$ is v_m-torsion free since its image $\vartheta_S^{K(m)}$ in $K(m)$ cohomology is nontrivial by Lemma 6-1. By (6-5), all the algebra generators of $K(m)^*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ come from BP cohomology via the canonical ring spectra map $BP \to K(m)$. Thus, by inverting v_m in BP cohomology, we obtain the surjection in (6-6).

Let x be an arbitrary element in $K(m)^*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$. By (6-5), x can be expressed as a power series in $\vartheta_S^{K(m)}$'s with coefficients in $K(m)^*$. If the degree of x is less than $2(p^m - 1)$, then powers of v_m^{-1} do not appear in the coefficients of this power series for degree reasons. Since all the generators $\vartheta_S^{K(m)}$ for $S \in \mathcal{S}_n^+(m)$ come from BP cohomology, it follows that x is in the image from BP cohomology. Thus, the canonical map $BP^*\big(K(\mathbb{Z}_{(p)}, n+2)\big) \to K(m)^*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ is surjective for $* < 2(p^m - 1)$. This completes the proof. □

For the Eilenberg–Mac Lane space $K(\mathbb{Z}, 1) = S^1$, obviously we have $BP^*(S^1) = \bigwedge_{BP^*}(\vartheta^{BP})$ and $K(m)^*(S^1) = \bigwedge_{K(m)^*}(\vartheta^{K(m)})$, where ϑ^{BP} and $\vartheta^{K(m)}$ are 1-dimensional generators and $\rho_*(\vartheta^{BP}) = \vartheta^{K(m)}$ under the Thom map. Thus, the statement in Proposition 6-5 is trivially true for this case.

In §3, for any $S \in \mathcal{S}_n^+(m)$ and for any positive integer j, a nontrivial element $\vartheta_S^{(j)} \in BP^*\big(K(\mathbb{Z}/p^j, n+1)\big)$ was defined by $\vartheta_S^{(j)} = \delta_j^*(\vartheta_S^{BP})$. So in $K(m)$ cohomology, we also have $\vartheta_S^{(j)} = \delta_j^*\big(\vartheta_S^{K(m)}\big)$. Pulling back the relation (6-4) by δ_j^* into $K(m)^*\big(K(\mathbb{Z}/p^j, n+1)\big)$, we have

$$\vartheta_S^{(j)} = \delta_j^*(x_S) + (\text{decomposables in } \delta_j^*(x_{S'})\text{'s with coefficients in } \mathbb{Z}_p[v_m]).$$

By (5-12), this can be rewritten as

$$\vartheta_S^{(j)} = y_S^{(j)} + (\text{decomposables in } y_{S'}^{(j)}\text{'s with coefficients in } \mathbb{Z}_p[v_m]). \tag{6-7}$$

This identity implies the following proposition.

Proposition 6-6. *Let $0 \le n < m$ and $j > 0$. Let p be any prime. The Morava K cohomology algebra $K(m)^*\big(K(\mathbb{Z}/p^j, n+1)\big)$ is generated by $\vartheta_S^{(j)}$'s for $S \in \mathcal{S}_n^+(m)$ as an algebra, and the canonical map $BP^*\big(K(\mathbb{Z}/p^j, n+1)\big) \to K(m)^*\big(K(\mathbb{Z}/p^j, n+1)\big)$ is surjective in the degree range $* < 2(p^m - 1)$.*

All the $K(m)^*$-algebra generators of $K(m)^*\big(K(\mathbb{Z}/p^j, n+1)\big)$ come from BP cohomology, and the following map is a surjection:

$$v_m^{-1} BP^*\big(K(\mathbb{Z}/p^j, n+1)\big) \to K(m)^*\big(K(\mathbb{Z}/p^j, n+1)\big). \tag{6-8}$$

Proof. Since $K(m)^*\big(K(\mathbb{Z}/p^j, n+1)\big)$ is a tensor product of finitely many truncated $K(m)^*$-polynomial algebras as described in Proposition 5-3, we can express $y_S^{(j)}$ in terms of $\vartheta_{S'}^{(j)}$'s by using (6-7) finitely many times. Thus, $K(m)^*\big(K(\mathbb{Z}/p^j, n+1)\big)$ is generated by $\vartheta_S^{(j)}$'s for $S \in \mathcal{S}_n^+(m)$ as an algebra. Since all of these generators come from $BP^*\big(K(\mathbb{Z}/p^j, n+1)\big)$, we get the surjection in (6-8) after inverting v_m in BP cohomology.

The proof of surjectivity of the canonical map from BP cohomology to Morava K theory of $K(\mathbb{Z}/p^j, n+1)$ in degrees less than $2(p^m - 1)$ is almost the same as the one given in the proof of Proposition 6-5 for $K(\mathbb{Z}_{(p)}, n+2)$. □

We can generalize Propositions 6-5 and 6-6 to generalized Eilenberg–Mac Lane spaces. Let $G = \bigoplus_{q=1}^r G_q$ be a finitely generated positively graded $\mathbb{Z}_{(p)}$-module. Here G_q is a finite direct sum of groups of the form $\mathbb{Z}_{(p)}$ or $\mathbb{Z}/p^j$ for some $j > 0$. Let $K(G) = \prod_{q=1}^r K(G_q, q)$ be the associated generalized Eilenberg–Mac Lane space.

Theorem 6-7. *Let $K(G)$ be the generalized Eilenberg–Mac Lane space associated to a finitely generated positively graded $\mathbb{Z}_{(p)}$-module $G = \bigoplus_{q=1}^r G_q$. For any integer $m \ge r$, the following canonical map is surjective*:

$$v_m^{-1} BP^*\big(K(G)\big) \to K(m)^*\big(K(G)\big). \tag{6-9}$$

Furthermore, $BP^*\big(K(G)\big) \to K(m)^*\big(K(G)\big)$ is surjective when $* < 2(p^m - 1)$.

If the highest degree group G_r is a torsion free group, then the above statements hold for $m \ge r - 1$.

Proof. Let $K(G) = X_1 \times \cdots \times X_h$, where each X_i is of the form $K(\mathbb{Z}_{(p)}, q)$ or $K(\mathbb{Z}/p^j, q)$ for some j and q. We have the following diagram:

$$\begin{aligned} BP^*(X_1) \otimes \cdots \otimes BP^*(X_h) &\to BP^*(X_1 \times \cdots \times X_h) \\ &\to K(m)^*(X_1 \times \cdots \times X_h) \cong K(m)^*(X_1) \hat{\otimes} \cdots \hat{\otimes} K(m)^*(X_h), \end{aligned}$$

where $\hat{\otimes}$ denotes the completed tensor product. In view of Propositions 6-5, 6-6, and the remark following Proposition 6-5, when $m \ge r$, all the $K(m)^*$-algebra generators of $K(m)^*(X_i)$ come from $BP^*(X_i)$ for $1 \le i \le h$. When G_r is torsion free, then by Proposition 6-6 we may take $m \ge r - 1$. Thus all the $K(m)^*$-algebra generators of $K(m)^*(X_1 \times \cdots \times X_h)$ come from $BP^*(X_1) \otimes \cdots \otimes BP^*(X_h)$, and consequently they come from $BP^*(X_1 \times \cdots \times X_h)$ in view of the above diagram. Hence after inverting v_m, we get the surjection (6-9).

The surjectivity of the map before localization in degrees less than $2(p^m-1)$ can be proved by an argument similar to the one in the proof of Proposition 6-5. $\square$

7. Determination of the Image of the BP Thom Map for Generalized Eilenberg–Mac Lane Spaces

In this section, we determine the precise image of the BP Thom map for generalized Eilenberg–Mac Lane spaces $K(G)$:

$$\rho_* : BP^*\big(K(G)\big) \to H\mathbb{Z}_p^*\big(K(G)\big).$$

In particular, we show that the inclusion relations (3-14) and (3-16) are actually equalities. The idea of the proof is to examine connective Morava K theories $k(m)^*\big(K(G)\big)$, through which the above ρ_* factors, in terms of its localization $K(m)^*\big(K(G)\big)$ whose structure is given in Propositions 5-3, 5-5, Lemma 6-4, and Proposition 6-6.

First we need one lemma describing torsion free elements in connective Morava K theories in a general setting.

Lemma 7-1. *Let m be a positive integer. Let X be a topological space. Then elements in $k(m)^t(X)$ are v_m-torsion free if $t \le 2p^m-1$. In other words, the localization map $\ell(m)_* : k(m)^*(X) \to K(m)^*(X)$ is injective when $* \le 2p^m-1$.*

Proof. Let $x \in k(m)^t(X)$ be an arbitrary nontrivial element with $t \le 2p^m-1$. We prove the above lemma by contradiction. Suppose x is a v_m-torsion element. Then there exists a positive integer r such that $v_m^r x = 0$, but $v_m^{r-1}x \ne 0$. Now consider the following portion of the Sullivan exact sequence:

$$\to H\mathbb{Z}_p^{t-2r(p^m-1)-1}(X) \xrightarrow{\Delta_m} k(m)^{t-2(r-1)(p^m-1)}(X) \xrightarrow{v_m\cdot} k(m)^{t-2r(p^m-1)}(X) \to .$$

The element $v_m^{r-1}x$ belongs to the group in the middle, and it is annihilated by multiplication by v_m by our assumption. By exactness, $v_m^{r-1}x$ is in the image of the connecting map Δ_m, say, $v_m^{r-1}x = \Delta_m(z)$ for some $z \in H\mathbb{Z}_p^{t-2r(p^m-1)-1}(X)$. However, since $t \le 2p^m-1$ and $r > 0$, we have $t-2r(p^m-1)-1 \le 0$.

When $t-2r(p^m-1)-1 < 0$, the mod-p cohomology group is zero. So $z=0$ and it follows that $v_m^{r-1}x = \Delta_m(z) = 0$, contradicting our assumption.

When $t-2r(p^m-1)-1 = 0$, we must have $t = 2p^m-1$ and $r=1$. In this case we choose a point from each path connected component of X and call the collection of these points "pts". Let $c : X \to \text{pts}$ be the locally constant map mapping any point in X to the chosen point in the same path connected component. We have the following commutative diagram:

$$\begin{array}{ccc} H\mathbb{Z}_p^0(X) & \xrightarrow{\Delta_m} & k(m)^{2p^m-1}(X) \\ {\scriptstyle c^*}\uparrow\cong & & \uparrow{\scriptstyle c^*} \\ H\mathbb{Z}_p^0(\text{pts}) & \xrightarrow{\Delta_m} & k(m)^{2p^m-1}(\text{pts}). \end{array}$$

Since the ring $k(m)^*(\text{pts})$ is even dimensional, $k(m)^{2p^m-1}(\text{pts}) = 0$. By commutativity of the above diagram, we see that the map $\Delta_m : H\mathbb{Z}_p^0(X) \to k(m)^{2p^m-1}(X)$ is a zero map. Thus, $v_m^{r-1}x = \Delta_m(z) = 0$, contradicting our hypothesis.

Hence any element in $k(m)^*(X)$ of degree less than or equal to $2p^m-1$ must be v_m-torsion free. This completes the proof. $\square$

Now we can determine the precise image of the Thom map. We use the notation ϑ_S^{BP} and $\vartheta_S^{K(m)}$ to distinguish elements ϑ_S in BP cohomology and their images in $K(m)$ theory of the Eilenberg–Mac Lane space $K(\mathbb{Z}_{(p)}, n+2)$.

Theorem 7-2. *Let p be a prime, even or odd. Let $n \geq 1$.*

(I) *The image of the Thom map $\rho_* : BP^*\big(K(\mathbb{Z}_{(p)}, n+2)\big) \to H\mathbb{Z}_p^*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ is an $\mathcal{A}(p)^*$-invariant polynomial subalgebra with infinitely many generators*:

$$\operatorname{Im}\rho_* = \mathbb{Z}_p[Q_S\tau_{n+2} \mid S \in \mathcal{S}_n^+]. \tag{7-1}$$

This subalgebra is annihilated by all Milnor primitives, and any polynomial generator $Q_S\tau_{n+2}$ is obtained by applying a Steenrod reduced power operation to the element $Q_n \cdots Q_1\tau_{n+2}$.

(II) *The image of the Thom map $\rho_* : BP^*\big(K(\mathbb{Z}/p^j, n+1)\big) \to H\mathbb{Z}_p^*\big(K(\mathbb{Z}/p^j, n+1)\big)$ is an $\mathcal{A}(p)^*$-invariant polynomial subalgebra with infinitely many generators*:

$$\operatorname{Im}\rho_* = \mathbb{Z}_p[Q_S\delta_j\iota_{n+1} \mid S \in \mathcal{S}_n^+], \tag{7-2}$$

where δ_j is the jth Bockstein operator. This subalgebra is annihilated by all Milnor primitives. Any polynomial generator $Q_S\delta_j\iota_{n+1}$ is obtained from $Q_n \cdots Q_1\delta_j\iota_{n+1}$ by applying a Steenrod reduced power operation.

Proof. (I) In Theorem 3-5, we have shown that $\operatorname{Im}\rho_* \supset \mathbb{Z}_p[Q_S\tau_{n+2} \mid S \in \mathcal{S}_n^+]$. To prove (7-1), we must show the opposite inclusion relation. Suppose a homogeneous element $z \in H\mathbb{Z}_p^*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ is in the image of the BP Thom map, say, $z = \rho_*(z')$ for some $z' \in BP^*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$. Choose a positive integer m such that $|z| = |z'| < 2(p^m - 1)$, and consider the following diagram:

(7-3)

$$\begin{array}{ccccc} BP^*\big(K(\mathbb{Z}_{(p)}, n+2)\big) & \xrightarrow{\rho_{(m)_*}} & k(m)^*\big(K(\mathbb{Z}_{(p)}, n+2)\big) & \xrightarrow{\rho_*^{(m)}} & H\mathbb{Z}_p^*\big(K(\mathbb{Z}_{(p)}, n+2)\big) \\ & & \big\downarrow{\scriptstyle \ell(m)_*} & & \\ & & K(m)^*\big(K(\mathbb{Z}_{(p)}, n+2)\big) & & \end{array}$$

Let $z'' = \rho_{(m)_*}(z') \in k(m)^*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$. By (6-5), $K(m)^*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ is generated by $\vartheta_S^{K(m)}$'s for $S \in \mathcal{S}_n^+(m)$. Thus under the localization map $\ell(m)_*$, the image $\ell(m)_*(z'') \in K(m)^*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ can be expressed as a power series in $\vartheta_S^{K(m)}$'s with $K(m)^*$ coefficients. Since $|\ell(m)_*(z'')| = |z| < 2(p^m - 1)$ by our choice of m, the coefficients of this power series expression are actually in $\mathbb{Z}_p[v_m]$ for degree reasons. Thus this power series is of the following form:

$$\ell(m)_*(z'') = f(\vartheta_S^{K(m)}\text{'s}) + v_m \cdot g(\vartheta_S^{K(m)}\text{'s}) \in K(m)^*\big(K(\mathbb{Z}_{(p)}, n+2)\big), \tag{7-4}$$

where f and g are power series in $\vartheta_S^{K(m)}$'s with coefficients in $\mathbb{Z}_p$ or in $\mathbb{Z}_p[v_m]$, respectively. But the elements $\vartheta_S^{K(m)}$ for $S \in \mathcal{S}_n^+(m)$ come from BP cohomology, and we have $\vartheta_S^{K(m)} = \ell(m)_* \circ \rho_{(m)_*}(\vartheta_S^{BP})$ for each $S \in \mathcal{S}_n^+(m)$. We put

$$\hat{z} = f(\vartheta_S^{BP}\text{'s}) + v_m \cdot g(\vartheta_S^{BP}\text{'s}) \in BP^*\big(K(\mathbb{Z}_{(p)}, n+2)\big),$$

replacing $\vartheta_S^{K(m)}$ by ϑ_S^{BP} in (7-4). Since $\rho_{(m)_*}$ and $\ell(m)_*$ are algebra maps, we have $\ell(m)_*(z'') = \ell(m)_* \circ \rho_{(m)_*}(\hat{z})$, or $\ell(m)_*\big(z'' - \rho_{(m)_*}(\hat{z})\big) = 0$. Since the degree of the difference element $z'' - \rho_{(m)_*}(\hat{z}) \in k(m)^*\big(K(\mathbb{Z}_{(p)}, n+2)\big)$ is less than $2(p^m - 1)$,

this element must be zero by the injectivity result on the localization map $\ell(m)_*$ due to Lemma 7-1. Thus,

$$\begin{aligned} z = \rho_*(z') = \rho_*^{(m)}(z'') = \rho_*^{(m)} \circ \rho_{(m)_*}(\hat{z}) &= \rho_*\big(f(\vartheta_S^{BP}\text{'s}) + v_m \cdot g(\vartheta_S^{BP}\text{'s})\big) \\ &= f(Q_S\tau_{n+2}\text{'s}) \in \mathbb{Z}_p[Q_S\tau_{n+2} \mid S \in \mathcal{S}_n^+]. \end{aligned}$$

This is because $\rho_*(\vartheta_S^{BP}) = Q_S\tau_{n+2}$, $\rho_*(v_m) = 0$, and ρ_* is a ring map (even when $p = 2$). This shows that any element in the image of the BP Thom map is contained in the subalgebra $\mathbb{Z}_p[Q_S\tau_{n+2} \mid S \in \mathcal{S}_n^+]$. This completes the proof of (7-1).

From the description of the mod-p cohomology of Eilenberg–Mac Lane spaces given in §2, all products of $n+1$ Milnor primitives act trivially on the fundamental class τ_{n+2}. So the subalgebra $\mathbb{Z}_p[Q_S\tau_{n+2} \mid S \in \mathcal{S}_n^+]$ is annihilated by all Milnor primitives. Since BP operations are lifts of Steenrod reduced power operations, the image from BP theory into mod-p theory is invariant under the action of Steenrod reduced power operations. Combining these two facts, we see that the image $\operatorname{Im}\rho_*$ is invariant under the action of the total Steenrod algebra.

By Lemma 3-4, any polynomial generator $Q_S\tau_{n+2}$ can be obtained from the element $Q_n \cdots Q_1\tau_{n+2}$ by the action of a Steenrod reduced power operation.

This completes the proof of part (I) of Theorem 7-2. Part (II) can be proved in a similar way. □

We can generalize Theorem 7-2 to generalized Eilenberg–Mac Lane spaces. For this, we need to introduce some notation. For $n \geq 0$, let

$$\begin{aligned} \mathcal{Q}\big(K(\mathbb{Z}_{(p)}, n+2)\big) &= \mathbb{Z}_p[Q_S\tau_{n+2} \mid S \in \mathcal{S}_n^+] \subset H\mathbb{Z}^*\big(K(\mathbb{Z}_{(p)}, n+2)\big), \\ \mathcal{Q}\big(K(\mathbb{Z}/p^j, n+1)\big) &= \mathbb{Z}_p[Q_S\delta_j\iota_{n+1} \mid S \in \mathcal{S}_n^+] \subset H\mathbb{Z}_p^*\big(K(\mathbb{Z}/p^j, n+1)\big). \end{aligned} \tag{7-5}$$

We let $\mathcal{Q}\big(K(\mathbb{Z}_{(p)}, 1)\big) = \bigwedge_{\mathbb{Z}_p}(\tau_1)$ be the exterior algebra generated by the fundamental class $\tau_1 \in H\mathbb{Z}_p^1(S^1_{(p)})$. Let $M = \bigoplus_i M_i$ be a finitely generated $\mathbb{Z}_{(p)}$-module where $M_i = \mathbb{Z}_{(p)}$ or $\mathbb{Z}/p^j$ for some positive integer j. We let $\mathcal{Q}\big(K(M, q)\big) = \bigotimes_i \mathcal{Q}\big(K(M_i, q)\big)$ for any $q \geq 1$. Now, let $G = \bigoplus_{q\geq 1} G_q$ be a finitely generated positively graded $\mathbb{Z}_{(p)}$-module. Let $K(G) = \prod_q K(G_q, q)$ be the associated generalized Eilenberg–Mac Lane space. Put

$$\mathcal{Q}\big(K(G)\big) = \bigotimes_{q\geq 1} \mathcal{Q}\big(K(G_q, q)\big) \subset H\mathbb{Z}_p^*\big(K(G)\big). \tag{7-6}$$

The generalized version of Theorem 7-2 can be stated as follows.

Theorem 7-3. *Let p be any prime, even or odd. Let G be a positively graded finitely generated $\mathbb{Z}_{(p)}$-module, and let $K(G)$ be the associated generalized Eilenberg–Mac Lane space. The image of the BP Thom map $\rho_* : BP^*\big(K(G)\big) \to H\mathbb{Z}_p^*\big(K(G)\big)$ is an $\mathcal{A}(p)^*$-invariant subalgebra given by*

$$\operatorname{Im}\rho_* = \mathcal{Q}\big(K(G)\big) \subset H\mathbb{Z}_p^*\big(K(G)\big). \tag{7-7}$$

This subalgebra is annihilated by all Milnor primitives.

Proof. An argument similar to the one given in Theorem 7-2 proves this theorem. We use the following diagram instead:

$$\begin{tikzcd} BP^*\big(K(G)\big) \arrow[r, "\rho_{(m)*}"] & k(m)^*\big(K(G)\big) \arrow[r, "\rho_*^{(m)}"] \arrow[d, "\ell(m)_*"] & H\mathbb{Z}_p^*\big(K(G)\big) \\ & K(m)^*\big(K(G)\big) & \end{tikzcd} \tag{7-8}$$

The point here is that both $K(m)$ theory and $H\mathbb{Z}_p$ theory have Künneth isomorphisms. So we can consider the image for each individual factor of $K(G)$. □

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