SIGNED QUASI-MEASURES

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Abstract. Let $X$ be a compact Hausdorff space and let $A$ denote the subsets of $X$ which are either open or closed. A quasi-linear functional is a map $\rho : C(X) \to \mathbb{R}$ which is linear on singly generated subalgebras and such that $|\rho(f)| \leq M\|f\|$ for some $M < \infty$. There is a one-to-one correspondence between the quasi-linear functional on $C(X)$ and the set functions $\mu : A \to \mathbb{R}$ such that i) $\mu(\emptyset) = 0$, ii) If $A, B, A \cup B \in A$ with $A$ and $B$ disjoint, then $\mu(A \cup B) = \mu(A) + \mu(B)$, iii) There is an $M < \infty$ such that whenever $\{U_\alpha\}$ are disjoint open sets, $\sum |\mu(U_\alpha)| \leq M$, and iv) if $U$ is open and $\varepsilon > 0$, there is a compact $K \subseteq U$ such that whenever $V \subseteq U \setminus K$ is open, then $|\mu(V)| < \varepsilon$. The space of quasi-linear functionals is investigated and quasi-linear maps between two $C(X)$ spaces are studied.

Let $X$ be a compact Hausdorff space and $C(X)$ the space of real-valued continuous functions on $X$. A map $\rho : C(X) \to \mathbb{R}$ is said to be a quasi-linear functional if $\rho$ is linear on singly generated subalgebras and bounded in the sense that there exists an $M < \infty$ such that $|\rho(f)| \leq M\|f\|_u$ for all $f \in C(X)$. Let $\|\rho\|$ be the minimal such $M$. If $\rho$ and $\eta$ are quasi-linear functionals, we define $\rho + \eta$ by pointwise action on functions. In this fashion, the collection of all quasi-linear functionals becomes a normed linear space. Call this space $QL(X)$.

Notice that if $\rho$ is quasi-linear, and $fg = 0$, then $\rho(f + g) = \rho(f) + \rho(g)$. In fact, if $f$ and $g$ are also positive, we have that the subalgebra generated by $f - g$ contains both $f$ and $g$. In general, we can break $f$ and $g$ into positive and negative parts to get the result. Also notice that if $c$ is a constant, $\rho(c + f) = \rho(c) + \rho(f)$. Thus, if $f$ is constant on the support of $g$, we still have that $\rho(f + g) = \rho(f) + \rho(g)$.

Our goal is to find set functions that produce all quasi-linear functionals on $C(X)$. We will use an approach inspired by the techniques in [1] where the theory of positive quasi-linear functionals is presented. We use the notation $f \prec U$ when $U$ is open to state that $0 \leq f \leq 1$ and $f$ has support contained in $U$. We also use the notation $sp\ f$ for the image of $f$.

Let $\mathcal{O}$ be the collection of open sets in $X$ and $\mathcal{C}$ the collection of closed sets. Also, let $A = \mathcal{O} \cup \mathcal{C}$. Thus $A$ is the collection of subsets of $X$ which are either open or closed.

Definition 1. A function $\mu : A \to \mathbb{R}$ is called a signed quasi-measure if the following hold:

1. $\mu(\emptyset) = 0$,
2. If $A, B \in A$ are disjoint with $A \cup B \in A$, then $\mu(A \cup B) = \mu(A) + \mu(B)$,
(iii) There is a constant $M < \infty$ such that whenever $\{U_n\}$ is a finite disjoint collection of open sets, then $\sum |\mu(U_n)| \leq M$.

(iv) If an open set $U$ and $\varepsilon > 0$ are given, there exists a closed set $K \subseteq U$ such that if $V$ is an open set with $V \subseteq U \setminus K$, we have $|\mu(V)| < \varepsilon$.

We define $||\mu||$ to be the minimal $M$ such that (iii) holds.

For future reference, we note that property (ii) above is equivalent to the following:

a) If $U$ and $V$ are disjoint open sets, then $\mu(U \cup V) = \mu(U) + \mu(V)$.

b) If $U$ and $V$ are open with $X = U \cup V$, then $\mu(U) + \mu(V) = \mu(X) + \mu(U \cap V)$.

c) If $U$ is open, then $\mu(X \setminus U) = \mu(X) - \mu(U)$.

This will allow us to define a quasi-measure by its action on only open sets.

Let $QM(X)$ denote the collection of all signed quasi-measures on $X$. If we define $\mu + \nu$ by action on sets, we see that $QM(X)$ is a normed linear space. We wish to show that in a natural way $QL(X)$ and $QM(X)$ are isomorphic as normed linear spaces, and are, in fact, Banach spaces.

Given a signed quasi-measure $\mu$, we may define a new set function $|\mu|$ on open sets by

$$|\mu|(U) = \sup \{ \sum |\mu(U_n)| : U_n \subseteq U \text{ are disjoint open sets} \}.$$ 

Then we see that $||\mu|| = |\mu|(X)$. It is important to note here that $|\mu|$ need not yield a quasi-measure. In particular, it is impossible to define $|\mu|$ on closed sets so that (ii) holds. An example of this will be seen later. It is clear, however, that (i) and (iii) hold for $|\mu|$. We will see later that (iv) does also.

**Proposition 2.** We have the following:

a) If $\{A_\alpha\}_{\alpha \in A} \subseteq A$ is a collection (possibly infinite) of disjoint subsets of $U \in \mathcal{O}$, then $\sum |\mu(A_\alpha)| \leq |\mu|(U)$.

b) If $U_1 \subseteq U_2 \subseteq \cdots$ are open, then $\mu(\bigcup_{n=1}^\infty U_i) = \lim_{n \to \infty} \mu(U_i)$.

c) If $\{U_\alpha\}$ is a collection (possibly infinite) of disjoint open sets, then $\mu(\bigcup U_\alpha) = \sum \mu(U_\alpha)$.

**Proof.** Both b) and c) are results of the regularity assumption (iv). For a), notice that there is a similar outer regularity for closed sets, so if $\{F_n\}_{n=1}^N$ are finitely many closed sets contained in $U$, and $\varepsilon > 0$, we may find disjoint open sets $U_n$ with $F_n \subseteq U_n \subseteq U$ and $|\mu(U_n \setminus F_n)| < \varepsilon/N$. Then $\sum |\mu(F_n)| \leq \sum |\mu(U_n)| + \varepsilon \leq |\mu|(U) + \varepsilon$.

For the general case, we may restrict to finitely many $A_\alpha$, and approximate by closed sets using inner regularity for open sets.

Now, let $f \in C(X)$ and $\alpha \in \mathbb{R}$, and define $\hat{f}(\alpha) = \mu(f^{-1}(\alpha, +\infty))$ and $\check{f}(\alpha) = \mu(f^{-1}[\alpha, +\infty))$. Notice that $\mu(f^{-1}(\alpha, \beta)) = \check{f}(\alpha) - \hat{f}(\beta)$ and $\mu(f^{-1}[\alpha, \beta]) = \hat{f}(\alpha) - \check{f}(\beta)$.

**Proposition 3.** Let $f \in C(X)$. Then

a) $\check{f}$ is continuous from the right and $\hat{f}$ is continuous from the left.

b) $\check{f}(\alpha^-) = \check{f}(\alpha)$ and $\hat{f}(\alpha^+) = \hat{f}(\alpha)$.

c) $\check{f}$ and $\hat{f}$ agree except at countably many $\alpha \in \mathbb{R}$.

d) If $\hat{f}(\alpha) = \check{f}(\alpha)$, then $\check{f}$ is continuous at $\alpha$.

e) $\check{f}$ is of bounded variation with variation less than $||\mu||$.  


Proof. a) If $\alpha_n$ decreases to $\alpha$, then

$$\tilde{f}(\alpha) = \mu(f^{-1}(\alpha, +\infty))$$

$$= \lim_{n \to \infty} \mu(f^{-1}(\alpha_n, +\infty))$$

$$= \lim_{n \to \infty} \tilde{f}(\alpha_n).$$

Thus $\tilde{f}$ is continuous from the right. Since $\tilde{f}(\alpha) = \mu(X) - (-f)^\vee(-\alpha)$, $\tilde{f}$ is continuous from the left.

b) Let $\alpha_0$ and $\varepsilon > 0$ be given and let $U = f^{-1}(\alpha_0, +\infty)$. Pick $K$ as in iv) of the definition of quasi-measure. Let $\beta$ be the minimum value of $f$ on $K$. Then $\alpha_0 < \beta$. If $\alpha_0 < \alpha < \beta$, we have that $f^{-1}(\alpha_0, \alpha) \subseteq U \setminus K$, so $|\tilde{f}(\alpha_0) - \tilde{f}(\alpha)| = \mu(f^{-1}(\alpha_0, \alpha)) < \varepsilon$. This shows that $\tilde{f}(\alpha_n^+) = \tilde{f}(\alpha_n)$. c) Notice that $\sum |\tilde{f}(\alpha) - \tilde{f}(\alpha)| = \sum \mu(f^{-1}(\{\alpha\})) < \|\mu\|$, by Proposition 2. Thus the set of $\alpha$ where $\tilde{f}(\alpha) \neq \tilde{f}(\alpha)$ is at most countable.

d) This follows from parts a) and b).

e) If $\{(\alpha_n, \beta_n)\}$ is a disjoint collection of intervals, then

$$\sum |\tilde{f}(\alpha_n) - \tilde{f}(\beta_n)| \leq \sum |\tilde{f}(\alpha_n) - \tilde{f}(\beta_n)| + \sum |\tilde{f}(\beta_n) - \tilde{f}(\beta_n)|$$

$$= \sum |\mu(f^{-1}(\alpha_n, \beta_n))| + \sum |\mu(f^{-1}(\beta_n))|$$

$$\leq \|\mu\|$$

by a) of Proposition 2. Thus $\tilde{f}$ is of bounded variation with variation at most $\|\mu\|$.

\[\square\]

Since $\tilde{f}$ is of bounded variation, there is a signed measure $\mu_f$ on $\mathbb{R}$ such that $\mu_f(\alpha, \beta) = \tilde{f}(\alpha) - \tilde{f}(\beta^-) = \tilde{f}(\alpha) - \tilde{f}(\beta) = \mu(f^{-1}(\alpha, \beta))$ and $\|\mu_f\| = \|\mu_f(\mathbb{R})\| \leq \|\mu\|$. If $O$ is any open set in $\mathbb{R}$, we may write $O$ as a disjoint union of open intervals to see that $\mu_f(O) = \mu(f^{-1}(O))$. It follows that $\mu_f$ is concentrated on $\text{sp } f$.

If $f \in C(X)$ and $\varphi \in C(\text{sp } f)$, we let $\varphi^*\mu_f$ denote the image measure of $\mu_f$ under the map $\varphi$. The following lemma simplifies the proof of Proposition 3.2 of [1].

Lemma 4. We have that $\mu_{\varphi^*f} = \varphi^*\mu_f$.

Proof. Let $O \subseteq \mathbb{R}$ be open. Then

$$\mu_{\varphi^*f}(O) = \mu((\varphi \circ f)^{-1}(O))$$

$$= \mu(f^{-1}(\varphi^{-1}(O)))$$

$$= \mu_f(\varphi^{-1}(O))$$

$$= (\varphi^*\mu_f)(O).$$

\[\square\]

Now we may define the functional $\rho_\mu(f) = \int_R i \, d\mu_f$ where $i : \mathbb{R} \to \mathbb{R}$ is the function $i(x) = x$. Since $\mu_f$ is concentrated on $\text{sp } f$, we have $|\rho_\mu(f)| \leq \int_R |i| \, d|\mu_f| \leq
\[ \|\mu\| \|f\|_u, \text{ so } \rho_\mu \text{ is bounded with } \|\rho_\mu\| \leq \|\mu\|. \] Also
\[
\rho_\mu(\varphi \circ f) = \int_R i \, d\mu_{\varphi \circ f} = \int_R i \, d\varphi^* \mu_f = \int_R \varphi \, d\mu_f,
\]
so \( \rho_\mu(\varphi \circ f + \psi \circ f) = \int_R \varphi + \psi \, d\mu_f = \rho_\mu(\varphi \circ f) + \rho_\mu(\psi \circ f). \) Thus \( \rho_\mu \) is a quasi-linear functional on \( C(X). \)

**Theorem 5.** The map \( \mu \to \rho_\mu \) is an isometric isomorphism of the normed linear space \( QM(X) \) onto \( QL(X). \)

**Proof.** It is easy to see that this map is linear. We show that it is onto \( QL(X). \)

Suppose \( \rho \) is a quasi-linear functional on \( C(X). \)

**Claim 1:** If \( U \) is open in \( X \) and \( \varepsilon > 0 \), there is a closed \( K \subseteq U \) such that if \( f \in C(X), \|f\|_u \leq 1, \) supp \( f \subseteq U, \) and \( f = 0 \) on \( K, \) then \( |\rho(f)| < \varepsilon. \)

Suppose no such \( K \) exists for some \( \varepsilon > 0. \) Pick \( f_1 \in C(X) \) such that \( \|f_1\|_u \leq 1, \) supp \( f_1 \subseteq U, \) and \( |\rho(f_1)| \geq \varepsilon. \) Pick \( |a_1| = 1 \) such that \( \rho(a_1 f_1) = |\rho(f_1)| \geq \varepsilon. \) Now \( K_1 = \text{supp } f_1 \) fails the conditions of the claim, so there is an \( f_2 \in C(X) \) supported in \( U \) such that \( \|f_2\| \leq 1, f_2 = 0 \) on \( K_1, \) and \( |\rho(f_2)| \geq \varepsilon. \) Pick \( |a_2| = 1 \) such that \( \rho(a_2 f_2) = |\rho(f_2)| \geq \varepsilon. \) Since \( f_2 = 0 \) on the support of \( f_1, \) we have that \( \rho(a_1 f_1 + a_2 f_2) = \rho(a_1 f_1) + \rho(a_2 f_2) \geq 2\varepsilon \) and \( \|a_1 f_1 + a_2 f_2\|_u \leq 1. \) Continuing by induction, we may find \( a_n \in C(X) \) supported in \( U \) that vanishes on the support of \( a_1 f_1 + a_2 f_2 + + a_n f_{n-1} \) and \( \|f_n\| \leq 1, \) while \( \rho(a_n f_n) = |\rho(f_n)| \geq \varepsilon. \) But then \( \rho(\sum a_i f_i) \geq n\varepsilon, \) which violates the boundedness of \( \rho \) for large \( n. \)

**Claim 2:** For \( U \) open, \( \lim_{f_n \to U} \rho(f) \) exists where the \( f \) are ordered pointwise.

We show that this net is a Cauchy net. In fact, let \( \varepsilon > 0 \) and let \( K \subseteq U \) be the closed set of Claim 1. Let \( f \) be any function such that \( f = 1 \) on \( K \) and \( f \not\in U. \) If \( f \leq g, h \ll U, \) then pick \( k \) with \( k = 1 \) on supp \( g \cup \text{supp } h, \) and \( k \ll U. \) Then \( g - k \) and \( h - k \) vanish on \( K, \) so we have that \( |\rho(g) - \rho(h)| \leq |\rho(g) - \rho(k)| + |\rho(h) - \rho(k)| = |\rho(g - k)| + |\rho(h - k)| \leq 2\varepsilon. \)

Define \( \mu(U) = \lim_{f_n \to U} \rho(f) \) for \( U \) open in \( X. \)

**Claim 3:** \( \mu \) is a signed quasi-measure on \( X. \)

Easily, \( \mu(U \cup V) = \mu(U) + \mu(V) \) if \( U \) and \( V \) are disjoint. Also \( \mu(\emptyset) = 0. \) Notice also that \( \mu(X) = \rho(1). \) We next show property b) after the definition of a signed quasi-measure.

Suppose that \( U \cup V = X. \) Pick \( C \subseteq U \) and \( K \subseteq V, \) closed such that \( C \cup K = X. \) Pick \( f_0 \ll U, g_0 \ll V \) with \( f_0 = 1 \) on \( C, g_0 = 1 \) on \( K, \) and such that \( f_0 \ll f \ll U \) implies \( |\rho(f) - \mu(U)| < \varepsilon \) and \( g_0 \ll g \ll V \) implies \( |\rho(g) - \mu(V)| < \varepsilon. \) Let \( h_0 \ll U \cap V \) with \( h_0 = 1 \) on \( C \cap K \) and such that \( h_0 \ll h \ll U \cap V \) implies that \( |\rho(h) - \mu(U \cap V)| < \varepsilon. \) Now, set \( f = \max\{f_0, h_0\} \) and \( g = \max\{g_0, h_0\}. \) Then \( f \ll U, g \ll V, \) and \( f \ll U \cap V, \) so \( |\rho(f) - \mu(U)| < \varepsilon, |\rho(g) - \mu(V)| < \varepsilon \) and \( |\rho(fg) - \mu(U \cap V)| < \varepsilon. \) Also, since \( f = 1 \) on \( C \) and \( g = 1 \) on \( K, \) and \( C \cup K = X, \) we have that \( (1 - f)(1 - g) = 0, \) so \( \rho((1 - f) + (1 - g)) = \rho(1 - f) + \rho(1 - g) \) and...
ρ(f + g) = ρ(1 + fg). This gives that ρ(f) + ρ(g) = ρ(f + g) = ρ(1) + ρ(fg), which shows that |μ(U) + μ(V) − μ(X) − μ(U ∩ V)| < 3ε.

Now suppose that \{U_n\}_{n=1}^N is a finite, disjoint collection of open sets. Let ε > 0 be given and choose \(f_n \prec U_n\) such that |ρ(f_n) − μ(U_n)| < ε. Now choose |a_n| = 1 such that |ρ(f_n)| = ρ(a_n f_n). Then |ρ \left( \sum a_n f_n \right)| ≤ ∥ρ∥, so ∥μ(U_n)∥ ≤ ∥ρ∥ + Nε. Now let ε → 0.

Finally, if U an open set and ε > 0 are given, choose K ⊆ U as in Claim 1, and argue as in the previous paragraph to show that if {U_n} is disjoint and open with \(U_n \subseteq U \setminus K\), then \(\sum |μ(U_n)| < ε\). In particular, if V ⊆ U \ K is open, |μ(V)| < ε.

Notice that \(∥μ∥ ≤ ∥ρ∥\).

**Claim 4:** We have that ρ = ρ_μ.

For each \(f ∈ C(X)\), the map \(ϕ → ρ(ϕ ∘ f)\) is bounded and linear on \(C(sp\ f)\). Thus, there is a signed measure ν_f on \(sp\ f\) such that

\[ ρ(ϕ ∘ f) = \int_R ϕ \, dν_f \]

for all \(ϕ ∈ C(sp\ f)\). Since \(ρ(f) = \int_R i \, dν_f\), we need only show that ν_f = μ_f for each \(f ∈ C(X)\). Notice that both measures are measures on \(R\).

Suppose that \(O\) is an open set in \(R\). Pick closed sets \(C_n \subseteq O\) such that \(C_n \subseteq \text{int}(C_{n+1})\) and \(O = \bigcup C_n\). Choose \(ϕ_n \prec O\) such that \(ϕ_n = 1\) on \(C_n\). Since \(X\) is compact, the sequence \(ϕ_n ∘ f\) is cofinal in the collection of functions \(g\) such that \(g \prec f^{-1}(O)\). Thus

\[ ν_f(O) = \lim_{n→∞} \int_R ϕ_n \, dν_f \]

\[ = \lim_{n→∞} ρ(ϕ_n ∘ f) \]

\[ = μ \left( f^{-1}(O) \right) \]

\[ = μ_f(O) \]

giving the required equality of measures. Thus \(∥μ∥ ≤ ∥ρ∥ = ∥ρ_μ∥ ≤ ∥μ∥\).

This shows that the map \(μ → ρ_μ\) is onto \(QL(X)\), and in fact, that any \(ρ ∈ QL(X)\) is the image of some \(μ ∈ QM(X)\) of the same norm. If we show that our map is one-to-one, we will be finished.

Assume ρ_μ = 0. Then μ_f = 0 for all \(f ∈ C(X)\). Thus \(\hat{f}(α) = μ_f(α, +∞) = 0\) for all \(α ∈ R\). If, now, \(U ⊆ X\) is open and ε > 0, pick \(K \subseteq U\) as in part (iv) of the definition of a signed quasi-measure. Choose any \(f ∈ C(X)\) with \(K \prec f ⊆ U\). Then

\[ |μ(U)| ≤ |μ(U) − μ(K)| + |\hat{f}(\frac{1}{2}) − μ(K)| + |\hat{f}(\frac{1}{2})| \]

\[ = |μ(U \setminus K)| + |μ \left( (f^{-1}(\frac{1}{2}, +∞)) \setminus K \right)| + |\hat{f}(\frac{1}{2})| \]

\[ ≤ 2ε. \]

Thus \(μ(U) = 0\) for all open sets, so μ = 0.

It should be noted that in Claim 2 we actually showed a stronger form of regularity. If \(U\) is open and ε > 0, then there is a closed set \(K ⊆ U\) with |μ|(U \ K) < ε. Thus |μ| obeys part (iv) of the definition of a quasi-measure.
There is yet another representation of $QL(X)$ that is sometimes useful. For each $f \in C(X)$, let $M(sp\ f)$ denote the collection of regular Borel measures on the compact set $sp\ f$ with the usual measure norm. Define $PM(X)$ to be

$$\{ (\nu_f) \in \prod_{f \in C(X)} M(sp\ f) : \nu_{\varphi f} = \varphi^* \nu_f \text{ for } \varphi \in C(sp\ f) \text{ and sup } \|\nu_f\| < \infty \}.$$  

Define a norm on $PM(X)$ by $\| (\nu_f) \| = \sup \|\nu_f\|$. Then it is easy to see that $PM(X)$ is a Banach space since this is true for each $M(sp\ f)$.

If $\mu$ is a signed quasi-measure, then the collection $(\mu_f)$ in the definition of $\rho_\mu$ is an element of $PM(X)$ with $\| (\mu_f) \| \leq \|\mu\|$. The induced map from $QM(X)$ to $PM(X)$ is evidently linear.

On the other hand, if $(\nu_f) \in PM(X)$, we may define $\rho(f) = \int_X i \ d \nu_f$. Then the argument just before the statement of the theorem shows that $\rho \in QL(X)$ with $\|\rho\| \leq \| (\nu_f) \|$. This, with the last paragraph shows that $PM(X)$ is isometrically isomorphic to both $QM(X)$ and $QL(X)$.

In particular $QM(X)$ is a Banach space. We can make it an ordered Banach space by taking the positive cone to be the collection of positive quasi-measures. The norm on this space is additive on the positive cone, but $QM(X)$ does not have to be a lattice. Thus, $QM(X)$ need not be an $L$-space. For example, in [3], Aarnes finds positive $[0,1]$-valued quasi-measures $\mu_1, \mu_2, \mu_3, \mu_4$ with $\mu_1 + \mu_3 = \mu_2 + \mu_4$. Since $[0,1]$-valued quasi-measures are extremal, there will then be no supremum of $\{\mu_1, \mu_2\}$.

For future convenience, we define the notation $(\mu, f) = \langle \rho_\mu, f \rangle = \rho_\mu(f)$.

Another aspect of the failure of the lattice property is that the set function $|\mu|$ need not have an extension to a positive quasi-measure on $X$. An example of this is given next.

**Example.** Let $X = [0,1] \times [0,1]$. In [2] and [6], it is shown how to construct the so-called three point quasi-measures. This is done as follows. A subset $A$ of $X$ is said to be solid if both $A$ and $X \setminus A$ are connected. If $C = \{x_1, x_2, x_3\}$ is a set with three elements, we define $\mu_C$ on solid sets by

$$\mu_C(A) = \begin{cases} 0 & \text{if } \text{card}(A \cap C) \leq 1, \\ 1 & \text{if } \text{card}(A \cap C) \geq 2. \end{cases}$$

There is then a unique extension of $\mu_C$ to a $[0,1]$-valued quasi-measure on $X$.

Now let $x_1 = (0,0), x_2 = (1,0), x_3 = (1,1)$, and $x_4 = (0,1)$. Let $C = \{x_1, x_2, x_3\}$, $D = \{x_2, x_3, x_4\}$, and $\mu = \mu_C - \mu_D$. If we let $U_1 = [0,1] \times [0,\frac{1}{2})$ and $U_2 = [0,1] \times (\frac{1}{2},1]$, we see that $2 = |\mu(U_1)| + |\mu(U_2)| \leq |\mu|(X) = \|\mu\| \leq 2$. Hence, $|\mu|(X) = 2$. We show that $|\mu|$ cannot be extended to closed sets in such a way that it is a positive quasi-measure. In particular, (ii) does not hold in the definition of a quasi-measure.

Assume such an extension exists. Let $V_1 = [0,\frac{3}{4}] \times [0,1]$ and $V_2 = (\frac{1}{4},1] \times [0,1]$. Write $K_1 = X \setminus V_1$ and $K_2 = X \setminus V_2$. Since $\mu_C(V_1) = \mu_D(V_1) = 0$, we see that $|\mu|(V_1) = 0 = |\mu|(V_1 \cap V_2)$. Hence $|\mu|(K_1) = |\mu|(K_1 \cup K_2) = 2$. Since $K_1$ and $K_2$ are disjoint, we must have that $|\mu|(K_2) = 0$, in other words that $|\mu|(V_2) = 2$. However this is not the case. In fact, $|\mu|(V_2) = 0$.

To see this we show that if $U \subseteq V_2$ is open, then $\mu(U) = 0$; that is $\mu_C(U) = \mu_D(U)$. Using symmetry and the fact that both $\mu_C$ and $\mu_D$ take on only 0 and
Proof. If each element, so \( \mu \topology \) if and only if \( \mu \topology \). Let \( f \in PM \) be the solid hull of \( K \), containing \( 2 \), which is connected. Hence, \( K \subseteq V_2 \), we have that \( x_2, x_3 \in W \). Since \( W \) is solid, we then have that \( W \topology \) if and only if \( \mu \topology \). Finally, \( \mu \topology \) is the supremum of \( \mu \topology \) as above, so \( \mu \topology = 0 \).

It would be nice to know that every signed quasi-measure is the difference of two positive quasi-measures. However, the failure of the lattice property and the fact that \( |\mu| \) need not be a quasi-measure brings this into question. At this point the issue remains open.

We now turn to another topology on \( QM(X) \) that is very useful.

**Definition 6.** The weak-* topology on \( QM(X) \) is the weakest topology making each map \( \mu \to \langle \mu, f \rangle \) continuous where \( f \) ranges over \( C(X) \).

Since each such map is linear and the collection of these maps separates points of \( QM(X) \), we obtain a locally convex topology on \( QM(X) \) where a net \( \mu_\alpha \) converges to \( \mu \) if and only if \( \langle \mu_\alpha, f \rangle \) converges to \( \langle \mu, f \rangle \) for every \( f \in C(X) \). This topology has been studied on the space of positive quasi-measures in [3].

**Proposition 7.** Let \( \mu_\alpha \) be a net in \( QM(X) \) and \( \mu \in QM(X) \). Let \( \langle \mu_\alpha, f \rangle \) and \( \langle \mu, f \rangle \) be the corresponding elements of \( PM(X) \). Then \( \mu_\alpha \) converges to \( \mu \) in the weak-* topology if and only if \( \mu_\alpha, f \) converges to \( \mu, f \) in the weak-* topology of \( M(\sp \ f) \) for each \( f \in C(X) \). Thus, the unit ball in \( QM(X) \) is weak-* compact.

**Proof.** If \( \mu_\alpha, f \) converges to \( \mu, f \) for each \( f \in C(X) \), then \( \langle \mu_\alpha, f \rangle = \int_\mathbb{R} i \ d\mu_\alpha, f \) converges to \( \int_\mathbb{R} i \ d\mu, f = \langle \mu, f \rangle \).

Conversely, if \( \mu_\alpha \) converges to \( \mu \) weak-* then for each \( \varphi \in C(\sp \ f) \), we have

\[
\lim_\alpha \int_\mathbb{R} \varphi \ d\mu_\alpha, f = \lim_\alpha \langle \mu_\alpha, \varphi \circ f \rangle = \langle \mu, \varphi \circ f \rangle = \int_\mathbb{R} \varphi \ d\mu, f.
\]

Thus \( \mu_\alpha, f \) converges to \( \mu, f \) in the weak-* topology.

Since the map \( \mu, f \rightarrow \varphi, f \) is weak-* continuous, the compactness of the unit ball of \( QM(X) \) follows from the compactness of the unit balls of \( M(\sp \ f) \).

**Definition 8.** Let \( X \) and \( Y \) be compact Hausdorff spaces. A quasi-linear map from \( C(X) \) to \( C(Y) \) is a map, \( T \), which is linear on each singly generated subalgebra of \( C(X) \) and which is bounded in the sense that there is an \( M < \infty \) with \( \|T(f)\| \leq M\|f\| \). If, in addition, \( T \) is multiplicative on each singly generated subalgebra, we say that \( T \) is a quasi-homomorphism.

For example, let \( \rho \) be a positive quasi-linear functional on \( X \) and let \( Y \) be any compact Hausdorff space. Define \( T_\rho : C(X \times Y) \rightarrow C(Y) \) by \( T_\rho(f)(y) = \rho(f^y) \) where \( f^y(x) = f(x, y) \). It is noted in [5] that \( T_\rho \) is a quasi-linear map. If \( \eta : C(Y) \rightarrow \mathbb{R} \)
is an additional quasi-linear functional, it is also noted there that the composition of $T_\rho$ and $\eta$ need not be quasi-linear.

**Proposition 9.** There is a one to one correspondence between quasi-linear maps from $C(X) \to C(Y)$ and norm-bounded functions $Y \to QM(X)$ which are weak-* continuous. Specifically, if $y \to \mu_y$ is a bounded, weak-* continuous map of $Y$ into $QM(X)$, the corresponding quasi-linear map is $T(f)(y) = (\mu_y, f)$.

The proof of this is evident. For comparison, there is a similar correspondence between the quasi-homomorphisms from $C(X) \to C(Y)$ and weak-* continuous functions $Y \to X^*$ where $X^*$ is the collection of $\{0,1\}$-quasi-measures on $X$. See [4] for details.

**Proposition 10.** Let $T: C(X) \to C(Y)$ be a quasi-homomorphism and $S: C(Y) \to C(Z)$ be a quasi-linear map. Then the composition $S \circ T: C(X) \to C(Z)$ is a quasi-linear map. If $y \to \mu_y$ is the map from $Y$ to $X^*$ corresponding to $T$, $z \to \nu_z$ the map from $Z$ into $QM(Y)$ corresponding to $S$, and $z \to \omega_z$ the map from $Z$ to $QM(X)$ corresponding to $S \circ T$, then for $U$ open in $X$,

$$\omega_z(U) = \nu_z(\{y \in Y : \mu_y(U) = 1\}).$$

**Proof.** That the composition is quasi-linear is evident.

Let $U \subseteq X$ be open and $\varepsilon > 0$. Let $W = \{y : \mu_y(U) = 1\}$.

**Claim:** For each $y \in W$ there is a compact $K_y \subseteq U$ and open set $y \in V_y$ such that $y' \in V_y$ implies $\mu_y(K_y) = 1$.

Since $\mu_y(U) = 1$, there is a compact $K_1 \subseteq U$ with $\mu_y(K_1) = 1$. Let $f \in C(X)$ with $K_1 \prec f \prec U$. Then $\langle \mu_y, f \rangle = 1$, so by weak-* continuity, there is an open set $y \in V$ such that $y' \in V$ implies that $\langle \mu_y, f \rangle > 2/3$. Let $K_y = \{x : f(x) \geq 1/3\}$. Since each $\mu_y$ is a $\{0,1\}$-quasi-measure, $\mu_y(K_y) = 1$ for $y' \in V$.

In particular, we see that $W$ is open in $Y$. This also shows that if $A \subseteq X$ is closed then $\{y : \mu_y(A) = 1\}$ is closed in $Y$. Now we can find compact sets $K \subseteq U$ and $C \subseteq W$ such that $K \prec f \prec U$ implies $|\omega_z(f) - \omega_z(U)| < \varepsilon$ and $C \prec g \prec W$ implies $|\nu_z(g) - \nu_z(W)| < \varepsilon$.

For each $y \in C$, pick $V_y$ and $K_y$ as in the claim. Choose finitely many $V_y$ to cover $C$, say $V_1, \ldots, V_n$. Let $K_1, \ldots, K_n$ be the corresponding $K_y$.

Set $L = K \cup K_1 \cup \cdots \cup K_n$ and find $L \prec f \prec U$. Recall $T(f)(y) = \langle \mu_y, f \rangle$ for $y \in Y$. If $y \in C$, say $y \in V_j$, $1 \leq \mu_y(K_j) \leq \mu_y(L) \leq \langle \mu_y, f \rangle \leq \mu_y(U) \leq 1$, so $C \prec T(f)$. Also, if $F$ is the support of $f$, then $E = \{y : \mu_y(F) = 1\}$ is closed and contained in $W$. If $y \not\in E$, $\mu_y(F) = 0$, so $\langle \mu_y, f \rangle = 0$. Thus, the support of $T(f)$ is contained in $E \subseteq W$. Hence $C \prec T(f) \prec W$.

Thus, $|\omega_z(U) - \langle \omega_z, f \rangle| < \varepsilon$ and $|\nu_z(W) - \langle \nu_z, T(f) \rangle| < \varepsilon$. However, $\langle \omega_z, f \rangle = S \circ T(f)(z) = \langle \nu_z, T(f) \rangle$. Hence $|\omega_z(U) - \nu_z(W)| < 2\varepsilon$. Now let $\varepsilon$ go to 0.

Another interpretation of this result may be obtained by noting that a quasi-homomorphism $T: C(X) \to C(Y)$ induces an image transformation $q : A(X) \to A(Y)$ (see [4] for details). In particular, for $A \in A(X)$, we have $q(A) = \{y : \mu_y(A) = 1\}$. For $\nu \in QM(Y)$, we can then define $q^*\nu \in QM(X)$ by $q^*\nu(A) = \nu(q(A))$. The above proposition then states that $\omega_z = q^*\nu_z$ for $z \in Z$.

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