WAVELET TRANSFORM AND ORTHOGONAL DECOMPOSITION OF $L^2$ SPACE ON THE CARTAN DOMAIN $BDI(q = 2)$

QINGTANG JIANG

Abstract. Let $G = (\mathbb{R}_+^* \times SO_0(1, n)) \ltimes \mathbb{R}^{n+1}$ be the Weyl-Poincaré group and $KAN$ be the Iwasawa decomposition of $SO_0(1, n)$ with $K = SO(n)$. Then the “affine Weyl-Poincaré group” $G_a = (\mathbb{R}_+^* \times AN) \ltimes \mathbb{R}^{n+1}$ can be realized as the complex tube domain $\Pi = \mathbb{R}^{n+1} + i\mathbb{C}$ or the classical Cartan domain $BDI(q = 2)$. The square-integrable representations of $G$ and $G_a$ give the admissible wavelets and wavelet transforms. An orthogonal basis $\{\psi_k\}$ of the set of admissible wavelets associated to $G_a$ is constructed, and it gives an orthogonal decomposition of $L^2$ space on $\Pi$ (or the Cartan domain $BDI(q = 2)$) with every component $A_k$ being the range of wavelet transforms of functions in $H^2$ with $\psi_k$.

1. Introduction

The wavelet transform is associated to the square-integrable representation of a locally compact group. Let $G$ be such a group with left Haar measure $dx$ and $x \rightarrow U(x)(x \in G)$ be an irreducible unitary representation of $G$ in a Hilbert space $\mathcal{H}$. A vector $\psi \in \mathcal{H}$ is said to be admissible if it satisfies the following “admissibility condition”:

$$0 < c_\psi := \int_G |(\psi, U(x)\psi)|^2 dx / (\psi, \psi) < \infty,$$

where $(\cdot, \cdot)$ is the inner product of $\mathcal{H}$. We denote the set of all such vectors by $AW$. If $AW \neq \emptyset$, then the representation $U$ is called square-integrable. For $\psi \in AW$, $f \rightarrow (f, U(x)\psi)$ is called the “continuous wavelet transform” of $f \in \mathcal{H}$ (cf. [4], [5]), and

$$f = \frac{1}{c_\psi} \int_G W_\psi f(x)U(x)\psi dx.$$

(1.2) is usually called the reconstructing formula, and it is one of the main motivations for the study of the wavelet transform. In this paper we will consider the wavelet transform associated to the Weyl-Poincaré group and its quotient group, and then give an orthogonal decomposition of $L^2$ space on the Cartan domain $BDI(q = 2)$.
Let \( C := \{(x_0, x_1, \cdots, x_n) : x_0^2 - x_1^2 - \cdots - x_n^2 > 0, x_0 > 0\} \) be the forward light cone (the Lorentz cone) and \( \Pi := \mathbb{R}^{n+1} + iC \). The complex tube domain \( \Pi \) is an (unbounded) realization of the Cartan domain \( BDI(q = 2) \ (SO_0(n, 2)/SO(n) \times SO(2)) \) (see [13], [15]). Let \( G := (\mathbb{R}_+^n \times SO_0(1, n)) \times \mathbb{R}^{n+1} \) be the Weyl-Poincaré group; it is the automorphism group of \( \Pi \), and \( G \) modulo a maximal compact subgroup \( SO(n) \) is isomorphic to \( \Pi \). Motivated by the work on quantization on the Cartan domain \( BDI(q = 2) \) in [13], [14] and on wavelet transform in [1], we considered in [7] the wavelet transform associated to \( G \) and its homogeneous space \( \Pi \).

Denote
\[
(1.3) \quad H^2 := \{ f(y) : \text{supp } \hat{f} \subset \overline{C}, f \in L^2(\mathbb{R}^{n+1}, dx) \},
\]
where \( \overline{C} \) is the closure of \( C \). In [7], from the square-integrable representation of the Weyl-Poincaré group \( G \) on \( H^2 \), we got the corresponding admissibility condition:
\[
0 < \int_C |\hat{\psi}(x)|^2 \frac{dx}{r(x)^{n+1}} < \infty.
\]
For such \( \psi \), (1.2) holds, i.e. for any \( f \in H^2 \), it is reconstructed from functions \( W_\psi f(y) \) on \( G \). In fact it doesn’t need so many variables for its reconstruction, which is very important in applications. On the other hand, from the viewpoint of the decomposition of \( L^2 \) on \( \Pi = \mathbb{R}^{n+1} + iC \) via the wavelet transform, we need a wavelet transform \( W_\psi \) such that \( W_\psi f(z) \) are functions on \( \Pi \). Therefore we considered there the wavelet transform associated to the homogeneous space \( \Pi \) (such a wavelet transform was just the one considered in [1]). In order that for any \( f \in H^2 \) it can be reconstructed from \( W_\psi f(z) \), the admissibility condition in this case is that
\[
(1.4) \quad \int_C |\hat{\psi}(\Lambda_y x)|^2 dy/r(y)^{\frac{n+1}{2}} = \int_C |\hat{\psi}(y)|^2 dy/r(y)^{\frac{n+1}{2}} \quad \text{for all } x \in C,
\]
and is finite, where \( \Lambda_y \) is an element in \( \mathbb{R}_+^n \times SO_0(1, n) \) such that \( \Lambda_y \omega = y \).

The condition in (1.4) is troublesome if we wish to give an orthogonal basis for the set of admissible wavelets which can give an orthogonal decomposition of \( L^2 \) on \( \Pi \) (or on the Cartan domain \( BDI(q = 2) \)). In this paper we will consider again the wavelet transform associated to \( G/SO(n) \) such that the condition (1.4) can be removed.

In §2 we introduce the wavelet transforms associated to \( G \) and to \( G/SO(n) \) and a kind of generalized wavelet transform. The main part of this paper is §3 and §4. In §3, we give a correspondence from \( \Pi \) to the quotient ("affine ") group \( G_a := (\mathbb{R}_+^n \times AN) \times \mathbb{R}^{n+1} \) of \( G \) from the Iwasawa decomposition of \( SO_0(1, n) = KAN \); then from the square-integrable representation of \( G_a \) on \( H^2 \) we define the associated wavelet transform and get that the admissibility condition is
\[
0 < \int_C |\hat{\psi}(y_0, y_1, y_n)|^2 \frac{dy_0 dy_1 dy_n}{(y_0 + y_1)^{n-1} r(y)} < \infty.
\]
In §4, by Laguerre polynomials, Jacobi polynomials and spherical harmonics we give an orthogonal basis for the set of the admissible wavelets which turns out to give an orthogonal decomposition of \( L^2 \) on \( \Pi \).

2. SQUARE-INTEGRABLE GROUP REPRESENTATION

Let \( ^tM \) denote the transpose of a matrix \( M \), and display a vector \( x \in \mathbb{R}^{n+1} \) formally as a column vector in the form \( x = ^t(x_0, x^*) = ^t(x_0, x_1, \cdots, x_n) \). Let \( I_n \)
denote the $n \times n$ identity matrix and let
\[ J := \begin{pmatrix} 1 & 0 \\ 0 & -I_n \end{pmatrix} \in GL(\mathbb{R}^{n+1}). \]
For any $x, y \in \mathbb{R}^{n+1}$, let $x \cdot y$ or $[x, y]$ be the bilinear form
\[ x \cdot y := [x, y] := t \hat{x} \hat{y} = x_0 y_0 - x_1 y_1 - \cdots - x_n y_n, \]
and $r(x) := x \cdot x$; also let $xy := (x, y) := \sum_{i=0}^{n} x_i y_i$ be the usual Euclidean inner product of $\mathbb{R}^{n+1}$.

The Lorentz group $SO(1, n)$ is the subgroup of $\Lambda \in GL(\mathbb{R}^{n+1})$ satisfying $\det \Lambda = 1$ and $t \Lambda J \Lambda = J$ (equivalently $[\Lambda x, \Lambda y] = [x, y]$ for any $x, y \in \mathbb{R}^{n+1}$). Let $SO_0(1, n)$ denote the connected component of the identity of $SO(1, n)$, and let $G_0 := \mathbb{R}_+^\times \times SO_0(1, n)$. Then the Weyl-Poincaré group $G$ is the semi-product $G_0 \ltimes \mathbb{R}^{n+1}$. Write $g \in G$ as $g = (a, \Lambda, b)$ with $a > 0, \Lambda \in SO(1, n), b \in \mathbb{R}^{n+1}$ then the group law of $G$ is given by
\[ (a, \Lambda, b)(a', \Lambda', b') = (aa', \Lambda \Lambda', a\Lambda' b + b) \]
for any $(a, \Lambda, b), (a', \Lambda', b') \in G$, and one can see the left invariant measure $d\mu(g)$ of $G$ is $a^{-(n+2)} da db$; here $d\Lambda$ is the invariant measure of $SO_0(1, n)$. As in § 1 let $C$ denote the forward light cone (the Lorentz cone) and $\Pi = \mathbb{R}^{n+1} + i \mathbb{C}$ the complex tube domain over $C$. The actions of $G_0$ on $C$ and $G$ on $\Pi$ are invariant under the actions of $G_0$ and $G$ respectively; here $dy, dx$ denote the Lebesgue measure on $\mathbb{R}^{n+1}$.

Still let $H^2$ denote the space defined by (1.3) and let $\langle \cdot, \cdot \rangle$ or $\langle \cdot, \cdot \rangle_{H^2}$ denote its inner product. Let $U_g$ be the unitary representation of $G$ on $H^2$ given by
\[ U_g f(x) := a^{\frac{n+1}{2}} f(\Lambda^{-1} x - \frac{b}{a}). \]
Taking the Fourier transform with respect to $x$ in both sides of (2.1), we get
\[ (U_g f)^\wedge(\xi) = a^{\frac{n+1}{2}} \hat{\psi}(a t \Lambda \xi) e^{-i t b}, \]
and therefore we know $U_g$ is irreducible on $H^2$. By the Plancherel formula,
\[ \int_C |\hat{\psi}(U_g \psi)|^2 d\mu = \int_{C} \int_{\mathbb{R}_+^\times SO_0(1, n) \times \mathbb{R}^{n+1}} |\hat{\psi}(\xi)\hat{\psi}(a t \Lambda \xi)|^2 \cdot \frac{a^{n+2} da db}{a^{n+2}} \]
\[ = \int_{\mathbb{R}_+^\times SO_0(1, n) \times \mathbb{R}^{n+1}} |\hat{\psi}(\xi)|^2 \cdot \frac{a^{n+2} da db}{a^{n+2}} \]
\[ = \int_{\mathbb{R}_+^\times SO_0(1, n)} |\hat{\psi}(a t \Lambda \xi)|^2 \cdot \frac{a^{n+2} da db}{a^{n+2}} \]
In the above equations, a constant $1/(2\pi)^{n+1}$ is dropped. In the following, for simplicity, we also drop this constant in some equations when we use the Plancherel formula.

For $\xi \in C$, there exists $\Lambda_\xi \in SO_0(1, n)$ such that $\Lambda_\xi \omega = \xi / r(\xi)^{\frac{1}{2}}$ (see [16, p.505]); here
\[ \omega := (1, 0, \cdots, 0). \]
Let Theorem 2.2. we have

\[ W \quad A \quad i.e. \quad L \]

Therefore the admissibility condition is

\[ \int_{\mathbb{R}^n} |\psi(a^t \Lambda \xi)|^2 \frac{d \Lambda \alpha}{a} = \int_{0}^{\infty} \int_{SO_0(1,n)} |\psi(a^t \Lambda \omega)|^2 \frac{d \Lambda \alpha}{a}. \]

From the “Euler angles” expression of \( \Lambda \in SO_0(1,n) \) as given in [16, p.508] and the “spherical coordinates” expression of \( x \in C \) as given in [16, p.504], we can get that for any \( \xi \in C \)

\[ \int_{\mathbb{R}^n} |\psi(a^t \Lambda \xi)|^2 \frac{d \Lambda \alpha}{a} = \int_{0}^{\infty} \int_{SO_0(1,n)} |\psi(a^t J^{-1} \omega)|^2 \frac{d \Lambda \alpha}{a} \]

\[ = \int_{C} |\tilde{\psi}(x)|^2 \frac{dx}{r(x)^{2+n}}. \]

Therefore the admissibility condition is

\[ 0 < c_\psi = \int_{C} |\tilde{\psi}(x)|^2 \frac{dx}{r(x)^{2+n}} < \infty. \]

Let \( AW \) denote the set of all admissible wavelets, i.e.

\[ AW := \{ \psi : \psi \in H^2, \ \psi \text{satisfies (2.2)} \}. \]

For \( \alpha > -1 \), let \( \psi \) be a function in \( H^2 \) defined by

\[ \hat{\psi}(\xi) = \begin{cases} r(\xi)^\alpha e^{-\xi_0}, & \text{for } \xi \in C, \\ 0, & \text{elsewhere}; \end{cases} \]

then \( \psi \in AW \) by a direct calculation. Thus the unitary irreducible representation given by (2.1) is square-integrable. For \( \psi \in AW \), the map \( f \rightarrow W_\psi f(g) := (f, U_g \psi) \) is the wavelet transform, and it is an isometry (up to a constant) from \( H^2 \) into \( L^2(G, d\mu(g)) \). One can get

**Theorem 2.1.** Let \( \psi, \phi \in AW \). Then for any \( f, h \in H^2 \),

\[ \langle W_\psi f, W_\psi h \rangle = (K^{-\frac{1}{2}} \phi, K^{-\frac{1}{2}} \psi)(f, h), \]

where \( \langle , \rangle \) is the inner product of \( L^2(G, d\mu(g)) \) and \( K \) is the positive operator given by

\[ (K \psi)^\wedge(\xi) = r(\xi)^{2+n} \tilde{\psi}(\xi). \]

**Theorem 2.2.** Let \( \psi \in AW \). Then for any \( f \in H^2 \),

\[ f(x) = \frac{1}{c_\psi} \int_G W_\psi f(g) U_g \psi(x) d\mu(g). \]

For \( \psi \in AW \), it is a (generalized) state in \( H^2 \) which can be written as \( \psi \). Then \( \{ U_g \psi \}_{g \in G} = \{ U_g |\psi\rangle \}_{g \in G} \) is a coherent state system [9]. Denote

\[ A_\psi := \{ \langle f | U_g \psi \rangle : f \in H^2 \} = \{ W_\psi f(g) : f \in H^2 \}, \]

i.e. \( A_\psi \) is the matrix coefficient space of the representation \( U_g \) or the range of the wavelet transform \( W_\psi \) of \( H^2 \) with \( \psi \). Then \( A_\psi \) is a Hilbert space with reproducing kernel.
Theorem 2.3. Let $K_\psi(g,g')$ denote the reproducing kernel of $A_\psi$. Then

$$K_\psi(g,g') = \frac{1}{c_\psi} (U_g^* \psi, U_g \psi).$$

Let $SO(n)$ denote the group of rotations of $\mathbb{R}^n$ and in this paper also let it denote the subgroup of $SO_0(1,n)$ fixing the point $\omega$, i.e. every element $u$ of this subgroup is given by

$$u = \begin{pmatrix} 1 & 0 \\ \bar{u} & \bar{u} \end{pmatrix}, \quad \text{with } \bar{u} \in SO(n).$$

We even let $SO(n)$ denote the subgroup $\{1 \times SO(n)\} \ltimes \{0\}$ of $G$. Then $SO(n)$ is a maximal subgroup of $G$ and the quotient space $G/\text{SO}(n)$ is isomorphic to the homogeneous space $\Pi$. From Theorem 2.2, we know for any $f \in H^2$ it is reconstructed from $W_\psi f(g)$. The wavelet transform $W_\psi f(g)$ is a function of $n + 2 + \frac{n(n+1)}{2}$ variables. In fact $f$ doesn’t need so many variables for its reconstruction. We consider the wavelet transform associated to $G/\text{SO}(n)$ or to $\Pi$ by a correspondence from $\Pi$ to $G/\text{SO}(n)$.

Recall that the action of $g = (a, \Lambda, b) \in G$ on $\Pi$ is given by

$$(a, \Lambda, b) z = a\Lambda z + b = a\Lambda (x + iy) + b.$$ Denote $z_0 := 0 + i\omega \in \Pi$. For $z = x + iy \in \Pi$, there exist a family of $g \in G$ such that $g z_0 = z$. In the following we choose $g = (r(y)^{1/2}, \Lambda_y', x)$, where $\Lambda_y' = r(y)^{-\frac{1}{2}}\Lambda_y$ and $\Lambda_y$ is given in [14]:

$$(2.4) \quad \Lambda_y := \begin{pmatrix} y_0 & -i y^* \\ y^* & r(y)^{\frac{1}{2}}I + (y_0 + r(y)^{1/2})^{-1} y^* t y^* \end{pmatrix}, \quad \text{with } y = (y_0, y^*).$$

Then $\Lambda_y' = \Lambda_y, \Lambda_y \omega = y$ and $\Lambda_y' \in SO_0(1,n), \Lambda_y \in G_0$. For each $z \in \Pi$, from (2.1), we define an operator $U_z$ on $H^2$ by

$$U_z \psi(t) := r(y)^{-\frac{n+1}{2}} \psi \left( r(y)^{-\frac{1}{2}} \Lambda_y^{-1}(t - x) \right)$$

or by

$$(U_z \psi)^{(\xi)} := r(y)^{\frac{n+1}{2}} \widehat{\psi} \left( r(y)^{1/2} \Lambda_y' \xi \right) e^{-ix\xi} = r(y)^{\frac{n+1}{2}} \widehat{\psi}(\Lambda_y \xi)e^{-ix\xi},$$

since $\Lambda_y' = \Lambda_y$ from (2.4). Then we define the associated wavelet transform for $f \in H^2$ by

$$(2.6) \quad W_\psi f(z) := (f, U_z \psi)$$

or by

$$(2.7) \quad (W_\psi f)^{(\xi,y)} := r(y)^{\frac{n+1}{2}} \overline{\widehat{\psi}(\Lambda_y \xi)f(\xi)},$$

where $(W_\psi f)^{(\xi,y)}$ denotes the Fourier transform of $W_\psi f(z)$ with respect to the variables $x$.

Any $f \in H^2$ also can be reconstructed from the wavelet transform $W_\psi f(z)$ for some $\psi$, i.e. the following formula holds:

$$(2.8) \quad f(\tau) = c \int_{\Pi} W_\psi f(z) U_z \psi(\tau) d\mu(z),$$
where \( d\mu(z) = dx dy/r(y)^{n+1} \) is the invariant measure on \( \Pi \) under \( G \). Let us give the corresponding admissibility condition for \( \psi \). By the Plancherel formula
\[
\int_{\Pi} |W\psi(z)|^2 d\mu(z) = \int_{\Pi} |\hat{\psi}(\xi)\hat{\psi}(\Lambda y \xi)|^2 d\xi dy/r(y)^{n+1} \geq 0.
\]
\[
= \int_{C} |\hat{\psi}(\xi)|^2 \int_{C} |\hat{\psi}(\Lambda y \xi)|^2 \frac{dy}{r(y)^{n+1}} d\xi.
\]
Thus in order that (2.8) holds, \( \psi \) must be such that \( \int_{C} |\hat{\psi}(\Lambda y \xi)|^2 dy/r(y)^{n+1} \) is independent of all \( \xi \in C \); especially, if \( \xi = \omega \), then it is \( \int_{C} |\hat{\psi}(\omega)|^2 dy/r(y)^{n+1} \). Thus the admissibility condition in this case is that \( \psi \) satisfies (2.2) and
\[
(2.9) \quad \int_{C} |\hat{\psi}(\Lambda y \xi)|^2 dy/r(y)^{n+1} = \int_{C} |\hat{\psi}(\xi)|^2 dy/r(y)^{n+1} \text{ for all } \xi \in C.
\]

If \( n > 1 \), (2.2) does not imply (2.9), i.e. there are many functions \( \psi \in H^2 \) satisfying (2.2) but not (2.9). If \( \psi \) is invariant under the action of \( SO(n) \) or “radial”, i.e. \( \psi(x_0, \rho x^*) = \psi(x_0, x^*) \) for all \( \rho \in SO(n) \), then it satisfies (2.9). In fact, in this case, \( \psi(\xi) \) is also “radial” and can be written as
\[
\hat{\psi}(\xi) = \phi(\rho_0, |\xi|^2) = \phi ((\xi, \omega), (\xi, \omega)^2 - r(\xi)),
\]
where \( \cdot, \cdot \) is the usual Euclidean inner product of \( \mathbb{R}^{n+1} \) as mentioned above. For \( \xi, y \in C \), let \( \Lambda_\xi, \Lambda_y \in G_0 = \mathbb{R}_+^* \times SO_0(1, n) \) be defined by (2.4). Then
\[
(\Lambda_y \xi, \omega) = (\xi, \Lambda_y \omega) = (\Lambda_\xi \omega, y) = (\omega, \Lambda_\xi y),
\]
since \( ^t \Lambda_y = \Lambda_y, ^t \Lambda_\xi = \Lambda_\xi \), and we have
\[
\int_{C} |\hat{\psi}(\Lambda_y \xi)|^2 \frac{dy}{r(y)^{n+1}} = \int_{C} |\phi ((\Lambda_y \xi, \omega), (\Lambda_y \xi, \omega)^2 - r(\Lambda_y \xi))|^2 \frac{dy}{r(y)^{n+1}}
\]
\[
= \int_{C} |\phi ((\omega, \Lambda y \xi), (\omega, \Lambda y \xi)^2 - r(\omega))|^2 \frac{dy}{r(y)^{n+1}}
\]
\[
= \int_{C} |\phi ((\omega, y), (\omega, y)^2 - r(y))|^2 \frac{dy}{r(y)^{n+1}}
\]
\[
= \int_{C} |\phi(y_0, y_0^2 - r(y))|^2 \frac{dy}{r(y)^{n+1}} = \int_{C} |\hat{\psi}(\xi)|^2 \frac{dy}{r(y)^{n+1}}.
\]
The third equality is from the invariant property of \( dy/r(y)^{n+1} \) under the action of \( G_0 \) and the fact that \( r(y)r(\xi) = r(\Lambda_\xi y) \). Thus when \( \psi \) is “radial”, the admissibility condition for (2.8) is (2.2).

For \( \psi \) satisfying (2.2) and (2.9), define the wavelet transform for \( f \in H^2 \) via (2.6); then for such a wavelet transform we can establish theorems similar to Theorems 2.1, 2.2, 2.3, but we won’t bother to list them here.

Condition (2.9) is troublesome if we want to give an orthogonal basis for the set of admissible wavelets. We now introduce, as in [8], a kind of generalized wavelet transform associated to \( \Pi \). In this case the admissibility condition is (2.2) and (2.9) will be removed.

For each \( z \in \Pi \), let us introduce an operator \( \bar{U}_z \) on \( H^2 \) similar to \( U_z \) defined above. For \( z \in \Pi \), define \( \bar{U}_z \) on \( H^2 \) by (compare with the definition of \( U_z \) given
in (2.5))
\begin{equation}
(\tilde{U}_z\psi)^\wedge(\xi) := r(y)^{\frac{n+1}{2}} \hat{\psi}(\Lambda_\xi y)e^{-i\xi},
\end{equation}
where $\psi \in H^2$ and $\Lambda_\xi$ is given by (2.4) corresponding to $\xi \in C$. Then from (2.10) we define a kind of generalized wavelet transform $\tilde{W}_\psi$ of functions $f$ in $H^2$ with some $\psi$ by
\begin{equation}
\tilde{W}_\psi f(z) := (f, \tilde{U}_z \psi),
\end{equation}
or by (compare with (2.7))
\begin{equation}
(\tilde{W}_\psi f)^\wedge(\xi, y) := r(y)^{\frac{n+1}{2}} \hat{\psi}(\Lambda_\xi y)\hat{f}(\xi).
\end{equation}

Let us give the condition on $\psi$ so that for any $f \in H^2$ it can be reconstructed from $\tilde{W}_\psi f(z)$. Of course, $\tilde{W}_\psi$ must satisfy $\int_\Pi |\tilde{W}_\psi f|^2 d\mu(z) = C\|f\|^2$, where $d\mu(z)$ is the invariant measure $dz dy/r(y)^{n+1}$. We have
\begin{equation}
\int_\Pi |\tilde{W}_\psi f|^2 d\mu(z) = \int_\Pi |\hat{\psi}(\Lambda_\xi y)\hat{f}(\xi)|^2 d\xi dy \frac{dy}{r(y)^{\frac{n+1}{2}}} = \int_C \int_C |\hat{\psi}(\Lambda_\xi y)|^2 dy \frac{dy}{r(y)^{\frac{n+1}{2}}} |\hat{f}(\xi)|^2 d\xi.
\end{equation}

Then by the invariant property of $dy/r(y)^{\frac{n+1}{2}}$ under actions of $G_0$, we have
\begin{equation}
\int_C |\hat{\psi}(\Lambda_\xi y)|^2 dy \frac{dy}{r(y)^{\frac{n+1}{2}}} = \int_C |\hat{\psi}(r(\xi)^2\Lambda_\xi y)|^2 dy \frac{dy}{r(y)^{\frac{n+1}{2}}} = \int_C |\hat{\psi}(r(y)^{\frac{n+1}{2}} y)|^2 dy \frac{dy}{r(y)^{\frac{n+1}{2}}}.
\end{equation}

Thus in this case, the admissibility condition is $\int_C |\hat{\psi}(y)|^2 dy/r(y)^{\frac{n+1}{2}} < \infty$. Let $AW$ denote the set of all the admissible wavelets, i.e. $AW$ consists of the $\psi$ satisfying (2.2). Then for $\psi \in AW$, we can prove that any $f \in H^2$ can also be reconstructed from $\tilde{W}_\psi f(z)$.

Though it is easy to check the admissibility condition for this kind of generalized wavelet transform and to give an orthogonal basis for $AW$ via Laguerre polynomials, Jacobi polynomials and spherical harmonics which will give an orthogonal decomposition of $L^2$ on $\Pi$, still such a wavelet transform does not associate to the square-integrable group representation and looks artificial. In next section, we will introduce another kind of wavelet transform from the square-integrable representation of a quotient group of the Weyl-Poincaré group.

3. SQUARE-INTEGRABLE REPRESENTATION MODULO A SUBGROUP

In this section we will introduce a quotient group of the Weyl-Poincaré group $G = (\mathbb{R}_+^* \times SO_0(1, n)) \ltimes \mathbb{R}^{n+1}$ and a kind of wavelet transform from the square-integrable representation of this group. Let $K = SO(n)$ be the rotation group of $\mathbb{R}^n$; as in §2 it also denotes a subgroup of $SO_0(1, n)$ and even a subgroup of $G$. Then $K$ is a maximal compact subgroup of $G$. The quotient group $G/K$ can realized as the complex tube domain $\Pi$ via the Iwasawa decomposition of $G$ (in fact the decomposition of $SO_0(1, n)$).

Let $g$ be the Lie algebra of $SO_0(1, n)$; then (see [12, p.222])
\begin{equation}
g = l \oplus a \oplus n,
\end{equation}
where \( l \) is the Lie algebra of \( K = SO(n) \), \( a \) is an one-dimensional algebra with generator

\[
a_0 = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}_{(n+1) \times (n+1)},
\]

and

\[
n = \{ x = \begin{pmatrix}
0 & 0 & t_s \\
0 & 0 & t_s \\
s & -s & O_{n-1}
\end{pmatrix} : s = t(s_2, \ldots, s_n) \in \mathbb{R}^{n-1} \}.
\]

Then \( SO_0(1, n) \) has the Iwasawa decomposition

\[
SO_0(1, n) = KAN
\]

with

\[
A = \{ a_r := \begin{pmatrix}
\text{chr} & shr & 0 \\
shr & \text{chr} & 0 \\
0 & 0 & I_{n-1}
\end{pmatrix} : r \in \mathbb{R}\},
\]

\[
N = \{ n_s := \begin{pmatrix}
\rho + 1 & -\rho & t_s \\
-\rho & 1 - \rho & t_s \\
s & -s & I_{n-1}
\end{pmatrix} : s = (s_2, \ldots, s_n) \in \mathbb{R}^{n-1}, \rho = \frac{1}{2}|s|^2 \}.
\]

The action of \( a_r \in A \) on \( N \) is given

\[
a_r : n_s \rightarrow a_r n_s a_r^{-1} = n_{e^{-r} s}.
\]

By a direct calculation, we have \( a_r a_{r'} = a_{r+r'}, n_s n_{s'} = n_{s+s'} \) and

\[
(3.1) \quad (a_r, n_s) (a_{r'}, n_{s'}) = a_{r+r'} n_{s+s-rr'}
\]

for any \( a_r, a_{r'} \in A, n_s, n_{s'} \in N \).

Let \( H_1 := \{ x \in C : r(x) = 1 \} \), the forward mass hyperboloid. We know \( SO_0(1, n)/K = AN \) is isomorphic to \( H_1 \), and for \( y' \in H_1 \) there exists a unique \( a_r, n_s \in AN \) such that

\[
a_r n_s \omega = y';
\]

also, we have

\[
(3.2) \quad \text{chr} + e^r o = y'_0, \quad \text{shr} + e^r o = y'_1, \quad s = t(y'_2, \ldots, y'_n).
\]

In this way we give a correspondence between \( AN \) and \( H_1 \) by

\[
(3.3) \quad y' = t(y'_0, y'_1, y'_s) \in H_1 \leftrightarrow a_r, n_s \in AN \quad \text{with} \quad e^{-r} = y'_0 - y'_1, \quad s = y'_s.
\]

Denote

\[
(3.4) \quad \Lambda_{r,s} := a_r n_s = \begin{pmatrix}
\text{chr} + e^r o & \text{shr} - e^r o & t_s e^r \\
\text{shr} + e^r o & \text{chr} - e^r o & t_s e^r \\
s & -s & I_{n-1}
\end{pmatrix};
\]

then \( \Lambda_{r,s} \in SO_0(1, n), \Lambda_{r,s} \Lambda_{r',s'} = \Lambda_{r+r',s+s-rr'}, \Lambda_{r,s}^{-1} = \Lambda_{-r,-s-rr'} \) and \( \Lambda_{r,s} \omega = y' \).

From (3.1), let us introduce a group, the “affine Weyl-Poincaré group”:

\[
G_a := \{(a, r, s, x) | a > 0, r \in \mathbb{R}, s \in \mathbb{R}^{n-1}, x \in \mathbb{R}^{n+1} \}
\]
The Fourier transform with respect to $\tau$

\[ (a, r, s, x)(a', r', s', x') = (aa', r + r', s' + se^{-r'}, x + a\Lambda_{r,s}x'). \]

We have $g^{-1} = (a, r, s, x)^{-1} = (\frac{1}{a}, -r, -se^{-r}, -\frac{1}{a}\Lambda_{r,s}^{-1}x)$, and $G_a$ has the left invariant measure

\[ d\mu(g) = a^{-(n+2)}dadrdsx. \]

The group $G_a$ defined above is a quotient group $(\mathbb{R}_+^* \times SO_0(1, n)) \ltimes \mathbb{R}^{n+1}/K$ of $G,$ and it is isomorphic to $\Pi$ by the following correspondence:

\[ z = x + iy \in \Pi \leftrightarrow (a, r, s, x) \in G_a, \]

with

\[ a = r(y)^\frac{1}{2}, \quad e^{-r} = r(y)^{-\frac{1}{2}}(y_0 - y_1), \quad s = r(y)^{-\frac{1}{2}}y_s \]

and $\Lambda_{r,s,\omega} = y/r(y)^\frac{1}{2}$ (here $\Lambda_{r,s}$ is given by (3.4)).

As usual, for $g \in G_a$, with $g^{-1} = \begin{pmatrix} A & B \\ 0 & 1 \end{pmatrix}, B \in \mathbb{R}^{n+1}, A \in \mathbb{R}_+^* \times SO_0(1, n), g$ acts on $\mathbb{R}^{n+1}$ by $g(\tau) = A\tau + B$. For $g = (a, r, s, x), g^{-1}$ can be written as

\[ \begin{pmatrix} \frac{1}{a}\Lambda_{r,s}^{-1} & -\frac{1}{a}\Lambda_{r,s}^{-1}x \\ 0 & 1 \end{pmatrix}, \]

and thus we have

\[ g(\tau) := \frac{1}{a}\Lambda_{r,s}^{-1}\tau - \frac{1}{a}\Lambda_{r,s}^{-1}x = \frac{1}{a}\Lambda_{r,s}^{-1}(\tau - x). \]

The above action of $g \in G_a$ on $\mathbb{R}^{n+1}$ induces a unitary representation of $G_a$ on $H^2,$ still denoted by $U_g$:

\[ U_g f(\tau) := \{g'(\tau)\}^\frac{1}{2} f(g(\tau)) = a^{-\frac{n+1}{2}} f\left(a^{-1}\Lambda_{r,s}^{-1}(\tau - x)\right). \]

Taking the Fourier transform with respect to $\tau$ in both sides of (3.8), we get

\[ (U_g f)(\xi) = a^{\frac{n+1}{2}} \int f(\Lambda_{r,s}^{-1}\tau)e^{-i\tau\xi} d\tau e^{-i\xi x} \]

\[ = a^{\frac{n+1}{2}} \int f(\Lambda_{r,s}^{-1}\tau)e^{-i(a\Lambda_{r,s})\xi}\Lambda_{r,s}^{-1}\tau d\tau e^{-i\xi x} = a^{\frac{n+1}{2}} \hat{f}(a\Lambda_{r,s}\xi)e^{-i\xi x}; \]

thus we know $U_g$ is irreducible on $H^2.$ We can get the admissibility condition (see the Appendix):

\[ 0 < C_\psi := \int_0^\infty \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\hat{\psi}(y_0, y_1, y_s)|^2 \frac{dy_0 dy_1 dy_s}{(y_0 + y_1)^n-1 f(y)} < +\infty. \]

When $n = 1$, (3.9) is just (2.2). Let

\[ AAW := \{\psi : \psi \in H^2, \psi \text{ satisfies (3.9)}\}. \]

Then by a direct calculation, we know the function $\psi$ defined by (2.3) is in $AAW$; thus the representation of $G_a$ given by (3.8) is square-integrable on $H^2.$ Let $\psi \in AAW,$ and define the wavelet transform for $f \in H^2$ from the square-integrable representation of $G_a$ by

\[ f \to (f, U_g\psi). \]
For such a wavelet transform, we can establish theorems similar to 2.1, 2.2 and 2.3; here the invariant measure of $G_\alpha$ is $d\mu(g)$ given in (3.5). We wouldn’t list them here.

By the correspondence given by (3.6) and (3.7), for each $z \in \Pi$, we define $U_z$ by (3.8), i.e.

$$U_z f(\tau) := r(y)^{-\frac{n+1}{4}} f \left( r(y)^{-\frac{1}{2}} \Lambda_{r,s}^{-1}(\tau - x) \right),$$

where $r, s$ are given by (3.7) and $\Lambda_{r,s} = a_r a_s$ by (3.4). Then for any fixed $\psi \in AAW$, or state $\psi$, applying $U_z$ to it, we get a coherent system $\{U_z|\psi\}_{z \in \Pi}$ [9]. Wavelet transforms of functions $f \in H^2$ defined by (3.10) are functions on $G_\alpha$. We now define the wavelet transform via $U_z$:

$$(3.11) \quad W_\psi f(z) := \langle f|U_z|\psi \rangle = (f, U_z \psi), \quad \text{for } f \in H^2,$$

then $W_\psi f(z)$ are functions on $\Pi$, and $f \in H^2$ can be reconstructed from $W_\psi f(z)$. Let $d\mu(z)$ be the invariant measure on $\Pi$:

$$d\mu(z) := dx dy/r(y)^{n+1}, \quad \text{with } z = x + iy;$$

then we have

**Theorem 3.1.** Let $\psi, \phi \in AAW$. Then for any $f, h \in H^2$,

$$(3.12) \quad \langle W_\psi f, W_\phi h \rangle = (K^{-\frac{1}{2}} \phi, K^{-\frac{1}{2}} \psi)(f, h),$$

where $\langle, \rangle$ is the inner product of $L^2(\Pi, d\mu(z))$ and $K$ is the positive operator given by

$$(K f)^\wedge(\xi) = (\xi_0 + \xi_1)^{n-1} r(\xi) \hat{f}(\xi).$$

**Theorem 3.2.** Let $\psi \in AAW$. Then for any $f \in H^2$,

$$(3.13) \quad f(\tau) = \frac{1}{C_\psi} \int_{\Pi} W_\psi f(z) U_z \psi(\tau) d\mu(z).$$

For $\psi \in AAW$, denote

$$(3.14) \quad A_\psi := \{ \langle f|U_z|\psi \rangle : f \in H^2 \} = \{ W_\psi f(z) : f \in H^2 \}.$$

Then $A_\psi$ is a subspace of $L^2(\Pi, d\mu(z))$ with a reproducing kernel.

**Theorem 3.3.** Let $K_\psi(z, z')$ denote the reproducing kernel of $A_\psi$. Then

$$K_\psi(z, z') = \frac{1}{C_\psi} (U_z \psi, U_z \psi).$$

In Theorem 3.1, (3.12) is also true for any $\psi, \phi \in H^2$ satisfying $(K^{-\frac{1}{2}} \psi, K^{-\frac{1}{2}} \phi) < \infty$. Clearly if $\psi, \phi \in AAW$, then $(K^{-\frac{1}{2}} \psi, K^{-\frac{1}{2}} \phi) < \infty$. With the identification (3.6) and (3.7), the tube domain $\Pi$ inherits the $G_\alpha$ group structure and (3.12) is in fact the Moyal formula; see [2]. For completeness, the proof of Theorem 3.1 will be also given in the Appendix.

In Theorem 3.2, (3.13) is true at least “in the weak sense”, i.e., taking the inner product of both sides of (3.13) with any $g \in H^2$ and commuting the inner product with the integral over $z$ in the right hand side leads to a true formula, which is
(3.12) with $\phi = \psi$. The convergence of the integral in (3.13) also holds in the following “strong sense”:

\[(3.15)\]
\[
\lim_{\delta_i \to 0, A, B \to +\infty} \|f(\tau) - C^{-1}_\psi \int_{|x| < A, \delta_1 < y < B, \delta_2 < r(y)/y_0^2} W_\psi f(z) U_z \psi(\tau) d\mu(z)\|_2 = 0.
\]

Here the integral stands for the unique element in $H^2$ that has inner product with $g \in H^2$ given by

\[
\int_{|x| < A, \delta_1 < y < B, \delta_2 < r(y)/y_0^2} W_\psi f(z) (U_z \psi, g) d\mu(z);
\]

since the absolute value of this is bounded by

\[
\int_{|x| < A, \delta_1 < y < B, \delta_2 < r(y)/y_0^2} \|f\|_2 \|U_z \psi\|_2^2 \|g\|_2 \frac{dy}{r(y)^{n+1}} \leq c_n A^{n+1} \|f\|_2 \|U_z \psi\|_2^2 \int_{|z| < y_0} \frac{dy}{r(y)^{n+1} (y_0^2 - y^2)^{n+1}} \leq \frac{c_n A^{n+1}}{n+1} \left( \frac{1}{\delta_1^{n+1}} - \frac{1}{B^{n+1}} \right) \|f\|_2 \|U_z \psi\|_2^2,
\]

where $c_n$ is a constant, the integral in (3.15) stands for a function in $H^2$ by the Riesz lemma. The proof of (3.15) will be given in the Appendix. Under some conditions on the decay properties at infinity of $\psi$, then for bounded $f(x) \in H^2$, (3.13) holds pointwise at every point $x$ where $f$ is continuous. We will not give the details on them here.

The reproducing kernel in Theorem 3.3 can be gotten from Theorem 3.2. In fact, from Theorem 3.2,

\[
W_\psi f(z) = (f, U_z \psi) = \frac{1}{C_\psi} \int_\Pi W_\psi f(z') (U_{z'} \psi, U_z \psi) d\mu(z')
\]

\[
= \frac{1}{C_\psi} \int_\Pi (U_{z'} \psi, U_z \psi) W_\psi f(z') d\mu(z').
\]

Thus $K_\psi(z, z') = \frac{1}{C_\psi} (U_{z'} \psi, U_z \psi)$.

In the next section, we will give an orthogonal decomposition of $L^2$ on $\Pi$ by giving an orthogonal basis of $AAW$.

4. Orthogonal decomposition of $L^2$ space on $BDI(q = 2)$

In this section we will give an orthogonal decomposition of the space $L^2$ on $\Pi$ (or on the Cartan domain $BDI(q = 2)$) in the form $\bigoplus_k A_k$, in which the $A_k$ are the ranges of wavelet transforms defined as in (3.14). First let us show that the Bergman space is such a range with a special choice of admissible wavelet.

For $\alpha \in \mathbb{R}$, let $L^{2^\alpha}(\Pi)$ denote the $L^2$ space on $\Pi$ defined by

\[
L^{2^\alpha}(\Pi) := \{F(x, y) : \int_\Pi |F(x, y)|^2 r(y)^{\alpha} dy < \infty\}.
\]

If $\alpha > -1$, then $L^{2^\alpha}(\Pi)$ contains holomorphic functions. In fact, $\alpha > -1$ is also a necessary condition for the existence of holomorphic functions in $L^{2^\alpha}(\Pi)$; see [3, p.260]. In this paper, we will assume $\alpha > -1$. Let $A^{2^\alpha}$ denote the subspace of $L^{2^\alpha}(\Pi)$ consisting of holomorphic functions; $A^{2^\alpha}$ is called the (weighted) Bergman
space. Let $H^{\alpha_2}(II)$ denote the subspace of $L^{\alpha_2}(II)$ of functions $F(x, y)$ such that $\text{supp} \hat{F}(\xi, y) \subset \mathcal{C} \times \mathcal{C}$; here $\hat{F}(\xi, y)$ denotes the Fourier transform of $F(x, y)$ with respect to the variables $x$ and $\mathcal{C}$ is the closure of $C$. Since the cone $C$ is homogeneous and the dual of $C$ is still itself, i.e. $C$ is a regular (symmetric) cone, then for every $F(z) \in A^{\alpha_2}$, there exists an $f \in H^2$ such that (refer to Theorem XIII 1.1 in [3])

\begin{equation}
F(z) = c \int_{\mathcal{C}} e^{i\xi z} \hat{f}(\xi) r(\xi) \frac{2^{n+1}}{1} d\xi,
\end{equation}

where $z = x + iy \in \Pi$. Conversely, for every $f \in H^2, F(z)$, defined by (4.1), belongs to $A^{\alpha_2}$. That is, the map $f \mapsto F$ is a linear isomorphism from $H^2$ onto $A^{\alpha_2}$. From (4.1), we know if $F(z) \in A^{\alpha_2}$, then supp$\hat{F}(\xi, y) \subset \overline{C} \times \overline{C}$, and therefore $A^{\alpha_2} \subset H^{\alpha_2}(II)$.

For $\psi \in AAW$, let $W_\psi f(z)$ be the wavelet transform defined by (3.11); by Theorem 3.1, it is an isometry (up to a constant) from $H^2$ into $L^2(II, d\mu(z))$. Define

\begin{equation}
W_\psi^\alpha f(z) := r(y)^{-\frac{n+1+\alpha}{2}} W_\psi f(z) = r(y)^{-\frac{n+1+\alpha}{2}} (f, U_z \psi);
\end{equation}

then $W_\psi^\alpha$ is an isometry (up to a constant) from $H^2$ into $L^{\alpha_2}(II)$, i.e.

\[\int_{\Pi} |W_\psi^\alpha f(z)|^2 r(y)^\alpha dx dy = C_\psi \|f\|^2_{L^2}.\]

About this transform $W_\psi^\alpha$, we also have theorems similar to Theorem 3.1, 3.2, 3.3. For example, for any $\psi, \phi \in AAW$ and $f, h \in H^2$,

\begin{equation}
\langle W_\psi^\alpha f, W_\phi^\alpha h \rangle_{L^{\alpha_2}(II)} = (f, h) \int_{\mathcal{C}} \frac{d\xi_0 d\xi_1 d\xi_2}{(\xi_0 + \xi_1)^{n-1}} r(\xi),
\end{equation}

which is just (3.12). For $\psi \in AAW$, let $A^\alpha_\psi$ be the space defined by

\begin{equation}
A^\alpha_\psi := \{ W_\psi^\alpha f(z) = r(y)^{-\frac{n+1+\alpha}{2}} (f, U_z \psi) : f \in H^2 \}.
\end{equation}

Since $W_\psi^\alpha$ is an isometry (up to a constant) from $H^2$ into $L^{\alpha_2}(II)$, we have $A^\alpha_\psi \subset L^{\alpha_2}(II)$. By (4.2),

\[\left( W_\psi^\alpha f \right)^\wedge (\xi, y) = r(y)^{-\frac{n+1+2\alpha}{2}} \hat{f}(\xi) \hat{\psi} \left( r(y) \frac{1}{2} \Lambda_{r,s} \xi \right) \quad \text{(with } \Lambda_{r,s} \omega = y/r(y)^{1/2})\]

and we know that supp$\left( W_\psi^\alpha f \right)^\wedge (\xi, y) \subset \overline{C} \times \overline{C}$ since supp$\hat{f} \subset \overline{C}$. Thus $A^\alpha_\psi$ is a subspace of $H^{\alpha_2}(II)$. We can see that $A^\alpha_\psi$ has a reproducing kernel, given by

\begin{equation}
K^\alpha_\psi (z, z') = C_\psi^{-1} (r(y) r(y'))^{-\frac{n+1+\alpha}{2}} (U_z \psi, U_{z'} \psi),
\end{equation}

with $z = x + iy, z' = x' + iy' \in \Pi$.

Let $\psi^\alpha_0 \in AAW$ be defined by

\begin{equation}\label{eq:psi_0^alpha}
\hat{\psi}^\alpha_0 (\xi) = \begin{cases} r(\xi) \frac{2^{n+1+1}}{1} e^{-\xi_0}, & \text{for } \xi \in C, \\ 0, & \text{elsewhere.} \end{cases}
\end{equation}

Then for all $f \in H^2$, the $W_\psi^\alpha f(z)$ are holomorphic on $\Pi$. In fact, let $\Lambda_{r,s}$ be the correspondence to $y \in C$ given by $\Lambda_{r,s} \omega = y/r(y)^{1/2}$; then $\omega, \Lambda_{r,s} \xi = (\Lambda_{r,s} \omega, \xi) = \ldots$
By the Plancherel formula,
\[ W^\alpha_{\psi_0^\alpha} f(z) = \int_C r(y)^{-\frac{n+1}{2}} \int_C r(y)^{\frac{n+1}{2}} \tilde{f}(\xi) \overline{\psi_0^\alpha(r(y)^{\frac{1}{2}} A_{r,\xi})} e^{iy\xi} d\xi \]

(4.7)
\[ = \int_C \tilde{f}(\xi)(\xi^{2n+1})^{1/4} e^{iy\xi} d\xi = \int_C \tilde{f}(\xi)(\xi^{2n+1})^{1/4} e^{iy\xi} d\xi; \]

thus \( W^\alpha_{\psi_0^\alpha} f(z) \) is holomorphic on \( \Pi \). From (4.7) and the relation between \( A^{\alpha 2} \) and \( H^2 \) given by (4.1) (i.e. Theorem XIII 1.1 in \( [3] \)), we know \( A_{\psi_0^\alpha} \) is just the Bergman space \( A^{\alpha 2} \). Such transforms \( W^\alpha_{\psi_0^\alpha} \) were also considered by Unterberger in \( [13], [14] \) for the quantizations of the Cartan domain \( BDI(q = 2) \).

Let us now give the expression of the Bergman kernel \( K^\alpha_{\psi_0^\alpha}(z, z') \) from (4.5). We have

\[ K^\alpha_{\psi_0^\alpha}(z, z') = C^{-1}_{\psi_0^\alpha}(r(y)^{-\frac{n+1}{2}} W^\alpha_{\psi_0^\alpha}(U_{r}\psi_0^\alpha) (z)) \]
\[ = C^{-1}_{\psi_0^\alpha}(r(y)^{-\frac{n+1}{2}} W^\alpha_{\psi_0^\alpha}(U_{r}\psi_0^\alpha) \tilde{f}(\xi) \overline{\psi_0^\alpha(r(y)^{\frac{1}{2}} A_{r,\xi})} e^{iy\xi} d\xi \]
\[ = C^{-1}_{\psi_0^\alpha} \int_C r(\xi)^{\frac{n+1}{2}} e^{iy\xi} d\xi. \]

Let \( A = -i(z - \overline{z}) \); then \( \text{Re} A = y + y' \in C \). To compute the integral

\[ I(A) = \int_C r(\xi)^{\frac{n+1}{2}} e^{-A\xi} d\xi, \]

we first assume \( \text{Im} A = 0 \), i.e. \( A \in C \); then for general \( A, I(A) \) can be gotten from the holomorphic property of \( I(A) \). For \( A \in C \), there exists \( \Lambda \in SO_0(1, n) \) such that \( a\Lambda\omega = A \) with \( a^2 = r(A) \). Then, by the invariance of the measure \( d\xi/r(\xi)^{\frac{n+1}{2}} \) under \( \mathbb{R}^*_+ \times SO_0(1, n) \),

\[ \int_C r(\xi)^{\frac{n+1}{2}} e^{-A\xi} d\xi = \int_C r(\xi)^{\frac{n+1}{2}} e^{[A, \xi]} d\xi \]

(with a change of variables \( \xi \rightarrow J\xi \))
\[ = \int_C r(\xi)^{\frac{n+1}{2}} e^{-a[\Lambda, \omega, \xi]} d\xi = \int_C r(\xi)^{\frac{n+1}{2}} e^{[\Lambda, \omega, \xi]} d\xi \]
\[ = \int_C r(\xi)^{\frac{n+1}{2}} e^{-a[\Lambda, \omega, \xi]} d\xi = \int_C r(\xi)^{\frac{n+1}{2}} e^{[\Lambda, \omega, \xi]} d\xi \]
\[ = a^{-2(n+1)} \int_C r(\xi)^{\frac{n+1}{2}} e^{-a\omega} d\xi = c_n/r(A)^{n+1}. \]

And therefore

**Proposition 4.1.** Let \( K(z, z) \) be the Bergman kernel of the tube domain \( \Pi \). Then

\[ K(z, z') = C_n/r(i(\overline{z} - z))^{n+1}, \]

where \( C_n \) is a constant.

\( II \) is an (unbounded) realization of \( BDI(q = 2) \) or the classical bounded domain of type IV (see \( [15] \)), and by a transform the kernel given by (4.8) it indeed coincides with the Bergman kernel given by Hua for the case \( \alpha = 0 \) (see \( [6], \text{p.88} \)).

In the rest of this section we want to construct \( \psi_{\xi}^\alpha \in AAW \) such that \( A_{\xi}^\alpha := A_{\psi_{\xi}^\alpha}^\alpha \)

given by (4.4) are orthogonal to each other and their sum is \( H^{\alpha 2}(\Pi) \) with \( A_{\xi}^\alpha = A^{\alpha 2} \).
From (4.3), we shall construct $\hat{\psi}_k \in \text{AAW}$ such that
\[
\int_C \hat{\psi}_k(y)\overline{\hat{\psi}_k(y)} \frac{dy_0dy_1dy_*}{(y_0+y_1)^{n-1}r(y)} = 0, \quad \text{if } \vec{k} \neq \vec{k}',
\]
with $\hat{\psi}_0 = \hat{\psi}_0^0$ given by (4.6).

Let $H_l$ be the space of all linear combinations of functions of the form $h(r)Y_l(x)$, where $h$ ranges over the radial functions and $Y_l$ ranges over the solid spherical harmonics of degree $l$. Then $L^2(\mathbb{R}^{n-1})$ can be decomposed as the orthogonal sum (see [10, p.151])
\[
L^2(\mathbb{R}^{n-1}) = \bigoplus_{l=0}^{\infty} H_l.
\]
Every element $h_l$ in $H_l$ can be written in the form $\sum_{j=1}^{a_l} h_{l,j}(r)Y_l^j(x)$, and
\[
\int_{\mathbb{R}^{n-1}} |h_l(x)|^2 dx = \sum_{j=1}^{a_l} \int_0^{\infty} |h_{l,j}(r)|^2 r^{n-2} dr,
\]
where $a_0 = 1$, $a_1 = n-1$, $a_l = \binom{n+l-2}{l} - \binom{n+l-4}{l-2}$, $l \geq 2$, and $\{Y_l^j(x)\}_{j=1}^{a_l}$ is an orthogonal basis of the space $H_l$ of surface spherical harmonics of degree $l$ (see [10, p.140]). It is well known for $n = 3$, $a_l = 2$ and $Y_l^1(x) = \frac{\sin l \theta}{\sqrt{\pi}}$, $Y_l^2(x) = \frac{\cos l \theta}{\sqrt{\pi}}$ with $x = e^{i\theta}$. For $n > 3$, an orthogonal basis $Y_l^j$ of $H_l$ can be given by the Gegenbauer polynomials; see [16], pp. 457–468.

If $\hat{\psi} \in \text{AAW}$, then for almost all $y_0, y_1$, $\hat{\psi}_{y_0,y_1}(y_*) := \hat{\psi}(y_0, y_1, y_*)$ is a function of $L^2(\mathbb{R}^{n-1})$, and can be written as
\[
(4.9) \quad \hat{\psi}(y) = \sum_{l=0}^{\infty} \sum_{j=1}^{a_l} r(y) \frac{2l+n+1}{4l+2} f_{j,l}(y_0, y_1) h_{j,l}(\frac{|y_*|^2}{y_0^2 - y_1^2}) Y_l^j(y_*).
\]
By the orthogonality of $Y_l^j$, we have
\[
\int_C |\hat{\psi}(y)|^2 \frac{dy_0dy_1dy_*}{(y_0+y_1)^{n-1}r(y)} = \sum_{l=0}^{\infty} \sum_{j=1}^{a_l} \int_{|y_0|, |y_1| > y_*} (y_0^2 - y_1^2 - r^2)^{\alpha + \frac{n-1}{2}}
\]
\[
\cdot \left| f_{j,l}(y_0, y_1) h_{j,l}(\frac{r^2}{y_0^2 - y_1^2}) \right|^2 \frac{dy_0dy_1}{(y_0+y_1)^{n-1}}
\]
\[
= \frac{1}{2} \sum_{l=0}^{\infty} \sum_{j=1}^{a_l} \int_{y_0 > |y_1|} |f_{j,l}(y_0, y_1)|^2 \frac{(y_0^2 - y_1^2)^{\alpha + \frac{n-1}{2}}}{(y_0+y_1)^{n-1}} dy_0dy_1
\]
\[
\cdot \int_0^1 \left| h_{j,l}(t) \right|^2 (1-t)^{\alpha + \frac{n-1}{2}} t^{n-3} dt \quad (\text{with } t = \frac{r^2}{y_0^2 - y_1^2})
\]
Let $\vec{k} = (m, \nu, k, l, j)$ and let $\psi_k \in \text{AAW}$ be defined by
\[
\hat{\psi}_k(y) = r(y) \frac{2\alpha + n+1}{4} e^{-y_0} L_{m,\nu}(y_0, y_1) P_{k}(\frac{|y_*|^2}{y_0^2 - y_1^2}) Y_l^j(y_*),
\]
where \(L_{m,\nu}, P_k\) are polynomials of degree \(m + \nu\) and \(k\) respectively. Then from the above calculation and the orthogonality of \(Y_j^l\), we have for \(\hat{k}, \hat{k}'\) with \(l = l', j = j'\),

\[
(4.10) \quad \int_C \hat{\psi}_k(y) \overline{\hat{\psi}_{k'}(y)} \frac{dy_0 dy_1 dy_s}{(y_0 + y_1)^{n-1}} = C \int_{y_0 > |y_1|} L_{m,\nu}(y_0, y_1)L_{m',\nu'}(y_0, y_1)e^{-2y_0} \cdot \frac{(y_0^2 - y_1^2)^{\alpha+n-1}}{(y_0 + y_1)^{n-1}} dy_0 dy_1 \int_0^1 \frac{P_k(t)P_{k'}(t)(1-t)^{\alpha+n-1}t^{n-2}}{t} dt
\]

\[
= C \int_0^\infty \int_0^\infty L_{m,\nu} \left( \frac{s + r}{2}, \frac{s - r}{2} \right)L_{m',\nu'} \left( \frac{s + r}{2}, \frac{s - r}{2} \right)e^{-s-r}r^{\alpha+n-1} ds dr \quad \text{(with} \quad s = \frac{y_0 + y_1}{2}, r = \frac{y_0 - y_1}{2}) \cdot \int_0^1 P_k(t)P_{k'}(t)(1-t)^{\alpha+n-1}t^{n-2} dt
\]

Choose

\[
L_{m,\nu}(y_0, y_1) = L_m^{(\alpha)}(y_0 + y_1)L_{\nu}^{(\alpha+n-1)}(y_0 - y_1),
\]

\[
P_k(t) = P_k^{(\alpha+n-1, \frac{\alpha+n-3}{2})}(2t - 1),
\]

where \(L_k^{(\alpha)}\), \(P_l^{(\alpha, \beta)}\) are the Laguerre polynomials of degree \(k\) and the Jacobi polynomials of degree \(l\) respectively; then, by the orthogonality of \(L_k^{(\alpha)}\), \(P_l^{(\alpha, \beta)}\) (see [11]) and \(Y_j^l\),

\[
\int_C \hat{\psi}_k(y) \overline{\hat{\psi}_{k'}(y)} \frac{dy_0 dy_1 dy_s}{(y_0 + y_1)^{n-1}} = C_{k, l, j} \delta_{kk'} \delta_{\nu\nu'} \int_0^\infty L_m^{(\alpha)}(s)L_{m'}^{(\alpha)}(s)e^{-s} s^{\alpha} ds \cdot \int_0^\infty P_{\nu}^{(\alpha+n-1)}(r)L_{\nu'}^{(\alpha+n-1)}(r)e^{-r} r^{\alpha+n-1} dr = C_k \delta_{kk'}.\]

Finally we give a series of orthogonal wavelets

\[
(4.11) \quad \hat{\psi}_k(y) = r(y)^{\frac{\alpha+n-1}{2}} e^{-y_0} L_m^{(\alpha)}(y_0 + y_1)L_{\nu}^{(\alpha+n-1)}(y_0 - y_1) \cdot P_k^{(\alpha+n-1, \frac{\alpha+n-3}{2})} \left( \frac{2||y_s||^2}{y_0^2 - y_1^2} - 1 \right) Y_j^l(y_s),
\]

and we have \(\hat{\psi}_0 = \psi_0^e\) given by (4.6). From the completeness properties of the sets \(\{L_k^{(\alpha)}(t)e^{-\frac{t}{2}}\}\) and \(\{P_l^{(\alpha, \beta)}\}\) for spaces \(L^2(\mathbb{R}^+, s^{\alpha} ds), L^2([-1, 1], (1-t)\alpha(1+t)^{\beta} dt)\) respectively, we know that \(f_{j, l}(y_0, y_1), h_{j, l} \left( \frac{2||y_s||^2}{y_0^2 - y_1^2} \right)\) in (4.9) can be written to be orthogonal sums of

\[
\{L_m^{(\alpha)}(y_0 + y_1)e^{-\frac{y_0 + y_1}{2}}, L_{\nu}^{(\alpha+n-1)}(y_0 - y_1)e^{-\frac{y_0 - y_1}{2}}\}_{m, \nu}
\]

and

\[
\{P_k^{(\alpha+n-1, \frac{\alpha+n-3}{2})} \left( \frac{2||y_s||^2}{y_0^2 - y_1^2} - 1 \right)\}_{k}
\]

respectively. Thus \(\{\hat{\psi}_k\}_{k}\) gives an orthogonal basis of \(AAW\) (here and in the following orthogonality of \(\hat{\psi}_k, \hat{\psi}_{k'}\) means that \(\hat{\psi}_k, \hat{\psi}_{k'}\) are orthogonal to each other with respect to the measure \(dy_0 dy_1 dy_s/(y_0 + y_1)^{n-1} r(y))\).

Let \(A_G\) denote the subspaces of \(H^{\alpha, 2}(\Pi)\) defined by (4.4) with \(\psi = \psi_G\); then the \(A_G\) are orthogonal to each other and in fact they give an orthogonal decomposition of \(H^{\alpha, 2}(\Pi)\):
**Theorem 4.1.** Let $A_\mathcal{E}$ be the subspaces of $H^{\alpha^2}(\Pi)$ defined above. Then

$$H^{\alpha^2}(\Pi) = \bigoplus_{\mathcal{E}} A_\mathcal{E},$$

with the first component $A_0 = A^{\alpha^2}$.

The proof of Theorem 4.1 will be given in the Appendix. In [7], we gave two orthogonal bases for $AAW$ in the case $n = 1$, and $\psi_\mathcal{E}$ given by (4.11) for $n = 1$ is just the suitable basis in [7] for studying the Toeplitz-Hankel type operators between $A_\mathcal{E}$. The orthogonal basis and the decomposition given here will be appropriate for the purpose of studying Toeplitz-Hankel type operators between $A_\mathcal{E}$ for $n \geq 2$.

**Appendix**

**A.1. Admissibility condition in \( \S 3 \).** Let us calculate here the admissibility condition for the wavelet transform defined by (3.10). By the Plancherel formula,

\[
\int_{G_a} |(\psi, U_g \psi)|^2 d\mu(y) = \int_{G_a} |\hat{\psi}(\xi)(U_g \psi)^\wedge(\xi)|^2 d\mu(y)
\]

\[
= \int_0^\infty \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n+1}} |\hat{\psi}(\xi)a^{\frac{n+1}{2}} \overline{\hat{\psi}(a^t \Lambda r,s \xi)}e^{2i\xi^t}d\xi|^2 \frac{d\mu(y)}{a^{n+2}}
\]

\[
= \int_0^\infty \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n+1}} |\hat{\psi}(\xi)\hat{\psi}(a^t \Lambda r,s \xi)|^2 \frac{d\xi}{a} \frac{dsdrda}{a} |\hat{\psi}(\xi)|^2 d\xi.
\]

Similarly to the correspondence between $AN$ and $H_1$ given by (3.3), we also can give a correspondence between $t^r(AN)$ and $H_1$. In fact for any $\xi \in C$, there exist unique $a_{r_0} \in A$ and $n_{s_0} \in N$ such that

\[
\begin{align*}
t^r_{\Lambda r_0,s_0} = t_{n_{s_0}} a_{r_0} & = \xi / r(\xi)^2, \\
\end{align*}
\]

with $r_{r_0} = (\xi_0 + \xi_1) / r(\xi)^2$, $s_0 = (\xi_2, \cdots, \xi_n) / (\xi_0 + \xi_1)$. Since $dsdrda/a$ is invariant under $AN$, then

\[
(A.0) \quad \int_0^\infty \int_{\mathbb{R}^{n-1}} |\hat{\psi}(a^t \Lambda r,s \xi)|^2 \frac{dsdrda}{a} = \int_0^\infty \int_{\mathbb{R}^{n-1}} |\hat{\psi}(a^t (\Lambda r_0,s_0 \Lambda r,s) \omega)|^2 \frac{dsdrda}{a} = \int_0^\infty \int_{\mathbb{R}^{n-1}} |\hat{\psi}(a^t \Lambda r,s \omega)|^2 \frac{dsdrda}{a}.
\]

Thus the admissibility condition is

\[
(A.1) \quad 0 < C_\psi = \int_0^\infty \int_{\mathbb{R}^{n-1}} |\hat{\psi}(a^t \Lambda r,s \omega)|^2 \frac{dsdrda}{a} < \infty.
\]

Taking the change of variables

\[
(A.2) \quad \begin{cases}
a = r(y)^\frac{1}{2}, \\
ae^{-r} = y_0 - y_1,
\end{cases}
\]

\[
as = y_\ast = (y_2, \cdots, y_n),
\]

with $(a, r, s) \in \mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_0, y_1, y_\ast) \in C$, by a direct calculation

\[
(A.3) \quad dadrds = a^{-n} dy_0 dy_1 dy_\ast = r(y)^{-\frac{1}{2}} dy_0 dy_1 dy_\ast.
\]
and
\[ a\Lambda_{r,s} \omega = t(a(\text{chr} + e^r \rho), a(\text{shr} + e^r \rho), as) = t(y_0, y_1, y_*) , \]
where \( \Lambda_{r,s} \) is given by (3.4) with \( \rho = \frac{1}{2}|s|^2 \). Thus for a function \( f \) on \( C \), we have
\[
(A.4) \quad \int_0^\infty \int_R \int_{R^{n-1}} f(a\Lambda_{r,s} \omega) \frac{dsdrda}{a} = \int_C f(y) \frac{dy}{r(y)^{\frac{n+1}{2}}} .
\]
From the definition of \( \Lambda_{r,s} \) in (3.4) and the change of variables by (A.2), we have
\[
\int_0^\infty \int_R \int_{R^{n-1}} \left| \widehat{\psi}(a^t\Lambda_{r,s} \omega) \right|^2 \frac{dsdrda}{a} = \int_0^\infty \int_R \int_{R^{n-1}} \left| \widehat{\psi}(a(\text{chr} + e^r \rho), a(\text{shr} - e^r \rho), ae^r s) \right|^2 \frac{dsdrda}{a} = \int_0^\infty \int_R \int_{R^{n-1}} \left| \widehat{\psi}(a(\text{chr} + e^r \rho), a(\text{shr} - e^r \rho), as) \right|^2 \frac{dsdrda}{r^{(n-1)}a} = \int_C |\widehat{\psi}(y_0, y_1, y_*)|^2 \frac{dy_0 dy_1}{(y_0 + y_1)^{n-1}r(y)}.
\]Thus from (A.1) and above calculation, we finally get the admissibility condition
\[ 0 < C_\psi = \int_0^\infty \int_R \int_{R^n} \left| \widehat{\psi}(y_0, y_1, y_*) \right|^2 \frac{dy_0 dy_1}{(y_0 + y_1)^{n-1}r(y)} < +\infty . \]

**A.2. Proof of Theorem 3.1 and (3.15).** The proof of Theorem 3.1 goes like the above calculation of the admissibility condition. In fact, by the Plancherel formula, the change of variables given in (A.2), and formula (A.4), we have
\[
\langle W_\psi f, W_\phi h \rangle = \int_R \int_R \left( W_\psi f \right)^*(\xi, y)(W_\phi h)^*(\xi, y) \frac{d\xi dy}{r(y)^{n+1}}
\]
\[
= \int_C \int_C \int_R \int_R \int_{R^{n-1}} \left( \hat{f}(\xi) \hat{\psi}(r(y)^{\frac{1}{2}}t\Lambda_{r,s} \xi) \hat{h}(\xi) \phi(r(y)^{\frac{1}{2}}t\Lambda_{r,s} \xi) \right) dsdrda \frac{d\xi dy}{r(y)^{n+1}}
\]
\[
= \int_C \int_R \int_{R^{n-1}} \left( \hat{f}(\xi) \hat{h}(\xi) \right) \left( \int_R \int_R \int_{R^{n-1}} \left| \hat{\psi}(a^t\Lambda_{r,s} \omega) \hat{\phi}(a^t\Lambda_{r,s} \xi) \right|^2 \frac{dsdrda}{a} \right) d\xi
dr \quad \text{(from the change of variables by (A.2))}
\]
\[
= \int_C \int_R \int_{R^{n-1}} \left( \hat{f}(\xi) \hat{h}(\xi) \right) \left( \int_R \int_R \int_{R^{n-1}} \left| \hat{\psi}(a^t\Lambda_{r,s} \omega) \hat{\phi}(a^t\Lambda_{r,s} \xi) \right|^2 \frac{dsdrda}{a} \right) d\xi
dr \quad \text{(similarly to (A.0))}
\]
\[
= \int_C \left( \hat{\psi}(y_0, y_1, y_*) \hat{\phi}(y_0, y_1, y_*) \right) \frac{dy_0 dy_1}{(y_0 + y_1)^{n-1}r(y)} \left( f, h \right)
\]
\[ = (K^{-\frac{1}{2}} \hat{\phi}, K^{-\frac{1}{2}} \hat{\psi})(f, h) . \]
Proof of (3.15). Denote \( E_{A,B,\delta_1,\delta_2} := \{(x, y) \in \Pi : |x| < A, \delta_1 < y_0 < B, \delta_2 < \rho(y)/y_0^\alpha \} \). Then we have

\[
\|f - C_\psi^{-1} \int_{E_{A,B,\delta_1,\delta_2}} \psi f(z) U_z \psi \, d\mu(z)\|_2 \\
= \sup_{\|g\|_2 = 1, g \in H^2} \left| \left( f - C_\psi^{-1} \int_{E_{A,B,\delta_1,\delta_2}} \psi f(z) U_z \psi \, g \right) \right| \\
\leq \sup_{\|g\|_2 = 1, g \in H^2} \left| C_\psi^{-1} \int_{\Pi \setminus E_{A,B,\delta_1,\delta_2}} \psi f(z) |W_\psi g(z)| \, d\mu(z) \right| \\
\leq \sup_{\|g\|_2 = 1, g \in H^2} \left| C_\psi^{-1} \int_{\Pi \setminus E_{A,B,\delta_1,\delta_2}} |W_\psi f(z)|^2 \, d\mu(z) \right| \left( \int_{\Pi} |\psi g(z)|^2 \, d\mu(z) \right)^{1/2} \\
\leq \left| C_\psi^{-1} \int_{\Pi \setminus E_{A,B,\delta_1,\delta_2}} |W_\psi f(z)|^2 \, d\mu(z) \right|^{1/2} \quad \text{(by Theorem 3.1).}
\]

Since the infinite integral \( \int_{\Pi} |W_\psi f(z)|^2 \, d\mu(z) \) converges (by Theorem 3.1 again),

\[
\int_{\Pi \setminus E_{A,B,\delta_1,\delta_2}} |W_\psi f(z)|^2 \, d\mu(z) \to 0, \quad \text{as } \delta_1, \delta_2 \to 0^+, \ A, B \to +\infty. \quad \text{•}
\]

A.3. Proof of Theorem 4.1. We have shown that \( \{C_\psi^{-1} \psi_\xi\}_\xi \) is an orthonormal basis of \( \text{AAW} \); let \( \{\psi_\xi\}_\xi \) denote this orthonormal basis for simplicity. Let \( \{f_\alpha\}_\alpha \) be an orthonormal basis of \( H^2 \). In order to prove Theorem 4.1, we need only to show that \( \{W_{\psi_\xi}^{\alpha} f_\alpha\}_{k,\xi} \) is an orthonormal basis of \( H^{2\alpha}(\Pi) \). The orthonormality of these functions is obvious (from (4.3)) and we need to show their completeness (closure) in \( H^{2\alpha}(\Pi) \), for which it is enough to show that

\[
\|F\|_{L^{2\alpha}(\Pi)}^2 = \sum_{k,\xi} \|F, W_{\psi_\xi}^{\alpha} f_\alpha\|_{L^{2\alpha}(\Pi)}^2
\]

for any \( F \in H^{2\alpha}(\Pi) \). We have for \( F \in H^{2\alpha}(\Pi) \)

\[
\left| \int_{\Pi} F(x, y) \overline{W_{\psi_\xi}^{\alpha} f_\alpha}(x, y) r(y)^\alpha \, dx \, dy \right| \\
= \left| \int_C \int_C \widehat{F}(\xi, \eta) \overline{f_\alpha(\eta)} r(y)^\alpha \frac{2\alpha+1}{2} \overline{\psi_\xi(r(y)^{1/2} \Lambda_{r,s} \xi)} r(y)^\alpha \, d\eta \, dy \right| \\
= \left| \int_C \int_0^\infty \int_{\mathbb{R}^{n-1}} \frac{2\alpha+1}{2} \overline{\psi_\xi(a \Lambda_{r,s} \xi)} \frac{\Lambda_{r,s} \xi}{a} \, da \, d\xi \right| \\
= \left| \int_C \int_0^\infty \int_{\mathbb{R}^{n-1}} \frac{2\alpha+1}{2} \overline{\psi_\xi(a \Lambda_{r,s} \xi)} \frac{\Lambda_{r,s} \xi}{a} \, da \, d\xi \right| \quad \text{(from the change of variables by (A.2))}
\]

\[
= \int_C \int_0^\infty \int_{\mathbb{R}^{n-1}} \frac{2\alpha+1}{2} \overline{\psi_\xi(a \Lambda_{r,s} \xi)} \frac{\Lambda_{r,s} \xi}{a} \, da \, d\xi \quad \text{(with } \xi = r(\xi)^{1/2} \Lambda_{r,s} \xi)\]
\]
\[
\begin{align*}
\int_C \int_0^\infty \int_{\mathbb{R}^{n-1}} f_n(\xi) \hat{F} \left( \xi, \frac{a}{r(\xi)z} \Lambda_{r_0,s_0}^{-1} \Lambda_{r,s,\omega} \right) \left( \frac{a}{r(\xi)z} \right)^{\frac{2n+n+1}{2}} \cdot \hat{\psi}_k(a^2 \Lambda_{r,s,\omega}) \frac{dadrds}{a} d\xi \\
= \left( \hat{M}_k, f_\kappa \right)_{H^2},
\end{align*}
\]
where the \( M_k \) are given by (for \( \xi \in C \))
\[
\hat{M}_k(\xi) = \frac{1}{r(\xi)^{\frac{2n+n+1}{2}}} \int_0^\infty \int_{\mathbb{R}^{n-1}} \hat{F} \left( \xi, \frac{a}{r(\xi)z} \Lambda_{r_0,s_0}^{-1} \Lambda_{r,s,\omega} \right) a^{\frac{2n+n+1}{2}} \hat{\psi}_k(a^2 \Lambda_{r,s,\omega}) \frac{dadrds}{a}.
\]
Since \( \{f_\kappa\}_n \) is an orthonormal basis of \( H^2 \), we have
\[
\sum_{k,n} |(F, W^0_{\psi_k} f_\kappa)|_{L^2(\Pi)}^2 = \sum_{k,n} \left( M_k, f_\kappa \right)^2 = \sum_k \| M_k \|_{H^2}^2.
\]
By (A.6) and the fact that \( \hat{\psi}_k \xi \) is an orthonormal basis of \( L^2(C, \frac{d\xi}{r(\xi)}) \) (or \( \{ \hat{\psi}_k(a^2 \Lambda_{r,s,\omega}) \}_k \) is an orthonormal basis of \( L^2(\mathbb{R}_+ \times \mathbb{R}^n, dadrds/a) \)), we have
\[
\sum_k |\hat{M}_k(\xi)|^2 = r(\xi)^{-\frac{2n+n+1}{2}} \int_0^\infty \int_{\mathbb{R}^{n-1}} |\hat{F}(\xi, \frac{a}{r(\xi)z} \Lambda_{r_0,s_0}^{-1} \Lambda_{r,s,\omega})|^2 a^{\frac{2n+n+1}{2}} \frac{dadrds}{a}
\]
\[
= r(\xi)^{-\frac{2n+n+1}{2}} \int_0^\infty \int_{\mathbb{R}^{n-1}} |\hat{F}(\xi, a \Lambda_{r,s,\omega})|^2 r(\xi)^{\frac{2n+n+1}{2}} a^{\frac{2n+n+1}{2}} \frac{dadrds}{a}
\]
\[
= \int_C |\hat{F}(\xi, y)|^2 r(y)^\alpha dy, \quad \text{(by (A.4) again)}
\]
and thus
\[
\sum_k \| M_k \|_{H^2}^2 = \sum_k \int_C |\hat{M}_k(\xi)|^2 d\xi = \int_C |\hat{F}(\xi, y)|^2 r(y)^\alpha dy dy dx dy.
\]
From (A.7) and (A.8), we get (A.5), and the proof of Theorem 4.2 is completed.\( \diamond \)

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References


Department of Mathematics, Peking University, Beijing 100871, P. R. China

Current address: Department of Mathematics, The National University of Singapore, Lower Kent Ridge Road, Singapore 119260

E-mail address: qjiang@haar.math.nus.sg