PRINCIPAL EIGENVALUES
WITH INDEFINITE WEIGHT FUNCTIONS

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ABSTRACT. Both existence and non-existence results for principal eigenvalues of an elliptic operator with indefinite weight function have been proved. The existence of a continuous family of principal eigenvalues is demonstrated.

1. INTRODUCTION AND RESULTS

Let $g(x)$ be a locally Hölder continuous function on R^n which changes sign. We consider the following eigenvalue problem with indefinite weight:

$$(P) \qquad\qquad -\Delta u(x) = \lambda g(x)u(x) \quad \text{for } x \in R^n,$$

where Δ is the standard Laplace operator. A principal eigenvalue of (P) is a positive constant λ_0, such that we can find a positive solution $u(x)$ for (P) if $\lambda = \lambda_0$.

Recently, a number of authors have investigated the existence of principal eigenvalues for (P). Part of the motivation to study (P) is to understand the bifurcation behavior of solutions for some nonlinear problems arising in population genetics [4], [5], [7]. Brown, Cosner and Fleckinger in [3] showed that if $\int_{R^n} g(x)dx > 0$, $n = 1, 2$, then (P) has no principal eigenvalues, while if $n \geq 3$ and $g(x)$ is negative and bounded away from 0 near ∞, then (P) has a principal eigenvalue. Brown and Tertikas [6] improved the results in [3] to conclude that (P) has a principal eigenvalue if $g^+(x) = \max\{g(x), 0\}$ has compact support. When $g(x)$ is bounded and $g^+(x) \in L^{\frac{n}{2}}(R^n)$, the existence of one principal eigenvalue and infinitely many other eigenvalues was proved by Allegretto in [1].

The goal of this paper is to continue to investigate when (P) has a principal eigenvalue. We shall show the existence of not only one principal eigenvalue, but also a continuous family of principal eigenvalues. Indeed by applying our results to the case that $g^+(x) = \max\{g(x), 0\}$ has compact support, we can conclude that there is a number $\lambda_0 > 0$, such that any number λ is a principal eigenvalue of (P) as long as $0 < \lambda \leq \lambda_0$.

The basic idea in the paper is to compare the different cases with different weight functions $g(x)$. First let us introduce a definition:

(G): A non-negative function $g_1(x)$ on R^n is said to have the property (G) if it is not identically zero and there is a principal eigenvalue μ for the problem

$$-\Delta u(x) = \mu g_1(x)u(x) \quad \text{for } x \in R^n.$$

Our first result says that (P) has a continuous family of principal eigenvalues if the weight function is dominated by a function having the property (G).

Theorem 1. *Let $n \geq 3$, and let $g_1(x)$ be a non-negative function having the property (G). If*

$$g(x) \leq g_1(x) \quad for \quad x \in R^n,$$

then any number λ is a principal eigenvalue of (P) as long as $0 < \lambda \leq \mu$, where μ is the number defined in the property (G).

By Theorem 1, if we can find a function $g_1(x)$ having the property (G), we can prove the existence of principal eigenvalues for all weight functions dominated by $g_1(x)$. If $n = 2$, the results in [3] showed that there are no functions having the property (G). When $n \geq 3$, there are a lot of functions having the property (G). Indeed the following result is part of Theorem 1 in [1] (since no information on the asymptotic behavior of eigenfunctions is given in the following result, one does not need to assume that $g(x)$ is globally bounded as in Theorem 1 in [1]).

Theorem 2 (Theorem 1 in [1]). *If $n \geq 3$, $g_1(x) \geq 0$ is not identically zero and*

$$\int_{R^n} |g_1(x)|^{\frac{n}{2}} dx < \infty,$$

then $g_1(x)$ has the property (G) and

$$\mu \geq c(n)(\int_{R^n} |g_1(x)|^{\frac{n}{2}} dx)^{-\frac{2}{n}},$$

where $c(n)$ is a constant depending only on the dimension n and μ is the principal eigenvalue defined in the property (G).

As an application of Theorems 1 and 2, we have (we recall $g^+(x) = \max\{g(x), 0\}$)

Theorem 3. *If $n \geq 3$, $g(x_0) > 0$ for some $x_0 \in R^n$, and*

$$\int_{R^n} |g^+(x)|^{\frac{n}{2}} dx < \infty,$$

then any number λ is a principal eigenvalue of (P) as long as

$$0 < \lambda \leq c(n)(\int_{R^n} |g^+|^{\frac{n}{2}} dx)^{-\frac{2}{n}},$$

where $c(n)$ is a constant depending only on the dimension n.

Proof. By the assumption and Theorem 2, $g^+(x)$ has the property (G) and $g(x)$ is naturally dominated by $g^+(x)$. Then Theorem 1 concludes the proof. $\square$

Comparing Theorem 3 with results in [1], [2], [5], we have generalized the existence results in [1], [2], [5] to the existence of a continuous family of principal eigenvalues. Furthermore, if $\int_{R^n} g(x)dx > 0$, $n \geq 3$, (P) still has principal eigenvalues provided the conditions of Theorem 3 are satisfied. This is in contrast to the results in [3], which showed non-existence of principal eigenvalues if $n = 1, 2$ and $\int_{R^n} g(x)dx > 0$.

To apply Theorem 1, it is important to find out which functions have the property (G). Here is a trivial way to produce more functions having the property (G). Let $g_1(x_1, x_2, x_3)$ be a function defined on R^3 having the property (G). Then there is a

positive constant μ and a positive function $u(x)$, such that $-\Delta u(x) = \mu g_1(x)u(x)$ for $x \in R^3$. Now we extend $u(x)$ and $g_1(x)$, $x \in R^3$, onto R^n trivially, that is,

$$u(x_1, x_2, x_3, \cdots, x_n) = u(x_1, x_2, x_3)$$

and

$$g_1(x_1, x_2, x_3, \cdots, x_n) = g_1(x_1, x_2, x_3).$$

Then we see that $-\Delta u(x) = \mu g_1(x)u(x)$ for $x \in R^n$. Thus $g_1(x)$ is a function on R^n having the property (G). Therefore, from Theorem 1, we have

Theorem 4. *Assume that $n \geq 3$, and there is a non-negative function $g_1(x_1, x_2, x_3)$ having the property (G), such that*

$$g(x_1, x_2, x_3, \cdots, x_n) \leq g_1(x_1, x_2, x_3) \quad on \quad R^n.$$

Then any number λ is a principal eigenvalue of (P) as long as $0 < \lambda \leq \mu$, where μ is defined in the property (G).

An immediate consequence of Theorem 4 is that there are functions having the property (G) and satisfying $\int_{R^n} |g^+|^{\frac{n}{2}} dx = \infty$ $(n \geq 4)$. Therefore there are functions having the property (G) and not covered by Theorem 2. The next result gives more functions having the property (G).

Theorem 5. *Let $n \geq 3$, and let $g_1(x)$ be a non-negative, not identically zero function on R^n and*

$$g_1(x) \leq \frac{c}{|x|^2} \quad for \quad x \in R^n.$$

Then $g_1(x)$ has the property (G), and we can take $\mu = \frac{(n-2)^2}{4c}$.

Remark 1.1. From Theorem 5, we see that

$$g_1(x') = (1 + |x'|^2)^{-1} \quad with \ x' = (x_1, x_2, x_3)$$

has the property (G). Then by applying Theorems 1, 4 and 5, we conclude that for all $g(x)$, if $g(x) \leq g_1(x')$ on R^n, then (P) has a continuous family of principal eigenvalues.

The next result says that the condition in Theorem 5 is quite sharp.

Theorem 6. *If $n \geq 3$, and*

$$\lim_{|x| \longrightarrow \infty} g(x)|x|^2 = \infty,$$

then (P) has no principal eigenvalues.

Remark 1.2. Theorem 6 is also true if $n = 2$. Indeed, in this case we automatically get $\int_{R^2} g(x)dx = \infty$; then we can use the result in [3].

In Theorems 1–5, we have no specific control on the asymptotic behavior of eigenfunctions. To obtain control on the asymptotic behavior of eigenfunctions, we need to add additional assumptions on the weight function.

Theorem 7. *Assume that $n \geq 3$, $\int_{R^n} |g^+|^{\frac{n}{2}} dx < \infty$, $g(x_0) > 0$ for some $x_0 \in R^n$, and there is a $p > \frac{n}{2}$ such that*

$$\int_{R^n} |g|^p (1 + |x|^2)^{2p-n} dx < \infty.$$

Then (P) has a principal eigenvalue λ_0 and an eigenfunction $u(x)$ such that

$$\lambda_0 \geq c(n)(\int_{R^n} |g_1^+|^{\frac{n}{2}} dx)^{-\frac{2}{n}}$$

where $c(n)$ is a constant depending only on the dimension n. Moreover,

(1.1) $u(x)|x|^{n-2} \longrightarrow c_0$ *as* $|x| \longrightarrow \infty$ *for a non-negative constant* c_0.

Remark 1.3. If there are $K > 0$ and $\alpha > 2$ such that $|g(x)| \leq K|x|^{-\alpha}$ for $|x|$ large, the existence of a principal eigenvalue with an eigenfunction $u(x)$ ($u(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$) was shown in Theorem 4.1 in [3]. In this case, the assumptions of Theorem 7 are clearly satisfied. If $g(x)$ is bounded and $g^+(x) \in L^{\frac{n}{2}}(R^n)$, the existence of a principal eigenvalue with an eigenfunction $u(x)$ ($u(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$) was shown in [1]. In Theorem 7, we do not assume $g(x)$ is bounded. Another comment is that Theorem 7 only yields the asymptotic behavior of one eigenfunction corresponding to a particular principal eigenvalue. By Theorem 3, under the assumptions of Theorem 7, (P) has a continuous family of principal eigenvalues. Most likely the asymptotic behavior of other eigenfunctions is not the same as the one described in (1.1) (see Remark 2.2 for some examples).

Finally, we consider the case $n = 2$. By a result in [3], a necessary condition for (P) having a principal eigenvalue is $\int_{R^2} g(x) < 0$. We would like to find out if this condition is sufficient. Although we believe the answer is yes, we only can prove a weaker result here.

Theorem 8. *Assume that $n = 2$, $\int_{R^2} g(x)dx < 0$, $g(x_0) > 0$ for some $x_0 \in R^2$, and there is a $p > 2$ such that*

$$\int_{R^2} |g^+|^p (1 + |x|^2)^{2(p-1)} dx < \infty.$$

Then (P) has a principal eigenvalue λ_0.

Remark 1.4. It would be interesting to know if there is also a continuous family of principal eigenvalues for the case $n = 2$. We can not do this here, since by a result in [3] there is no function having the property (G) for $n = 2$.

The paper is organized as follows. In section 2, we consider the case $n \geq 3$, where Theorems 1,2,5,6, and 7 are proved and a generalization to more general elliptic equations is given. In section 3, we consider the case $n = 2$, where Theorem 8 is proved.

Acknowledgement

2. The case that $n \geq 3$

We first prove Theorem 1.

Proof of Theorem 1. By assumption, there is a positive number μ and positive function $w(x)$ on R^n, such that

(2.1) $-\Delta w = \mu g_1(x)w$ on R^n.

Let $B_k(0)$ be the ball centered at origin with radius k, $k = 1, 2, 3, \cdots$. For each fixed number $0 < \lambda \leq \mu$, we consider a family of boundary value problems:

(P_k)
$$\begin{cases} -\Delta u = \lambda g(x)u & \text{on} \quad B_k(0); \\ u = k & \text{on} \quad \partial B_k(0). \end{cases}$$

We claim that (P_k) has a positive solution $u_k(x)$ for all k.

We set $u_k(x) = v_k(x)w(x)$; then it is straightforward to check (using (2.1) and (P_k)) that $u_k(x)$ is a positive solution of (P_k) if and only if $v_k(x)$ is a positive solution of

$(P_{k,v})$
$$\begin{cases} -\Delta v - 2\nabla \ln w \cdot \nabla v = (\lambda g(x) - \mu g_1(x))v & \text{on} \quad B_k(0); \\ v = kw^{-1} & \text{on} \quad \partial B_k(0). \end{cases}$$

Set $c(x) = \lambda g(x) - \mu g_1(x)$. Since $g(x) \leq g_1(x)$ on R^n, $g_1(x) \geq 0$, $0 < \lambda \leq \mu$, we see that

$$c(x) = \lambda g(x) - \mu g_1(x) \leq \lambda g_1(x) - \mu g_1(x) \leq \mu g_1(x) - \mu g_1(x) = 0.$$

Thus we only need to show that the following problem has a positive solution $v_k(x)$:

$(P_{k,v})$
$$\begin{cases} \Delta v + 2\nabla \ln w \cdot \nabla v + c(x)v = 0 & \text{on} \quad B_k(0); \\ v = kw^{-1} & \text{on} \quad \partial B_k(0). \end{cases}$$

The existence of a solution follows immediately from Theorem 6.8 on p.100 in [8]. The positivity of the solution follows from Hopf's maximum principle (Theorem 3.5 on p. 35 in [8]) since $v_k > 0$ on the boundary of $B_k(0)$. Therefore (P_k) has a positive solution $u_k(x)$. We normalize $u_k(x)$ by setting $u_k(0) = 1$. Then

(2.2)
$$\begin{cases} -\Delta u_k = \lambda g(x)u_k & \text{on} \quad B_k(0); \\ u_k(x) > 0 & \text{on} \quad B_k(0); \\ u_k(0) = 1. \end{cases}$$

Now we apply the Harnack inequality for positive solutions of elliptic equations (Theorem 8.20 on p. 199 in [8]) to (2.2); we see that for any compact set Ω on R^n, there are constants K and M (where M depends only on Ω, λ, and $g(x)$; K depends only on Ω), such that

$$0 < u_k(x) \leq M \quad \text{on} \quad \Omega \quad \text{for} \quad k \geq K.$$

Then an application of the interior Schauder estimates (Theorem 6.2 on p. 90 in [8]) for solutions of elliptic equations yields

$$|u_k|_{C^{2,\beta}(\Omega)} \leq M_1 \quad \text{for} \quad k \geq K_1 > K,$$

where $|u_k|_{C^{2,\beta}(\Omega)}$ is the $C^{2,\beta}$ norm of u_k, M_1, K_1 and β are constants (where M_1 depends only on Ω, λ and $g(x)$; β depends only on λ and $g(x)$; K_1 depends only on Ω). Then it is clear that $u_k(x)$ has a subsequence which converges to a C^2 non-negative function $u(x)$ on R^n on any compact set. Therefore

$$-\Delta u(x) = \lambda g(x)u(x) \quad \text{on} \quad R^n, \quad u(0) = 1, \quad u(x) \geq 0.$$

Now an application of a maximum principle concludes that $u(x)$ is positive everywhere on R^n. $\square$

As we mentioned before, Theorem 2 is a result in [1]. For the convenience of our readers, we present a proof here using a slightly different approach. For a domain Ω on R^n, we denote $H_0^1(\Omega)$ the Banach space of all functions that have square integrable first order derivatives and are zero on the boundary.

Proof of Theorem 2. Let $B_k(0)$ be the ball centered at the origin with radius k, $k = 1, 2, 3, \cdots$. By assumption, $g_1(x_0) > 0$ for some $x_0 \in R^n$. We may assume $x_0 \in B_1(0)$. We consider a family of eigenvalue problems:

$$(D_k) \qquad \begin{cases} -\Delta u = \lambda g_1(x)u & \text{on} \quad B_k(0); \\ u = 0 & \text{on} \quad \partial B_k(0). \end{cases}$$

Let

$$\lambda_1(k) = \inf\{ \int_{B_k(0)} |\nabla u|^2 dx \quad | \quad u \in H_0^1(B_k(0)), \int_{B_k(0)} g_1(x)u^2 = 1\}.$$

Since $g(x_0) > 0$, we see that $\lambda_1(k)$ is well defined, non-increasing on k and $\lambda_1(k) \geq 0$. Thus $\mu = \lim_{k \longrightarrow \infty} \lambda_1(k)$ exists. Furthermore it is easy to see that there is a non-negative function $u_k(x) \in H_0^1(B_k(0))$ that achieves the infimum. By the regularity theory for solutions of elliptic equations (Theorem 3.55 on p.86 in [2]) and the maximum principle, we see that $u_k(0)$ is actually a C^2 positive solution of (D_k). We normalize $u_k(x)$ so that $u_k(0) = 1$. Then

$$(D_k) \qquad \begin{cases} -\Delta u_k = \lambda_1(k)g_1(x)u_k & \text{on} \quad B_k(0); \\ u_k(x) > 0 & \text{on} \quad B_k(0); \\ u_k(0) = 1. \end{cases}$$

Now exactly as we did in the proof of Theorem 1, we see that $u_k(x)$ has a subsequence which converges to a C^2 non-negative function $u(x)$ on R^n on any compact set. And $u(x)$ is a positive eigenfunction with the principal eigenvalue μ.

To complete the proof, we only need to show that μ is bounded away from zero by the bound given in the statement of Theorem 2.

By the Hölder inequality, we have

$$\int_{B_k(0)} g(x)u^2 dx \leq \int_{B_k(0)} g^+(x)u^2 dx$$

$$\leq (\int_{B_k(0)} |g^+(x)|^{\frac{n}{2}} dx)^{\frac{2}{n}} (\int_{B_k(0)} |u|^{\frac{2n}{n-2}} dx)^{\frac{n-2}{n}}.$$

Then if $\int_{B_k(0)} g(x)u^2 dx = 1$, we have

$$(\int_{B_k(0)} |g^+(x)|^{\frac{n}{2}} dx)^{-\frac{2}{n}} \leq (\int_{B_k(0)} |u|^{\frac{2n}{n-2}} dx)^{\frac{n-2}{n}}.$$

On the other hand, by the Sobolev imbedding inequality (Theorem 2.14 on p. 39 in [2]), there is a constant $c_1(n)$, depending only on the dimension n, such that for all $u \in H_0^1(B_k(0))$,

$$(\int_{B_k(0)} |u|^{\frac{2n}{n-2}} dx)^{\frac{n-2}{2n}} \leq c_1(n)(\int_{B_k(0)} |\nabla u|^2 dx)^{\frac{1}{2}}.$$

Thus

$$\left(\int_{B_k(0)} |g^+(x)|^{\frac{n}{2}} dx\right)^{-\frac{2}{n}} \leq c_1^2(n) \left(\int_{B_k(0)} |\nabla u|^2 dx\right).$$

This immediately yields

$$\lambda_1(k) \geq c(n) \left(\int_{B_k(0)} |g^+(x)|^{\frac{n}{2}} dx\right)^{-\frac{2}{n}} \geq c(n) \left(\int_{R^n} |g^+(x)|^{\frac{n}{2}} dx\right)^{-\frac{2}{n}}.$$

Therefore

$$\mu \geq c(n) \left(\int_{R^n} |g^+(x)|^{\frac{n}{2}} dx\right)^{-\frac{2}{n}}.$$

$\square$

Remark 2.1. The proofs of Theorems 1 and 2 can also be modified to cover more general cases. Let Ω be an unbounded domain in R^n ($n \geq 3$) with $C^{2,\beta}$ boundary. We consider the principal eigenvalue problem for a second order linear elliptic operator in divergent form:

$$(P_1) \qquad \begin{cases} -\dfrac{\partial}{\partial x_i}\left(a_{ij}(x)\dfrac{\partial u}{\partial x_j}\right) = \lambda g(x)u & \text{on} \quad \Omega; \\ u = 0 & \text{on} \quad \partial\Omega; \\ u(x) > 0 & \text{on} \quad \Omega. \end{cases}$$

Here the summation convention is used. We also assume that the coefficients are Hölder continuous and the equation is uniformly elliptic, that is, there are positive constants m_1 and m_2, such that

$$m_1|\eta|^2 \leq a_{ij}(x)\eta_i\eta_j \leq m_2|\eta|^2 \quad \text{for} \quad \eta = (\eta_1, \eta_2, \cdots, \eta_n) \in R^n, \quad x \in R^n.$$

Then a direct modification of the proofs of Theorems 1 and 2 yields

Theorem 9. *If $n \geq 3$, $g^+(x_0) > 0$ for some $x_0 \in \Omega$, and*

$$\int_\Omega |g^+|^{\frac{n}{2}} dx < \infty,$$

then any number λ is a principal eigenvalue of (P_1) as long as

$$0 < \lambda \leq c(n, m_1, m_2)\left(\int_\Omega |g^+|^{\frac{n}{2}} dx\right)^{-\frac{2}{n}},$$

where $c(n, m_1, m_2)$ is a constant depending only on n, m_1, m_2.

Sketch of the proof. In the proof of Theorem 1, we replace $B_k(0)$ in (P_k) by $\Omega_k = B_k(0) \cap \Omega$. Then we choose a function $h_k(x)$ on $\partial\Omega_k$ such that $h_k(x) \geq 0$, $h_k(x) = 0$ on $\partial\Omega \cap \partial\Omega_k$, and $h_k(x) > 0$ on part of $B_k(0) \cap \partial\Omega_k$. Also, let $u = h_k(x)$ on $\partial B_k(0)$ in (P_k).

In the proof of Theorem 2, we replace $B_k(0)$ in (D_k) by $\Omega_k = B_k(0) \cap \Omega$.

Those are all the changes needed to carry out the complete proof of Theorem 9. (Actually one might need to smooth out the edges of Ω_k; we leave it to reader.) $\square$

Now we prove Theorem 5.

Proof of Theorem 5. We choose a radially symmetric smooth positive function $\eta(x)$ such that $\frac{c}{|x|^2} \geq \eta(x) \geq g_1(x)$ on R^n and $\eta(x) = \frac{c}{|x|^2}$ for $|x|$ large. If we can show Theorem 5 for $\eta(x)$, we can apply Theorem 1 to $g_1(x)$ to get the conclusion.

It is easy to see that the Euler equation

$$v''(r) + \frac{n-1}{r}v'(r) + \frac{(n-2)^2}{4r^2}v(r) = 0$$

has a solution $v(r) = r^{-\frac{n-2}{2}}$ for $0 < r < \infty$. Thus $v(x) = v(|x|)$ satisfies

(2.3) $-\Delta v = \frac{(n-2)^2}{4|x|^2}v \quad \text{on} \quad R^n \setminus \{0\}.$

Now for any number $0 < \lambda \leq \frac{(n-2)^2}{4c}$, we consider an initial value problem for the ordinary differential equation:

(2.4)
$$\begin{cases} \psi''(r) + \dfrac{n-1}{r}\psi'(r) + \lambda\eta(r)\psi(r) = 0, \\ \psi(0) = 1, \psi'(0) = 0. \end{cases}$$

Since $\eta(r)$ is smooth, it is clear that (2.4) has a solution $\psi(r)$ defined on $(0, T)$ for some $T > 0$. Also $\eta(r) > 0$ implies that $\psi(r)$ is decreasing on $(0, T)$. We claim that $\psi(r) > 0$ on $(0, T)$. Assume the claim for the moment. Then $\psi(r)$ can be extended beyond T, that is, $\psi(r)$ exists on $0 < r < \infty$. Thus $\psi(r) > 0$ is an eigenfunction corresponding to the principal eigenvalue λ.

We now prove that claim. Indeed if $\psi(r) < 0$ for some $0 < r < T$, since $\psi(0) = 1$, we can find a r_0 such that $\psi(r_0) = 0$, $\psi(r) > 0$ for $0 < r < r_0$. Then for $\psi(x) = \psi(|x|)$

(2.5)
$$\begin{cases} -\Delta\psi = \lambda\eta(|x|)\psi \quad \text{on} \quad 0 < |x| < r_0; \\ \psi(0) = 1, \psi(x) = 0 \quad \text{if} \quad |x| = r_0. \end{cases}$$

We then set $\psi(|x|) = \phi(|x|)v(|x|)$ for a new function $\phi(|x|)$. (2.3), (2.5) and a calculation shows that ϕ satisfies

(2.6)
$$\begin{cases} -\Delta\phi - 2\nabla\ln v \cdot \nabla\phi = (\lambda\eta(|x|) - \dfrac{(n-2)^2}{4|x|^2})\phi(|x|), \\ \phi(0) = 0, \phi(|x|) = 0 \quad \text{if} \quad |x| = r_0; \\ \phi(x) > 0 \quad \text{for} \quad 0 < |x| < r_0. \end{cases}$$

But $\lambda \leq \frac{(n-2)^2}{4c}$ and the construction of $\eta(x)$ imply

$$\lambda\eta(|x|) \leq \frac{\lambda c}{|x|^2} \leq \frac{(n-2)^2}{4|x|^2} \quad \text{for} \quad 0 < |x| < r_0.$$

Then we apply Hopf's maximum principle (Theorem 3.71 on p.96 in [2]) to (2.6) to conclude that $\phi(|x|)$ can not take a positive maximum on $0 < |x| < r_0$ unless ϕ is a constant. This is a contradiction. $\square$

Remark 2.2. Since $\eta(r) = \frac{c}{r^2}$ for r large, from (2.4) and the formula for general solutions of Euler equation, we see that if $0 < \lambda < \frac{(n-2)^2}{4c}$,

$$\psi(r) = c_1 r^{\alpha_1} + c_2 r^{\alpha_2} \quad \text{for} \quad r \quad \text{large,}$$

where c_1, c_2 are constants and

$$\alpha_1 = \frac{1}{2}(-(n-2) + \sqrt{(n-2)^2 - 4\lambda c}), \quad \alpha_2 = \frac{1}{2}(-(n-2) - \sqrt{(n-2)^2 - 4\lambda c}).$$

Therefore for different λ, the corresponding principal eigenfunction has different asymptotic behavior. For some particular principal eigenvalue, the asymptotic behavior of the corresponding eigenfunction is investigated in Theorem 7.

Now we show Theorem 6, which says that the assumptions in Theorem 5 are quite sharp.

Proof of Theorem 6. We prove the conclusion by contradiction. Suppose that there is a $\lambda_0 > 0$ and a function $u(x) > 0$, such that

$$-\Delta u(x) = \lambda_0 g(x) u(x) \quad \text{on} \quad R^n.$$

Fix a number $M > \frac{(n-2)^2}{4}$; then by the assumption of the theorem we can choose a number M_1 such that

$$(2.7) \qquad \lambda_0 g(x) \geq \frac{M}{|x|^2} \quad \text{for} \quad |x| \geq M_1.$$

Let k be an integer greater than M_1. We consider a boundary value problem for an ordinary differential equation

$$(2.8) \qquad \begin{cases} -v''(r) - \dfrac{n-1}{r} v'(r) = \dfrac{\mu}{r^2} v(r) & \text{for} \quad M_1 < r < k; \\ v(M_1) = 0, \quad v(k) = 0; \\ v(r) > 0 \quad \text{for} \quad M_1 < r < k. \end{cases}$$

Since the equation is an Euler equation, we see that only for

$$\mu = \frac{(n-2)^2}{4} + \pi^2 \ln^{-2}\left(\frac{k}{M_1}\right),$$

(2.8) has a solution of the form

$$(2.9) \qquad v_k(r) = C r^{-\frac{(n-2)}{2}} \sin\left(\sqrt{\mu - \frac{(n-2)^2}{4}} \ln \frac{r}{M_1}\right).$$

Thus $\mu \longrightarrow \frac{(n-2)^2}{4}$ as $k \longrightarrow \infty$. Now we fix a k such that the corresponding $\mu < M$. Let $v_k(r)$ be the solution defined by (2.9) with $C = 1$. We set $v_k(x) = v_k(|x|) = \phi_k(x) u(x)$ for a new function $\phi_k(x)$; a calculation shows that $\phi_k(x)$ satisfies

$$\begin{cases} -\Delta \phi_k(x) - 2\nabla \ln u(x) \cdot \nabla \phi_k(x) = \left(\dfrac{\mu}{|x|^2} - \lambda_0 g(x)\right)\phi_k(x); \\ \phi_k(x) = 0, \quad \text{for} \quad |x| = M_1; \\ \phi_k(x) = 0, \quad \text{for} \quad |x| = k; \\ \phi_k(x) > 0 \quad \text{for} \quad M_1 < |x| < k. \end{cases}$$

But (2.7) and the choice of μ imply that

$$\frac{\mu}{|x|^2} - \lambda_0 g(x) \leq 0 \quad \text{for} \quad M_1 \leq |x| \leq k.$$

Now an application of Hopf's maximum principle (Theorem 3.71 on p.96 in [2]) lets us conclude that $\phi_k(x)$ can not take an interior positive maximum unless $\phi_k(x)$ is a constant. This is a contradiction. $\qquad \square$

Now we prove Theorem 7. The proof is carried out by pulling back the problem (P) to an equivalent one on the unit sphere. Indeed, the proof of Theorem 7 includes that of Theorem 2. We have given a proof for Theorem 2 separately just for simplicity. Before we start the proof of Theorem 7, let us fix some notation. Let $s_0 = (0, 0, \cdots, -1)$ be the south pole of the unit sphere S^n in R^{n+1}, ds^2 the standard metric on S^n, dx^2 the Euclidean metric on R^n. Use the stereographic projection $F : R^n \longrightarrow S^n \setminus \{s_0\}$,

$$F(x) = (\frac{2x}{1 + |x|^2}, \frac{1 - |x|^2}{1 + |x|^2}), \quad x \in R^n.$$

$S^n \setminus \{s_0\}$ is diffeomorphic to R^n, and under this coordinate chart

$$ds^2 = \frac{4}{(1 + |x|^2)^2} dx^2, \quad x \in R^n.$$

We denote $\psi(x) = 4^{\frac{(n-2)}{4}}(1 + |x|^2)^{-\frac{(n-2)}{2}}$; then $ds^2 = \psi(x)^{\frac{4}{(n-2)}} dx^2$. The following lemma says that the eigenvalue problem (P) can be translated into an equivalent eigenvalue problem on S^n. (For a function $v(x)$ on R^n, we still denote the corresponding function $v \circ F^{-1}$ by $v(x)$. The notation should be clear from the context.)

Lemma 2.1. $u(x) > 0$ *satisfies*

$$(P) \qquad -\frac{4(n-1)}{n-2}\Delta u(x) = \lambda g(x)u(x) \quad on \quad R^n$$

if and only if the function $w(x) = u(x)\psi^{-1}(x)$ satisfies

$$(P_s) \qquad -\frac{4(n-1)}{n-2}\Delta_s w + n(n-1)w = \lambda g\psi^{-\frac{4}{n-2}}w \quad on \quad S^n \setminus \{s_0\},$$

where Δ_s is the Laplace-Beltrami operator on S^n with respect to the metric ds^2.

Proof. $u(x) > 0$ satisfies (P) if and only if

$$-\frac{4(n-1)}{n-2}\Delta u(x) = \lambda g(x)u^{-\frac{4}{n-2}}(x)u^{\frac{n+2}{n-2}}(x) \quad on \quad R^n.$$

That is, the metric $u^{\frac{4}{n-2}}(x)dx^2$ has the scalar curvature function $\lambda g(x)u^{-\frac{4}{n-2}}(x)$. Therefore the metric $u^{\frac{4}{n-2}}(x)\psi^{-\frac{4}{n-2}}ds^2$ has the scalar function $\lambda g(x)u^{-\frac{4}{n-2}}(x)$. For $w(x) = u(x)\psi^{-1}$, this is equivalent to (see p. 126 in [2])

$$-\frac{4(n-1)}{n-2}\Delta_s w + n(n-1)w = \lambda g u^{-\frac{4}{n-2}}w^{\frac{n+2}{n-2}} \quad on \quad S^n \setminus \{s_0\}.$$

By the definition of $w(x)$, this is just (P_s). $\qquad \square$

Now we are ready to prove Theorem 7. The proof is more or less the same as that of Theorem 2. The difference here is that we are now working on the unit sphere S^n. Then we can use the fact that S^n is a compact manifold.

Proof of Theorem 7. Let $B_{\frac{1}{k}}(s_0)$ be the geodesic ball on S^n centered at the south pole s_0 with radius $\frac{1}{k}$, $k = 10, 11, \cdots$. Denote $\Omega_k = S^n \setminus B_{\frac{1}{k}}(s_0)$. We may assume that $g(x_0) > 0$ for some $x_0 \in \Omega_{10}$. We consider

$$(P_{s,k}) \qquad \begin{cases} -\dfrac{4(n-1)}{n-2}\Delta_s w + n(n-1)w = \lambda g \psi^{-\frac{4}{n-2}}w & \text{on} \quad \Omega_k; \\ w = 0, \quad \text{on} \quad \partial\Omega_k. \end{cases}$$

Let

$$\lambda_1(k) = \inf\{\frac{1}{2}\int_{\Omega_k}(\frac{4(n-1)}{n-2}|\nabla_s w|^2 + n(n-1)w^2)dVol, \quad \text{where}$$

$$w \in H_0^1(\Omega_k), \int_{\Omega_k} g(x)\psi^{-\frac{4}{n-2}}w^2 dVol = 1\}.$$

where $dVol$ is the volume form on S^n induced from the metric ds^2 and ∇_s is the gradient with respect to the metric ds^2. Since $g(x_0) > 0$ for some x_0, $\lambda_1(k)$ is well defined and there is a non-negative function $w_k(x) \in H_0^1(\Omega_k)$ that achieves the infimum. By the regularity theory for solutions of elliptic equations (Theorem 3.55 on p.86 in [2]) and the maximum principle, we see that $w_k(x)$ is actually a C^2 positive solution of $(P_{s,k})$. Furthermore, from the definition of $\lambda_1(k)$, we see that $\lambda_1(k)$ is non-increasing on k and $\lambda_1(k) \geq 0$; thus $\lambda_0 = \lim_{k\to\infty}\lambda_1(k)$ exists. Now we have

$$\lambda_1(k) = \frac{1}{2}\int_{\Omega_k}(\frac{4(n-1)}{n-2}|\nabla_s w_k|^2 + n(n-1)w_k^2)dVol$$

and

$$\int_{\Omega_k} g(x)\psi^{-\frac{4}{n-2}}w_k^2 dVol = 1.$$

Thus a subsequence of $w_k(x)$ converges weakly in $H^1(S^n)$ to a non-negative function w in $H^1(S^n)$. Then w_k converges strongly to w in $L^q(S^n)$ for any $q < \frac{2n}{n-2}$. Since for some $p > \frac{n}{2}$,

$$\int_{R^n} |g(x)|^p(1+|x|^2)^{2p-n}dx < \infty,$$

we have

$$\int_{S^n} |g|^p\psi^{-\frac{4p}{n-2}}dVol < \infty.$$

Then by the Hölder inequality,

$$1 = \int_{\Omega_k(0)} g(x)\psi^{-\frac{4}{n-2}}w_k^2 dVol \leq (\int_{S^n}|g|^p\psi^{-\frac{4p}{n-2}}dVol)^{\frac{1}{p}}(\int_{S^n}w_k^{2q}dVol)^{\frac{1}{q}}$$

for some $q < \frac{n}{n-2}$. Thus

$$\int_{S^n} g(x)\psi^{-\frac{4}{n-2}}w^2 dVol = \lim_{k\to\infty}\int_{\Omega_k} g(x)\psi^{-\frac{4}{n-2}}w_k^2 dVol = 1.$$

On the other hand, it is easy to see that w is a weak solution of

$$\begin{cases} -\dfrac{4(n-1)}{n-2}\Delta_s w + n(n-1)w = \lambda_0 g\psi^{-\frac{4}{n-2}}w & \text{on} \quad S^n; \\ w \geq 0 \quad \text{on} \quad S^n. \end{cases}$$

Since $g\psi^{-\frac{4}{n-2}} \in L^p(S^n)$ for some $p > \frac{n}{2}$, $w \in L^{\frac{2n}{n-2}}(S^n)$, we have $g\psi^{-\frac{4}{n-2}}w \in L^q(S^n)$ for some $q > \frac{n}{2}$ (this can be seen by Theorem 9.15 on p.241 in [8] and the bootstrap argument). Then the regularity theory for solutions of elliptic equations (Theorem 3.55 on p.86 in [2]) conclude that w is smooth on $S^n \setminus \{s_0\}$ and continuous on S^n. An application of a maximum principle lets us conclude that $w(x) > 0$ on $S^n \setminus \{s_0\}$. Therefore if $\lambda_0 > 0$, then λ_0 is a principal eigenvalue for (P_s) and w is a corresponding eigenfunction. We notice that for (P), λ_0 is a principal eigenvalue and $u(x) = w(x)\psi(x)$ is a corresponding principal eigenfunction. Moreover,

$$u(x)|x|^{n-2} = w(x)\psi(x)|x|^{n-2} \longrightarrow w(s_0) \geq 0 \quad \text{as} \quad |x| \longrightarrow \infty.$$

Now we still need to prove that λ_0 is bounded by the lower bound given in the statement of Theorem 7. To do so, we only need to show that

$$(2.10) \qquad \lambda_1(k) \geq c(n)\left(\int_{R^n} |g^+|^{\frac{n}{2}}dx\right)^{-\frac{2}{n}}.$$

For any $w \in H^1(S^n)$ (see Proposition 6.4 on p.126 and Theorem 2.14 on p.39 in [2]),

$$(2.11) \qquad \int_{S^n}\left(\frac{4(n-1)}{n-2}|\nabla_s w|^2 + n(n-1)w^2\right)dVol \geq c_2(n)\left(\int_{S^n} w^{\frac{2n}{n-2}}dVol\right)^{\frac{n-2}{n}},$$

where $c_2(n)$ is a constant depending only on n. Also, by the Hölder inequality,

$$(2.12) \qquad \int_{S^n} g(x)\psi^{-\frac{4}{n-2}}w^2 dVol \leq \int_{S^n} g^+(x)\psi^{-\frac{4}{n-2}}w^2 dVol$$

$$\leq \left(\int_{S^n} |g^+(x)|^{\frac{n}{2}}\psi^{-\frac{2n}{n-2}}dVol\right)^{\frac{2}{n}}\left(\int_{S^n} |w|^{\frac{2n}{n-2}}dVol\right)^{\frac{n-2}{n}},$$

and finally we notice that

$$(2.13) \quad \int_{S^n} |g^+(x)|^{\frac{n}{2}}\psi^{-\frac{2n}{n-2}}dVol = \int_{R^n} |g^+(x)|^{\frac{n}{2}}\psi^{-\frac{2n}{n-2}}\psi^{\frac{2n}{n-2}}dx = \int_{R^n} |g^+(x)|^{\frac{n}{2}}dx.$$

Then (2.10) is a direct consequence of (2.11), (2.12) and (2.13). $\qquad\qquad\square$

Remark 2.3. As we pointed out in Remark 2.2, Theorem 7 provides information on the asymptotic behavior only for a particular eigenfunction. The asymptotic behavior of other eigenfunctions (corresponding to other principal eigenvalues) is unknown.

If we in addition assume that $p > n$ in Theorem 7, then the principal eigenfunction $u(x)$ obtained in the proof will satisfy

$$u(x)|x|^{n-2} \longrightarrow c_0 \geq 0 \quad \text{as} \quad |x| \longrightarrow \infty$$

and

$$|\nabla u||x|^{n-1} \longrightarrow c_1 \quad \text{as} \quad |x| \longrightarrow \infty$$

for some constants c_0 and c_1.

3. The case that $n = 2$

Now we prove Theorem 8. As we did in the proof of Theorem 7, we shall pull the equation in (P) back onto the two dimensional unit sphere S^2. Let s_0 be the south pole of the unit sphere S^2, ds^2 the standard metric on S^2, dx^2 the Euclidean metric on R^2. By the stereographic projection $F : R^2 \longrightarrow S^2 \setminus \{s_0\}$, $S^2 \setminus \{s_0\}$ is diffeomorphic to R^2, and under this coordinate chart

$$ds^2 = \frac{4}{(1 + |x|^2)^2} dx^2, \quad x \in R^2.$$

We denote $\psi(x) = 4(1 + |x|^2)^{-2}$; then $ds^2 = \psi(x) dx^2$. The following lemma says that the eigenvalue problem can be translated into an equivalent eigenvalue problem on S^2. (Once again, we shall not distinguish the notation for functions on R^2 and $S^2 \setminus \{s_0\}$ as we did in the proof of Theorem 7.)

Lemma 3.1. $u(x)$ satisfies

$$(P) \qquad\qquad -\Delta u(x) = \lambda g(x) u(x) \quad on \quad R^2$$

if and only if the function $u(x)$ satisfies

$$(P_s) \qquad\qquad -\Delta_s u = \lambda g \psi^{-1} u \quad on \quad S^2 \setminus \{s_0\},$$

where Δ_s is the Laplace-Beltrami operator on S^2.

Proof. Under the coordinate chart (stereographic projection) $F : R^2 \longrightarrow S^2 \setminus \{s_0\}$,

$$ds^2 = \psi(x) dx^2, \quad x \in R^2.$$

A direct calculation from the definition of the Laplace-Beltrami operator yields that

$$\Delta_s u = \psi^{-1} \Delta u,$$

where $\psi(x) = 4(1 + |x|^2)^{-2}$. $\qquad\qquad\qquad\qquad\qquad\qquad \square$

Proof of Theorem 8. We write $g(x) = g^+(x) - g^-(x)$, where $g^+(x) = \max\{g(x), 0\}$ and $g^-(x) = \max\{-g(x), 0\}$. Now take a function $\phi(x) \in C_0^\infty(R^2)$ such that $\phi(x) = 0$ for $|x|$ large, $0 \le \phi(x) \le 1$ on R^2, and

$$(3.1) \qquad\qquad \int_{R^2} (g^+(x) - \phi(x) g^-(x)) dx < 0.$$

(3.1) is possible since we have assumed $\int_{R^2} g(x) dx < 0$. We denote $g_1(x) = g^+(x) - \phi(x) g^-(x)$; then $g_1^+(x) = g^+(x)$ and $g_1^-(x) = \phi(x) g^-(x)$, and (3.1) implies that

$$(3.2) \qquad\qquad \int_{S^2} g_1 \psi^{-1} dVol = \int_{R^2} g_1 \psi^{-1} \psi dx = \int_{R^2} g_1 dx < 0.$$

Also the assumptions on g imply

$$\int_{R^2} |g_1(x)|^p (1 + |x|^2)^{2(p-1)} dx < \infty,$$

for some $p > 2$. Then

$$(3.3) \qquad \int_{S^2} |g_1(x) \psi^{-1}|^p dVol = 4^{1-p} \int_{R^2} |g_1(x)|^p (1 + |x|^2)^{2p} (1 + |x|^2)^{-2} dx$$

$$= \int_{R^2} |g_1(x)|^p (1 + |x|^2)^{2(p-1)} < \infty.$$

Then by Theorem 4.7 on p.104 in [2], (3.2) and (3.3), there is a function z on $W^{2,p}(S^2) \cap C^{1,\beta}(S^2) \cap C^2(S^2 \setminus \{s_0\})$, such that

$$\Delta_s z = g^* - g_1 \psi^{-1} \quad \text{on} \quad S^2 \setminus \{s_0\},$$

where

$$g^* = \frac{1}{4\pi} \int_{S^2} g_1 \psi^{-1} dVol.$$

Thus we can choose a constant $\delta_1 > 0$ such that if $0 < \delta < \delta_1$, then

$$\Delta_s(\delta z) + |\nabla_s(\delta z)|^2 + \delta g(x)\psi^{-1}(x) = \delta \Delta_s z + \delta^2 |\nabla_s z|^2 + \delta g(x)\psi^{-1}(x)$$

$$= \delta g^* - \delta g_1 \psi^{-1} + \delta^2 |\nabla_s z|^2 + \delta g(x)\psi^{-1}(x)$$

$$= \delta g^* - \delta(g_1^+(x) - g_1^-(x))\psi^{-1} + \delta^2 |\nabla_s z|^2 + \delta(g^+(x) - g^-(x))\psi^{-1}(x)$$

$$= \delta g^* - \delta(g^+(x) - g^-(x)\phi(x))\psi^{-1} + \delta^2 |\nabla_s z|^2 + \delta(g^+(x) - g^-(x))\psi^{-1}(x)$$

$$= \delta g^* + \delta^2 |\nabla_s z|^2 + \delta(\phi(x) - 1)g^-(x)\psi^{-1}(x)$$

$$\leq \delta g^* + \delta^2 |\nabla_s z|^2 \leq 0 \quad \text{on} \quad S^2 \setminus \{s_0\};$$

here we have used the fact that $0 \leq \phi(x) \leq 1$. Then it is easy to check that $v = e^{\delta z}$ satisfies

(3.4) $-\Delta_s v \geq \delta g(x)\psi^{-1} v \quad \text{on} \quad S^2 \setminus \{s_0\} \quad \text{for} \quad 0 < \delta < \delta_1.$

Now we start the same procedure as in the proof of Theorem 7. That is, let $B_{\frac{1}{k}}(s_0)$ be the geodesic ball on S^2 centered at the south pole s_0 with radius $\frac{1}{k}$, $k = 10, 11, \cdots$. Denote $\Omega_k = S^2 \setminus B_{\frac{1}{k}}(s_0)$. We may assume that $g(x_0) > 0$ for some $x_0 \in \Omega_{10}$. We consider

$(P_{s,k})$
$$\begin{cases} -\Delta_s u = \lambda g \psi^{-1} u & \text{on} \quad \Omega_k; \\ u = 0, & \text{on} \quad \partial \Omega_k. \end{cases}$$

Let

$$\lambda_1(k) = \inf\{\int_{\Omega_k} |\nabla_s u|^2 dVol \,|\, u \in H_0^1(\Omega_k), \int_{\Omega_k} g(x)\psi^{-1} u^2 dVol = 1\}.$$

As we saw before, $\lambda_1(k)$ is well defined and achieved by a positive C^2 function $u_k(x) \in H_0^1(\Omega_k)$. Furthermore, $\lambda_0 = \lim_{k \to \infty} \lambda_1(k)$ exists. Now we normalize $u_k(x)$ such that $u_k(y_0) = 1$ at the north pole y_0. As we did in the proof of Theorem 2, we see that a subsequence of $u_k(x)$ converges in C^2 to a positive C^2 function $u(x)$ on any compact set on $S^2 \setminus \{s_0\}$. Therefore if $\lambda_0 > 0$, λ_0 is a principle eigenvalue and $u(x)$ is a corresponding eigenfunction.

Now we show that $\lambda_0 > 0$. Let δ_1 and $v(x)$ be defined in (3.4). We claim that $\lambda_0 \geq \delta_1$. We only need to show that $\lambda_1(k) \geq \delta_1$ for k large. Indeed if $\lambda_1(k) < \delta_1$ for some k, let $\delta = \lambda_1(k)$, $v_k = e^{\lambda_1(k)z}$; (3.4) is

(3.5) $-\Delta_s v_k \geq \lambda_1(k)g(x)\psi^{-1} v_k \quad \text{on} \quad S^2 \setminus \{s_0\}.$

Set $u_k = v_k h_k$ for a new function h_k. Then the definition of $u_k(x)$ and (3.5) implies that h_k satisfies

(3.6)
$$\begin{cases} -\Delta_s h_k - 2\langle \nabla \ln v_k, \nabla h_k \rangle_s \leq 0 & \text{on} \quad \Omega_k; \\ h_k = 0 & \text{on} \quad \partial\Omega_k; \\ h_k > 0 & \text{on} \quad \Omega_k. \end{cases}$$

where $\langle \nabla \ln v_k, \nabla h_k \rangle_s$ is the inner product of the vectors $\nabla \ln v_k$ and ∇h_k under the metric ds^2. Now an application of Hopf's maximum principle (Theorem 3.71 on p.96 in [2]) to (3.6) concludes that h_k can not take an interior positive maximum on Ω_k unless $h_k(x)$ is a constant. This is a contradiction. $\qquad\square$

Remark 3.1. The idea of constructing $v_k(x)$ in (3.5) is borrowed from [9].

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