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A FIXED POINT INDEX FOR GENERALIZED INWARD MAPPINGS OF CONDENSING TYPE

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Abstract. A fixed point index is defined for mappings defined on a cone K which do not necessarily take their values in K but satisfy a weak type of boundary condition called generalized inward. This class strictly includes the well-known weakly inward class. New results for existence of multiple fixed points are established.

1. Introduction

Many fixed point principles for maps involving cones are easily proved if there is a theory of fixed point index for the class of mappings involved. Such a theory can be used to establish the existence of several solutions of nonlinear equations.

The fixed point index is often defined via a theory of degree employing retractions onto the cone. A key restriction is that the image points of the maps should belong to the cone. However, some fixed point theorems are known to be valid for maps whose values need not lie in the cone, such as the weakly inward mappings. (The precise definition of concepts mentioned in the introduction can be found later in the paper.)

Recently there has been progress in extending the fixed point index to weakly inward maps that are are defined on compact convex sets [15], or are compact [8]. [15] employs the metric projection r (nearest point retraction) in a strictly convex space and defines the fixed point index by approximating on shrinking neighbourhoods of the compact convex set. If K is a cone, or more generally a closed convex set in a Banach space X, Ω is a bounded subset of K that is open relative to K, $A : \overline{\Omega} \to X$ is compact, weakly inward, and $Ax \neq x$ for $x \in \partial\Omega_K$, a retraction with a certain property (P) is used in [8] to define the index via the degree $d(I - Ar, r^{-1}(\Omega), 0)$ where $r : X \to K$ is the continuous retraction with property (P). Although it is shown that such retractions often exist, it is not shown that two retractions with property (P) give the same definition of index.

In this paper we adopt a different approach to defining the index, which enables us to deal with more general mappings and to give an unambiguous definition. Firstly we work with a class of mappings which we call generalized inward and which strictly includes the weakly inward maps. Some fixed point theory for these maps has been studied in Hilbert spaces [1], [16] and in locally convex spaces in [13]. Secondly, we use the theory of condensing mappings, compact mappings being but a special case. However, we employ metric projections and utilize their properties,

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and this imposes restrictions (made precise later in the paper) on the convex set K and on the mappings we can deal with. Generally, the more we can say about the metric projection, the less we need on the map A. For example, in any Hilbert space and for every closed, convex set our map need only be condensing, but in more general Banach spaces A may need to be compact. The theory does not apply to all closed convex sets in arbitrary Banach spaces, even for compact mappings, but we show that it is applicable on cones that arise frequently in applications.

Our results for generalized inward maps are new; we know of no other results concerning the existence of multiple fixed points for these mappings. In fact we obtain new results even for weakly inward compact maps on reflexive Banach spaces.

In section 2 of this paper we first introduce the concept of a generalized inward map relative to a closed convex set K. We give several equivalent definitions (see Proposition 2.2 below) which show, in particular, that weakly inward maps are generalized inward. We exhibit a generalized inward map that is not weakly inward. The fixed point index for generalized inward maps is defined using the theory for condensing maps [10]. We show that our definition is unambiguous and that most of the usual properties remain valid.

In section 3 we obtain new fixed point theorems by using the index theory developed in section 2. Our results establish the existence of at least one fixed point and of multiple fixed points under appropriate assumptions. In section 4 new results are given for the special case of weakly inward maps, where homotopy arguments are more easily established.

We give an application to an integral equation similar to one considered in [2] but in a situation where the abstract results of [2] are not applicable.

2. A fixed point index for generalized inward maps

Let X be a real Banach space. For a bounded set Q in X, $\gamma(Q)$ will stand for either the set measure of noncompactness $\alpha(Q)$ defined by

$$\alpha(Q) = \inf\{d > 0 : Q \text{ admits a finite cover by sets of diameter at most } d\},$$

or the ball measure $\beta(Q)$ defined analogously when the covering sets are balls. For properties of $\gamma(Q)$ see [2], [10], [14]. A continuous map $A : \mathrm{dom}(A) \subset X \to X$ is called k-γ-contractive if there is $k \geq 0$ such that $\gamma(A(Q)) \leq k\gamma(Q)$ for each bounded $Q \subset \mathrm{dom}(A)$; γ-condensing if $\gamma(A(Q)) < \gamma(Q)$ for each bounded $Q \subset \mathrm{dom}(A)$ with $\gamma(Q) \neq 0$. It is readily seen that a compact map A is 0-γ-contractive, where A is said to be compact if A is continuous and $\overline{A(Q)}$ (the closure) is compact in X for every bounded set $Q \subset \mathrm{dom}(A)$. Every k-γ-contractive map with $k < 1$ is γ-condensing, but there are γ-condensing maps that are not k-γ-contractive for any $k < 1$. Now let D be a bounded open set in X, let K be a closed convex set and suppose that $D_K = D \cap K \neq \emptyset$. Denote by $\overline{D}_K$ the closure and ∂D_K the boundary of D_K relative to K. When $A : \overline{D}_K \to K$ is γ-condensing and $x \neq Ax$ for $x \in \partial D_K$, there is defined in [10], [14] an integer $i_K(A, D_K)$, called the fixed point index of A on D_K, which has the following properties.

(P_1) (Existence property) If $i_K(A, D_K) \neq 0$, then A has a fixed point in D_K.

(P_2) (Normalization) If $u \in D_K$, then $i_K(\hat{u}, D_K) = 1$, where $\hat{u}(x) = u$ for $x \in \overline{D}_K$.

(P_3) (Additivity property) If W^1, W^2 are disjoint relatively open subsets of D_K such that $x \neq Ax$ for $x \in \overline{D}_K \setminus (W^1 \cup W^2)$, then

$$i_K(A, D_K) = i_K(A, W^1) + i_K(A, W^2).$$

(P_4) (Homotopy property) If $H : [0,1] \times \overline{D}_K \to K$ is continuous and such that $\gamma(H([0,1] \times Q)) < \gamma(Q)$ for each $Q \subset D$ with $\gamma(Q) \neq 0$, and if $x \neq H(t,x)$ for $x \in \partial D_K$ and $t \in [0,1]$, then

$$i_K(H(0,\cdot), D_K) = i_K(H(1,\cdot), D_K).$$

(P_5) If $A : K \to K$ is such that $A(K)$ is bounded, then there exists $\rho_0 > 0$ such that $i_K(A, B_K(\rho)) = 1$ for all $\rho \geq \rho_0$, where $B(\rho) = \{x \in X : \|x\| < \rho\}$.

(P_5) is a slight variant of a property in [12] (also see [11]).

Let K be a closed convex set. For $x \in K$, the *inward set* of x relative to K (see for example [2]) is the set $I_K(x) = \{x + c(z - x) : z \in K \text{ and } c \geq 0\}$. Geometrically it is the union of all rays beginning at x and passing through some other point of K. Because of the convexity of K, it follows that $I_K(x) = \{x + c(z - x) : z \in K \text{ and } c \geq 1\}$. Its closure is written $\overline{I}_K(x)$. Recall that K is called a *wedge* if $\lambda x \in K$ for $x \in K$ and $\lambda \geq 0$. If a wedge K also satisfies $K \cap (-K) = \{0\}$, then K is called a *cone*. The following properties are easily verified.

(i) $\overline{I}_K(x)$ is a convex set containing K for each $x \in K$.

(ii) If K is a cone or a wedge in X, then $\overline{I}_K(x)$ is a wedge for each $x \in K$.

A map $A : \Omega \subset K \to X$ is said to be *inward* (respectively, *weakly inward*) on Ω relative to K if $Ax \in I_K(x)$ (resp. $Ax \in \overline{I}_K(x)$) for $x \in \Omega$. Other equivalent definitions of weakly inward are possible, for example in terms of support functionals. We shall utilize the following in our application in section 5. We refer to [2], §18.3 and §20.4, for details. $A : K \to X$ is weakly inward on a cone K if

$$x \in \partial K,\ x^* \in K^* \text{ and } x^*(x) = 0 \text{ together imply } x^*(Ax) \geq 0.$$

Here $K^* = \{x^* \in X^* : x^*(x) \geq 0 \text{ on } K\}$, the so-called dual cone (but actually a wedge).

Weakly inward mappings have been well studied by many authors and a number of fixed point theorems obtained, see for example [2]–[5].

We now define the generalized inward maps. Several equivalent definitions will be established which show, in particular, that all weakly inward maps are generalized inward. We give an example to show that the converse does not hold, even in finite dimensional Hilbert space. Conditions very similar to our generalized inward condition have been previously used in the study of fixed points of maps, for example, in Hilbert space by Browder and Petryshyn [1], and for set-valued maps in locally convex spaces by Reich [13].

Definition 2.1. Let K be a closed convex set. A map $A : \Omega \subset K \to X$ is said to be *generalized inward* on Ω relative to K if the following condition is satisfied:
(*G-I*): $\quad d(Ax, K) < \|x - Ax\| \quad$ for $x \in \Omega$ with $Ax \notin K$,
where $d(y, K) = \inf\{\|y - u\| : u \in K\}$.

Proposition 2.2. *The condition G-I is equivalent to each of the following conditions:*
(H_1) *for every $x \in \Omega$ with $Ax \notin K$, there exists $y \in I_K(x)$ such that $\|Ax - y\| < \|x - Ax\|$;*
(H_2) *$d(Ax, I_K(x)) < \|x - Ax\|$ for $x \in \Omega$ with $Ax \notin K$;*
(H_3) *$d(Ax, \overline{I}_K(x)) < \|x - Ax\|$ for $x \in \Omega$ with $Ax \notin K$;*
(H_4) *for every $x \in \Omega$ with $Ax \notin K$,*

$$d((1-t)x + tAx, K) < t\|x - Ax\| \ \textit{ for all } t \in (0,1);$$

(H_5) *for every* $x \in \Omega$ *with* $Ax \notin K$*, there exists* $t \in (0,1)$ *such that*

$$d((1-t)x + tAx, K) < t\|x - Ax\|;$$

(H_6) $\liminf_{t\to 0^+} t^{-1} d((1-t)x + tAx, K) < \|x - Ax\|$ *for* $x \in \Omega$ *with* $Ax \notin K$.

Proof. We first prove that G-I and H_1–H_3 are equivalent. Obviously H_1 is equivalent to each of H_2 and H_3. It is easy to see that G-I implies H_1. Conversely, if H_1 holds and also $y \in K$, then $d(Ax, K) \le \|Ax - y\| < \|x - Ax\|$ and thus (G-I) holds. If $y \notin K$ there exist $t \in (0,1)$ and $z \in K$ such that $z = ty + (1-t)x$. Hence by H_1 we have

$$d(Ax, K) \le \|Ax - z\| \le t\|Ax - y\| + (1-t)\|Ax - x\| < \|x - Ax\|.$$

Next we prove that G-I and H_4–H_6 are equivalent. Firstly, H_4 implies H_5 which is equivalent to H_6, because if $\|(1-t)x + tAx - y\| < t\|x - Ax\|$, then

$$\|(1-st)x + stAx - [(1-s)x + sy]\| < st\|x - Ax\|, \quad \text{for } 0 \le s \le 1.$$

Now we prove that G-I implies H_4 and that H_5 implies G-I. Assume that G-I holds. Since K is convex, $(1-t)x + ty \in K$ for each $y \in K$ and $t \in (0,1)$, and thus we have

$$d((1-t)x + tAx, K) \le \|(1-t)x + tAx - (1-t)x - ty\| = t\|y - Ax\|,$$

Consequently, we obtain $t^{-1}d((1-t)x + tAx, K) \le d(Ax, K)$. Thus H_4 holds. Now assume that H_5 holds. There exist $t \in (0,1)$ and $u \in K$ such that

$$\|(1-t)x + tAx - u\| < t\|x - Ax\|.$$

Hence we have

$$\begin{aligned} d(Ax, K) &\le \|Ax - u\| \le \|Ax - (1-t)x - tAx\| + \|(1-t)x + tAx - u\| \\ &< (1-t)\|x - Ax\| + t\|x - Ax\| = \|x - Ax\|, \end{aligned}$$

that is, G-I holds. □

When $\Omega = K$ much of this proposition is known. H_6 has been studied by Cramer and Ray amongst others, see for example [16]. Also Williamson [16] shows G-I is equivalent to H_6 (and other conditions) in Hilbert spaces. If $\Omega = K$ is a ball then it is easy to see that G-I implies the Leray Schauder condition

$$(LS) \qquad Ax \ne \lambda x \text{ for all } \lambda > 1 \text{ and all } x \in \partial\Omega.$$

It is easy to check that the converse also holds if the space is strictly convex.

By H_3 in Proposition 2.2 we immediately obtain

Lemma 2.3. *If* $A : \Omega \subset K \to X$ *is weakly inward on* Ω *relative to* K*, then* A *is generalized inward on* Ω *relative to* K.

The converse of Lemma 2.3 is false.

Example 2.4. Let $H = \mathbb{R}^2$, $K = \{(x,0) : x \ge 0\}$, and D the open disk of radius 5, so that $\overline{D}_K = \{(x,0) : 0 \le x \le 5\}$. We define a map $A : \overline{D}_K \to H$ as follows:

$$A(x,0) = \begin{cases} (x/2, x), & x \in [0,2], \\ (2x-3, 6-2x), & x \in [2,3], \\ (x/2 + 3/2, x/2 - 3/2), & x \in [3,5]. \end{cases}$$

It is readily checked that A is generalized inward but not weakly inward. It is possible to give even simpler examples, but this one also satisfies other hypotheses used later in the paper (see Corollary 3.3).

Recall that a map r from X to K is called a metric projection if r satisfies $\|x - rx\| = d(x, K)$ for each $x \in X$, that is, rx is a point of K nearest to x. A nearest point exists for every closed convex set K if X is reflexive. If r_1, r_2 are metric projections from X into K, then $tr_1 + (1-t)r_2$ is also a metric projection for each $t \in [0,1]$. Hence there is at most one metric projection if the space X is strictly convex. Furthermore, if X is locally uniformly convex, this unique metric projection is continuous.

We shall consider convex sets K for which there is a continuous metric projection r that is a k-γ-contraction. We define $\gamma(r) = \inf\{k : r \text{ is a } k\text{-}\gamma\text{-contraction}\}$.

Definition 2.5. K is said to be an M_l-set for some $1 \le l < +\infty$ if there exists a continuous metric projection r from X to K such that $\gamma(r) = l$. K is called an M_∞-set if the metric projection is only continuous. For convenience, in this last case we write $\gamma(r) = \infty$.

We will have greatest flexibility in the choice of mappings when K is an M_1-set. We therefore give some examples.

Example 2.6. If X is a Hilbert space, then any closed convex set K in X is an M_1-set, because the metric projection from X into K is nonexpansive (see Proposition 9.2 in [2]).

Example 2.7. In any Banach space X, let K be a ball. Then K is an M_1-set.

If $K = \{x : \|x - x_0\| \le a\}$, it is readily verified that

$$r(x) = \begin{cases} x & \text{if } \|x - x_0\| \le a, \\ x_0 + \dfrac{a}{\|x - x_0\|}(x - x_0) & \text{if } \|x - x_0\| > a, \end{cases}$$

is the metric projection. It is a 1-set-contraction since $r(\Omega) \subseteq \mathrm{co}(x_0 \cup \Omega)$, (cf. [10]).

Example 2.8. Let Ω be a bounded domain in $\mathbb{R}^n$ and let $X = C(\overline{\Omega})$ be the space of continuous functions endowed with the norm $\|u\| = \sup_{x \in \overline{\Omega}} |u(x)|$. Then the solid cone $K = \{u \in X : u(x) \ge 0 \text{ for all } x \in \overline{\Omega}\}$ is an M_1-set.

The metric projection onto K is given by $ru = u^+ = \max\{u, 0\}$. To see this note that $|u(x) - v(x)|$ is minimized for $v \in K$ by taking $v(x) = u(x)$ if $u(x) \ge 0$ and $v(x) = 0$ if $u(x) < 0$, that is, taking $v(x) = u^+(x)$. Hence $|u(x) - u^+(x)| \le |u(x) - v(x)|$ for every v satisfying $v(x) \ge 0$. Now r is nonexpansive because

$$\begin{aligned} |u^+(x) - v^+(x)| &= \frac{1}{2}\{|\, u(x) + |u(x)| - v(x) - |v(x)|\,|\} \\ &\le \frac{1}{2}\{|\, u(x) - v(x)| + |\,|u(x)| - |v(x)|\,|\} \\ &\le |u(x) - v(x)|. \end{aligned}$$

Example 2.9. Let Ω be as in Example 2.8, let $X = L^p(\Omega), 1 \le p \le \infty$, and let $K = \{u \in X : u(x) \ge 0 \text{ a.e.}\}$. Then K is an M_1-set.

In this space, K is a cone with empty interior. The metric projection is again given by $ru = u^+$, and is nonexpansive because $|u^+(x) - v^+(x)| \leq |u(x) - v(x)|$ holds a.e. Raising to the power p and integrating the inequality when $p < \infty$, or taking the essential supremum when $p = \infty$, proves that r is nonexpansive.

Remark 2.10. If X is reflexive and locally uniformly convex, then any closed convex set in X is an M_∞-set. Every reflexive space can be renormed so as to be locally uniformly convex. For weakly inward maps we can often change to an equivalent norm. However, the definition of G-I maps depends directly on the norm employed.

The following result will be used to show that our definition of index does not depend on the choice of suitable metric projection if more than one exists.

Lemma 2.11. *If K is an M_l-set in a Banach space X for some $l \in [1, \infty]$ and r_1, r_2 are l-γ-contractive metric projections from X into K, then $tr_1 + (1-t)r_2$ is also an l-γ-contractive metric projection for each $t \in [0, 1]$.*

The next result is an important step in developing the index theory for generalized inward maps; it shows that rA has the same fixed points as A.

Lemma 2.12. *Let K be a closed convex set in a Banach space X and suppose $A : \Omega \subset K \to X$ is a generalized inward map on Ω relative to K such that $x \neq Ax$ for $x \in \Omega$. If r is a metric projection from X to K, then $x \neq rAx$ for all $x \in \Omega$.*

Proof. The proof is by contradiction. Assume that there exists $x \in \Omega$ such that $x = rAx$. If $Ax \in K$, then $x = rAx = Ax$, a contradiction. If $Ax \notin K$, then

$$\|Ax - x\| = \|Ax - rAx\| = d(Ax, K) < \|x - Ax\|$$

since r is a metric projection and A is generalized inward, another contradiction. □

Notation. In the following we require rA to be γ-condensing. If $\gamma(A) = k$ and $\gamma(r) = l$ this is so if $kl < 1$. We will write $kl < 1$ to mean either the case just mentioned or $l = 1$ and A is condensing or A is compact and r is continuous.

Lemma 2.13. *Let K be an M_l-closed convex set in a Banach space X for some $l \in [1, \infty]$ and let D be a bounded open set in X such that $D_K \neq \emptyset$. Let $A : \overline{D}_K \to X$ be a k-γ-contractive map with $kl < 1$ such that A is generalized inward on ∂D_K relative to K. Suppose that $x \neq Ax$ for all $x \in \partial D_K$. Then for any l-γ-contractive metric projections r_1, r_2 from X to K, we have $i_K(r_1 A, D_K) = i_K(r_2 A, D_K)$.*

Proof. Let $H(t, x) = tr_1 Ax + (1-t)r_2 Ax$ for $x \in \overline{D}_K$ and $t \in [0, 1]$. Obviously $H : [0, 1] \times \overline{D}_K \to K$ is continuous and such that $\gamma(H([0, 1] \times Q)) < \gamma(Q)$ for each $Q \subset D$ with $\gamma(Q) \neq 0$. By Lemmas 2.11 and 2.12 we have $x \neq tr_1 Ax + (1-t)r_2 Ax$ for $x \in \partial D_K$ and $t \in [0, 1]$. It follows from (P_4) that $i_K(r_1 A, D_K) = i_K(r_2 A, D_K)$. □

We now define the fixed point index for generalized inward γ-condensing maps.

Definition 2.14. Let K be an M_l-closed convex set in a Banach space X for some $l \in [1, \infty]$, and D a bounded open set in X such that $D_K \neq \emptyset$. Let $A : \overline{D}_K \to X$ be a k-γ-contractive map with $kl < 1$ such that A is generalized inward on ∂D_K relative to K and $x \neq Ax$ for all $x \in \partial D_K$. Then we define the fixed point index of A over D_K with respect to K as follows:

$$i_K(A, D_K) = i_K(rA, D_K)$$

where $r : X \to K$ is any l-γ-contractive metric projection and $i_K(rA, D_K)$ is the fixed point index for condensing mappings defined in [10].

By Lemmas 2.12 and 2.13 we see that $i_K(A, D_K)$ makes sense and is independent of the choice of l-γ-contractive metric projection. Note also that the new index coincides with the usual one when $A(\overline{D_K}) \subseteq K$.

The fixed point index in Definition 2.14 has most of the usual properties of fixed point index. However, although we can define the index for a map A that is generalized inward on ∂D_K, we need A to be generalized inward on $\overline{D}_K$ to ensure that a nonzero index implies the existence of fixed points.

Theorem 2.15. *Let K be an M_l-closed convex set in a Banach space X for some $l \in [1, \infty]$ and D a bounded open set in X such that $D_K \neq \emptyset$. Assume that $A : \overline{D}_K \to X$ is a generalized inward k-γ-contractive map with $kl < 1$ and $x \neq Ax$ for $x \in \partial D_K$. Then the index satisfies properties (P_1)–(P_3) as listed above and (P_4) in the following form.*
(P_4) (Homotopy property) If $H : [0, 1] \times \overline{D}_K \to X$ is continuous and such that for each $t \in [0, 1]$, $H(t, .) : \partial D_K \to X$ is a generalized inward map and either
(a) when $l \neq 1$, there exist $k \geq 0$ such that $kl < 1$ and $\gamma(H([0, 1] \times Q)) \leq k\gamma(Q)$ for each $Q \subset \overline{D}_K$, or
(b) when $l = 1$, $\gamma(H([0, 1] \times Q)) < \gamma(Q)$ for each $Q \subset \overline{D}_K$ with $\gamma(Q) \neq 0$.
Then, if $x \neq H(t, x)$ for $x \in \partial D_K$ and $t \in [0, 1]$,

$$i_K(H(0, .), D_K) = i_K(H(1, .), D_K).$$

Proof. (P_1) If $i_K(A, D_K) \neq 0$, then we have $i_K(rA, D_K) \neq 0$ by Definition 2.14. The earlier version of (P_1) implies that rA has a fixed point in D_K and, by Lemma 2.12, A has too.

(P_2) is obvious.

(P_3) follows easily from the definition and use of Lemma 2.12.

(P_4) If $x \neq H(t, x)$ for $x \in \partial D_K$ and $t \in [0, 1]$, by Lemma 2.12 we have $x \neq rH(t, x)$ for $x \in \partial D_K$ and $t \in [0, 1]$. The homotopy property for condensing maps implies the result. □

3. Fixed point theorems for generalized inward maps

In this section we shall obtain some new fixed point theorems by using the fixed point index developed in section 2.

Theorem 3.1. *Let K be an M_l-closed convex set in a Banach space X for some $l \in [1, \infty]$, and let $A : K \to X$ be a k-γ-contractive generalized inward map with $kl < 1$ and such that $A(K)$ is bounded. Then there is $\rho_0 > 0$ such that $i_K(A, B_K(\rho)) = 1$ for all $\rho \geq \rho_0$, where $B(\rho) = \{x \in X : \|x\| < \rho\}$. Hence, A has a fixed point in K.*

Proof. Let r be an l-γ-contractive metric projection from X to K. Since $A(K)$ is bounded, $rA(K)$ is bounded too. Let $\rho_0 > 0$ be such that $\overline{A(K) \cup rA(K)} \subset B(\rho_0)$. Then, for each $\rho \geq \rho_0$, $\overline{rA(K)} \subset B_K(\rho)$ and $x \neq Ax$ for $x \in \partial B_K(\rho)$. By Definition 2.14 and property (P_5) we have $i_K(A, B_K(\rho)) = i_K(rA, B_K(\rho)) = 1$. □

This result extends Theorem 18.3 in [2].

Theorem 3.2. *Let K be an M_l-closed convex set in a Banach space X for some $l \in [1, \infty]$, and D a bounded open set such that $D_K \neq \emptyset$. Assume that $A : \overline{D}_K \to X$ is a k-γ-contractive generalized inward map with $kl < 1$ such that*
(1) there exists $x_0 \in D_K$ such that $tA(x) + (1 - t)x_0$ is a generalized inward map on ∂D_K for each $t \in (0, 1)$, and

(LS) $x \neq tA(x) + (1-t)x_0$ *for all* $x \in \partial D_K$ *and* $t \in (0,1)$.
Then A *has a fixed point in* $\overline{D}_K$*, and if* $x \neq A(x)$ *for* $x \in \partial D_K$*, then* $i_K(A, D_K) = 1$.

Proof. Assume without loss of generality that $x \neq A(x)$ for $x \in \partial D_K$. Let $H(t,x) = tAx + (1-t)x_0$ for $x \in \overline{D}_K$ and $t \in [0,1]$. By hypothesis $H(t,.) : \partial D_K \to X$ is a generalized inward map for each $t \in [0,1]$, and $x \neq H(t,x)$ for $x \in \partial D_K$ and $t \in [0,1]$. Hence, from Theorem 2.15, $i_K(A, D_K) = i_K(\hat{x}_0, D_K) = 1$. □

The special case of Theorem 3.2 when A is weakly inward is an improvement of Theorem 1 in [3].

Note that if $0 \in D_K$ and A is a generalized inward on ∂D_K it does not follow in general that tA is generalized inward on ∂D_K for $t \in [0,1]$, though it does for weakly inward maps. This limits the use of homotopy arguments. The next result gives hypotheses that ensure that tA is generalized inward.

Corollary 3.3. *Let* K *be a closed convex set in a Hilbert space* H *and* D *a bounded open set such that* $0 \in D_K$*. Assume that* $A : \overline{D}_K \to H$ *is a* γ*-condensing generalized inward map such that the following conditions hold:*
(2) $(x, Ax) \leq \|x\|^2$ *for* $x \in \partial D_K$ *with* $Ax \notin K$*;*
(LS) $x \neq tAx$ *for* $x \in \partial D_K$ *and* $t \in (0,1)$.
Then A *has a fixed point in* $\overline{D}_K$.

Proof. It is sufficient to prove that tA is a generalized inward map on ∂D_K for each $t \in (0,1)$. In fact, for any fixed $x \in \partial D_K$ and $t \in (0,1)$ such that $tAx \notin K$, we have $Ax \notin K$ since $0 \in K$ and K is convex. Since $(x, Ax) \leq \|x\|^2$ and $0 < t < 1$, we have $2t(1-t)(x, Ax) \leq (1-t^2)\|x\|^2$. This implies

$$\begin{aligned} t^2\|x - Ax\|^2 &= t^2\|x\|^2 + t^2\|Ax\|^2 - 2t^2(x, Ax) \\ &\leq \|x\|^2 + t^2\|Ax\|^2 - 2t(x, Ax) \leq \|x - tAx\|^2. \end{aligned}$$

Thus we have $t\|x - Ax\| \leq \|x - tAx\|$. Since A is generalized inward on ∂D_K, there exists $y \in K$ such that $\|Ax - y\| < \|x - Ax\|$. Hence

$$d(tAx, K) \leq \|tAx - ty\| < t\|x - Ax\| \leq \|x - tAx\|.$$

□

Note that, if in Corollary 3.3, (2) holds for all $x \in \partial D_K$, then *(LS)* holds too. Further if D is a ball and $K = \overline{D}$ and $A : K \to H$ is generalized inward, then *(LS)* holds.

Example 2.4 given earlier satisfies all of the conditions in Corollary 3.3 but is not weakly inward.

Remark 3.4. The following norm-type boundary condition implies *(LS)*.
(B_1) : $\|Ax\| < \|x\| + \|x - Ax\|$ for each $x \in \partial D_K$ with $\|Ax\| > \|x\|$.
When X is a Hilbert space, (B_1) is equivalent to the following condition:
$(B_2) : (x, Ax) < \|x\|\,\|Ax\|$ for each $x \in \partial D_K$ with $\|Ax\| > \|x\|$.
This condition is then equivalent to *(LS)* because of the known criterion for equality in the Schwarz inequality.

4. Nonzero fixed point theorems for weakly inward maps

In this section we discuss the existence of nonzero fixed points for weakly inward maps. As remarked earlier, it is often possible to change to an equivalent locally uniformly convex norm; the weakly inward property is preserved. Therefore, for

compact maps, for most of the results of this section one can change to such an equivalent norm if necessary.

Theorem 4.1. *Let K be an M_l-wedge in a Banach space X for some $l \in [1, \infty]$, and D a bounded open set in X such that $D_K \neq \emptyset$. Assume that $A : \overline{D}_K \to X$ is a k-γ-contractive map with $kl < 1$ and A is weakly inward on $\overline{D}_K$ relative to K. If there exists $e \in K \setminus \{0\}$ such that $x \neq Ax + \lambda e$ for $x \in \partial D_K$ and $\lambda \geq 0$, then $i_K(A, D_K) = 0$.*

Proof. Assume, for a contradiction argument, that $i_K(A, D_K) \neq 0$ and let $\lambda > 0$. Let $H(t, x) = Ax + t\lambda e$ for $x \in \overline{D}_K$ and $t \in [0, 1]$. By hypothesis, $x \neq H(t, x)$ for $x \in \partial D_K$ and $t \in [0, 1]$. It is obvious that H satisfies (P_4) in Theorem 2.15 when $l > 1$ and when $l = 1$. Also $H(t, .) : \overline{D}_K \to X$ is weakly inward on $\overline{D}_K$ for each $t \in [0, 1]$. By Theorem 2.15 we have $i_K(A, D_K) = i_K(A + \lambda \hat{e}, D_K) \neq 0$. Hence for each $n \in N$, there exists x_n in the bounded set $\overline{D}_K$ such that $x_n = Ax_n + ne$; this is impossible since $e \neq 0$. □

We use Theorem 4.1 to obtain nonzero solutions when K is a wedge.

Theorem 4.2. *Let K be an M_l-wedge in a Banach space X for some $l \in [1, \infty]$, and D^1, D bounded open sets in X such that $0 \in D^1$ and $\overline{D^1_K} \subset D_K$. Suppose $A : \overline{D}_K \to X$ is a k-γ-contractive map with $kl < 1$ and A is weakly inward on $\overline{D}_K$. Suppose the following conditions hold.*
(LS) $x \neq tAx$ for $x \in \partial D^1_K$ and $t \in (0, 1)$.
(E) There exists $e \in K \setminus \{0\}$ such that $x \neq Ax + \lambda e$ for $x \in \partial D_K$ and $\lambda > 0$.
Then A has a fixed point in $\overline{D}_K \setminus D^1_K$.

The same assertion is valid if (LS) holds on D_K while (E) holds on D^1_K.

Proof. We may assume that A has no fixed point in $\partial D_K \cup \partial D^1_K$. It follows from Theorem 3.1 that $i_K(A, D^1_K) = 1$ and from Theorem 4.1 that $i_K(A, D_K) = 0$. By the additivity property of index we have

$$i_K(A, D_K \setminus \overline{D^1_K}) = i_K(A, D_K) - i_K(A, D^1_K) = 0 - 1 = -1$$

and thus A has a fixed point in $\overline{D}_K \setminus D^1_K$.

The result can be proved similarly when the hypotheses are interchanged. □

Theorem 4.2 improves Theorem 2 in [3] and Theorem 3 in [6].

Theorem 4.3. *Let K, D^1, D and A be as in Theorem 4.2 and suppose there exists a weakly inward compact map $C : \overline{D}_K \to X$ such that the following conditions hold.*
(LS) $x \neq tAx$ for $x \in \partial D^1_K$ and $t \in (0, 1)$.
(A_1) $\partial K_1 = \{x \in K : \|x\| = 1\}$ is not compact.
(A_2) $\alpha := \inf\{\|Cx\| : x \in \partial D_K\} > 0$.
(A_3) $x \neq Ax + \lambda Cx$ for $x \in \partial D_K$ and $\lambda > 0$.
Then A has a fixed point in $\overline{D}_K \setminus D^1_K$.

The same assertion is valid if we assume that (LS) holds on D_K while (A_2) and (A_3) hold on D^1_K.

Proof. We may assume that A has no fixed point in $\partial D_K \cup \partial D^1_K$. We first show that there exists $e \in K$ with $\|e\| = 1$ such that $\{te : t \geq 0\} \cap -\overline{C(\partial D_K)} = \emptyset$. If this is false, for each $x \in K$ with $\|x\| = 1$, there exists t_x such that $t_x x \in -\overline{C(\partial D_K)}$. Thus the set $Q := \{t_x x : \|x\| = 1\}$ is relatively compact and hence the set $\overline{\text{co}}(Q \cup \{0\})$ is compact. By hypothesis (A_2), we have $t_x \geq \alpha$, so that

$\overline{\text{co}}(Q \cup \{0\}) \supseteq K \cap \{\|x\| = \alpha\}$, a contradiction. Next we assert that there exists $\lambda_0 > 1$ such that $x \neq Ax + \lambda_0 Cx + \beta e$ for $x \in \partial D_K$ and $\beta \geq 0$. In fact, if not, there exist sequences $\{\lambda_n\}$, $\lambda_n \to \infty$, $\{\beta_n\}$ and $\{x_n\} \subset \partial D_K$ such that $x_n = Ax_n + \lambda_n Cx_n + \beta_n e$. Hence, $Cx_n + \beta_n e/\lambda_n \to 0$, and as $\{Cx_n\}$ is bounded we may assume that $\beta_n/\lambda_n \to b \in [0, +\infty)$. By (A_2), $b \neq 0$. It follows that $Cx_n \to -be$, contradicting the part just shown. Also $A + \lambda_0 C$ is weakly inward, as $\overline{I}_K(x)$ is a wedge. By Theorem 4.1 we have $i_K(A + \lambda_0 C, D_K) = 0$. Let $H(t, x) = Ax + t\lambda_0 Cx$ for $x \in \overline{D}_K$ and $t \in [0, 1]$. By hypothesis, we have $x \neq H(t, x)$ for $x \in \partial D_K$ and $t \in [0, 1]$. By (P_4) in Theorem 2.15 we get $i_K(A, D_K) = 0$. It follows from (P_3) that $i_K(A, D_K \setminus \overline{D_K^1}) = -1$, and thus A has a fixed point in $D_K \setminus \overline{D_K^1}$. □

The arguments used in this proof are similar to some of those used in [7]. Theorem 4.3 improves Theorem 1 in [9]. Even the special case of $A = C$ in Theorem 4.3 improves Theorem 3 in [3].

Theorem 4.4. *Under the same hypotheses as Theorem 4.3 the same conclusions hold if* (A_1) *is replaced by*
$(A_1)'$ $\quad \overline{-C(\partial D_K)} \cap \{K \setminus -K\} = \emptyset$.

Proof. By a contradiction argument we see that for $e \in K \setminus -K$ with $\|e\| = 1$ there is $\lambda_0 > 1$ such that $x \neq Ax + \lambda_0 Cx + \beta e$ for $x \in \partial D_K$ and $\beta \geq 0$. The proof then proceeds as before. □

To conclude this section we give conditions that assure the existence of at least two nonzero fixed points in K.

Theorem 4.5. *Let K, D, D^1 be as in Theorem 4.2 and let $A : K \to X$ be a k-γ-contractive weakly inward map with $kl < 1$ and $A(K)$ bounded. Suppose that (LS) holds on $\partial(D_K^1)$ and (E) holds on $\partial(D_K)$. Then A has at least two nonzero fixed points in K.*

Proof. Since $A(K)$ is bounded, by Theorem 3.1 there exists a bounded open set D^2 such that $\overline{D}_K \subset D_K^2$ and $i_K(A, D_K^2) = 1$. If A has no fixed point on ∂D_K, then by (E) and the additivity property we have $i_K(A, D_K^2 \setminus \overline{D}_K) = i_K(A, D_K^2) - i_K(A, D_K) = 1 - 0 = 1$. Hence A has a fixed point x_1 in $D_K^2 \setminus D_K$. If A has a fixed point in ∂D_K^1, then the conclusion holds. If $x \neq Ax$ for $x \in \partial D_K^1$, by the proof of Theorem 4.2 we have $i_K(A, D_K \setminus \overline{D_K^1}) = -1$, and thus A has a fixed point x_2 in $D_K \setminus \overline{D_K^1}$. □

By an argument similar to that of Theorem 4.5, and using the proof of Theorem 4.3, we obtain

Theorem 4.6. *Under the hypotheses of Theorem 4.3, suppose that $A : K \to X$ is weakly inward on all of K and with $A(K)$ bounded. Then A has at least two nonzero fixed points in K.*

5. Application

In this section we consider the perturbed Volterra integral equation

$$x(t) = g(t, x(t)) + \int_0^t f(s, x(s))\, ds, \quad t \in [0, 1]. \tag{5.1}$$

We make the following hypotheses:

(C_1) $g : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$ is continuous and there exists $L \in (0,1)$ such that

$$|g(t,x) - g(t,y)| \le L|x-y| \quad \text{for } t \in [0,1] \text{ and } x, y \in \mathbb{R}^+.$$

(C_2) $g(t,0) \ge 0$ for $t \in [0,1]$.
(C_3) $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$ satisfies Carathéodory conditions and there exists $b > 0$ such that

$$|f(t,x)| \le b(1+x) \quad \text{for } t \in [0,1] \text{ and } x \in \mathbb{R}^+.$$

(C_4) For every $x \in L^2[0,1]$ with $x(t) \ge 0$ a.e. on $[0,1]$ there exists $M_x \ge 0$ such that

$$\int_0^t f(s, x(s))\, ds \ge -M_x\, x(t) \quad \text{for } t \in [0,1].$$

Remark 5.1. (C_4) holds if $f(t,x) \ge 0$ for $x \ge 0$. If we used the standard theory of maps with values in a cone, then in place of (C_2) we would have to assume that $g(t,x) \ge 0$ for all $x \ge 0$.

Theorem 5.2. *Assume that* (C_1)–(C_4) *hold. Then equation* (5.1) *has a solution* x *in* $L^2[0,1]$ *such that* $x(t) \ge 0$ *for a.e.* $t \in [0,1]$.

Proof. Let $X = L^2[0,1]$ and $K = \{x \in X : x(t) \ge 0 \text{ a.e. on } [0,1]\}$. Then K is a cone in X. We define G, F and $A : K \to X$ by

$$Gx(t) = g(t, x(t)), \quad Fx(t) = \int_0^t f(s, x(s))\, ds \quad \text{and} \quad A = G + F.$$

(C_1) implies that $|g(t,x)| \le M + Lx$ for $x \in K$, where M is a bound for $|g(t,0)|$, so that G is continuous and maps bounded sets in K into bounded sets in X. Also (C_1) implies that G is an L-set-contraction. Let $x \in K$ be such that $x^*(x) = 0$ for some $x^* \in K^*$. Then x^* can be identified with an L^2 function, $x^*(t) \ge 0$ a.e. and $\int_0^1 x(t)\, x^*(t)\, dt = 0$. Thus $x^*(t) = 0$ a.e. on the set $\{t : x(t) \ne 0\}$. Therefore, we have

$$x^*(Gx) = \int_0^1 g(s, x(s)) x^*(s)\, ds = \int_{\{s : x(s) = 0\}} g(s,0) x^*(s)\, ds \ge 0.$$

This shows that G is weakly inward (see section 2). [A longer direct argument may also be used.]

From (C_3), F is compact. For any $x \in K$, by (C_4) we have $Fx + M_x\, x \in K$. Therefore F is weakly inward, since $-x \in I_K(x)$ and $\overline{I}_K(x)$ is a wedge. Hence A is a weakly inward L-set-contraction.

We assert that the set $\{x \in K : x = \lambda Ax,\ 0 \le \lambda \le 1\}$ is bounded. Assume that $x \in K$ is such that $x = \lambda Ax$ for some $\lambda \in (0,1]$. By using (C_1), (C_3) and a routine application of Gronwall's inequality we obtain $\|x\| \le m$ for a suitable constant m, independent of λ, proving our assertion. By Theorem 3.2, A has a fixed point in K. □

Remark 5.3. Note that K has empty interior, so Theorem 20.4 in [2] cannot be used to treat the above problem. Also, since $A(K)$ may be unbounded, Theorem 3.1 (hence Theorem 18.3 in [2]) cannot be used either.

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