THE NONEXISTENCE OF EXPANSIVE HOMEOMORPHISMS
OF A CLASS OF CONTINUA WHICH CONTAINS ALL
DECOMPOSABLE CIRCLE-LIKE CONTINUA

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Abstract. A homeomorphism \( f : X \to X \) of a compactum \( X \) with metric \( d \) is expansive if there is \( c > 0 \) such that if \( x, y \in X \) and \( x \neq y \), then there is an integer \( n \in \mathbb{Z} \) such that \( d(f^n(x), f^n(y)) > c \). It is well-known that \( p \)-adic solenoids \( S_p \) \( (p \geq 2) \) admit expansive homeomorphisms, each \( S_p \) is an indecomposable continuum, and \( S_p \) cannot be embedded into the plane. In case of plane continua, the following interesting problem remains open: For each \( 1 \leq n \leq 3 \), does there exist a plane continuum \( X \) so that \( X \) admits an expansive homeomorphism and \( X \) separates the plane into \( n \) components? For the case \( n = 2 \), the typical plane continua are circle-like continua, and every decomposable circle-like continuum can be embedded into the plane. Naturally, one may ask the following question: Does there exist a decomposable circle-like continuum admitting expansive homeomorphisms? In this paper, we prove that a class of continua, which contains all chainable continua, some continuous curves of pseudo-arcs constructed by W. Lewis and all decomposable circle-like continua, admits no expansive homeomorphisms. In particular, any decomposable circle-like continuum admits no expansive homeomorphism. Also, we show that if \( f : X \to X \) is an expansive homeomorphism of a circle-like continuum \( X \), then \( f \) is itself weakly chaotic in the sense of Devaney.

1. Introduction

All spaces considered in this paper are assumed to be separable metric spaces. Maps are continuous functions. By a compactum we mean a nonempty compact metric space. A continuum is a connected compactum. A homeomorphism \( f : X \to X \) of a compactum \( X \) with metric \( d \) is called expansive ([20], [1] and [2]) if there is \( c > 0 \) such that for any \( x, y \in X \) with \( x \neq y \), there is an integer \( n \in \mathbb{Z} \) such that

\[
d(f^n(x), f^n(y)) > c.
\]

A homeomorphism \( f : X \to X \) of a compactum \( X \) is continuum-wise expansive [8] if there is \( c > 0 \) such that if \( A \) is a nondegenerate subcontinuum of \( X \), then there is an integer \( n \in \mathbb{Z} \) such that

\[
diam f^n(A) > c,
\]

where \( diam B = \sup\{d(x, y)| x, y \in B \} \) for a set \( B \). Such a positive number \( c \) is called an expansive constant for \( f \). Note that each expansive homeomorphism

\[
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\]
is continuum-wise expansive, but the converse assertion is not true. There are many continuum-wise expansive homeomorphisms which are not expansive (e.g., see [8], [9] and [11]). In fact, there are many decomposable circle-like continua admitting continuum-wise expansive homeomorphisms. By the definitions, we see that expansiveness and continuum-wise expansiveness do not depend on the choice of the metric $d$ of $X$. These notions have been extensively studied in the area of topological dynamics, ergodic theory and continuum theory (e.g., see [1], [2], [7]–[12], [20], and [21]).

Let $f : X \to X$ be a homeomorphism of a compactum $X$. A (nonempty) closed subset $M$ of $X$ is a minimal set of $f$ if $M$ is $f$-invariant, i.e., $f(M) = M$, and for any $x \in M$, the orbit $O(f) = \{f^n(x) \mid n \in \mathbb{Z}\}$ is dense in $M$. Note that every homeomorphism of a compactum has a minimal set. For a point $x \in X$, the $\omega$-limit set of $x$ is the set

$$\omega f(x) = \omega(x) = \{ y \in X \mid \text{there is a sequence } n_1 < n_2 < \ldots \text{ of natural numbers such that } \lim_{i \to \infty} f^{n_i}(x) = y \}.$$ 

Similarly, the $\alpha$-limit set of $x$ is the set $\omega f^{-1}(x)$.

Let $X$ be a compactum. Let $2^X$ be the set of all nonempty closed sets of $X$ and $C(X)$ the set of all nonempty subcontinua of $X$. Suppose that $U_1, \ldots, U_n$ are nonempty open sets of $X$. Put

$$\langle U_1, \ldots, U_n \rangle = \{ A \in 2^X \mid A \cap U_i \neq \emptyset, A \subset \bigcup_{i=1}^n U_i \}.$$ 

Then

$$\beta = \{ \langle U_1, U_2, \ldots, U_n \rangle \mid n \geq 1 \text{ and } U_i (i \leq n) \text{ are nonempty open sets of } X \}$$

is a base of $2^X$, and it is called the Vietoris topology. Then $2^X$ and $C(X)$ are compacta. The spaces $2^X$ and $C(X)$ are called the hyperspaces of $X$. For a map $f : X \to X$, we define a map $f_* : 2^X \to 2^X$ by $f_*(A) = f(A) = \{ f(a) \mid a \in A \}$ for $A \in 2^X$. Also, put $C(f) = f_*|C(X) : C(X) \to C(X)$. Then $X$ is identified with the closed invariant subset of singletons, i.e., degenerate subcontinua.

For the map $C(f) : C(X) \to C(X)$, we shall deal with $\omega(E) = \omega(f)(E)$ and $\alpha(E) = f C(f)^{-1}(E)$ for $E \in C(X)$.

For a homeomorphism $f : X \to X$, if $Z \subset X$ is a closed invariant subset for $X$, then $Z$ is isolated if for some neighborhood $U$ of $Z$ in $X$ any orbit lying entirely in $U$ is in fact in $Z$, i.e., $Z = \bigcap_{n=1}^{\infty} f^n(U)$. Then $f$ is expansive (resp. continuum-wise expansive) if and only if $X$ is isolated in $2^X$ for $f_*$ (resp. in $C(X)$ for $C(f)$) (see [1]).

Let $A$ and $B$ be closed $C(f)$-invariant sets in $C(X)$. Then we define the orderings $\ast <, <, \ast$ and $\ast <, <, \ast$ as follows: Define $A, B$ (resp. $A, B$) iff for any $A \in A$ there is $B \in B$ (resp. for any $B \in B$ there is $A \in A$) such that $A \subset B$. Also, define $A <, B$ if $A <, B$ and $A <, B$. Example: for $E_0, E_1 \in C(X), E_0 \subset E_1$ implies $\omega(E_0) <, \omega(E_1)$ and $\alpha(E_0) <, \alpha(E_1)$.
For a homeomorphism $f : X \to X$, we define sets of stable and unstable nondegenerate subcontinua of $X$ as follows (see [9]):

$$V^s = \{ A | A \text{ is a nondegenerate subcontinuum of } X \text{ such that } \lim_{n \to \infty} \text{diam } f^n(A) = 0 \}.$$  

$$V^u = \{ A | A \text{ is a nondegenerate subcontinuum of } X \text{ such that } \lim_{n \to \infty} \text{diam } f^{-n}(A) = 0 \}.$$  

For each $0 < \delta < \epsilon$, put

$$V^s(\delta; \epsilon) = \{ A \in C(X) | \text{diam } A \geq \delta, \text{ and diam } f^n(A) \leq \epsilon \text{ for each } n \geq 0 \},$$  

$$V^u(\delta; \epsilon) = \{ A \in C(X) | \text{diam } A \geq \delta, \text{ and diam } f^{-n}(A) \leq \epsilon \text{ for each } n \geq 0 \}.$$  

Then $V^s(\delta; \epsilon)$ is closed in $C(X)$. Note that if $f : X \to X$ is a continuum-wise expansive homeomorphism with an expansive constant $c > 0$, then for each $0 < \delta < \epsilon < c$ we have $V^s(\delta; \epsilon) \subseteq V^s$, and $V^s$ is an $F_\sigma$-set in $C(X)$.

A chain $C = [C_1, C_2, \ldots, C_m]$ of $X$ is a finite collection of open sets of $X$ satisfying the following property:

$$\text{Cl}(C_i) \cap \text{Cl}(C_j) \neq \emptyset \text{ if and only if } |i - j| \leq 1.$$  

Each $C_i$ is called a link of the chain $C$. Moreover, if for each $i = 1, \ldots, m$, diam $(C_i) < \epsilon$, i.e., mesh $(C) < \epsilon$, then we say that the chain $C$ is an $\epsilon$-chain. For a chain $C = [C_1, C_2, \ldots, C_m]$ and two points $p, q \in X$, if $p \in C_i$ and $q \in C_m$, we say that $C = [C_1, C_2, \ldots, C_m]$ is a chain from $p$ to $q$. A continuum $X$ is chainable if for any $\epsilon > 0$ there is an $\epsilon$-chain covering of $X$.

If $n$ is a natural number, let $I(n) = \{1, 2, \ldots, n\}$. A surjective function $f : I(m) \to I(n)$ is called a pattern provided that $|f(i + 1) - f(i)| \leq 1$ for each $i = 1, \ldots, m - 1$. Let $C = [C_1, C_2, \ldots, C_n]$ and $D = [D_1, D_2, \ldots, D_m]$ be chain coverings of $X$ and let $f : I(m) \to I(n)$ be a pattern. We say that $D$ follows the pattern $f$ in $C$ provided that $D_i \subseteq C_{f(i)}$ for each $i \in I(m)$.

Let $\mathcal{P}$ be a family of compact polyhedra. A continuum $X$ is called a $\mathcal{P}$-like continuum if for any $\epsilon > 0$ there is an onto map $g : X \to P$ such that $P \in \mathcal{P}$ and diam $g^{-1}(y) < \epsilon$ for each $y \in P$. Note that $X$ is chainable if and only if $X$ is arc-like. A circular chain differs from a chain in that the first and last links intersect. Then a continuum $X$ is circle-like if and only if for any $\epsilon > 0$, there is an $\epsilon$-circular chain covering of $X$.

Concerning expansive homeomorphisms, we have the following general problem:

**Problem 1.1.** What kinds of (plane) continua admit expansive homeomorphisms?

Note that $p$-adic solenoids $S_p$ ($p \geq 2$) are indecomposable circle-like continua admitting expansive homeomorphisms (see [21]), and they cannot be embedded into the plane $R^2$. On the other hand, each decomposable circle-like continuum $X$ can be embedded into $R^2$, and $R^2 - X$ has at most 2 components. It is known that for each $n \geq 4$ there is a plane continuum $X$ which is called a Lake of Wada, such that $X$ admits an expansive homeomorphism and $R^2 - X$ has $n$ components. It is not known whether there exists a plane continuum $X$ such that $X$ admits an expansive homeomorphism and $X$ separates the plane $R^2$ into $n$ components ($n \leq 3$), or not. For the case $n = 2$, the typical continua are circle-like continua.

In [7, 8], we proved that if $X$ is a tree-like continuum admitting a continuum-wise expansive homeomorphism, it must contain an indecomposable subcontinuum.
Also, in [10], we proved that chainable continua admit no expansive homeomorphisms. Naturally, we are interested in the following problem:

**Problem 1.2.** Does there exist a decomposable circle-like continuum admitting an expansive homeomorphism?

In this paper, we prove that some kinds of continua, including all chainable continua, some continuous curves of pseudo-arcs constructed by W. Lewis and all decomposable circle-like continua, admit no expansive homeomorphisms. In particular, any decomposable circle-like continuum admits no expansive homeomorphism. For example, we know that a solenoid of pseudo-arcs and the circle of pseudo-arcs is characterized [4] as a (nondegenerate) hereditarily indecomposable chainable continuum. The pseudo-arc has many remarkable properties in topology and chaotic dynamics (e.g., see [3]–[6] and [13]–[16]). For example, the pseudo-arc is homogeneous [3], each onto map of the pseudo-arc is a near homeomorphism [15], and the pseudo-arc admits chaotic homeomorphisms in the sense of Devaney (see [13]). Also, there is an onto map from the pseudo-arc to each chainable continuum (see [6] and [14]).

From the proof of [8, Proposition 2.3] we have

**Lemma 2.1.** Let \( f : X \to X \) be a continuum-wise expansive homeomorphism of a compactum \( X \) with an expansive constant \( c > 0 \), and let \( 0 < \epsilon < c/2 \). Then there is a positive number \( \delta < \epsilon \) such that if \( A \) is a subcontinuum of \( X \) with \( \text{diam} \ A \leq \delta \) and \( \text{diam} \ f^n(A) \geq \epsilon \) for some integer \( m \geq 0 \) (resp. \( m < 0 \)), then for each \( n \geq m \) and for each \( x \in f^n(A) \), there is a subcontinuum \( B \) of \( A \) such that \( x \in f^n(B) \), \( \text{diam} \ f^j(B) \leq \epsilon \) for \( 0 \leq j \leq n \) and \( \text{diam} \ f^n(B) = \delta \) (resp. for each \( n \leq -m \) and for each \( x \in f^{-n}(A) \), there is a subcontinuum \( B \) of \( A \) such that \( x \in f^{-n}(B) \), \( \text{diam} \ f^{-j}(B) \leq \epsilon \) for \( 0 \leq j \leq n \), and \( \text{diam} \ f^{-n}(B) = \delta \).

**Corollary 2.2.** Let \( f : X \to X, c, \epsilon, \delta \) be as in Lemma 2.1.

(a) For every nondegenerate subcontinuum \( A \) of \( X \) with \( \text{diam} \ A \leq \delta \), exactly one of the following assertions holds:

1. For all \( n \geq 0 \), \( \text{diam} \ f^n(A) \leq \epsilon \), in which case \( A \in \mathcal{V}^s \) and \( \omega(A) \subset X \subset C(X) \).
2. For \( n \geq 0 \) sufficiently large, \( \text{diam} \ f^n(A) \geq \delta \).

(b) For every subcontinuum \( A \), either \( \omega(A) \subset X \subset C(X) \) or \( \text{diam} \ E \geq \delta \) for all \( E \in \omega(A) \).

For \( n \leq 0, \mathcal{V}^u \) and \( \alpha(A) \), the similar properties are satisfied.

**Lemma 2.3** ([8, Corollary 2.4]). Let \( f : X \to X \) be a continuum-wise expansive homeomorphism of a compactum \( X \) with \( \dim X > 0 \). Then the following are true.

1. \( \mathcal{V}^u \neq \phi \) or \( \mathcal{V}^s \neq \phi \).
2. If \( \delta > 0 \) is as in the above lemma, then for each \( \gamma > 0 \) there is a natural number \( N(\gamma) \) such that if \( A \) is a subcontinuum of \( X \) with diam \( A \geq \gamma \), then either \( \text{diam } f^n(A) \geq \delta \) for each \( n \geq N(\gamma) \) or diam \( f^{-n}(A) \geq \delta \) for each \( n \geq N(\gamma) \).

From the above lemma, we see that \( V^s \cap V^u = \phi \) and moreover if \( A \in V^u, B \in V^s \), then \( \text{dim } (A \cap B) \leq 0 \).

**Lemma 2.4.** Under the same hypothesis as in Lemma 2.3, let \( E_0, E_1 \) be nondegenerate subcontinua of \( X \) with \( E_1 \in \omega(E_0) \). Then one of the following holds:

1. Every nondegenerate subcontinuum \( A_0 \) of \( E_0 \) with diam \( A_0 < \delta \) lies in \( V^s \).
2. There is a subcontinuum \( A_1 \) of \( E_1 \) with diam \( A_1 = \delta \) lying in \( V^u \). Moreover, if \( E_0 \in V^u \), then for any \( x \in E_1 \) there is a subcontinuum \( A \) of \( E_1 \) such that diam \( A = \delta \) and \( x \in A \in V^u \).

**Proof.** If the first condition is not true, then there is a subcontinuum \( B \) of \( E_0 \) with \( 0 < \gamma = \text{diam } B < \delta \) and a natural number \( n \) such that \( \text{diam } f^n(B) > \epsilon \). Choose a sequence \( 0 = n_0 < n_1 < \ldots \) of natural numbers such that \( n_{i+1} - n_i \geq N(\gamma) \) (see Lemma 2.3) and \( \lim_{i \to \infty} f^{n_i}(E_0) = E_1 \). By using Lemmas 2.1 inductively, we can construct a sequence \( B_0, B_1, \ldots \) of subcontinua with \( B_0 = B \), diam \( B_0 = \gamma < \delta \), \( B_{i+1} \subset f^{n_{i+1}-n_i}(B_i) \), diam \( B_i = \delta \) (\( i \geq 1 \)), and \( \text{diam } f^{-j}(B_i) \leq \epsilon \) for each \( 0 \leq j \leq n_i \). We may assume that \( \lim_{i \to \infty} B_i = A_1 \). Then \( A_1 \in V^u \) and \( A_1 \subset E_1 \).

Moreover, suppose that \( E_0 \notin V^u \). For any \( x \in E_1 \), we choose a sequence \( x_0, x_1, \ldots \) of points such that \( x_i \in f^{n_i}(E_0) \) and \( \lim_{i \to \infty} x_i = x \). Choose a subcontinuum \( B \) of \( E_0 \) such that \( x_0 \in B \) and diam \( B = \gamma < \delta \). By Lemma 2.1, we can choose a sequence \( B_0, B_1, \ldots \) satisfying the above conditions with \( x_i \in B_i \) for each \( i \). Then \( x \in A_1 \subset V^u \).

**Corollary 2.5.** Under the same hypothesis as in Lemma 2.3, let \( A \) be a minimal set of \( C(f) \). Assume that there is a nondegenerate subcontinuum \( A \in A \).

(a) For all \( A \in A \), diam \( A \geq \delta \).
(b) Exactly one of the following three conditions holds for \( A \):
   1. For all \( A \in A \) and all subcontinua \( B \) of \( A \) with diam \( B < \delta \), \( B \in V^s \).
   2. For all \( A \in A \) and all subcontinua \( B \) of \( A \) with diam \( B \leq \delta \), \( B \in V^u \).
   3. For all \( A \in A \) there are subcontinua \( B_0, B_1 \) of \( A \) with diam \( B_0 = \text{diam } B_1 = \delta \) and \( B_0 \in V^s, B_1 \in V^u \).
(c) If \( A \in A \) and \( B \) is a nondegenerate subcontinuum of \( A \) with \( B \notin V^s \), then \( \text{diam } E \geq \delta \) for each \( E \in \omega(B) \), \( \omega(B)_* <_* A \), and if \( A_0 \) is a minimal set in \( \omega(B) \), then \( A_0 * <_* A \) as well.

The following propositions are used in the sequel.

**Proposition 2.6.** Let \( f : X \to X \) be a continuum-wise expansive homeomorphism of a compactum \( X \) with \( \text{dim } X > 0 \). Suppose that \( B \) is a \( C(f) \)-invariant set such that some element of \( B \) is nondegenerate. Then there exists a minimal set \( A_* < B \) of \( C(f) \) such that each element of \( A_* \) is nondegenerate, and such that for each \( A \in A_* \) and each nondegenerate subcontinuum \( B \) of \( A \) either \( B \in V^s \) or \( A \subset \omega(B) \), and either \( B \in V^u \) or \( A \subset \alpha(B) \).

**Proof.** For pairs \( (A, A) \) such that \( A \) is minimal, \( A_* < B \), \( A \in A \) and \( A \) is a nondegenerate subcontinuum, we consider the order by inclusion of the \( A_* \)'s. By Corollary 2.2.(b), there exists such a pair. If \( \{(A_\alpha, A_\alpha)\} \) is a totally ordered family,
then \( B = \bigcap_{i=0}^{\alpha} A_{i} \) is a nondegenerate subcontinuum and so either \( \omega(B) \) or \( \alpha(B) \) contains a minimal subset \( A \) such that its elements are nondegenerate subcontinua and \( A \prec \prec A_{i} \) for all \( i \). For each \( i \) choose \( B_{i} \subset A_{i} \) with \( B_{i} \prec A_{i} \). Then any limit point \( A \) of the net \( \{ B_{i} \} \) is an element of \( A \) contained in all the \( A_{i} \)'s. So Zorn's lemma applies to the pairs. If \( (A, A) \) is minimal with respect to this ordering, then \( A \) satisfies the conclusion. In fact, if \( A \in A \) and \( B \) is a nondegenerate subcontinuum of \( A \) not in \( V^{*} \), then \( \omega(B) \prec \prec A \) and \( A_{0} \prec \prec A \) for any minimal subset \( A_{0} \) of \( A \). Then there is \( A_{0} \in A_{i} \) such that \( A_{0} \subset A_{i} \), and so by minimality we see that \( A_{0} \in A_{i} \) and so \( A = \omega(A_{0}) = A_{0} \subset \omega(B) \).

This completes the proof.

**Proposition 2.7.** Under the same assumption as in the above proposition, the minimal set \( A \) satisfies one of the following conditions:

1. If some \( A_{0} \in A \) contains an element of \( V^{*} \), then for any \( x \in A \in A \), there is a nondegenerate subcontinuum \( A_{x} \) of \( A \) such that \( x \in A_{x} \in V^{*} \), and if \( A' \) is a nondegenerate subcontinuum of \( A \in A \) with \( A' \notin V^{*} \), then for each \( H \in A \) there is a sequence \( n_{1} < n_{2} < \ldots \) of natural numbers such that
   \[
   \lim_{i \to \infty} f^{n_{i}}(A) = \lim_{i \to \infty} f^{n_{i}}(A') = H.
   \]

2. If some \( A_{0} \in A \) contains an element of \( V^{*} \), then for any \( x \in A \in A \), for any \( x \in A \in A \), there is a nondegenerate subcontinuum \( A_{x} \) of \( A \) such that \( x \in A_{x} \in V^{*} \), and if \( A' \) is a nondegenerate subcontinuum of \( A \in A \) with \( A' \notin V^{*} \), then for each \( H \in A \) there is a sequence \( n_{1} < n_{2} < \ldots \) of natural numbers such that
   \[
   \lim_{i \to \infty} f^{-n_{i}}(A) = \lim_{i \to \infty} f^{-n_{i}}(A') = H.
   \]

**Proof.** We shall show the first case. Let \( B \in V^{*} \) and \( B \in A_{0} \in A \). By the above proposition, we see that \( A \subset \omega(B) \). By Lemma 2.4, we see that for any \( x \in A \in A \), there is \( A_{x} \in V^{*} \) such that \( x \in A_{x} \subset A \). Since \( A \) is closed in \( C(X) \), \( A \) contains an maximal element in order by inclusion. In fact, for a Whitney map \( \mu : C(X) \to [0, 1] \) (see [18]), we can choose an element \( T \) of \( A \) such that \( \mu(T) = \max(\mu(E) | E \in A) \). Then \( T \) is a maximal element of \( A \). Suppose that \( A' \) is a nondegenerate subcontinuum of \( A \in A \) with \( A' \notin V^{*} \). Let \( H \in A \). Since \( \omega(A') \supset A \) (see Proposition 2.6), \( T \in \omega(A') \). Hence there is a sequence \( i_{1} < i_{2} < \ldots \) of natural numbers such that \( \lim_{i \to \infty} f^{i_{k}}(A') = T \). We may assume that \( \{ f^{i_{k}}(A) \}_{k=1}^{\infty} \) is convergent. Since \( T \) is maximal in \( A \), we see that \( \lim_{i \to \infty} f^{i_{k}}(A) = T \). Since \( A \) is minimal, \( H \in \omega(T) \). Then we can choose a sequence \( n_{1} < n_{2} < \ldots \) of natural numbers such that
   \[
   \lim_{i \to \infty} f^{n_{i}}(A') = \lim_{i \to \infty} f^{n_{i}}(A) = H.
   \]

This completes the proof.

Let \( f : X \to X \) be a continuum-wise expansive homeomorphism of a compactum \( X \) with \( \dim X > 0 \). Note that every minimal set of \( f \) is 0-dimensional (see [8, Theorem 5.2]). Consider the following sets (see [12]):

1. \( I(f) = \{ A \subset 2^X \mid A \text{ is } f \text{-invariant} \} \).
2. \( I^{+}(f) = \{ A \in I(f) \mid \dim A > 0 \} \).
3. \( D(f) \) is the set of all minimal elements of \( I^{+}(f) \) in the order by inclusion.
Note that $D(f) \neq \phi$ (see [12, Proposition 2.4]) and if $Y \in D(f)$, then $f_Y = f|Y : Y \to Y$ is weakly chaotic in the sense of Devaney, i.e., $f_Y$ has sensitive dependence on initial conditions, $f_Y$ is topologically transitive and the union of all minimal sets of $f_Y$ is dense in $Y$ ([12, Theorem 2.7]), i.e., the min-center of $f_Y$ is $Y$ (see [1, p. 70]).

**Proposition 2.8.** Let $f : X \to X$ be a continuum-wise expansive homeomorphism of a compactum $X$ with $\dim X > 0$. If $Y \in D(f)$, then there is a minimal set $A$ of $C(f)$ satisfying one of the conditions (1) and (2) as in Proposition 2.7 and $\bigcup \{A | A \in A\} = Y$.

**Proof.** Consider the map $f|Y : Y \to Y$. Then there is a minimal set $A$ of $C(f|Y)$ as in Proposition 2.7. Put $Y' = \bigcup \{A | A \in A\}$. Then $Y'$ is $f$-invariant and $\dim Y' > 0$. Hence $Y = Y'$.

The following lemma follows from [3, Theorem 6] (see also [15, Lemmas 2 and 1.1]).

**Lemma 2.9.** Let $P$ be the pseudo-arc. Let $C = [C_1, C_2, \ldots, C_n]$ be a chain covering of $P$ and $f : (I(m)) \to (I(n))$ a pattern with $f(1) = 1$. Let $p \in C_1$. Then there is a chain covering $D = [D_1, D_2, \ldots, D_m]$ such that $D$ refines the chain $C$, $p \in D_1$ and $D$ follows the pattern $f$ in $C$.

The following lemma is a simple modification of the uniformization theorem of Mioduszewski (see [17] and [19]). For completeness, we give the proof.

**Lemma 2.10.** Let $I = [0,1]$ be the unit interval. Suppose that $f, g : I \to I$ are piecewise linear onto maps. If $f(0) = g(0) = 0$, then there are onto maps $a, b : I \to I$ such that $f \cdot a = g \cdot b$ and $a(0) = b(0) = 0$.

**Proof.** Let $\psi : I^2 \to R$ be the map defined by $\psi(x, y) = f(x) - g(y)$. Note that $I^2$ is unicoherent (i.e., if $A$ and $B$ are continua with $A \cup B = I^2$, then $A \cap B$ is connected). In [17], Mioduszewski proved that there is a component $K$ of $\psi^{-1}(0)$ such that $K$ meets all four sides of $I^2$ (see also [19]). Note that each component of $\psi^{-1}(0)$ is a polyhedron. Let $L$ be a component of $\psi^{-1}(0)$ containing the point $p = (0,0) \in I^2$.

Suppose, on the contrary, that $L \cap (I \times \{1\} \cup \{1\} \times I) = \phi$. Then there is an arc $\alpha : I \to I^2$ such that $\alpha(0) = (x_1, 0) \in I \times \{0\}$, $\alpha(1) = (0, y_1) \in \{0\} \times I$, and $\psi^{-1}(0) \cap \alpha(I) = \phi$. Note that $g(0) = 0 < f(x_1)$ and $g(y_1) > f(0)$. Hence we can see that there is a point $q = (q_1, q_2) \in \alpha(I)$ such that $f(q_1) = g(q_2)$, which implies that $q \in \phi^{-1}(0)$. This is a contradiction. Hence $K$ contains $L$. By using this fact, we can choose desired maps $a, b : I \to I$.

3. **The nonexistence of expansive homeomorphisms of certain continua**

The following is the main theorem in this paper.

**Theorem 3.1.** Let $f : X \to X$ be a homeomorphism of a compactum $X$. If there is a minimal set $A$ of $C(f)$ such that some element $A$ of $A$ is a (nondegenerate) chainable continuum, then $f$ is not expansive.

**Proof.** Suppose, on the contrary, that $f$ is expansive. Replace $A$ if necessary by an $A_0 < A$ which satisfies the condition (1) of Proposition 2.7. Since every subcontinuum of a chainable continuum is also chainable, we may assume that $A$ satisfies the conditions of Proposition 2.7, (1).
Let $c > 0$ be an expansive constant for $f$ and $c/2 > \epsilon > 0$. Now, we shall prove the following property

(3.1.1)

For any $0 < \tau < \epsilon$ there are two points $x, y$ of $X$ and a natural number $n(\tau)$ such that $d(x, y) \leq \tau$, $d(f^n(\tau)(x), f^n(\tau)(y)) \leq \tau$, and

$$\epsilon \leq \sup \{d(f^i(x), f^i(y)) | 0 \leq j \leq n(\tau)\} \leq 2\epsilon.$$

Let $A \in A$ be a chainable continuum. Since $A$ is chainable, there is a $\tau/4$-chain $C = [C_1, C_2, \ldots, C_r]$ in $X$ which is an open covering of $A$. We can choose a subcontinuum $B_1$ of $A$ such that $B_1 \in V^n(\tau; \epsilon)$ (see (1) of Proposition 2.7), and we may assume that $\text{diam}(B_1) = \tau$. Since $B_1 \in V^n$ and $f$ is expansive, we can choose a natural number $N_1$ such that if $x, y \in B_1$ and $d(x, y) \geq \tau/4$, then

$$\sup \{d(f^j(x), f^j(y)) | 0 \leq j \leq N_1\} > 2\epsilon.$$

Choose a subcontinuum $B_2$ of $B_1$ such that $\text{diam}B_2 = \tau/2$. By the assumption, there is a sequence $n_1 < n_2 < \ldots$ of natural numbers such that $\lim_{n \to \infty} f^{n_i}(B_2) = \lim_{n \to \infty} f^n(A) = A$ (see Proposition 2.7). Hence, we can choose a natural number $N > N_1$ such that $f^N(B_2), f^N(B_1) \in (C_1, \ldots, C_r)$. Choose a point $e \in B_2$ such that $f^N(e) \in C_1$. Since $B_1, B_2$ are chainable, by [6] or [14] there are onto maps $\psi_k : P \to B_k$ ($k = 1, 2$) from the pseudo-arc $P$ onto $B_k$. Let $p \in P$. Since $P$ is homogeneous [3], we may assume that $\psi_k(p) = e$ for each $k = 1, 2$. Choose a chain covering $D = [D_1, \ldots, D_s]$ of $P$ such that its mesh is sufficiently small. We may assume that if $x, y \in D_i \cup D_{i+1}$, then

$$\sup \{d(f^j(\psi_k(x)), f^j(\psi_k(y))) | 0 \leq j \leq N\} < \epsilon/2 \quad (3.1.2)$$

for each $k = 1, 2$. We may assume that $p \in D_1$ (see the proof of [3, Theorem 13]). Also we may assume that $D$ is a refinement of the chains $C^k = (f^N \cdot \psi_k)^{-1}(C)$ ($k = 1, 2$). Let $f_k : I(s) \to I(r) (k = 1, 2)$ be patterns such that $D$ follows the patterns $f_k$ in $C^k$ ($k = 1, 2$). Then the patterns $f_k$ ($k = 1, 2$) induce maps $f_k : N(D) = N(D_1, \ldots, D_s) \to N(C) = N(C_1, \ldots, C_r)$ which are natural simplicial maps from the nerve $N(D)$ of $D$ to $N(C)$ of $C$ with $f_k(D_j) = C_{f_k(j)}$ for each $j$.

Since the above nerves are arcs, we can consider that $f_k$ is a map from the unit interval $I$ onto $I$ such that $f_k(0) = 0 (k = 1, 2)$. By Lemma 2.10, there are onto maps $g_k : I \to I$ such that $f_1 \cdot g_1 = f_2 \cdot g_2$ and $g_0(0) = 0$.

By using $g_k$ ($k = 1, 2$), we obtain patterns $g_k : I(l) \to I(s)$ satisfying the inequality $|f_1g_1(j) - f_2g_2(j)| \leq 1$ for each $j = 1, 2, \ldots, l$. By Lemma 2.9, we can choose chain coverings $E = \{E_1, E_2, \ldots, E_l\}$ and $F = \{F_1, F_2, \ldots, F_l\}$ of $P$ such that $E$ follows the pattern $g_1$ in $D$ and $F$ follows the pattern $g_2$ in $D$. We may assume that $p \in E_1 \cap F_1$.

Choose points $a_1, \ldots, a_i, b_1, \ldots, b_i$ of $P$ beginning with $p = a_1 = b_1$ and such that $a_j \in E_j, b_j \in F_j$. Note that

$$d(f^N(\psi_1(a_j)), f^N(\psi_2(b_j))) \leq \tau.$$

For each $i = 1, 2, \ldots, l$, put

$$r_i = \sup \{d(f^j(\psi_1(a_i)), f^j(\psi_2(b_i))) | 0 \leq j \leq N\}.$$

Since the chain cover $D$ is sufficiently small (see (3.1.2)), we may assume that

$$|r_i - r_{i+1}| < \epsilon.$$
Note that \( r_1 = 0 \). Since \( \psi_1 \) is surjective, there is a point \( a_u \ (u \leq l) \) such that \( d(\psi_1(a_u), B_2) \geq \tau/4 \), and hence \( d(\psi_1(a_u), \psi_2(b_u)) \geq \tau/4 \). Thus \( r_u > 2\epsilon \). Then we can choose \( i \leq u \) such that \( \epsilon \leq r_i \leq 2\epsilon \). The two points \( a_i, b_i \) satisfy the condition (3.1.1).

Let \( \{\epsilon_i\}_{i=1}^{\infty} \) be a sequence of positive numbers such that \( \lim_{i \to \infty} \epsilon_i = 0 \). By the condition (3.1.1), there are two points \( x_i, y_i \in X \) and a natural number \( n(i) \) such that

\[
d(x_i, y_i) < \epsilon_i, \quad d(f^{n(i)}(x_i), f^{n(i)}(y_i)) < \epsilon_i
\]

and

\[
\epsilon \leq \sup\{d(f^j(x_i), f^j(y_i))| 0 \leq j \leq n(i)\} \leq 2\epsilon.
\]

Choose \( 0 < m(i) < n(i) \) such that \( d(f^{m(i)}(x_i), f^{m(i)}(y_i)) \geq \epsilon \). We may assume that \( \{f^{m(i)}(x_i)\} \) and \( \{f^{m(i)}(y_i)\} \) are convergent to \( x_0 \) and \( y_0 \), respectively. Note that

\[
\lim_{i \to \infty} (n(i) - m(i)) = \infty = \lim_{i \to \infty} m(i).
\]

Then \( x_0 \neq y_0 \) and \( d(f^n(x_0), f^n(y_0)) \leq 2\epsilon < c \) for all \( n \in \mathbb{Z} \). This is a contradiction.

**Corollary 3.2.** If \( X \) is a decomposable circle-like continuum, then \( X \) admits no expansive homeomorphism.

**Proof.** Suppose, on the contrary, that there is an expansive homeomorphism \( f : X \to X \). Since \( X \) is decomposable, there are two proper nonempty subcontinua \( A, B \) of \( X \) such that \( A \cup B = X \). Since \( X \) is circle-like, \( A \) and \( B \) are chainable. Note that \( A \cap B \) has at most 2 components [5, Theorem 5]. By [11, Theorem 3.6], there is a \( \sigma \)-chaotic continuum \( C \) of \( f \). We may assume that \( \sigma = u \). Then \( C \) is indecomposable (see [11, Corollary 5.3]) and is a proper subcontinuum of \( X \). Note that \( f^n(C) \cap A \) and \( f^n(C) \cap B \) have at most 2 components. Since \( f^n(C) \) is indecomposable, we can easily see that for each \( n = 0, 1, \ldots, f^n(C) \subset A \) or \( f^n(C) \subset B \). Hence we see that there is a minimal set \( A \) of \( C(f) \) satisfying the condition of Theorem 3.1. By Theorem 3.1, \( f \) is not expansive.

**Corollary 3.3.** Let \( f : X \to X \) be a homeomorphism of a compactum \( X \). Suppose that there are maps \( \psi : X \to Y \) and \( g : Y \to Y \) such that \( \psi \cdot f = g \cdot \psi \) and for each \( y \in Y \) \( \psi^{-1}(y) \) is a (nondegenerate) chainable continuum. Then \( f \) is not expansive.

**Proof.** Let \( y_0 \in Y \). By Corollary 2.2, we may assume each element of \( \omega(\psi^{-1}(y_0)) \) is nondegenerate. Since \( \psi \cdot f = g \cdot \psi \) and the collection \( \{ \psi^{-1}(y) | y \in Y \} \) is an upper semi-continuous decomposition of \( X \), each element of \( \omega(\psi^{-1}(y_0)) \) is contained in some \( \psi^{-1}(y) \), and hence it is chainable.

Take a minimal set \( A \) of \( \omega(\psi^{-1}(y_0)) \). Then each element of \( A \) is a chainable continuum. By Theorem 3.1, \( f \) is not expansive.

**Corollary 3.4.** Let \( f : X \to X \) be an expansive homeomorphism of a circle-like continuum \( X \), and let \( \delta > 0 \) be a positive number as in Lemma 2.1. Then one of the following conditions is satisfied:

(i) If \( A \in C(X) \) and \( 0 < \text{diam} \ A < \delta \), then \( A \in V_u \), and if \( B \) is a nondegenerate subcontinuum of \( X \), then \( X \in \omega(B) \).

(ii) If \( A \in C(X) \) and \( 0 < \text{diam} \ A < \delta \), then \( A \in V_s \), and if \( B \) is a nondegenerate subcontinuum of \( X \), then \( X \in \alpha(B) \).
Proof. By Lemma 2.3, we may assume that $V^u \neq \phi$. Let $A \in V^u$. Suppose, on the contrary, that $\omega(A)$ does not contain $X$. Then we obtain a minimal set $A \subset \omega(A)$ of $C(f)$ satisfying the condition of Theorem 3.1. Hence $f$ is not expansive, which is a contradiction. Since $X \in \omega(A)$, by Lemma 2.4 we see that for each $x \in X$ there is $x \in A_x \in V^u(\delta; \varepsilon)$, where $\delta, \varepsilon$ are as in Lemma 2.1. Suppose, on the contrary, that $V^* \neq \phi$. Then we see also that for each $x \in X$ there is $x \in B_x \in V^*(\delta; \varepsilon)$. Since $f$ is expansive, we know that $A_x \cap B_x = \{x\}$. Choose $A_x$ and two points $y, z \in A_x$ such that $x, y$ and $z$ are different. Then there are three subcontinua $B_x, B_y, B_z$ such that their diameters are small and $B_x, B_y$ and $B_z$ are mutually disjoint. Then $T = A_x \cup B_x \cup B_y \cup B_z$ is a triod. Since $X$ is atriodic, this is a contradiction. Hence $V^* = \phi$. Let $A$ be a nondegenerate subcontinuum of $X$ with $\text{diam} A = \gamma < \delta$. Suppose, on the contrary, that $\text{sup}\{\text{diam} f^{-n}(A) | n \geq 0\} \geq \varepsilon$. By using Lemmas 2.1 and 2.3 inductively, we have a sequence $n_1 < n_2 < \ldots$ of natural numbers and a sequence $B_1, B_2, \ldots$ of subcontinua such that $B_i \subset f^{-n_i}(A)$, $\text{diam} B_i = \delta$ and $\text{diam} f(B_i) \leq \varepsilon$ for each $0 \leq j \leq n_i$. We may assume that $\lim_{i \to \infty} B_i = B \in V^*$. This is a contradiction. Hence we see that $A \in V^u$. Clearly, if $B$ is a nondegenerate subcontinuum, then $X \in \omega(B)$, because $B \notin V^*$.

Corollary 3.5. If $f : X \to X$ is an expansive homeomorphism of a circle-like continuum $X$, then $f$ is itself weakly chaotic in the sense of Devaney.

Proof. Consider the set $D(f) \neq \phi$. Let $Y \in D(f)$. Since $\dim Y > 0$, $Y$ contains a nondegenerate subcontinuum. By Corollary 3.4, we see that $Y = X$. Hence $f$ is weakly chaotic in the sense of Devaney.

Remark. In [16], Lewis showed that, for every 1-dimensional continuum $M$ there exists a 1-dimensional continuum $\hat{M}$ such that $M$ has a continuous decomposition $\psi : \hat{M} \to M$ into pseudo-arcs such that the decomposition space is homeomorphic to $M$ and the decomposition elements are all terminal continua in $\hat{M}$, i.e., every subcontinuum of $\hat{M}$ either is contained in a single decomposition element or is a union of decomposition elements. More generally, let $\hat{N}$ be a compactum that has an upper semi-continuous decomposition $\varphi$ into indecomposable chainable continua such that the decomposition elements are all terminal, and let $N$ be the decomposition space. Moreover, if each proper subcontinuum of $\hat{N}$ is decomposable, then for any homeomorphism $\hat{h} : \hat{N} \to \hat{N}$ there is a homeomorphism $h : N \to N$ such that $\varphi \cdot \hat{h} = h \cdot \varphi$. By Corollary 3.3, $\hat{N}$ admits no expansive homeomorphism. The typical continua are solenoids of pseudo-arcs and hence they admit no expansive homeomorphisms.

Problem 3.6. Does there exist an indecomposable plane circle-like continuum which admits an expansive homeomorphism? In particular, does the pseudo-circle admit an expansive homeomorphism?

Problem 3.7. Does there exist a hereditarily indecomposable continuum which admits an expansive homeomorphism?

The author wishes to thank the referee for very helpful suggestions and comments, in particular, the proof of Proposition 2.6.
References


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