FOUR-MANIFOLDS WITH SURFACE FUNDAMENTAL GROUPS

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Abstract. We study the homotopy type of closed connected topological 4-manifolds whose fundamental group is that of an aspherical surface \( F \). Then we use surgery theory to show that these manifolds are \( s \)-cobordant to connected sums of simply-connected manifolds with an \( S^2 \)-bundle over \( F \).

1. Introduction

In this paper we shall study closed connected oriented topological 4-manifolds \( M^4 \) such that \( \Pi_1(M) \cong \Pi_1(F) \), where \( F \) is a closed oriented aspherical surface, i.e. \( F = K(\Pi_1,1) = B\Pi_1 \). The easiest examples of such manifolds are connected sums of the type \( E \# M' \), where \( E \to F \) is an \( S^2 \)-bundle over \( F \) and \( M' \) is a simply-connected 4-manifold. There are reasons to conjecture that any such manifold is topologically homeomorphic to some \( E \# M' \). Other natural examples of 4-manifolds with surface fundamental groups are given by certain elliptic surfaces as communicated to us by Matsumoto in [9]. Recall that a compact complex manifold of complex dimension two is said to be an elliptic surface if it is fibered over a Riemann surface with general fiber an elliptic curve, i.e. a 2-torus \( T^2 \cong S^1 \times S^1 \). It may admit certain (possibly multiple) singular fibers (for details see [10]). It was proved in [10] that an elliptic surface is a 4-manifold whose fundamental group is isomorphic to that of a closed surface if it has positive Euler number and does not have multiple fibers (see [10], Remark 2, p. 563).

For simplicity, we will assume that \( M \) is a spin manifold, i.e. \( w_2(M) = 0 \), where \( w_2 \) denotes the second Stiefel-Whitney class. As a consequence, the sphere-bundle \( E \) will be trivial. However, a condition weaker than \( w_2(M) = 0 \) would suffice to prove Theorem 1.1 below; in fact, \( w_2(u) = 0 \) is sufficient. Here \( u \in H_2(M;\mathbb{Z}) \) is defined in Section 2.

The referee suggested that we treat also the case \( w_2(u) \neq 0 \). The proof is similar to that of Theorem 1.1, but for technical reasons we will give it in the appendix.

In Section 2 we define a map of degree 1, \( \psi: M \to F \times S^2 \), which gives rise to the split exact sequence

\[ 0 \to K_2(\psi,\Lambda) \to H_2(M;\Lambda) \to H_2(F \times S^2;\Lambda) \to 0. \]
where \( \Lambda = \mathbb{Z}[\Pi_1(M)] \) is the integral group ring.

Similarly, there is a split exact sequence
\[
0 \to K_2(\psi, \mathbb{Z}) \to H_2(M; \mathbb{Z}) \to H_2(F \times S^2; \mathbb{Z}) \to 0.
\]
The splittings respect the intersection pairings. By the result of M. Freedman (see [4] and [5]) the induced intersection form on \( K_2(\psi, \mathbb{Z}) \) can be realized as intersection form of a closed simply-connected topological 4-manifold \( M' \). Let \( M_1 \) denote the connected sum of \( F \times S^2 \) and \( M' \), and let
\[
c: M_1 = (F \times S^2) \# M' \to F \times S^2
\]
be the collapsing map. Since \( c \) is of degree 1, we have short split exact sequences as above; in particular,
\[
0 \to K_2(c, \Lambda) \to H_2(M_1; \Lambda) \to H_2(F \times S^2; \Lambda) \to 0.
\]
In Section 2 we are going to construct a map from the 3-skeleton of \( M_1 \) into \( M \). Furthermore, we prove that it can be extended over \( M_1 \) if the \( \Lambda \)-intersection forms on \( K_2(\psi, \Lambda) \) and on \( K_2(c, \Lambda) \) coincide.

More precisely, we have

**Theorem 1.1.** Let \( M^4 \) be a closed connected oriented TOP 4-manifold with \( w_2(M) = 0 \) and \( \Pi_1(M) \cong \Pi_1(F) \), where \( F \) is a closed aspherical surface. Then \( M \) is simple homotopy equivalent to the connected sum \( M_1 = (F \times S^2) \# M' \) if and only if the \( \Lambda \)-intersection forms on \( K_2(\psi, \Lambda) \) and on \( K_2(c, \Lambda) \) are isomorphic.

In particular, if \( \chi(M) = 2\chi(F) \), then \( K_2(\psi, \Lambda) \cong 0 \), hence \( M \) is simple homotopy equivalent to \( F \times S^2 \).

We observe that in our case any homotopy equivalence is simple because the Whitehead group of \( \Pi_1(F) \) vanishes (see [11]). Furthermore, the manifold \( M' \) is unique, up to TOP homeomorphism, because its intersection form over \( \mathbb{Z} \) must be even (see for example [5]). We also note that the second part of the statement in Theorem 1.1 gives a simple alternative proof of Theorem 3 of [6].

Using recent results of Hillman ([6], [7]) and of Cochran and Habegger ([3]), we also prove that the homotopy type classifies our manifolds, up to TOP s-cobordism.

**Theorem 1.2.** With the above notation, if \( M \) is simple homotopy equivalent to \( E \# M' \), then \( M \) and \( E \# M' \) are topologically s-cobordant.

The assertion was first proved for the case when \( M \) is simple homotopy equivalent to \( E \) by Hillman (see [6]). We also note that TOP s-cobordant 4-manifolds \( M \) and \( N \) are stably homeomorphic (see for example [5]), i.e. \( M \# k(S^2 \times S^2) \) is TOP homeomorphic to \( N \# \ell(S^2 \times S^2) \) for some integers \( k, \ell \geq 0 \). Thus Theorem 1.2 extends a well-known result of Wall (see [12]) to the non-simply-connected case.

In a particular case, i.e. \( \Pi_1(M) \cong \mathbb{Z} \oplus \mathbb{Z} \), the fact that the fundamental group is elementary amenable implies that s-cobordisms are always topologically products (see [5]). Thus we have the following characterization result.

**Theorem 1.3.** Let \( M^4 \) be a closed connected oriented TOP 4-manifold with \( \Pi_1(M) \cong \mathbb{Z} \oplus \mathbb{Z} \). Let \( M' \) be the simply-connected 4-manifold defined in the discussion preceding the statement of Theorem 1.1. Then \( M \) is TOP homeomorphic to the connected sum of \( M' \) with an \( S^2 \)-bundle over the torus if and only if the homological condition of Theorem 1.1 holds.

If further \( \chi(M) = 0 \), then \( K_2(\psi, \Lambda) \cong 0 \), hence \( M \) is homeomorphic to an \( S^2 \)-bundle over the torus.
Although we work in the topological category, we occasionally use “transversality” and “regular values”. This is possible by for example [5]. Moreover, we assume that $M$ has a CW-structure. For a general reference on combinatorial homotopy of 4-complexes see [1]. For surgery theory we refer to [2] and [13].

2. Homotopy type

Let $M^4$ be a manifold with the properties described in Section 1. Since $F$ is an aspherical closed surface, we have that $F = K(\Pi_1(F), 1) = B\Pi_1(F)$. For the proof of Theorem 1.1 it will not be important which isomorphism $\Pi_1(M) \cong \Pi_1(F)$ one chooses. This isomorphism is realized by a classifying map $f : M \to F$, i.e. $f$ classifies the universal covering $\tilde{M}$ of $M$.

**Lemma 2.1.** There exists a map $j : F \to M$ such that the composition

$$f \circ j : F \to F$$

is homotopic to the identity.

**Proof.** There is an embedding $j_0 : F \setminus D^2 \simeq \mathbb{S}^1 \to M$ such that $f \circ j_0$ is homotopic to the inclusion $F \setminus D^2 \to F$. Here $g$ denotes the genus of $F$. The obstruction to extending $j_0$ is the homotopy class $[j_0]_{\partial D^2} \in \Pi_1(M)$, and it is mapped to the obstruction to extending $f \circ j_0$ via the isomorphism $f_* : \Pi_1(M) \to \Pi_1(F)$; hence it must be zero. Therefore $j_0$ extends to a map $j : F \to M$. It is now easy to see that $\deg(f \circ j) = 1$; hence $f \circ j$ is homotopic to the identity map of $F$.

We define two elements of $H_2(M)$, by setting $u = j_*[F]$ and $v = [F']$, where $[F] \in H_2(F)$ is the fundamental class of $F$, $F' = f^{-1}(x_0)$ and $x_0 \in F$ is a regular value of $f$.

**Lemma 2.2.** The homology classes $u, v \in H_2(M)$ have the following intersection numbers:

1. $u \circ v = 1$; and
2. $v \circ v = 0$.

**Proof.** (1) Let $PD : H^2(M) \to H_2(M)$ denote the Poincaré duality isomorphism and let $\omega_F \in H^2(M)$ be the dual class of $[F]$. Then we have that

$$PD^{-1}(v) = PD^{-1}[F'] = f^*(\omega_F).$$

So we obtain that

$$u \circ v = (PD^{-1}(u) \cup PD^{-1}(v)) \cap [M] = PD^{-1}(v) \cap j_*[F] = f^*(\omega_F) \cap j_*[F] = j^* \circ f^*(\omega_F) \cap [F] = 1,$$

since $j^* \circ f^* = (f \circ j)^*$ is identity.

(2) Choosing a regular value $x_0'$ near to $x_0$ yields $[f^{-1}(x_0')] = [f^{-1}(x_0)] = v$. But obviously, $f^{-1}(x_0') \cap f^{-1}(x_0)$ is empty, hence $v \circ v = 0$.

Set $a = u \circ u$. The intersection matrix of the pair $(u, v)$ is

$$\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}.$$
The hypothesis $w_2(M) = 0$ implies that $a \equiv 0 \pmod{2}$, i.e. $a = 2k$, for some integer $k$. The change $u \to u - kv$ produces the intersection matrix
\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
\]

**Lemma 2.3.** There exists a map $j' : F \to M$ with the following properties:

1. $f \circ j'$ is homotopic to the identity; and
2. $j'_*[F] = u - kv$.

**Proof.** First, we represent the homology class $v = [F']$ by an immersed 2-sphere $\varphi : S^2 \to M$. We choose a collection of embedded circles in $F'$ whose homology classes form a symplectic basis for $H_1(F')$. Then from this basis we choose a single generator for each handle of $F'$. Next, we note that $\Pi_1(F') \to \Pi_1(M)$ is the trivial homomorphism. Therefore, by the general position each of the chosen circles bounds a 2-disc immersed into $M$ (see [5]). We use these immersed discs to surger $F'$ and the result is an immersed sphere $\Sigma^2$ which represents the homology class $v$. Then $j(F) \# k(-\Sigma^2)$ is the image of a map $k' : F \to M$ which satisfies properties (1) and (2) of the statement. If $\varphi : S^2 \to M$ represents the immersed 2-sphere $\Sigma^2 \subset M$, we have $j' = j \# k\varphi$ as required. \qed

**Remark.** Obviously, we can always assume that the map $j : F \to M$ is an immersion. Thus $\Sigma^2 \subset M$ is an algebraic dual of $j(F)$.

From now on we shall assume that $j : F \to M$ is already chosen so that it satisfies the properties of the following corollary.

**Corollary 2.4.** There is a map $j : F \to M$ such that:

1. $f \circ j$ is homotopic to the identity; and
2. the intersection matrix of the pair $u = j_*[F]$, $v = [F']$ is
\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} \text{ if } w_2(u) = 0 \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \text{ if } w_2(u) = 1.
\]

Recall that $PD^{-1}(v) = f^*(\omega_F)$ and $f_*(u) = [F]$. The next goal is to construct a map $g : M \to S^2$ such that $g_*(v) = [S^2]$ generates $H_2(S^2)$. But the property $g_*(v) = [S^2]$ follows from the relation $g^*(\omega_{S^2}) = PD^{-1}(u)$, where $\omega_{S^2} \in H^2(S^2)$ is the dual of $[S^2]$. This holds because
\[
1 = u \circ v = (PD^{-1}(u) \cup PD^{-1}(v)) \cap [M] = PD^{-1}(u) \cap v = g^*(\omega_{S^2}) \cap v
= g_*(g^*(\omega_{S^2}) \cap v) = \omega_{S^2} \cap g_*(v),
\]
i.e. $g_*(v) = [S^2]$ (note that $g^*(\omega_{S^2}) \cap v \in H_0(M)$ and $g_* = \text{Id} : H_0(M) \to H_0(S^2)$).

**Lemma 2.5.** There exists a map $g : M \to S^2$ such that $g^*(\omega_{S^2}) = PD^{-1}(u)$, where $\omega_{S^2}$ is the generator of $H^2(S^2)$.

**Proof.** Let $g' : M \to K(Z,2) = CP^\infty$ be a map which represents the cohomology class $PD^{-1}(u) \in H^2(M) \cong [M, K(Z, 2)]$. Since $M$ has dimension four, we can assume $g' : M \to CP^2 = CP^1 \cup_D D^4$, where $\eta : S^3 \to CP^1 = S^2$ is the Hopf map.

Now $PD^{-1}(u^2) = a\omega_M = 0$, where $\omega_M$ is the dual of the fundamental class of $M$. Thus $g'$ factors over $g : M \to CP^1 = S^2$. \qed

Note that the map $\psi = f \times g : M \to F \times S^2$ has degree one. We use this map to prove the following result.
Proposition 2.6. There exists a map \( \alpha : F \times S^2 \setminus D^4 \to M \) such that \( \psi \circ \alpha \) is homotopic to the inclusion \( F \times S^2 \setminus D^4 \to F \times S^2 \).

Proof. Recall that we have constructed \( j : F \to M \) and \( \varphi : S^2 \to M \), i.e. we have a map \( j \vee \varphi : F \vee S^2 \to M \). The first obstruction to extending \( j \vee \varphi \) to \( F \times S^2 \) lies in the cohomology group \( H^3(F \times S^2; \Pi_2(M)) \) with local coefficients. Poincaré duality now implies that \( H^3(F \times S^2; \Pi_2(M)) \cong H_1(F \times S^2; \Pi_2(M)) \). By a result of Hillman (see [6], p. 279), one has that

\[
\Pi_2(M) \cong H_2(M; \Lambda) \cong \mathrm{Ext}^2_\Lambda(H_0(M; \Lambda), \Lambda) \oplus \mathrm{Ext}^0_\Lambda(H_2(M; \Lambda), \Lambda)
\]

where the \( \Lambda \)-module \( Q = \mathrm{Ext}^0_\Lambda(H_2(M; \Lambda), \Lambda) \) is stably \( \Lambda \)-free. Here \( \Lambda \) is as usual the group ring \( \mathbb{Z}[\Pi_1(M)] \). The fact that \( \ker(\psi_\ast : H_2(M; \Lambda) \to H_2(F \times S^2; \Lambda)) \) is stably \( \Lambda \)-free follows from [13]. Since \( Q \) is stably \( \Lambda \)-free, we have

\[
H_1(F \times S^2; Q) \cong \mathrm{Tor}_1^\Lambda(\mathbb{Z}, Q) \cong 0.
\]

Hence we obtain

\[
H_1(F \times S^2; \Pi_2(M)) \cong H_1(F \times S^2; H^2(F)) \cong H_1(F \times S^2; \mathbb{Z}),
\]

i.e. \( H^3(F \times S^2; \Pi_2(M)) \cong H^3(F \times S^2; \mathbb{Z}) \). Since \( F \) is aspherical, \( \Pi_2(F \times S^2) \cong \mathbb{Z} \) and so the map \( \psi : M \to F \times S^2 \) induces an isomorphism

\[
\psi_\ast : H^3(F \times S^2; \Pi_2(M)) \to H^3(F \times S^2; \Pi_2(F \times S^2)).
\]

By naturality, the image of the obstruction under \( \psi_\ast \) is the obstruction to extending \( \psi \circ (j \vee \varphi) : F \vee S^2 \to F \times S^2 \). But the last obstruction vanishes as \( \psi \circ (j \vee \varphi) \) is homotopic to the inclusion map (use Corollary 2.4). Therefore \( j \vee \varphi \) extends to the 3-skeleton \( (F \times S^2)^{(3)} \cong F \times S^2 \setminus D^4 \), and the extension \( \alpha : F \times S^2 \setminus D^4 \to M \) satisfies the property \( \psi \circ \alpha \simeq \text{inclusion} \).

Since the map \( \psi : M \to F \times S^2 \) has degree one, it induces a splitting of the integral intersection form \( \lambda_M : H_2(M) \times H_2(M) \to \mathbb{Z} \), i.e.

\[
\lambda_M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \lambda'.
\]

By Freedman’s theorems (see [4] and [5]) we can realize \( \lambda' \) as the intersection form of a topological simply-connected 4-manifold \( M' \), i.e. \( \lambda' = \lambda_{M'} \). Recall that \( H_2(M; \Lambda) \cong H_2(F) \oplus \mathrm{Ext}^1_\Lambda(H_2(M; \Lambda), \Lambda), \) where \( Q = \mathrm{Ext}^0_\Lambda(H_2(M; \Lambda), \Lambda) \) is stably \( \Lambda \)-free. Using the universal coefficient spectral sequence

\[
\mathrm{Tor}_q^\Lambda(H_q(M; \Lambda), \mathbb{Z}) \to H_{p+q}(M; \mathbb{Z}),
\]

we obtain that

\[
H_2(M; \mathbb{Z}) \cong \mathrm{Tor}_0^\Lambda(H_2(M; \Lambda), \mathbb{Z}) \oplus \mathrm{Tor}_2^\Lambda(H_0(M; \Lambda), \mathbb{Z}) \cong H_2(M; \Lambda) \otimes \mathbb{Z} \oplus H_2(\Pi_1; \mathbb{Z}) \cong (H_2(F; \mathbb{Z}) \oplus Q) \otimes \mathbb{Z} \oplus H_2(F; \mathbb{Z}) \cong \mathbb{Z} \oplus Q \otimes \mathbb{Z}.
\]

Note that \( Q \otimes \mathbb{Z} \cong \oplus_r \mathbb{Z} \), where \( r = \text{rank } Q \). In particular, we have

\[
Q \otimes \mathbb{Z} \cong H_2(M'; \mathbb{Z}),
\]
Figure 1

and the above decomposition of $H_2(M;\mathbb{Z})$ into a direct sum gives the splitting

$$\lambda_M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \lambda_{M'}$$

of the intersection form over $\mathbb{Z}$. In summary, we have

$$\Pi_2(M) \otimes_\Lambda \mathbb{Z} \cong (\mathbb{Z} \oplus \mathbb{Q}) \otimes_\Lambda \mathbb{Z} \cong \mathbb{Z} \oplus H_2(M'),$$

i.e. the $r$ generators of $H_2(M')$ can be represented by maps of 2-spheres. In other words, we have a map $\beta : M' \setminus D^4 \simeq \bigvee_r S^2 \to M$. Now we observe that $((F \times S^2)\# M')\setminus D^4$ is homotopy equivalent to the wedge $(F \times S^2\setminus D^4) \lor (M'\setminus D^4)$, as shown in Figure 1.

Thus the map $\alpha \# \beta : (F \times S^2\# M')\setminus D^4 \to M$ induces isomorphisms on $\Pi_1$ and on $H_2(M;\mathbb{Z})$. Let us denote $M_1 = F \times S^2\# M'$. The above arguments also imply that the $\Lambda$-ranks of $H_2(M;\Lambda)$ and $H_2(M_1;\Lambda)$ coincide. Next we want to extend $\alpha \# \beta : M_1\setminus D^4 \to M$ to a map $M_1 \to M$. The obstruction for extending $\alpha \# \beta$ is

$$\theta = [\partial(M_1\setminus D^4) \xrightarrow{\alpha \# \beta} M] \in \Pi_3(M),$$

i.e. $\theta$ is the homotopy class of the restriction of $\alpha \# \beta$ to the boundary of $M_1\setminus D^4$.

Obviously, $\theta$ is the image of the generator of

$$\Pi_4(M_1, M_1\setminus D^4) \cong H_4(M_1, M_1\setminus D^4;\Lambda) \cong \Lambda$$

under the composition

$$\Pi_4(M_1, M_1\setminus D^4) \xrightarrow{\partial_*} \Pi_3(M_1\setminus D^4) \xrightarrow{(\alpha \# \beta)_*} \Pi_3(M).$$

Therefore the existence of an extension $h : M_1 \to M$ of $\alpha \# \beta$ follows from the following result.

**Proposition 2.7.** With the above notation, the composition $(\alpha \# \beta)_* \circ \partial_*$ is the trivial homomorphism.

Using this proposition, we can complete the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Since the obstruction $\theta$ is zero, there exists a map $h : M_1 \to M$ which extends $\alpha \# \beta$. Obviously, $h$ induces an isomorphism on $\Pi_1$. It
suffices to prove that \( h_\ast : H_4(M_1; \Lambda) \to H_q(M; \Lambda) \) is an isomorphism for \( q = 2, 3, 4 \).

Since

\[
h_\ast : H_2(M_1; \mathbb{Z}) \xrightarrow{\cong} H_2(M; \mathbb{Z})
\]

and \( H_2(M; \mathbb{Q}) \neq 0 \), the map \( h \) has degree one if one chooses the appropriate orientations. Hence \( h_\ast : H_2(M_1; \Lambda) \to H_2(M; \Lambda) \) is onto. The kernel \( K_2(h, \Lambda) \) of \( h_\ast : H_2(M_1; \Lambda) \to H_2(M; \Lambda) \) is \( \Lambda \)-projective (see [13]); in fact, it is stably \( \Lambda \)-free.

Since the \( \Lambda \)-ranks of \( H_2(M_1; \Lambda) \) and \( H_2(M; \Lambda) \) coincide, the \( \Lambda \)-rank of \( K_2(h, \Lambda) \) is zero. Therefore \( K_2(h, \Lambda) \cong 0 \), by Kaplansky’s lemma (see for example [6] and [8]). By Poincaré duality we obtain isomorphisms for all \( q \), i.e. \( h \) is a homotopy equivalence, as asserted.

**Proof of Proposition 2.7.** Note first that \( \alpha \# \beta : M_1 \setminus D^4 \to M \) factors over the 3-skeleton of \( M \), i.e.

\[
\alpha \# \beta : M_1 \setminus D^4 \to M \setminus D^4 \subset M.
\]

Here we have used the identifications \( M \setminus D^4 = M^{(3)} \) and \( M_1 \setminus D^4 = M_1^{(3)} \), where \( M^{(q)} \) and \( M_1^{(q)} \) denote the \( q \)-skeletons of \( M \) and \( M_1 \), respectively. We can also assume that \( \alpha \# \beta \) is a cellular map. Consider the following diagram

\[
\begin{array}{ccc}
\Pi_4(M_1, M_1 \setminus D^4) & \xrightarrow{\partial_*} & \Pi_3(M_1 \setminus D^4) \\
\downarrow \alpha \# \beta_\ast & & \downarrow \alpha \# \beta_\ast \\
\Pi_4(M, M \setminus D^4) & \xrightarrow{} & \Pi_3(M \setminus D^4) \\
\end{array}
\]

The proof will be completed once we construct a homomorphism

\[
\gamma : \Pi_3(M_1) \to \Pi_3(M)
\]

such that the diagram (/) commutes. For this, we consider the Whitehead exact sequence for a 4-dimensional CW-complex \( X \) (see [1], [14] and [15]):

\[
\begin{array}{ccc}
H_4(\tilde{X}) & \xrightarrow{} & \Gamma(\Pi_2(X)) \\
\xrightarrow{} & \Pi_3(X) & \xrightarrow{} & H_3(\tilde{X}) \\
\end{array}
\]

This sequence is natural with respect to maps \( X \to Y \). Here \( \tilde{X} \) is the universal covering of \( X \), \( \Pi_3(X) \to H_3(\tilde{X}) \) is the Hurewicz homomorphism and \( \Gamma \) denotes the quadratic functor on abelian groups. We recall that \( \Gamma(\Pi_2(X)) \) is equal to \( \text{Im} (\Pi_3(X^{(2)}) \to \Pi_3(X^{(3)})) \). In our case, we have \( H_4(\tilde{M}) \cong H_4(M_1) \cong 0 \) because \( \Pi_1(M) \cong \Pi_1(M_1) \) is an infinite group. Moreover,

\[
H_2(\tilde{M}) \cong H_2(M; \Lambda) \cong H^1(M; \Lambda) \cong H^1(\Pi_1; \Lambda) \cong H^1(F; \Lambda) \cong H_1(F; \Lambda) \cong 0
\]

as \( F \) is an aspherical surface. Similarly, \( H_3(\tilde{M}_1) \cong 0 \). Hence the above sequence implies that \( \Gamma(\Pi_2(M)) \cong \Pi_3(M) \) and \( \Gamma(\Pi_2(M_1)) \cong \Pi_3(M_1) \). Now the map

\[
\alpha \# \beta : M_1 \setminus D^4 \to M \text{ induces } (\alpha \# \beta)_\ast : \Pi_2(M_1) \cong \Pi_2(M_1 \setminus D^4) \to \Pi_2(M), \text{ hence } (\alpha \# \beta)_\ast : \Gamma(\Pi_2(M_1)) \to \Gamma(\Pi_2(M)).
\]

Then the homomorphism \( \gamma \) is defined by the
following diagram:

\[
\begin{array}{ccc}
\Gamma(\Pi_2(M_1)) & \xrightarrow{(\alpha \# \beta)\ast} & \Gamma(\Pi_2(M)) \\
\text{iso} & & \text{iso} \\
\Pi_3(M_1) & \xrightarrow{\gamma} & \Pi_3(M).
\end{array}
\]

The commutativity of (\(\Gamma\)) follows from the second interpretation of \(\Gamma(\Pi_2)\) looking at the diagram shown below:

\[
\begin{array}{ccc}
\Pi_3(M_1^{(2)}) & \xrightarrow{\circ} & \Pi_3(M_1 \setminus D^4) \\
\downarrow (\alpha \# \beta) \ast & & \downarrow (\alpha \# \beta) \ast \\
\Pi_3(M_1^{(2)}) & \xrightarrow{\circ} & \Pi_3(M_\ast) \\
\end{array}
\]

This completes the proof. \(\square\)

Remarks. (1) As a corollary we obtain that in the decomposition \(\Pi_2(M) \cong \mathbb{Z} \oplus Q\), the \(\Lambda\)-module \(Q\) is actually free. This improves the result of Hillman [6].

(2) The proof of Proposition 2.7 shows that \(\gamma : \Pi_3(M_1) \to \Pi_3(M)\) is an isomorphism, and hence the sequence

\[
\Pi_3(M_1, M_1 \setminus D^4) \xrightarrow{\partial} \Pi_3(M_1 \setminus D^4) \xrightarrow{(\alpha \# \beta)\ast} \Pi_3(M) \xrightarrow{\gamma} 0
\]

is exact.

(3) The proof of Proposition 2.7 can be most easily seen as follows. We write the obstruction \(\theta = \theta_1 + \theta_2 + \theta_3\) according to the splitting

\[
\Pi_3(M) \cong \Gamma(\Pi_2(F \times S^2)) \oplus \Pi_2(F \times S^2) \otimes K_2(\psi, \Lambda) \oplus \Gamma(K_2(\psi, \Lambda))
\]

induced by \(H_2(M; \Lambda) \cong H_2(F \times S^2; \Lambda) \oplus K_2(\psi, \Lambda)\).

Now \(\theta_1 \in \Gamma(\Pi_2(F \times S^2))\) is zero because it is the obstruction for extending \(\psi \circ \alpha\), hence vanishes by Proposition 2.6.

The addendum \(\theta_2 \in \Pi_2(F \times S^2) \otimes K_2(\psi, \Lambda)\) is determined by intersection numbers of elements of the submodule \(A\), generated by \(\text{Im}(\alpha) \subset H_2(M; \Lambda)\), and elements of \(K_2(\psi, \Lambda)\). But they are all zero by construction.

Finally, \(\theta_3 \in \Gamma(K_2(\psi, \Lambda))\) is zero by hypothesis.

3. s-COBORDISM TYPE

In this section we are going to prove Theorem 1.2. In Section 2 we have constructed a simple homotopy equivalence \(h : M \to F \times S^2 \# M'\). To obtain Theorem 1.2, it suffices to prove the following two results.

**Proposition 3.1.** The pair \((M, h)\) is normally cobordant to a self-homotopy equivalence

\[
g : F \times S^2 \# M' \to F \times S^2 \# M'.
\]

The following is well-known (see [6], Lemma 6, p. 282).

**Proposition 3.2.** The surgery obstruction map

\[
\theta : [(F \times S^2 \# M') \times I, (F \times S^2 \# M') \times \partial I \times G/\text{TOP}] \to L_5(\Pi_1)
\]

is surjective.
Now one can use the 5-dimensional surgery theory to construct an s-cobordism between $M$ and $M_1 = F \times S^2 \# M'$. In fact, let $W \to M_1 \times I$ be a normal cobordism between $(M,h)$ and $(M_1,g)$ guaranteed by Proposition 3.1, i.e. the normal invariants of $(M,h)$ and $(M_1,g)$ coincide. Using the surgery sequence (see [5] and [13]) and Proposition 3.2, it follows that $M_1$ and $M$ are topologically s-cobordant. This proves Theorem 1.2.

Since Proposition 3.2 is well-known, it only remains to prove Proposition 3.1. In the case $M' \cong S^4$, the result was proved by Hillman (see [7]). To prove Proposition 3.1 we use this result and “paste it together” with the corresponding result for simply-connected topological 4-manifolds (see [3]).

Let us first recall the description of normal invariants (for more details we refer to [2]). Let $\delta : M_1 = F \times S^2 \# M' \to BTOP$ be the classifying map of the stable normal (micro) bundle of $M_1$ and let $\rho : BTOP \to BG$ be the principal fibration with fiber $G/TOP$. Here $BG$ is the classifying space of stable spherical fibrations, i.e. $\xi = \rho \circ \delta : M_1 \to BG$ classifies the Spivak fibration of the Poincaré 4-complex $M_1$. Any normal cobordism class of normal maps $N \to M_1$ is determined by a linearization of $\xi$, i.e. by a lifting $\delta'$ of $\xi = \rho \circ \delta$ via the Thom construction. This means, fixing the lifting $\delta'$, that the normal cobordism classes of normal maps correspond uniquely to the elements of $[M_1,G/TOP]$, i.e. $\delta'(x) = g(x)\delta(x)$ with $g : M_1 \to G/TOP$. Let $\Sigma^3 \subset M_1 = F \times S^2 \# M'$ be the 3-sphere along which the manifolds $F \times S^2$ and $M'$ are glued together. Then $[g]_{\Sigma^3} \in \Pi_3(G/TOP) = 0$. Consequently, $g|_{F \times S^2 \setminus D^4}$ and $g|_{M' \setminus D^4}$ extend to maps $g_1 : F \times S^2 \to G/TOP$ and $g_2 : M' \to G/TOP$, respectively. The values of $g_1$ and $g_2$ coincide on the 4-ball $D^4$. Two extensions of $g|_{\Sigma^3}$ over the 4-ball $D^4$ differ by an element of $\Pi_4(G/TOP) \cong \mathbb{Z}$. We use the unique extension of $g|_{\Sigma^3}$ such that the surgery obstruction of $g_2$ is zero. In other words, we have constructed a map

$$\mu : [F \times S^2 \# M',G/TOP] \to [F \times S^2,G/TOP] \oplus [M',G/TOP]$$

which sends $[g]$ into $([g_1],[g_2])$.

On the other hand, attaching a 4-ball $D^4$ to $\Sigma^3$ yields a map $t : F \times S^2 \# M' \to F \times S^2 \# M' \cup_{\Sigma^3} D^4 \simeq F \times S^2 \vee M'$ which induces

$$t_* : [F \times S^2 \vee M',G/TOP] \cong [F \times S^2,G/TOP] \oplus [M',G/TOP]$$

$$\quad \to [F \times S^2 \# M',G/TOP].$$

Now it is very easy to see that $t_* \circ \mu$ is the identity, hence $t_*$ is surjective. On the other hand, the connected sum with $(M',g_2)$ gives the following commutative
The intersection matrix of $F$ Ker($\text{Id}$ of the results of the quoted papers show that there are representatives in HE invariants. This proves Proposition 3.1.

Theorem 3.4. arguments prove the following more general result.

The map induced on $L_4(\Pi_1)$ is the identity because the surgery obstruction of $(M', g_2)$ is zero. If $g : F \times S^2 \# M' \to G/\text{TOP}$ is the normal invariant of a given (simple) homotopy equivalence $h : M \to F \times S^2 \# M'$ and $\mu([g]) = ([g_1], [g_2])$, then $\theta_1(g_1) = 0$. This follows from the diagram (//) and the fact that $\theta_2(g_2) = 0$, where $\theta_2 : [M', G/\text{TOP}] \to L_4(1)$.

In summary, we have proved the following result.

Proposition 3.3. Any element $[g] \in [F \times S^2 \# M', G/\text{TOP}]$, coming from a (simple) homotopy equivalence $h : M \to F \times S^2 \# M'$, belongs to $\text{Im} t_s$.

More precisely, there are elements

$[g_1] \in \text{Ker}(\theta_1 : [F \times S^2, G/\text{TOP}] \to L_4(\Pi_1))$,
$[g_2] \in \text{Ker}(\theta_2 : [M', G/\text{TOP}] \to L_4(1))$

such that $t_s([g_1], [g_2]) = [g]$.

To finish the proof of Proposition 3.1 we recall that the elements of Ker($\theta_1$) and Ker($\theta_2$) come from elements of HE$_{4d}(F \times S^2)$ and HE$_{4d}(M')$, respectively (see [3] and [7]). Here HE$_{4d}$ denotes the set of homotopy classes of simple self-homotopy equivalences inducing the identities on $\Pi_1$ and on $H_s$. More precisely, the proofs of the results of the quoted papers show that there are representatives in HE$_{4d}$ leaving a 4-ball fixed. Therefore, if $h_1 : F \times S^2 \to F \times S^2$ and $h_2 : M' \to M'$ are such representatives of $g_1$ and $g_2$, then $h_{1|D^4_i} = \text{identity}$ for $i = 1, 2$. Thus we can form the map $h_1 \# h_2 : M_1 \to M_1$. Obviously, $h$ and $h_1 \# h_2$ have the same normal invariants. This proves Proposition 3.1.

In this section we did not use the hypothesis that $w_2(M) = 0$. In fact, our arguments prove the following more general result.

Theorem 3.4. Let $M^4$ be a closed connected oriented (TOP) 4-manifold homotopy equivalent to $E \# M'$, where $E$ is an $S^2$-bundle over a closed oriented aspherical surface $F$ and $M'$ is a simply-connected 4-manifold. Then $M$ is topologically s-cobordant to $E \# M'$.

4. Appendix

As announced in the introduction, here we will treat the case $w_2(u) \neq 0$. First recall that there is only one twisted $S^2$-bundle over an oriented closed surface $F$, denoted by $F \times S^2$, because these bundles are determined by the first and second Stiefel-Whitney classes. It can be obtained from $(F \setminus D^2) \times S^2$ by attaching $D^2 \times S^2$ with a map $\alpha : \partial D^2 \times S^2 \to \partial D^2 \times S^2$ associated to the generator of $\Pi_1(SO(3))$. The intersection matrix of $F \times S^2$ is

$\begin{pmatrix}
1 & 1 \\
1 & 0
\end{pmatrix}$
with respect to \( x, y \in H_2(F \times S^2) \), where \( y \) represents the fiber and \( \pi_*(x) = [F] \), 
\( \pi : F \times S^2 \rightarrow F \) being the fiber projection.

**Proposition 4.1.** Let \( M^4 \) be a closed connected oriented 4-manifold with \( \Pi_1(M) \cong \Pi_1(F) \). Assume that \( w_2(u) \neq 0 \) (notation as in Section 2). Then there is a map \( \phi : M \rightarrow F \times S^2 \) of degree 1.

**Proof.** Let \( f : M \rightarrow F \) and \( g' : M \rightarrow \mathbb{C}P^\infty \) be as in the proof of Lemma 2.5.

Then the restriction
\[
f \times g'|_{M \setminus \partial D^4} : M \setminus \partial D^4 \rightarrow F \times \mathbb{C}P^\infty
\]
factors as follows:
\[
\begin{array}{ccc}
M \setminus \partial D^4 & \xrightarrow{f \times g'|_{M \setminus \partial D^4}} & F \times \mathbb{C}P^\infty \\
\phi' \downarrow & & \uparrow \!
\end{array}
\]
\[
[F \times \mathbb{C}P^\infty]^{(3)} \rightarrow [F \times \mathbb{C}P^\infty]^{(3)}.
\]

But note that \([F \times \mathbb{C}P^\infty]^{(3)} = F \times S^2 \setminus B^4, B^4 \) being a 4-ball. Hence we have a map
\[
\phi' : M \setminus \partial D^4 \rightarrow F \times S^2 \setminus \partial D^4.
\]

Obviously \( \phi' \) extends to \( \phi : M \rightarrow (F \times S^2 \setminus \partial D^4) \cup_\lambda D^4 \), where \( \lambda = \phi'|_{\partial D^4} \). Therefore it remains to show that \((F \times S^2 \setminus \partial D^4) \cup_\lambda D^4 \) is homotopy equivalent to \( F \times S^2 \). If \( F \) were \( S^2 \), then \((F \times S^2 \setminus \partial D^4) \cup_\lambda D^4 \) is a Poincaré complex with intersection matrix
\[
\begin{pmatrix}
1 & 1 & 0 \\
1 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

It is homotopy equivalent to \( S^2 \times S^2 \). The general case can be reduced to that of a sphere \( S^2 \) by considering the collapsing map
\[
c : F \times S^2 \rightarrow \sim (F/(\sim F \setminus \sim D^2)) \times S^2 \cong S^2 \times S^2.
\]

Here \( D^2_1 \subset F \) is a 2-disc which contains in its interior the 2-disc \( D^2_2 \subset F \) used at the beginning of the section to describe
\[
F \times S^2 \cong ((F \setminus \partial D^2) \times S^2) \cup_\alpha (D^2 \times S^2).
\]

The general case will follow from the fact that \( \phi' \) can be homotoped in a collar of the boundary of \( M \setminus \partial D^4 \) such that \( \phi'|_{\partial D^4} \subset \partial D^4 \times S^2 \). To extend \( \phi'|_{\partial D^4} \) over \( D^4 \) we need to reglue \( D^2 \times S^2 \subset D^2_1 \times S^2 \) by the twist \( \alpha : \partial D^2 \times S^2 \rightarrow \partial D^2 \times S^2 \), i.e. we have to form
\[
F \times S^2 \cong ((F \setminus \partial D^2) \times S^2) \cup_\alpha (D^2 \times S^2).
\]
To see that we can assume \( \partial' (\partial D^4) \subset D_1^2 \times S^2 \) we consider the short exact homotopy sequence (recall that \( F \) is now aspherical, so \( \Pi_3 (F \times S^2) \cong \mathbb{Z} \)):

\[
0 \to \Pi_4 (F \times S^2, F \times S^2 \backslash B^4) \cong \Lambda \to \Pi_3 (F \times S^2 \backslash B^4) \to \Pi_3 (F \times S^2) \cong \mathbb{Z} \to 0.
\]

This sequence splits because

\[
\text{Ext}_\Lambda^1 (Z, \Lambda) \cong H^1 (F; \Lambda) \cong H_1 (F; \Lambda) \cong 0.
\]

Then we have \( [\lambda] = [\lambda_1] + [\lambda_2] \in \Lambda \oplus \mathbb{Z} \). Therefore \( [\lambda_2] = k [\eta] \), where \( k \in \mathbb{Z} \), \( \eta : S^1 \to \{ * \} \times S^2 \) is the Hopf map and \( * \in D_2^4 \). It follows that \( \lambda_2 (\partial D^4) \subset D_1^2 \times S^2 \). On the other hand we choose \( B^4 = D^2 \times D^2 \subset F \times S^2 \), where \( D^2 \) is the lower hemisphere. Hence a generator \( \tau \in \Lambda \subset \Pi_3 (F \times S^2 \backslash B^4) \) has image in \( D_1^2 \times S^2 \). Since \( [\lambda_1] = a \tau \), where \( a \in \Lambda \), the image of \( \lambda_1 \) belongs to \( D_1^2 \times S^2 \), up to some arcs running through \( (F \backslash D^2) \times S^2 \). This completes the proof. \( \square \)

Since the other arguments are the same as in the case \( w_2 (u) = 0 \), we have completed Theorem 1.1 with the following result involving twisted \( \mathbb{S}^2 \)-bundles over aspherical surfaces.

**Theorem 1.1'.** Let \( M^4 \) be a closed connected oriented 4-manifold with \( \Pi_1 (M) \cong \Pi_1 (F) \). Assume that \( w_2 (u) \neq 0 \) (notation as in Section 2). Then \( M \) is simple homotopy equivalent to the connected sum \( M_1 = (F \times S^2) \# M' \), where \( M' \) is the simply-connected 4-manifold defined in the discussion preceding the statement of Theorem 1.1, if and only if the \( \lambda \)-intersection forms on \( K_2 (\phi, \Lambda) \) and on \( K_2 (c', \Lambda) \) are isomorphic, where \( c' \) denotes the collapsing map from \( M_1 \) to \( F \times S^2 \). Moreover, the manifolds \( M \) and \( (F \times S^2) \# M' \) are topologically s-cobordant.

**References**

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