

ON THE CONJECTURE OF BIRCH AND SWINNERTON-DYER

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ABSTRACT. In this paper we complete Rubin's partial verification of the conjecture for a large class of elliptic curves with complex multiplication by $\mathbb{Q}(\sqrt{-7})$.

1. INTRODUCTION

In this paper we prove the full Birch and Swinnerton-Dyer conjecture for a class of elliptic curves with complex multiplication by $\mathbb{Q}(\sqrt{-7})$.

This paper is an expanded version of the author's doctoral thesis [11], which was completed at The Ohio State University under the supervision of Karl Rubin. It is to be the first in a series of papers aimed at completing Rubin's partial verification of one important case of the conjecture (see Theorem 11.1(i) of [21]).

The setting is as follows.

Let E be an elliptic curve defined over the field $K = \mathbb{Q}(\sqrt{-7})$ (which is one of the 9 imaginary quadratic fields of class number 1), with complex multiplication by the ring of integers \mathcal{O} of K , and with minimal period lattice generated by $\Omega \in \mathbb{C}^\times$. Suppose that the L -function of E over K does not vanish at $s = 1$, i.e. $L(E/K, 1) \neq 0$. Then $E(K)$ is finite [5] and the Tate-Shafarevich group $\text{III}(E/K)$ is finite [20]. Now for each prime \mathfrak{q} of K let $c_{\mathfrak{q}} = [E(K_{\mathfrak{q}}) : E_0(K_{\mathfrak{q}})]$, where $E_0(K_{\mathfrak{q}})$ is the subgroup of $E(K_{\mathfrak{q}})$ of points with non-singular reduction modulo \mathfrak{q} . In this work we prove the following theorem.

Theorem A. *Suppose $L(E/K, 1) \neq 0$. Then*

$$L(E/K, 1) = \Omega \bar{\Omega} \cdot (\#E(K))^{-2} \cdot \#\text{III}(E/K) \cdot \prod c_{\mathfrak{q}}.$$

In other words, the full Birch and Swinnerton-Dyer conjecture is true for E .

In addition, we will deduce from Theorem A the following result concerning curves defined over \mathbb{Q} . For any $d \in \mathbb{Z} - \{0\}$, let E^d be the elliptic curve $y^2 = x^3 + 21dx^2 + 112d^2x$, which has complex multiplication by $\mathcal{O} = \mathbb{Z}[(1 + \sqrt{-7})/2]$.

Theorem B. *If $L(E_{\mathbb{Q}}^d, 1) \neq 0$, then the full Birch and Swinnerton-Dyer conjecture is true for $E_{\mathbb{Q}}^d$.*

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The following is a summary of the paper.

Write ψ for the Hecke character of K attached to E . Then $L(\psi, 1)/\Omega \in K$ and the L -function of E over K factors as

$$(1) \quad L(E/K, s) = L(\psi, s)L(\bar{\psi}, s).$$

Now write $\text{III} = \text{III}(E/K)$ and let B be the set of primes of K where E has bad reduction. We will see below (Proposition 2.6) that $c_{\mathfrak{q}} = 4$ for all $\mathfrak{q} \in B$, so $\prod c_{\mathfrak{q}} = 4^b$ where $b = \#B$. Further, the theorem of Rubin alluded to above ([21], Theorem 11.1(i)) together with the fact that $\#(\text{III}_{\mathfrak{q}^\infty}) = \#(\text{III}_{\bar{\mathfrak{q}}^\infty})$ for every prime \mathfrak{q} of K ([14], p. 228) shows that if $\mathfrak{q} \nmid \#\mathcal{O}^\times$, then

$$(2) \quad L(\bar{\psi}, 1)/\Omega \sim_{\mathfrak{q}} 2^b \cdot (\#E(K))^{-1} \cdot \sqrt{\#\text{III}},$$

where $\sim_{\mathfrak{q}}$ means “equal up to a unit of $K_{\mathfrak{q}}$ ”.

In this paper we prove (2) for the primes \mathfrak{q} that divide $\#\mathcal{O}^\times = 2$. Since 2 splits in K (and we note that $K = \mathbb{Q}(\sqrt{-7})$ is the only imaginary quadratic field of class number 1 with this property), it will be sufficient to show that

$$(3) \quad L(\bar{\psi}, 1)/\Omega \sim_{\mathfrak{p}} 2^b \cdot (\#E(K)_{2^\infty})^{-1} \cdot \#\text{III}_{\mathfrak{p}^\infty}$$

for $\mathfrak{p} \mid 2$. It will then follow that Gross’ refinement [14] of the Birch and Swinnerton-Dyer conjecture for E is true, i.e.

$$(4) \quad L(\bar{\psi}, 1)/\Omega = \pm 2^b \cdot (\#E(K))^{-1} \cdot \sqrt{\#\text{III}}.$$

Using (1), this formula immediately implies Theorem A.

For convenience, we will exclude from the remainder of this discussion a small number of “exceptional” curves (these are defined at the beginning of §2 and will be studied in §8).

To establish (3) for the remaining, non-exceptional, curves, we will first show that

$$(5) \quad \#\text{Hom}(X_\infty, E_{\mathfrak{p}^\infty})^{\mathcal{G}} = 2^{b^*} \cdot \#E(K_{\mathfrak{p}})_{\bar{\mathfrak{p}}^\infty} \cdot (\#E(K)_{2^\infty})^{-1} \cdot \#\text{III}_{\mathfrak{p}^\infty},$$

where $b^* = \#(B - \{\mathfrak{p}\})$, X_∞ is the Galois group of the maximal abelian 2-extension of $K_\infty = K(E_{\mathfrak{p}^\infty})$ which is unramified outside of the primes above \mathfrak{p} , and $\mathcal{G} = \text{Gal}(K_\infty/K)$. The essential ingredient in the derivation of this result is a theorem of Bashmakov [1], which describes the image of a certain localization map. See Theorem 3.2 below. The rest of the argument leading to (5) is likely to seem familiar to those readers acquainted with Coates’ paper [4] (but also a little more complicated, because here we deal with the troublesome prime 2 and we do not assume good reduction at 2).

Now Rubin [21] has shown how to relate the integer on the left-hand side of formula (5) to $\text{ord}_{\mathfrak{p}}(L(\bar{\psi}, 1)/\Omega)$ when $\mathfrak{p} \nmid 2$, as an application of the “main conjecture” of Iwasawa theory for K . In this work we prove a main conjecture for the extension K_∞/K (see Theorem 4.1 below) which has similar applications. Since 2 splits in K , we are in the setting of a one-variable main conjecture, which makes the case $K = \mathbb{Q}(\sqrt{-7})$ the simplest of all. We will then use the main conjecture to show that

$$\#\text{Hom}(X_\infty, E_{\mathfrak{p}^\infty})^{\mathcal{G}} \sim_{\mathfrak{p}} 2^{b^* - b} \cdot \#E(K_{\mathfrak{p}})_{\bar{\mathfrak{p}}^\infty} \cdot L(\bar{\psi}, 1)/\Omega.$$

This formula together with (5) yields (3), thereby completing the verification of the Birch and Swinnerton-Dyer conjecture in the present case.

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2. PRELIMINARIES

Let $K = \mathbb{Q}(\sqrt{-7})$, and write \mathcal{O} for the ring of integers of K . Fix a prime \mathfrak{p} of K lying above 2, and let $\bar{\mathfrak{p}}$ denote the complex conjugate of \mathfrak{p} . Then $\mathfrak{p} \neq \bar{\mathfrak{p}}$. Now fix an elliptic curve E defined over K with complex multiplication by \mathcal{O} .

As explained in the Introduction, to prove Theorem A it suffices to check formula (3) at both primes \mathfrak{p} and $\bar{\mathfrak{p}}$. In fact, we need only check (3) at the prime \mathfrak{p} , for after this is done we can certainly repeat the entire argument replacing \mathfrak{p} by $\bar{\mathfrak{p}}$ throughout to obtain a proof of (3) for the prime $\bar{\mathfrak{p}}$.

Now let B denote the set of primes of K where E has bad reduction. The theory of complex multiplication shows that B is never empty. We shall say that E is *exceptional* if E has bad reduction at \mathfrak{p} and good reduction at all other primes, i.e. if $B = \{\mathfrak{p}\}$. (We remark here that there are very few exceptional curves.) As it turns out, it is convenient to prove formula (3) for exceptional and non-exceptional curves separately, so throughout this and the next several sections we will work under the assumption that E is non-exceptional, deferring the study of the exceptional curves to §8.

We now introduce some additional notations. If F is any field, we will write G_F for $\text{Gal}(\bar{F}/F)$, where \bar{F} denotes the algebraic closure of F . Further, if M is an \mathcal{O} -module and \mathfrak{a} is an ideal of \mathcal{O} , we will write $M_{\mathfrak{a}}$ for the \mathfrak{a} -torsion in M and $M_{\mathfrak{a}^\infty}$ for $\bigcup_{n \geq 1} M_{\mathfrak{a}^n}$. There will be two exceptions to this rule: if \mathfrak{q} is a prime of K , $\mathcal{O}_{\mathfrak{q}}$ (resp. $\bar{K}_{\mathfrak{q}}$) will denote the completion of \mathcal{O} (resp. K) at \mathfrak{q} . Also, for convenience, we will write $E_{\mathfrak{a}}$ for $E(\bar{K})_{\mathfrak{a}}$.

Now, for each n with $1 \leq n \leq \infty$, let $K_n = K(E_{\mathfrak{p}^n})$. Further, for any ideal $\mathfrak{a} \subset \mathcal{O}$ set $\mathbf{N}(\mathfrak{a}) = \#(\mathcal{O}/\mathfrak{a})$ and write $K(\mathfrak{a})$ for the ray class field of K modulo \mathfrak{a} . The theory of complex multiplication shows that $K(\mathfrak{p}^n) \subset K_n$ for all $n \leq \infty$, where $K(\mathfrak{p}^\infty)$ is defined as $\bigcup_{n \geq 1} K(\mathfrak{p}^n)$.

Lemma 2.1. (i) *If $n \geq 2$, then $E_{/K_n}$ has good reduction at every prime of K_n not lying above \mathfrak{p} .*

(ii) $E(K)_{\mathfrak{p}^\infty} = E_{\mathfrak{p}}$.

Proof. For (i), see for example [5] Theorem 2. The proof is a variant of the criterion of Néron-Ogg-Shafarevich, using the facts that E has potential good reduction everywhere and $\text{Gal}(K_\infty/K_n) \subset 1 + \mathfrak{p}^n \mathcal{O}_{\mathfrak{p}}$ is torsion-free if $n \geq 2$. To prove (ii) we note first that $\#E_{\mathfrak{p}} = \mathbf{N}(\mathfrak{p}) = 2$, so G_K acts trivially on $E_{\mathfrak{p}}$. Thus $E_{\mathfrak{p}} \subset E(K)$. Now if we had $E_{\mathfrak{p}^2} \subset E(K)$, then (i) would show that E is an exceptional curve, contravening our hypothesis. \square

Remark. For each result in this section which depends on the choice of \mathfrak{p} , there is a corresponding result with the prime \mathfrak{p} replaced by $\bar{\mathfrak{p}}$, provided $B \neq \{\bar{\mathfrak{p}}\}$. We will make use of this fact at various places below.

Lemma 2.2. (i) *Every prime in $B - \{\mathfrak{p}\}$ is ramified in K_2/K .*

(ii) *If $2 \leq n \leq \infty$, then $[K_n : K(\mathfrak{p}^n)] = \#\mathcal{O}^\times = 2$.*

Proof. For (i) see [22], Corollary 2 of Theorem 2 (note that the set $B - \{\mathfrak{p}\}$ is non-empty, because E is non-exceptional). As regards assertion (ii), it is shown in [19] (Lemma 21(iv)) that $[K_n : K(\mathfrak{p}^n)] \leq \#\mathcal{O}^\times = 2$ for all $n < \infty$. On the other hand (i) shows that $K_n \not\subset K(\mathfrak{p}^\infty)$ for all $n \geq 2$, and (ii) follows. \square

Let $\mathcal{G} = \text{Gal}(K_\infty/K)$ and define an injective map $\chi_E : \mathcal{G} \rightarrow \mathcal{O}_\mathfrak{p}^\times$ by $P^\sigma = \chi_E(\sigma)P$ for all $P \in E_{\mathfrak{p}^\infty}$ and all $\sigma \in \mathcal{G}$.

Proposition 2.3. *The map χ_E is an isomorphism.*

Proof. The theory of complex multiplication shows that $\mathcal{O}^\times \chi_E(\mathcal{G}) = \mathcal{O}_\mathfrak{p}^\times$ (see [23], Theorem 5.4). On the other hand Lemma 2.2(ii) shows that \mathcal{G} contains a subgroup of order 2, namely $\text{Gal}(K_\infty/K(\mathfrak{p}^\infty))$. Consequently $\{\pm 1\} = \mathcal{O}^\times \subset \chi_E(\mathcal{G})$, which completes the proof. \square

Define $\tau = \chi_E^{-1}(-1)$, i.e. τ is that element of \mathcal{G} which acts as multiplication by -1 on $E_{\mathfrak{p}^\infty}$. Now let $\langle \tau \rangle$ be the cyclic group generated by τ .

Corollary 2.4. (i) $\text{Gal}(K_\infty/K(\mathfrak{p}^\infty)) = \langle \tau \rangle$.

(ii) $\text{Gal}(K_2/K)$ is cyclic, generated by the restriction of τ to K_2 .

Proof. Assertion (i) follows from the proof of Proposition 2.3. Now by Proposition 2.3, χ_E induces an isomorphism $\text{Gal}(K_2/K) \simeq \mathcal{O}_\mathfrak{p}^\times / (1 + \mathfrak{p}^2\mathcal{O}_\mathfrak{p})$. Noting that $\mathcal{O}_\mathfrak{p}^\times = 1 + \mathfrak{p}\mathcal{O}_\mathfrak{p} = \{\pm 1\} \times (1 + \mathfrak{p}^2\mathcal{O}_\mathfrak{p})$, (ii) follows at once. \square

Lemma 2.5. *Suppose $\mathfrak{q} \in B - \{\mathfrak{p}\}$. Then:*

(i) *The inertia group of \mathfrak{q} in K_∞/K is $\langle \tau \rangle$.*

(ii) $E(K_\mathfrak{q})_{\mathfrak{p}^\infty} = E_\mathfrak{p}$.

Proof. By Lemma 2.2(i), \mathfrak{q} ramifies in K_∞/K . Further $\mathfrak{q} \neq \mathfrak{p}$, so (i) follows from Corollary 2.4(i). Now if $E(K_\mathfrak{q})_{\mathfrak{p}^\infty} = E_{\mathfrak{p}^j}$, then \mathfrak{q} splits completely in K_j/K . But \mathfrak{q} ramifies in K_2/K (Lemma 2.2(i)), so $j \leq 1$. Since $j \geq 1$ by Lemma 2.1(ii), the proof is complete. \square

For each $\mathfrak{q} \in B$ let

$$c_\mathfrak{q} = [E(K_\mathfrak{q}) : E_0(K_\mathfrak{q})],$$

where $E_0(K_\mathfrak{q})$ is the subgroup of $E(K_\mathfrak{q})$ of points with non-singular reduction modulo \mathfrak{q} .

Proposition 2.6. *For every $\mathfrak{q} \in B$, $c_\mathfrak{q} = 4$.*

Proof. If $\mathfrak{q} \in B$ and $\mathfrak{q} \nmid 2$, then $c_\mathfrak{q} = \#E(K_\mathfrak{q})_2$ ([14], Proposition 4.9). Thus by Lemma 2.5(ii) and its analogue for $\bar{\mathfrak{p}}$ (see the remark following the proof of Lemma 2.1), we have $c_\mathfrak{q} = 4$ for such \mathfrak{q} . It remains to show that $c_\mathfrak{p} = 4$ if $\mathfrak{p} \in B$ (the equality $c_{\bar{\mathfrak{p}}} = 4$ when $\bar{\mathfrak{p}} \in B$ is proved similarly).

It is shown in [14] (proof of Proposition 4.5) that the only possible Kodaira types for E over $K_\mathfrak{p}$ are I_ν^* (in which case $c_\mathfrak{p} = 4$), II and II^* (which have $c_\mathfrak{p} = 1$, i.e. $E(K_\mathfrak{p}) = E_0(K_\mathfrak{p})$). To see that the last two types cannot occur, simply note that $E_0(K_\mathfrak{p})$ is $\bar{\mathfrak{p}}$ -divisible (cf. [25]) but $E(K_\mathfrak{p})$ is not, by the analogue of Lemma 2.5(ii) for $\bar{\mathfrak{p}}$. \square

We devote the remainder of this section to proving a number of results on the Galois cohomology of E .

If F is any field, we will write $H^1(F, E)$ for $H^1(G_F, E(\bar{F}))$.

Let $\Gamma = \text{Gal}(K_\infty/K_2)$. Then by Proposition 2.3, $\Gamma \simeq 1 + \mathfrak{p}^2\mathcal{O}_\mathfrak{p} \simeq \mathbb{Z}_2$.

Lemma 2.7. (i) *The restriction map induces an isomorphism*

$$H^1(K_2, E_{\mathfrak{p}^\infty}) \simeq H^1(K_\infty, E_{\mathfrak{p}^\infty})^\Gamma.$$

(ii) $\#H^1(\mathcal{G}, E_{\mathfrak{p}^\infty}) = \#E(K)_{\mathfrak{p}^\infty}$.

Proof. A standard calculation shows that $H^i(\Gamma, E_{\mathfrak{p}^\infty}) = 0$ for all $i \geq 1$ (cf. Lemma 6 of [4]). This fact together with the appropriate inflation-restriction exact sequences gives (i), and shows in addition that

$$H^1(\mathcal{G}, E_{\mathfrak{p}^\infty}) \simeq H^1(G_2, E_{\mathfrak{p}^2}),$$

where $G_2 = \text{Gal}(K_2/K)$. Now using Corollary 2.4(ii) and Lemma 2.1(ii), we have

$$H^1(G_2, E_{\mathfrak{p}^2}) \simeq E_{\mathfrak{p}^2}/2E_{\mathfrak{p}^2} \simeq E_{\mathfrak{p}} = E(K)_{\mathfrak{p}^\infty}.$$

□

If Ω is a prime of K_∞ and $n < \infty$, we will write $K_{n,\Omega}$ for the completion of K_n at the prime below Ω . Now let $K_{\infty,\Omega} = \bigcup_{n \geq 1} K_{n,\Omega}$.

Lemma 2.8. *Let Ω be a prime of K_∞ and let \mathfrak{q} be the prime of K lying below Ω . Then, if $\mathfrak{q} \notin B \cup \{\mathfrak{p}\}$,*

$$H^1(\text{Gal}(K_{\infty,\Omega}/K_{\mathfrak{q}}), E(K_{\infty,\Omega})) = 0.$$

Proof. This is well-known, coming from the facts that E has good reduction over $K_{\mathfrak{q}}$ and $K_{\infty,\Omega}/K_{\mathfrak{q}}$ is unramified. See for example [17], Corollary 4.4. □

Lemma 2.9. *Let Ω be a prime of K_∞ and let \mathfrak{q} be the prime of K lying below Ω . Then, if $\mathfrak{q} \in B$ and $\mathfrak{q} \nmid 2$,*

$$\#H^1(\text{Gal}(K_{\infty,\Omega}/K_{\mathfrak{q}}), E(K_{\infty,\Omega}))_{\mathfrak{p}^\infty} = \#E(K_{\mathfrak{q}})_{\bar{\mathfrak{p}}^\infty}.$$

Proof. E has good reduction over $K_{2,\Omega}$ by Lemma 2.1(i), so $K_{\infty,\Omega}/K_{2,\Omega}$ is unramified (see [24], §VII.4). Thus by an analogue of Lemma 2.8 and the usual inflation-restriction exact sequence, there is an isomorphism

$$H^1(\text{Gal}(K_{\infty,\Omega}/K_{\mathfrak{q}}), E(K_{\infty,\Omega})) \simeq H^1(G_{2,\Omega}, E(K_{2,\Omega})),$$

where $G_{2,\Omega} = \text{Gal}(K_{2,\Omega}/K_{\mathfrak{q}})$. Now since $\Omega \nmid \mathfrak{p}$ there is a decomposition $E(K_{2,\Omega}) \simeq E(K_{2,\Omega})_{\mathfrak{p}^\infty} \oplus A$, where A is a uniquely \mathfrak{p} -divisible $\mathcal{O}[G_{2,\Omega}]$ -module (see §VII.6.3 of [24]). It follows that

$$H^1(G_{2,\Omega}, E(K_{2,\Omega}))_{\mathfrak{p}^\infty} \simeq H^1(G_{2,\Omega}, E(K_{2,\Omega})_{\mathfrak{p}^\infty}).$$

Now since the quadratic extension K_2/K is ramified at \mathfrak{q} (see Corollary 2.4(ii) and Lemma 2.2(i)), we have $G_{2,\Omega} \simeq \text{Gal}(K_2/K)$, whence $H^1(G_{2,\Omega}, E(K_{2,\Omega})_{\mathfrak{p}^\infty}) \simeq E_{\mathfrak{p}}$ (cf. the proof of Lemma 2.7). Finally, using the analogue of Lemma 2.5(ii) for $\bar{\mathfrak{p}}$, we have $\#E_{\mathfrak{p}} = \#E_{\bar{\mathfrak{p}}} = \#E(K_{\mathfrak{q}})_{\bar{\mathfrak{p}}^\infty}$, and the lemma follows. □

Lemma 2.10. *Suppose $\bar{\mathfrak{p}} \in B$. Then there is a unique prime of K_∞ lying above $\bar{\mathfrak{p}}$.*

Proof. We must show that the unique prime of K_2 lying above $\bar{\mathfrak{p}}$, say $\bar{\varrho}$, is inert in K_∞/K_2 . To this end let $m \leq \infty$ be such that $\bar{\varrho}$ splits completely in K_m/K_2 , so $E_{\mathfrak{p}^m} \subset E(K_{2,\bar{\varrho}})$. Since $\bar{\varrho} \nmid \mathfrak{p}$ the reduction-modulo- $\bar{\varrho}$ map sends $E_{\mathfrak{p}^m}$ injectively into $\tilde{E}(\mathcal{O}/\bar{\mathfrak{p}})$, where \tilde{E} denotes the reduction of E modulo $\bar{\varrho}$. As $\#\tilde{E}(\mathcal{O}/\bar{\mathfrak{p}}) \leq 5$, we conclude that $m = 2$, which proves the lemma. □

3. THE INFINITE DESCENT

In this section we prove formula (5) of the Introduction.

Keep the notation and assumptions of §2. In addition, assume that our elliptic curve E satisfies $L(E/K, 1) \neq 0$. In this case the finiteness of $E(K)$ and of the Tate-Shafarevich group of E over K have been demonstrated by Coates and Wiles [5] and Rubin [20], respectively.

Let $\text{III}_{\mathfrak{p}^\infty}$ and S denote, respectively, the \mathfrak{p} -power torsion in the Tate-Shafarevich group of E over K and the direct limit of the Selmer groups of E relative to powers of \mathfrak{p} . Thus

$$\text{III}_{\mathfrak{p}^\infty} = \ker \left[H^1(K, E)_{\mathfrak{p}^\infty} \rightarrow \bigoplus_{\mathfrak{q}} H^1(K_{\mathfrak{q}}, E)_{\mathfrak{p}^\infty} \right]$$

and

$$S = \ker \left[H^1(K, E_{\mathfrak{p}^\infty}) \rightarrow \bigoplus_{\mathfrak{q}} H^1(K_{\mathfrak{q}}, E)_{\mathfrak{p}^\infty} \right].$$

We note that the \mathfrak{q} -component $\lambda_{\mathfrak{q}} : H^1(K, E_{\mathfrak{p}^\infty}) \rightarrow H^1(K_{\mathfrak{q}}, E)_{\mathfrak{p}^\infty}$ of the map appearing in the definition of S is the restriction homomorphism to $H^1(K_{\mathfrak{q}}, E_{\mathfrak{p}^\infty})$ followed by the canonical map from this group to $H^1(K_{\mathfrak{q}}, E)_{\mathfrak{p}^\infty}$.

Lemma 3.1. *There is an isomorphism*

$$\text{III}_{\mathfrak{p}^\infty} \simeq S.$$

Proof. Galois cohomology gives us an exact sequence

$$0 \rightarrow E(K) \otimes_{\mathcal{O}} (K_{\mathfrak{p}}/\mathcal{O}_{\mathfrak{p}}) \rightarrow S \rightarrow \text{III}_{\mathfrak{p}^\infty} \rightarrow 0.$$

See §1 of [20]. Since $E(K)$ is finite, the group on the left is zero, which gives the lemma. \square

Recall the set B of primes of K where E has bad reduction. Let $B' = B \cup \{\mathfrak{p}\}$, and define a modified Selmer group $S(B') \supset S$ by

$$S(B') = \ker \left[H^1(K, E_{\mathfrak{p}^\infty}) \rightarrow \bigoplus_{\mathfrak{q} \notin B'} H^1(K_{\mathfrak{q}}, E)_{\mathfrak{p}^\infty} \right].$$

There is a natural exact sequence

$$(6) \quad 0 \rightarrow S \rightarrow S(B') \xrightarrow{\lambda_{B'}} \bigoplus_{\mathfrak{q} \in B'} H^1(K_{\mathfrak{q}}, E)_{\mathfrak{p}^\infty},$$

where $\lambda_{B'}$ is the restriction of $\bigoplus_{\mathfrak{q} \in B'} \lambda_{\mathfrak{q}}$ to $S(B')$. The image of $\lambda_{B'}$ has been described by Bashmakov [1] in terms of the local Tate pairing, and we now proceed to state his result.

For any field $F \supset K$ let $E^*(F) = \varprojlim E(F)/\bar{\mathfrak{p}}^n E(F)$, where the inverse limit is taken with respect to the natural maps. Note that $E^*(K) = E(K)_{\bar{\mathfrak{p}}^\infty}$ injects into $E^*(K_{\mathfrak{q}})$ for any prime \mathfrak{q} . Now for each $\mathfrak{q} \in B'$ write $\langle \cdot, \cdot \rangle_{\mathfrak{q}}$ for the non-degenerate pairing $E^*(K_{\mathfrak{q}}) \times H^1(K_{\mathfrak{q}}, E)_{\mathfrak{p}^\infty} \rightarrow \mathbb{Q}/\mathbb{Z}$ which is induced by the Tate pairing.

Theorem 3.2. (Bashmakov) *Suppose $\text{III}_{\bar{\mathfrak{p}}^\infty}$ is finite. Then a necessary and sufficient condition for an element $(\xi_{\mathfrak{q}}) \in \bigoplus_{\mathfrak{q} \in B'} H^1(K_{\mathfrak{q}}, E)_{\mathfrak{p}^\infty}$ to be in the image of*

$\lambda_{B'}$ is that

$$\sum_{\mathfrak{q} \in B'} \langle x, \xi_{\mathfrak{q}} \rangle_{\mathfrak{q}} = 0$$

for every $x \in E(K)_{\bar{\mathfrak{p}}^\infty}$. In particular,

$$\#\text{coker}(\lambda_{B'}) = \#E(K)_{\bar{\mathfrak{p}}^\infty}.$$

Proof. See §3.3 of [1]. □

Viewing $H^1(K_{\mathfrak{p}}, E)_{\mathfrak{p}^\infty}$ as a subgroup of $\bigoplus_{\mathfrak{q} \in B'} H^1(K_{\mathfrak{q}}, E)_{\mathfrak{p}^\infty}$ in a natural way, we have the following:

Corollary 3.3. $H^1(K_{\mathfrak{p}}, E)_{\mathfrak{p}^\infty} \not\subset \text{image}(\lambda_{B'})$.

Proof. This is immediate from Theorem 3.2 and the non-degeneracy of $\langle \cdot, \cdot \rangle_{\mathfrak{p}}$. □

Lemma 3.4. Suppose $\mathfrak{q} \in B' - \{\bar{\mathfrak{p}}\}$. Then

$$\#H^1(K_{\mathfrak{q}}, E)_{\mathfrak{p}^\infty} = \#E(K_{\mathfrak{q}})_{\bar{\mathfrak{p}}^\infty}.$$

Proof. The groups $H^1(K_{\mathfrak{q}}, E)_{\mathfrak{p}^\infty}$ and $E^*(K_{\mathfrak{q}})$ are dual to one another under $\langle \cdot, \cdot \rangle_{\mathfrak{q}}$. On the other hand, as $E(K_{\mathfrak{q}}) \simeq E(K_{\mathfrak{q}})_{\text{torsion}} \oplus \mathcal{O}_{\mathfrak{q}}$ (see [24] §VII.6.3) and $\mathfrak{q} \neq \bar{\mathfrak{p}}$, we have $E^*(K_{\mathfrak{q}}) = E(K_{\mathfrak{q}})_{\bar{\mathfrak{p}}^\infty}$, which proves the lemma. □

Remark. The proof of the above lemma shows that $H^1(K_{\bar{\mathfrak{p}}}, E)_{\mathfrak{p}^\infty}$ is infinite, so $S(B')$ is infinite if $\bar{\mathfrak{p}} \in B'$ (see (6) and Theorem 3.2).

Proposition 3.5. Suppose E has good reduction at $\bar{\mathfrak{p}}$, i.e. $\bar{\mathfrak{p}} \notin B'$. Then

$$\#S(B') = 2^{b^*} \cdot \#E(K_{\bar{\mathfrak{p}}})_{\bar{\mathfrak{p}}^\infty} \cdot (\#E(K)_{\bar{\mathfrak{p}}^\infty})^{-1} \cdot \#\text{III}_{\mathfrak{p}^\infty},$$

where $b^* = \#(B - \{\mathfrak{p}\})$.

Proof. This follows from (6), Lemma 3.1 and Theorem 3.2, using Lemmas 3.4 and 2.5(ii) (for $\bar{\mathfrak{p}}$) to compute the order of $\bigoplus_{\mathfrak{q} \in B'} H^1(K_{\mathfrak{q}}, E)_{\mathfrak{p}^\infty}$. □

The above result brings us closer to a proof of (5) for those curves which have a good reduction at $\bar{\mathfrak{p}}$. We have yet to relate $S(B')$ to $\text{Hom}(X_\infty, E_{\mathfrak{p}^\infty})^{\mathcal{G}}$, as well as deal with the curves that have a bad reduction at $\bar{\mathfrak{p}}$. To these ends, we now introduce Selmer groups over the field K_∞ .

For any set T of primes of K , we write \tilde{T} for the set of primes of K_∞ which lie above the primes in T , and define

$$S_\infty(T) = \ker \left[H^1(K_\infty, E_{\mathfrak{p}^\infty}) \rightarrow \bigoplus_{\Omega \notin \tilde{T}} H^1(K_\infty, \Omega, E)_{\mathfrak{p}^\infty} \right].$$

Now recall $B' = B \cup \{\mathfrak{p}\}$ and set $\mathcal{T} = B' \cap \{\mathfrak{p}, \bar{\mathfrak{p}}\}$. Thus $\mathcal{T} = \{\mathfrak{p}\}$ if E has good reduction at $\bar{\mathfrak{p}}$ and $\mathcal{T} = \{\mathfrak{p}, \bar{\mathfrak{p}}\}$ otherwise. In what follows we shall be concerned with the groups $S_\infty(\mathfrak{p})$ and $S_\infty(\mathcal{T})$. Clearly $S_\infty(\mathcal{T}) \supset S_\infty(\mathfrak{p})$ and $S_\infty(\mathcal{T}) = S_\infty(\mathfrak{p})$ if E has good reduction at $\bar{\mathfrak{p}}$. Recall $\mathcal{G} = \text{Gal}(K_\infty/K)$.

Lemma 3.6. Let X_∞ denote the Galois group of the maximal abelian 2-extension of K_∞ which is unramified outside of the primes above \mathfrak{p} . Then there is a canonical \mathcal{G} -isomorphism

$$S_\infty(\mathfrak{p}) \simeq \text{Hom}(X_\infty, E_{\mathfrak{p}^\infty}).$$

Proof. This is well-known. See for example [4], Theorem 12. □

Now consider the standard inflation-restriction exact sequence

$$(7) \quad 0 \rightarrow H^1(\mathcal{G}, E_{\mathfrak{p}^\infty}) \rightarrow H^1(K, E_{\mathfrak{p}^\infty}) \xrightarrow{\text{Res}} H^1(K_\infty, E_{\mathfrak{p}^\infty})^{\mathcal{G}}.$$

In contrast to the situation prevalent in the case $\mathfrak{p} \nmid 2$, the restriction map Res is neither injective (see Lemma 2.7(ii)) nor surjective. The following result is all we need, however.

Lemma 3.7. $S_\infty(\mathfrak{p})^{\mathcal{G}} \subset \text{image}(\text{Res})$.

Proof. Let $G_2 = \text{Gal}(K_2/K)$, and let $r : H^1(K, E_{\mathfrak{p}^\infty}) \rightarrow H^1(K_2, E_{\mathfrak{p}^\infty})^{G_2}$ and $\rho : H^1(K_2, E_{\mathfrak{p}^\infty})^{G_2} \rightarrow H^1(K_\infty, E_{\mathfrak{p}^\infty})^{\mathcal{G}}$ be the natural restriction maps. Then ρ is an isomorphism by Lemma 2.7(i), and Res is the composition of r and ρ . Thus to prove the lemma it suffices to check that $\rho^{-1}(S_\infty(\mathfrak{p})^{\mathcal{G}}) \subset \text{image}(r)$. Choose a prime $\mathfrak{q} \in B - \{\mathfrak{p}\}$, fix a prime of \bar{K} lying above \mathfrak{q} , and write $I_{\mathfrak{q}}$ for the corresponding inertia group. Now recall the automorphism $\tau \in \mathcal{G}$ which acts as multiplication by -1 on $E_{\mathfrak{p}^\infty}$. Since τ generates the inertia group of \mathfrak{q} in K_∞/K (see Lemma 2.5(i)), we can find an element $\bar{\tau} \in I_{\mathfrak{q}}$ whose restriction to K_∞ is τ . Then $\bar{\tau}^2 \in G_{K_\infty} \cap I_{\mathfrak{q}}$. Now using the fact that the elements of $S_\infty(\mathfrak{p})$ are unramified outside of \mathfrak{p} by Lemma 3.6, it is not difficult to see that every cohomology class $\{\xi\}$ in $\rho^{-1}(S_\infty(\mathfrak{p})^{\mathcal{G}})$ satisfies $\xi(\bar{\tau}^2) = 0$. It is now a simple matter to check that the map $c = c_\xi : G_K \rightarrow E_{\mathfrak{p}^\infty}$ given by $c(\sigma\bar{\tau}^i) = \xi(\sigma)$ ($\sigma \in G_{K_2}$, $i = 0, 1$) is a 1-cocycle whose cohomology class in $H^1(K, E_{\mathfrak{p}^\infty})$ is mapped to $\{\xi\}$ by r . \square

Define $S_\infty^*(\mathcal{T}) = S_\infty(\mathcal{T}) \cap \text{image}(\text{Res})$. The above lemma shows that $S_\infty^*(\mathcal{T}) \supset S_\infty(\mathfrak{p})^{\mathcal{G}}$ and $S_\infty^*(\mathcal{T}) = S_\infty(\mathfrak{p})^{\mathcal{G}}$ if E has good reduction at $\bar{\mathfrak{p}}$ (since $S_\infty(\mathcal{T}) = S_\infty(\mathfrak{p})$ for such curves). Now recall the group $S(B') \subset H^1(K, E_{\mathfrak{p}^\infty})$ defined at the beginning of this section.

Proposition 3.8. *The inflation-restriction exact sequence (7) induces an exact sequence*

$$0 \rightarrow H^1(\mathcal{G}, E_{\mathfrak{p}^\infty}) \rightarrow S(B') \xrightarrow{\text{Res}} S_\infty^*(\mathcal{T}) \rightarrow 0.$$

Proof. That the inflation homomorphism maps $H^1(\mathcal{G}, E_{\mathfrak{p}^\infty})$ into $S(B')$ follows easily from Lemma 2.8. Now for each prime \mathfrak{Q} of K_∞ we have a commutative diagram

$$\begin{array}{ccc} H^1(K, E_{\mathfrak{p}^\infty}) & \xrightarrow{\lambda_{\mathfrak{q}}} & H^1(K_{\mathfrak{q}}, E)_{\mathfrak{p}^\infty} \\ \text{Res} \downarrow & & \downarrow \text{Res}_{\mathfrak{Q}} \\ H^1(K_\infty, E_{\mathfrak{p}^\infty}) & \xrightarrow{\lambda_{\infty, \mathfrak{Q}}} & H^1(K_{\infty, \mathfrak{Q}}, E)_{\mathfrak{p}^\infty} \end{array}$$

where \mathfrak{q} is the prime of K lying below \mathfrak{Q} , $\lambda_{\mathfrak{q}}$ is the localization map defined before the statement of Lemma 3.1, $\lambda_{\infty, \mathfrak{Q}}$ is the analogue of $\lambda_{\mathfrak{q}}$ for the field K_∞ , and $\text{Res}_{\mathfrak{Q}}$ is the local restriction map.

If \mathfrak{Q} lies above a prime $\mathfrak{q} \in B' - \mathcal{T} = B - \{\mathfrak{p}, \bar{\mathfrak{p}}\}$, then Lemmas 2.9 and 3.4 show that $\text{Res}_{\mathfrak{Q}}$ is the zero map. If $\mathfrak{Q}|\mathfrak{q}$ with $\mathfrak{q} \notin B'$, then Lemma 2.8 shows that $\text{Res}_{\mathfrak{Q}}$ is injective. Now let $c \in S(B')$. Then, using the above diagram,

$$\lambda_{\infty, \mathfrak{Q}}(\text{Res}(c)) = \text{Res}_{\mathfrak{Q}}(\lambda_{\mathfrak{q}}(c)) = 0$$

if $\mathfrak{q} \notin B'$ (by definition of $S(B')$) or if $\mathfrak{q} \in B' - \mathcal{T}$ (since then $\text{Res}_{\mathfrak{Q}}$ is the zero map). Thus $\text{Res}(S(B')) \subset S_\infty^*(\mathcal{T})$. To prove the reverse inclusion, select an $f \in S_\infty^*(\mathcal{T})$

and find an element $c \in H^1(K, E_{\mathfrak{p}^\infty})$ such that $f = \text{Res}(c)$. Then, for $\Omega|\mathfrak{q}$ with $\mathfrak{q} \notin B'$,

$$\text{Res}_\Omega(\lambda_{\mathfrak{q}}(c)) = \lambda_{\infty, \Omega}(f) = 0,$$

whence $\lambda_{\mathfrak{q}}(c) = 0$ because Res_Ω is injective for such Ω . We conclude that $c \in S(B')$, which completes the proof. \square

We are now in a position to prove formula (5) of the Introduction.

Theorem 3.9. *Let $b^* = \#(B - \{\mathfrak{p}\})$. Then*

$$\#\text{Hom}(X_\infty, E_{\mathfrak{p}^\infty})^{\mathcal{G}} = 2^{b^*} \cdot \#E(K_{\bar{\mathfrak{p}}})_{\mathfrak{p}^\infty} \cdot (\#E(K)_{2^\infty})^{-1} \cdot \#\text{III}_{\mathfrak{p}^\infty}.$$

Proof. By Lemmas 3.1 and 3.6, the above formula is equivalent to

$$(8) \quad \#S_\infty(\mathfrak{p})^{\mathcal{G}} = 2^{b^*} \cdot \#E(K_{\bar{\mathfrak{p}}})_{\mathfrak{p}^\infty} \cdot (\#E(K)_{2^\infty})^{-1} \cdot \#S.$$

Case I. E has good reduction at $\bar{\mathfrak{p}}$.

In this case $S_\infty(\mathfrak{p})^{\mathcal{G}} = S_\infty^*(T)$, and Proposition 3.8 yields

$$\#S_\infty(\mathfrak{p})^{\mathcal{G}} = (\#H^1(\mathcal{G}, E_{\mathfrak{p}^\infty}))^{-1} \cdot \#S(B').$$

Formula (8) now follows from Proposition 3.5 and Lemma 2.7(ii).

Case II. E has bad reduction at $\bar{\mathfrak{p}}$.

The argument in this case is more involved, due to the fact that $S(B')$ is infinite (see the remark preceding the statement of Proposition 3.5). To circumvent this difficulty, consider the commutative diagram

$$\begin{array}{ccc} S(B') & \xrightarrow{\text{Res}} & S_\infty^*(T) \\ \lambda_{B'} \downarrow & & \downarrow \lambda_{\bar{\mathfrak{p}}} \\ \bigoplus_{\mathfrak{q} \in B'} H^1(K_{\mathfrak{q}}, E)_{\mathfrak{p}^\infty} & \xrightarrow{\varphi} & H^1(K_{\infty, \bar{\mathfrak{p}}}, E)_{\mathfrak{p}^\infty} \end{array}$$

where $\bar{\mathfrak{p}}$ is the unique prime of K_∞ lying above $\bar{\mathfrak{p}}$ (see Lemma 2.10), $\lambda_{\bar{\mathfrak{p}}}$ is the restriction to $S_\infty^*(T)$ of the natural map $H^1(K_\infty, E_{\mathfrak{p}^\infty}) \rightarrow H^1(K_{\infty, \bar{\mathfrak{p}}}, E)_{\mathfrak{p}^\infty}$, and φ is the composition of the projection map $\bigoplus_{\mathfrak{q} \in B'} H^1(K_{\mathfrak{q}}, E)_{\mathfrak{p}^\infty} \rightarrow H^1(K_{\bar{\mathfrak{p}}}, E)_{\mathfrak{p}^\infty}$ and the restriction map $H^1(K_{\bar{\mathfrak{p}}}, E)_{\mathfrak{p}^\infty} \rightarrow H^1(K_{\infty, \bar{\mathfrak{p}}}, E)_{\mathfrak{p}^\infty}$. By Proposition 3.8, the map Res in the above diagram is surjective with kernel $H^1(\mathcal{G}, E_{\mathfrak{p}^\infty})$, and the definitions together with Lemma 3.7 show that $\ker(\lambda_{B'}) = S$ and $\ker(\lambda_{\bar{\mathfrak{p}}}) = S_\infty(\mathfrak{p})^{\mathcal{G}}$. Applying the snake lemma to the above diagram then yields the formula

$$(9) \quad \#H^1(\mathcal{G}, E_{\mathfrak{p}^\infty}) \cdot \#\text{coker}(\lambda_{B'}) \cdot \#S_\infty(\mathfrak{p})^{\mathcal{G}} = \#\ker(\varphi) \cdot \#\text{image}(\tilde{\varphi}) \cdot \#S,$$

where $\tilde{\varphi} : \text{coker}(\lambda_{B'}) \rightarrow \text{coker}(\lambda_{\bar{\mathfrak{p}}})$ is the map induced by φ . Now Lemma 2.7(ii) and Theorem 3.2 show that $\#H^1(\mathcal{G}, E_{\mathfrak{p}^\infty}) \cdot \#\text{coker}(\lambda_{B'}) = \#E(K)_{2^\infty}$. On the other hand, the order of

$$\ker(\varphi) = H^1(\text{Gal}(K_{\infty, \bar{\mathfrak{p}}}/K_{\bar{\mathfrak{p}}}), E(K_{\infty, \bar{\mathfrak{p}}}))_{\mathfrak{p}^\infty} \oplus \left(\bigoplus_{\mathfrak{q} \in B^*} H^1(K_{\mathfrak{q}}, E)_{\mathfrak{p}^\infty} \right),$$

where $B^* = B' - \{\bar{\mathfrak{p}}\}$, may be computed as follows: the proof of Lemma 2.9 shows that $\#H^1(\text{Gal}(K_{\infty, \bar{\mathfrak{p}}}/K_{\bar{\mathfrak{p}}}), E(K_{\infty, \bar{\mathfrak{p}}}))_{\mathfrak{p}^\infty} = \#E_{\mathfrak{p}} = 2$, and Lemmas 3.4 and

2.5(ii) together show that $\#(\bigoplus_{\mathfrak{q} \in B^*} H^1(K_{\mathfrak{q}}, E)_{\mathfrak{p}^\infty}) = 2^{b^*-1} \cdot \#E(K_{\mathfrak{p}})_{\bar{\mathfrak{p}}^\infty}$, where $b^* = \#(B - \{\mathfrak{p}\})$. Thus

$$\#\ker(\varphi) = 2^{b^*} \cdot \#E(K_{\mathfrak{p}})_{\bar{\mathfrak{p}}^\infty}.$$

Finally, we claim that $\tilde{\varphi}$ is the zero map, so $\#\text{image}(\tilde{\varphi}) = 1$. To prove our claim, we use the above diagram to obtain the equivalent statement

$$\bigoplus_{\mathfrak{q} \in B'} H^1(K_{\mathfrak{q}}, E)_{\mathfrak{p}^\infty} = \ker(\varphi) + \text{image}(\lambda_{B'}).$$

This must hold since $\#\text{coker}(\lambda_{B'}) = 2$ by Theorem 3.2 and Lemma 2.1(ii), and $\ker(\varphi) \not\subset \text{image}(\lambda_{B'})$ by Corollary 3.3.

Using all of the above in (9) gives (8), thereby completing the proof of the theorem. □

4. THE MAIN CONJECTURE: STATEMENT AND BEGINNING OF THE PROOF

Keep the notation of §§2 and 3. Thus E is an elliptic curve defined over K with complex multiplication by the ring of integers \mathcal{O} of K , \mathfrak{p} is a prime of K lying above 2, and $K_n = K(E_{\mathfrak{p}^n})$ for $1 \leq n \leq \infty$. We continue to assume that E is non-exceptional.

We begin this section by defining the elliptic units of K_n that we will use in this paper.

Fix a minimal model of E over K , let L denote the corresponding period lattice, and write $I(6)$ for the set of ideals of \mathcal{O} which are prime to 6. For each $\mathfrak{a} \in I(6)$ define a function

$$\Theta_0(z; \mathfrak{a}) = \eta(\mathfrak{a}) \prod_u (\wp(z; L) - \wp(u; L))^{-1},$$

where $\wp(z; L)$ is the Weierstrass \wp -function for the lattice L , the product is taken over representatives of the non-zero classes u in $(\mathfrak{a}^{-1}L/L)/\pm 1$, and $\eta(\mathfrak{a}) \in K$ is the 12-th root of $\Delta(L)^{\mathbf{N}(\mathfrak{a})}/\Delta(\mathfrak{a}^{-1}L)$ constructed by Robert in [18], where Δ is the usual Ramanujan Δ -function. This function Θ_0 is the unique 12-th root of the function $\Theta(z; L, \mathfrak{a})$ used in §II.2 of [8] which satisfies a certain distribution relation. See [18] for more details.

Write \mathfrak{f} for the conductor of the Hecke character of K attached to E , and let \mathfrak{f}' be the least common multiple of the prime-to- \mathfrak{p} part of \mathfrak{f} and $\bar{\mathfrak{p}}^2$ (so \mathfrak{f}' is prime to \mathfrak{p} and $\mathfrak{f}' \not\propto 2 = \#\mathcal{O}^\times$). Now, for every $n \leq \infty$, let $F_n = K(E_{\mathfrak{f}'}K_n)$. Further, set $\mathfrak{f}_n = \mathfrak{f}'\mathfrak{p}^n$ and fix a point $v \in \mathbb{C}/L$ of order exactly \mathfrak{f}_n . Then $\Theta_0(v; \mathfrak{a}) \in K(\mathfrak{f}_n) \subset F_n$ if $(\mathfrak{a}, 6\mathfrak{f}) = 1$ ([18], no. 12), and we let $\mathcal{C}_{\mathfrak{f}_n}$ denote the group generated by all norms $\mathbf{N}_{F_n/K_n}(\Theta_0(v; \mathfrak{a}))$ ($(\mathfrak{a}, 6\mathfrak{f}) = 1$) and by all roots of unity in K_n . Then $\mathcal{C}_{\mathfrak{f}_n}$ is a $\mathbb{Z}[\text{Gal}(K_n/K)]$ -submodule of the global units of K_n ([8], §II.2.4) whose definition is independent of the choice of v .

We will also need the elliptic units of conductor \mathfrak{p}^n , whose definition we now recall. Choose a point $w \in \mathbb{C}/L$ of order exactly \mathfrak{p}^n . Then $\Theta_0(w; \mathfrak{a}) \in K(\mathfrak{p}^n)$ for all $\mathfrak{a} \in I(6)$, and we write $\mathcal{C}_{\mathfrak{p}^n}$ for the group generated by all products $\prod \Theta_0(w; \mathfrak{a})^{m(\mathfrak{a})}$ with $\sum m(\mathfrak{a})(\mathbf{N}(\mathfrak{a}) - 1) = 0$ ($\mathfrak{a} \in I(6)$) and by all roots of unity in $K(\mathfrak{p}^n)$. Then $\mathcal{C}_{\mathfrak{p}^n}$ is a Galois-stable subgroup of the global units of $K(\mathfrak{p}^n)$ whose definition is independent of the choice of w .

We are now ready to define the various Iwasawa modules that enter into the statement of the main conjecture.

Write U_n for the group of local units of $K_n \otimes_K K_{\mathfrak{p}}$ which are congruent to 1 modulo the primes above \mathfrak{p} . Let \bar{C}_{f_n} and $\bar{C}_{\mathfrak{p}^n}$ denote the closures of $C_{f_n} \cap U_n$ and $C_{\mathfrak{p}^n} \cap U_n$, respectively, in U_n . Define

$$U_\infty = \varprojlim U_n, \quad \bar{C}_{f'} = \varprojlim \bar{C}_{f_n}, \quad \bar{C}_1 = \varprojlim \bar{C}_{\mathfrak{p}^n} \quad \text{and} \quad \bar{C}_\infty = \bar{C}_{f'} \bar{C}_1,$$

all inverse limits being taken with respect to the norm maps. Global class field theory gives us a map

$$U_\infty / \bar{C}_\infty \rightarrow X_\infty,$$

where X_∞ denotes, as before, the Galois group of the maximal abelian 2-extension of K_∞ which is unramified outside of the primes above \mathfrak{p} .

Now recall $\Gamma = \text{Gal}(K_\infty/K_2) \simeq \mathbb{Z}_2$, and consider the standard Iwasawa algebra

$$\Lambda = \mathbb{Z}_2[[\Gamma]] = \varprojlim \mathbb{Z}_2[\text{Gal}(K_n/K_2)],$$

inverse limit over $n \geq 2$. Then X_∞ and U_∞/\bar{C}_∞ are finitely generated torsion Λ -modules. See [21].

It follows from the well-known classification theorem for Λ -modules that for every finitely generated torsion Λ -module Y we can find elements $f_i \in \Lambda$ and a finite Λ -module Z such that there is an exact sequence

$$0 \rightarrow \bigoplus \Lambda/f_i\Lambda \rightarrow Y \rightarrow Z \rightarrow 0.$$

We will write $\text{char}(Y)$ for the characteristic ideal $(\prod f_i)\Lambda$ of Y .

We can now state the “main conjecture” of Iwasawa theory for the extension K_∞/K (E non-exceptional).

Theorem 4.1. *We have*

$$\text{char}(X_\infty) = \text{char}(U_\infty/\bar{C}_\infty).$$

We will now show how the proof of Theorem 4.1 reduces to the verification of the equality of the Iwasawa invariants of U_∞/\bar{C}_∞ and X_∞ .

Write A_n for the 2-primary part of the ideal class group of K_n , let \mathcal{E}_n denote the group of global units of K_n , and write $\bar{\mathcal{E}}_n$ for the closure of $\mathcal{E}_n \cap U_n$ in U_n . Define

$$A_\infty = \varprojlim A_n \quad \text{and} \quad \bar{\mathcal{E}}_\infty = \varprojlim \bar{\mathcal{E}}_n,$$

inverse limits with respect to the norm maps. Global class field theory gives us an exact sequence

$$(10) \quad 0 \rightarrow \bar{\mathcal{E}}_\infty/\bar{C}_\infty \rightarrow U_\infty/\bar{C}_\infty \rightarrow X_\infty \rightarrow A_\infty \rightarrow 0.$$

Proposition 4.2. *There is an integer $r \geq 0$ such that*

$$\text{char}(A_\infty) \text{ divides } 2^r \text{char}(\bar{\mathcal{E}}_\infty/\bar{C}_\infty).$$

Proof. This result is similar to a theorem of Rubin ([21], Theorem 8.3) and may be proved using methods analogous to those of §§1, 2 and 8 of [21]. For the details see §3.8 of [11]. □

Corollary 4.3. *There is an integer $r \geq 0$ such that*

$$\text{char}(X_\infty) \text{ divides } 2^r \text{char}(U_\infty/\bar{C}_\infty).$$

Proof. This is immediate from (10) and Proposition 4.2. □

The above corollary shows that in order to prove Theorem 4.1 it is sufficient to verify that X_∞ and U_∞/\bar{C}_∞ have the same Iwasawa invariants. This verification is carried out below.

5. THE MAIN CONJECTURE: CONCLUSION OF THE PROOF

Recall that \mathfrak{f} denotes the conductor of the Hecke character of K attached to E , \mathfrak{f}' is the least common multiple of the prime-to- \mathfrak{p} part of \mathfrak{f} and \mathfrak{p}^2 , and $F_n = K(E_{\mathfrak{f}'})K_n$ for $n \leq \infty$. For each $n < \infty$, let \mathcal{U}_n be the group of local units of $F_n \otimes_K K_{\mathfrak{p}}$ which are congruent to 1 modulo the primes above \mathfrak{p} , and define

$$\mathcal{U}(\mathfrak{f}') = \varprojlim \mathcal{U}_n \quad \text{and} \quad V_{\infty} = \mathbf{N}_{F_{\infty}/K_{\infty}}(\mathcal{U}(\mathfrak{f}')) \subset U_{\infty},$$

inverse limit with respect to the norm maps. Local class field theory shows that U_{∞}/V_{∞} is finite, so $\text{char}(U_{\infty}/\bar{\mathcal{C}}_{\infty}) = \text{char}(V_{\infty}/\bar{\mathcal{C}}_{\infty} \cap V_{\infty})$.

In this section we will generalize arguments from §III.2 of [8] to prove that the Iwasawa invariants of X_{∞} and $V_{\infty}/(\bar{\mathcal{C}}_{\infty} \cap V_{\infty})$ are equal. As explained above, this will complete the proof of Theorem 4.1.

Recall that $\mathcal{G} = \text{Gal}(K_{\infty}/K)$, $\Gamma = \text{Gal}(K_{\infty}/K_2)$ and τ is the element of \mathcal{G} which acts as multiplication by -1 on $E_{\mathfrak{p}^{\infty}}$. Then $\mathcal{G} = \langle \tau \rangle \times \Gamma$ (see Corollary 2.4(ii)). We now define, for any ideal \mathfrak{g} of \mathcal{O} , $K(\mathfrak{g}\mathfrak{p}^{\infty}) = \bigcup_{n \geq 1} K(\mathfrak{g}\mathfrak{p}^n)$ and $\mathcal{G}(\mathfrak{g}) = \text{Gal}(K(\mathfrak{g}\mathfrak{p}^{\infty})/K)$. Using Corollary 2.4(i), we will often identify $\mathcal{G}(1)$ with Γ . Further, it is shown in §II.1.6 of [8] (for example) that $F_{\infty} = K(\mathfrak{f}'\mathfrak{p}^{\infty})$, so \mathcal{G} is a quotient of $\mathcal{G}(\mathfrak{f}')$.

Let \mathbf{D} be the ring of integers of the completion of the maximal unramified extension of \mathbb{Q}_2 , and let $\text{Res} : \mathbf{D}[[\mathcal{G}(\mathfrak{f}'))]] \rightarrow \mathbf{D}[[\mathcal{G}]]$ be the map induced by the restriction map $\mathcal{G}(\mathfrak{f}') \rightarrow \mathcal{G}$. Define $m(\mathfrak{f}') \in \mathbf{D}[[\Gamma]]$ by the equality

$$(1 - \tau)m(\mathfrak{f}') = (1 - \tau)\text{Res}(\nu(\mathfrak{f}')),$$

where $\nu(\mathfrak{f}') \in \mathbf{D}[[\mathcal{G}(\mathfrak{f}'))]]$ is the 2-adic integral measure constructed in §II.4.12 of [8]. Further, let $\nu(1)$ denote the “pseudo-measure” defined there, so that $(\gamma - 1)\nu(1) \in \mathbf{D}[[\mathcal{G}(1)]]$ for every $\gamma \in \mathcal{G}(1)$. Now fix a topological generator γ_0 of $\Gamma \simeq \mathbb{Z}_2$. Then identifying $\mathcal{G}(1)$ with Γ , we define $m(1) = (\gamma_0 - 1)\nu(1) \in \mathbf{D}[[\Gamma]]$.

Now recall that if $R = \mathbb{Z}_2$ or \mathbf{D} and $h \in R[[\Gamma]]$, the Iwasawa invariants $\mu(h)$ and $\lambda(h)$ of h are defined as follows: $\mu(h)$ is the largest non-negative integer such that $2^{\mu(h)}$ divides h , and $\lambda(h)$ is the degree of the “distinguished polynomial” part of h given by the Weierstrass preparation theorem. Recall also that $\Lambda = \mathbb{Z}_2[[\Gamma]]$. Let

$$\mathfrak{g} = m(\mathfrak{f}')m(1) \in \mathbf{D}[[\Gamma]].$$

Proposition 5.1. *Let $f \in \Lambda$ be any generator of $\text{char}(X_{\infty})$. Then the Iwasawa invariants of f and \mathfrak{g} are equal.*

Proof. This may be proved using straightforward adaptations of arguments from §§III.2.2–2.11 of [8]. See §§3.9 and 3.10 of [11] for the details. \square

Thus it remains to show that the Iwasawa invariants of \mathfrak{g} agree with those of $\text{char}(V_{\infty}/\bar{\mathcal{C}}_{\infty} \cap V_{\infty})$. In fact, we will show that \mathfrak{g} and $\text{char}(V_{\infty}/\bar{\mathcal{C}}_{\infty} \cap V_{\infty})$ generate the same ideal in $\mathbf{D}[[\Gamma]]$.

For each n with $2 \leq n \leq \infty$, let τ_n denote the restriction of $\tau = \tau_{\infty}$ to K_n , and write K_n^+ for the fixed field of $\langle \tau_n \rangle$ in K_n .

Proposition 5.2. *For all $n \geq 2$,*

$$K_n^+ = K(\mathfrak{p}^n).$$

Proof. The proposition holds for $n = \infty$ by Corollary 2.4(i), so $K_n^+ = K_n \cap K_{\infty}^+ \supset K(\mathfrak{p}^n)$ for every n . But $[K_n : K(\mathfrak{p}^n)] = [K_n : K_n^+] = 2$ if $n \geq 2$ by Lemma 2.2(ii), and the proposition follows. \square

Recall $V_\infty = \mathbf{N}_{F_\infty/K_\infty}(\mathcal{U}(f'))$. Let $i(f') : \mathcal{U}(f') \rightarrow \mathbf{D}[[\mathcal{G}(f')]]$ be the injective $\mathcal{G}(f')$ -homomorphism defined in §§II.4.6 and 4.7 of [8] (which is available to us since $f' \nmid \#\mathcal{O}^\times = 2$). Since $(1 + \tau)V_\infty = \mathbf{N}_{K_\infty/K_\infty^+}(V_\infty)$ and $K_\infty^+ = K(\mathfrak{p}^\infty)$ by the above proposition, we may define, as on p. 100 of [8], maps $i : V_\infty \rightarrow \mathbf{D}[[\mathcal{G}]]$ and $j : (1 + \tau)V_\infty \rightarrow \mathbf{D}[[\mathcal{G}(1)]]$ so that the following diagram commutes:

$$\begin{array}{ccccc} \mathcal{U}(f') & \xrightarrow{\mathbf{N}_{F_\infty/K_\infty}} & V_\infty & \xrightarrow{\mathbf{N}_{K_\infty/K_\infty^+}} & (1 + \tau)V_\infty \\ i(f') \downarrow & & \downarrow i & & \downarrow j \\ \mathbf{D}[[\mathcal{G}(f')]] & \xrightarrow{\text{Res}} & \mathbf{D}[[\mathcal{G}]] & \xrightarrow{\text{Res}^+} & \mathbf{D}[[\mathcal{G}(1)]] \end{array}$$

where Res is as defined above and Res⁺ is the obvious analogue of Res. Once again identifying $\mathcal{G}(1)$ with Γ , we may view j as a map from $(1 + \tau)V_\infty$ into $\mathbf{D}[[\Gamma]]$.

For any \mathbb{Z}_2 -module M , we will write $M \widehat{\otimes}_{\mathbb{Z}_2} \mathbf{D}$ for the completion of $M \otimes_{\mathbb{Z}_2} \mathbf{D}$. Now recall the Iwasawa modules $\bar{\mathcal{C}}_{f'}$ and $\bar{\mathcal{C}}_1$ defined in the preceding section.

Proposition 5.3. (i) *The map i induces an isomorphism of $\mathbf{D}[[\Gamma]]$ -modules*

$$\{(1 - \tau)V_\infty / (1 - \tau)\bar{\mathcal{C}}_{f'}\} \widehat{\otimes}_{\mathbb{Z}_2} \mathbf{D} \simeq \mathcal{A}/m(f')\mathcal{B}$$

where \mathcal{A} and \mathcal{B} are ideals of height 2 in $\mathbf{D}[[\Gamma]]$.

(ii) *The map j embeds $(1 + \tau)V_\infty \widehat{\otimes}_{\mathbb{Z}_2} \mathbf{D}$ in $\mathbf{D}[[\Gamma]]$ as an ideal of height 2. Under this embedding,*

$$\{(1 + \tau)V_\infty \cap \bar{\mathcal{C}}_1\} \widehat{\otimes}_{\mathbb{Z}_2} \mathbf{D} \subset m(1)\mathbf{D}[[\Gamma]].$$

Proof. Both parts of the proposition follow from direct analogues of Propositions III.1.3 and 1.4 of [8]. See [11], §3.10. □

Corollary 5.4. *We have*

$$\text{char}((1 - \tau)V_\infty / (1 - \tau)\bar{\mathcal{C}}_{f'}) \mathbf{D}[[\Gamma]] = m(f') \mathbf{D}[[\Gamma]].$$

Proof. This is immediate from part (i) of the above proposition. □

We will show next that $m(1)$ is a unit of $\mathbf{D}[[\Gamma]]$ and that the characteristic ideal appearing in the statement of the above corollary is equal to $\text{char}(V_\infty / \bar{\mathcal{C}}_\infty \cap V_\infty)$. These facts and the equality of the corollary will show that $\text{char}(V_\infty / \bar{\mathcal{C}}_\infty \cap V_\infty)$ and $\mathfrak{g} = m(f')m(1)$ generate the same ideal in $\mathbf{D}[[\Gamma]]$, thereby completing the proof of the main conjecture.

Recall that $\text{Gal}(K(\mathfrak{p}^\infty)/K) = \mathcal{G}(1) \simeq \Gamma \simeq \mathbb{Z}_2$.

Lemma 5.5. *The class number of K_n^+ is odd for all $n < \infty$.*

Proof. Since $K_\infty^+ = K(\mathfrak{p}^\infty)$ by Proposition 5.2, K_∞^+/K is a \mathbb{Z}_2 -extension in which only \mathfrak{p} ramifies, and this prime is totally ramified since K has class number 1. The lemma is thus a special case of a well-known result. See [26], Theorem 13.22. □

For every $n \leq \infty$ and any $\text{Gal}(K_n/K)$ -module Y , we will write Y^+ for the submodule of Y of all elements fixed by τ_n . Now recall the Iwasawa module of global units $\bar{\mathcal{E}}_\infty = \varprojlim \bar{\mathcal{E}}_n$.

Lemma 5.6. *We have*

$$\bar{\mathcal{C}}_1 = \bar{\mathcal{E}}_\infty^+.$$

Proof. Fix an n with $2 \leq n < \infty$. Noting that $K(\mathfrak{p}^n) = K_n^+$ is a cyclic extension of K , one can easily see that the group of elliptic units of $K(\mathfrak{p}^n)$ defined by Gillard in §6 of [10] agrees with our group $\mathcal{C}_{\mathfrak{p}^n}$. Then Théorème 5 of [10] gives

$$[\mathcal{E}_n^+ : \mathcal{C}_{\mathfrak{p}^n}] = h(K_n^+),$$

where $h(K_n^+)$ is the class number of K_n^+ . Since $h(K_n^+)$ is odd by Lemma 5.5 and the \mathfrak{p} -adic analogue of Leopoldt’s conjecture is true for K_n , we conclude that

$$\bar{\mathcal{C}}_{\mathfrak{p}^n} = \mathcal{C}_{\mathfrak{p}^n} \otimes \mathbb{Z}_2 = \mathcal{E}_n^+ \otimes \mathbb{Z}_2 = \bar{\mathcal{E}}_n^+.$$

Since $\bar{\mathcal{C}}_1 = \varprojlim \bar{\mathcal{C}}_{\mathfrak{p}^n}$, the lemma follows. □

Write $M(K_\infty^+)$ for the maximal abelian 2-extension of K_∞^+ which is unramified outside of the prime above \mathfrak{p} , and let $X(K_\infty^+) = \text{Gal}(M(K_\infty^+)/K_\infty^+)$.

Lemma 5.7. (i) $X(K_\infty^+) = 0$.
 (ii) $\mathcal{E}_\infty^+ = U_\infty^+$.

Proof. (ii) is immediate from (i) and the inclusion $U_\infty^+/\bar{\mathcal{E}}_\infty^+ \subset X(K_\infty^+)$ of global class field theory. Now set $\mathcal{G}^+ = \text{Gal}(K_\infty^+/K)$ and write $I(\mathcal{G}^+)$ for the augmentation ideal of $\mathbb{Z}_2[[\mathcal{G}^+]]$. Then

$$X(K_\infty^+)/I(\mathcal{G}^+)X(K_\infty^+) = \text{Gal}(M_1/K_\infty^+),$$

where M_1 is the maximal abelian extension of K in $M(K_\infty^+)$. But $K_\infty^+ = K(\mathfrak{p}^\infty)$ is the maximal abelian 2-extension of K which is unramified outside of $\{\mathfrak{p}\}$, so $M_1 = K_\infty^+$ and $X(K_\infty^+)/I(\mathcal{G}^+)X(K_\infty^+) = 0$. Now an application of Nakayama’s lemma gives (i). □

Proposition 5.8. (i) $m(1)$ is a unit of $\mathbf{D}[[\Gamma]]$.
 (ii) $\text{char}((1 - \tau)V_\infty/(1 - \tau)\bar{\mathcal{C}}_{\mathfrak{f}'}) = \text{char}(V_\infty/\bar{\mathcal{C}}_\infty \cap V_\infty)$.

Proof. Lemmas 5.6 and 5.7(ii) show that $\bar{\mathcal{C}}_1 = U_\infty^+$, so $(1 + \tau)V_\infty \subset \bar{\mathcal{C}}_1$. Then Proposition 5.3(ii) implies that $m(1)\mathbf{D}[[\Gamma]]$ contains an ideal of height 2, which gives (i). Now $U_\infty^+ \subset \bar{\mathcal{C}}_{\mathfrak{f}'}\bar{\mathcal{C}}_1 = \bar{\mathcal{C}}_\infty$, so the natural map

$$(V_\infty + \bar{\mathcal{C}}_\infty)/\bar{\mathcal{C}}_\infty \longrightarrow (1 - \tau)(V_\infty + \bar{\mathcal{C}}_\infty)/(1 - \tau)\bar{\mathcal{C}}_\infty$$

is an isomorphism. Noting that $(1 - \tau)\bar{\mathcal{C}}_\infty = (1 - \tau)\bar{\mathcal{C}}_{\mathfrak{f}'}$, (ii) follows easily. □

It is immediately clear from the above proposition and Corollary 5.4 that $\mathfrak{g} = m(\mathfrak{f}')m(1)$ and $\text{char}(V_\infty/\bar{\mathcal{C}}_\infty \cap V_\infty)$ generate the same ideal in $\mathbf{D}[[\Gamma]]$. This concludes the proof of Theorem 4.1.

6. THE BIRCH AND SWINNERTON-DYER CONJECTURE OVER K

In this section we will use the results of §§4 and 5 to relate $\#\text{Hom}(X_\infty, E_{\mathfrak{p}^\infty})^{\mathcal{G}}$ to $L(\psi, 1)/\Omega$ when E is non-exceptional. We will then combine this information with the main result of §3 (Theorem 3.9) to establish formula (3) of the Introduction for these curves.

Recall that ψ denotes the Hecke character of K attached to E and $\Omega \in \mathbb{C}^\times$ is a generator of the period lattice of a minimal model of E over K . Also recall that $X_\infty = \text{Gal}(M_\infty/K_\infty)$, where M_∞ is the maximal abelian 2-extension of K_∞ which is unramified outside of the primes above \mathfrak{p} .

Let $\kappa : \Gamma \rightarrow \mathbb{Z}_2^\times$ denote the character giving the action of Γ on $E_{\mathfrak{p}^\infty}$. If $a, b \in K^\times$, we will write $a \sim b$ to signify that a/b is a unit at \mathfrak{p} .

Proposition 6.1. *If $L(\bar{\psi}, 1) \neq 0$ then*

$$\#\text{Hom}(X_\infty, E_{\mathfrak{p}^\infty})^\Gamma \sim (1 - \psi(\mathfrak{p})/\mathbf{N}(\mathfrak{p})) L(\bar{\psi}, 1)/\Omega.$$

Proof. Arguing as in [21] (proof of Theorem 11.4), the main conjecture (Theorem 4.1 above) implies that for any generator $g \in \Lambda$ of $\text{char}(V_\infty/\bar{\mathcal{C}} \cap V_\infty)$,

$$\#\text{Hom}(X_\infty, E_{\mathfrak{p}^\infty})^\Gamma \sim \kappa(g).$$

Now Theorem II.4.12 of [8], Corollary 5.4 and Proposition 5.8(ii) above show that $\text{char}(V_\infty/\bar{\mathcal{C}} \cap V_\infty)$ has a generator g such that $\kappa(g) = (1 - \psi(\mathfrak{p})/\mathbf{N}(\mathfrak{p}))L(\bar{\psi}, 1)/\Omega$, which gives the proposition. \square

Recall that $\mathcal{G} = \langle \tau \rangle \times \Gamma$. Also recall that for any \mathcal{G} -module Y , Y^+ denotes the submodule of Y of elements fixed by τ . The next proposition shows that $\text{Hom}(X_\infty, E_{\mathfrak{p}^\infty})^\Gamma = \text{Hom}(X_\infty, E_{\mathfrak{p}^\infty})^\mathcal{G}$.

Proposition 6.2. *We have*

$$X_\infty^+ = 0.$$

Proof. We will prove that $X_\infty/(1 - \tau)X_\infty$ is finite. This will show that X_∞^+ is finite, hence zero because X_∞ has no non-zero finite submodules (see [12], Proposition 3 and the comments at the end of §4).

Since $X_\infty/(1 - \tau)X_\infty$ is the largest quotient of X_∞ on which τ acts trivially,

$$X_\infty/(1 - \tau)X_\infty = \text{Gal}(L/K_\infty),$$

where L is the maximal extension of K_∞ in M_∞ which is abelian over K_∞^+ . We will now show that $\text{Gal}(L/K_\infty^+)$ is finite. By Lemma 5.7(i) there is no non-trivial abelian 2-extension of K_∞^+ which is unramified outside of \mathfrak{p} , so

$$\text{Gal}(L/K_\infty^+) = \prod_{v \nmid \mathfrak{p}} I_v,$$

where the product extends over all primes v of K_∞^+ not lying above \mathfrak{p} and I_v is the inertia group of v in $\text{Gal}(L/K_\infty^+)$. Since L/K_∞ is unramified outside of \mathfrak{p} , I_v injects into the inertia group of v in $\text{Gal}(K_\infty/K_\infty^+)$ for each $v \nmid \mathfrak{p}$. This inertia group is clearly finite, and non-trivial only when v lies above one of the finitely many primes of K where E has bad reduction. Finally, class field theory shows that the primes of K other than \mathfrak{p} are finitely decomposed in $K_\infty^+ = K(\mathfrak{p}^\infty)$, and the proposition follows. \square

Now recall the set B of primes of K where E has bad reduction, and write $b = \#B$ and $b^* = \#(B - \{\mathfrak{p}\})$.

Lemma 6.3. *We have*

$$1 - \psi(\mathfrak{p})/\mathbf{N}(\mathfrak{p}) \sim 2^{b^* - b} \cdot \#E(K_{\mathfrak{p}})_{\bar{\mathfrak{p}}^\infty}.$$

Proof. When E has good reduction at \mathfrak{p} this follows from Lemma 1 of [4]. If E has bad reduction at \mathfrak{p} then $\psi(\mathfrak{p}) = 0$ and $\#E(K_{\mathfrak{p}})_{\bar{\mathfrak{p}}^\infty} = 2$ by Lemma 2.5(ii) for $\bar{\mathfrak{p}}$, so the assertion of the lemma is the trivial statement $1 \sim 1$. \square

We can now prove formula (3) of the Introduction for non-exceptional curves.

Theorem 6.4. *Suppose E is non-exceptional. Let b denote the number of primes of K where E has bad reduction, and let $\text{III}_{\mathfrak{p}^\infty}$ denote the \mathfrak{p} -power torsion in the Tate-Shafarevich group of E over K . Then, if $L(\bar{\psi}, 1) \neq 0$,*

$$L(\bar{\psi}, 1)/\Omega \sim 2^b \cdot (\#E(K)_{2^\infty})^{-1} \cdot \#\text{III}_{\mathfrak{p}^\infty}.$$

Proof. Propositions 6.1 and 6.2 together with Lemma 6.3 show that

$$2^{b^* - b} \cdot \#E(K_{\mathfrak{p}})_{\bar{\mathfrak{p}}^\infty} \cdot L(\bar{\psi}, 1)/\Omega \sim \#\text{Hom}(X_\infty, E_{\mathfrak{p}^\infty})^{\mathcal{G}}.$$

On the other hand, Theorem 3.9 gives

$$\#\text{Hom}(X_\infty, E_{\mathfrak{p}^\infty})^{\mathcal{G}} = 2^{b^*} \cdot \#E(K_{\mathfrak{p}})_{\bar{\mathfrak{p}}^\infty} \cdot (\#E(K)_{2^\infty})^{-1} \cdot \#\text{III}_{\mathfrak{p}^\infty},$$

which completes the proof. □

7. THE CONJECTURE OVER \mathbb{Q}

For any non-zero, square-free integer d , let E^d denote the elliptic curve with equation

$$y^2 = x^3 + 21dx^2 + 112d^2x,$$

which has discriminant $-2^{12}7^3d^6$. The family of curves E^d is exactly the class of elliptic curves over \mathbb{Q} with complex multiplication by the ring of integers \mathcal{O} of $K = \mathbb{Q}(\sqrt{-7})$ (see [15]). Suppose now that $L(E^d_{/\mathbb{Q}}, 1) \neq 0$. In this section we will show that Birch and Swinnerton-Dyer’s conjectural formula for $L(E^d_{/\mathbb{Q}}, 1)$ is valid, i.e. we will show that

$$(11) \quad L(E^d_{/\mathbb{Q}}, 1) = W_d \cdot (\#E^d(\mathbb{Q}))^{-2} \cdot \#\text{III}(E^d_{/\mathbb{Q}}) \cdot \prod c_p^{(d)}$$

where $c_p^{(d)} = [E^d(\mathbb{Q}_p) : E_0^d(\mathbb{Q}_p)]$ is the Tamagawa factor for the rational prime p , W_d is the fundamental real period of E^d , and

$$\text{III}(E^d_{/\mathbb{Q}}) = \ker \left[H^1(\mathbb{Q}, E^d) \rightarrow \bigoplus_v H^1(\mathbb{Q}_v, E^d) \right],$$

where the sum extends over all places v (including the archimedean one) of \mathbb{Q} .

First we note that if d is divisible by 7 the curves E^d and $E^{-d/7}$ are isogenous over \mathbb{Q} (see [13]). Thus E^d and $E^{-d/7}$ have the same L -function. Further, Cassels [3] has shown that the right-hand side of (11) is an isogeny invariant of E^d . From these facts it follows that we need only consider values of d which are prime to 7. We may further assume that d is positive, for if d is prime to 7, then the sign in the functional equation of $L(E^d_{/\mathbb{Q}}, s)$ is $d/|d|$ ([13] §19), so $L(E^d_{/\mathbb{Q}}, 1) = 0$ if $d < 0$.

So let d be positive and prime to 7, and let \mathcal{D}_d and L^d denote, respectively, the discriminant ideal and period lattice of a minimal model of E^d over \mathbb{Q} . Define

$$I_d = \int_0^\infty \frac{dx}{\sqrt{x^3 + 21dx^2 + 112d^2x}}.$$

Then $(2^{m_d})^{12} \mathcal{D}_d = (-2^{12}7^3d^6)$ with $m_d = 0$ or 1, the fundamental real period W_d equals $2^{m_d}I_d$, and (using the fact that +1 and -1 are the only roots of unity in $\mathbb{Q}(\sqrt{d})$ since $d > 0$)

$$(12) \quad L^d = \left(2^{m_d - m_1} / \sqrt{d} \right) L^1.$$

- Lemma 7.1.** (i) $W_d = W_{-7d}$.
 (ii) For any d , $E_{/K}^d$ has bad reduction at $\sqrt{-7}$.
 (iii) $L^d = W_d \cdot \mathcal{O}$.

Proof. E^{-7d} is the twist of E^d by the non-trivial character of $\text{Gal}(\mathbb{Q}(\sqrt{-7})/\mathbb{Q})$. Thus \mathcal{D}_d and \mathcal{D}_{-7d} differ by a power of (7) (see [7]). In particular $\text{ord}_2(\mathcal{D}_d) = \text{ord}_2(\mathcal{D}_{-7d})$, so $m_d = m_{-7d}$. On the other hand formula 241.00 of [2] shows that $I_{-1} = \sqrt{7} I_1$, so $I_d = I_1/\sqrt{d} = I_{-1}/\sqrt{7d} = I_{-7d}$. This proves (i). Assertion (ii) is clear since $\text{ord}_{\sqrt{-7}}(-2^{12}7^3d^6) \not\equiv 0 \pmod{12}$. Finally, (iii) is known to hold for $d = 1$ ([13], p. 82). This fact together with (12) gives (iii) for all $d > 0$. \square

Lemma 7.2. Let b denote the number of primes of K where $E_{/K}^d$ has bad reduction. Then, for all d ,

$$\prod c_p^{(d)} = 2^b.$$

In particular, E^d and E^{-7d} have the same Tamagawa product.

Proof. An application of Tate’s algorithm [25] to compute the $c_p^{(d)}$ terms yields the following: if $E_{/Q}^d$ has bad reduction at p , then $c_p^{(d)} = 2^{n(p)}$, where $n(p)$ is the number of primes of K lying above p . The first assertion of the lemma now follows easily, using the semi-stable reduction theorem ([24], Proposition VII.5.4) and Lemma 7.1(ii). As to the second, simply note that the K -isomorphic curves E^d and E^{-7d} have the same b . \square

Let ψ_d denote the Hecke character of K attached to E^d .

- Lemma 7.3.** (i) $L(E_{/Q}^d, s) = L(\psi_d, s) = L(\bar{\psi}_d, s)$.
 (ii) For any d , $E^d(\mathbb{Q}) \simeq \mathbb{Z}/2\mathbb{Z}$ and $E^d(K) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
 (iii) $L(\psi_d, 1) > 0$.

Proof. Statement (i) is due to Deuring [9], and (ii) is proved in §14 of [13]. As regards to (iii), Theorem 2 of [16] shows that $L(\psi_d, 1) \geq 0$, whence (iii) follows since $L(\psi_d, 1) \neq 0$ by hypothesis. \square

Proposition 7.4. We have

$$L(E_{/Q}^d, 1) = W_d \cdot (\#E^d(\mathbb{Q}))^{-2} \cdot \sqrt{\#\text{III}(E_{/K}^d)} \cdot \prod c_p^{(d)}.$$

Proof. If B denotes the set of primes of K where $E_{/K}^d$ has bad reduction, then $B \neq \{\mathfrak{p}\}$ by Lemma 7.1(ii). Thus $E_{/K}^d$ is a non-exceptional curve, so formula (3) of the Introduction is valid for the prime \mathfrak{p} (see Theorem 6.4 above). Similarly $B \neq \{\bar{\mathfrak{p}}\}$, so (3) is valid at the prime $\bar{\mathfrak{p}}$ as well (see the comments at the beginning of §2). As explained in the Introduction, this gives Gross’ refined Birch and Swinnerton-Dyer formula (4). Now since d is prime to 7, a minimal model of E^d over \mathbb{Q} is also minimal over K , so by Lemma 7.1(iii) we can take $\Omega = W_d$ in (4). Finally, using Lemmas 7.2 and 7.3 in (4) yields the formula of the proposition. \square

To complete the proof of (11) we will show that for every rational prime p

$$(13) \quad \#\text{III}(E_{/K}^d)_{p^\infty} = (\#\text{III}(E_{/Q}^d)_{p^\infty})^2.$$

For convenience, we will write $G_{K/\mathbb{Q}}$ for $\text{Gal}(K/\mathbb{Q})$.

Proposition 7.5. *For all d , the restriction map $H^1(\mathbb{Q}, E^d) \rightarrow H^1(K, E^d)^{G_{K/\mathbb{Q}}}$ is an isomorphism.*

Proof. A trite calculation based on Lemma 7.3(ii) shows that for all $i \geq 1$,

$$H^i(G_{K/\mathbb{Q}}, E^d(K)) = 0.$$

The lemma now follows easily. □

Proposition 7.6. *For all d , the restriction map $H^1(\mathbb{Q}, E^d) \xrightarrow{\sim} H^1(K, E^d)^{G_{K/\mathbb{Q}}}$ induces an isomorphism*

$$\text{III}(E_{\mathbb{Q}}^d) \simeq \text{III}(E_{/K}^d)^{G_{K/\mathbb{Q}}}.$$

Proof. It suffices to check that if v is a place of \mathbb{Q} and w is a place of K lying above v , then the local restriction map $H^1(\mathbb{Q}_v, E^d) \rightarrow H^1(K_w, E^d)$ is injective. The kernel of this map is $H^1(\text{Gal}(K_w/\mathbb{Q}_v), E^d(K_w))$, which is clearly zero if $K_w = \mathbb{Q}_v$ or if v is a prime of good reduction for $E_{/\mathbb{Q}}^d$ (since then $v \neq 7$, so K_w/\mathbb{Q}_v is unramified; cf. Lemma 2.8). Suppose now that v is a (finite) prime of bad reduction for $E_{/\mathbb{Q}}^d$ which does not split in K/\mathbb{Q} (hence $v \neq 2$). We will identify $\text{Gal}(K_w/\mathbb{Q}_v)$ with $G_{K/\mathbb{Q}}$.

There is an isomorphism $E^d(K_w) \simeq E^d(K_w)_{2^\infty} \oplus A$, where A is a uniquely 2-divisible $G_{K/\mathbb{Q}}$ -module. Further, since $E_{/K}^d$ has bad reduction at w (see Lemma 7.1(ii) and Proposition VII.5.4(a) of [24]), Lemma 2.5(ii) shows that $E^d(K_w)_{2^\infty} = E_2^d$. It follows that there is an isomorphism

$$H^1(G_{K/\mathbb{Q}}, E^d(K_w)) \simeq H^1(G_{K/\mathbb{Q}}, E_2^d).$$

As $H^i(G_{K/\mathbb{Q}}, E_2^d) = 0$ for all $i \geq 1$ by Lemma 7.3(ii), the proof for finite v is complete. If v is the infinite place, the last-mentioned fact shows that the multiplication-by-2 map $H^1(G_{K/\mathbb{Q}}, E^d(\mathbb{C})) \rightarrow H^1(G_{K/\mathbb{Q}}, E^d(\mathbb{C}))$ is an isomorphism. But $H^1(G_{K/\mathbb{Q}}, E^d(\mathbb{C}))$ is annihilated by 2, so $H^1(G_{K/\mathbb{Q}}, E^d(\mathbb{C})) = 0$. □

We can now prove formula (13). Let c denote the non-trivial element of $G_{K/\mathbb{Q}}$.

Case I. $p = 2$.

Recall that 2 splits in K as $\mathfrak{p}\bar{\mathfrak{p}}$ with $\bar{\mathfrak{p}} \neq \mathfrak{p}$ ($\bar{\mathfrak{p}} = \mathfrak{p}^c$). We have $\#\text{III}(E_{/K}^d)_{2^\infty} = (\#\text{III}(E_{/K}^d)_{\bar{\mathfrak{p}}^\infty})^2$, and multiplication by $(1 + c)$ on $\text{III}(E_{/K}^d)_{\bar{\mathfrak{p}}^\infty}$ induces an isomorphism

$$\text{III}(E_{/K}^d)_{\bar{\mathfrak{p}}^\infty} \simeq \text{III}(E_{/K}^d)_{2^\infty}^{G_{K/\mathbb{Q}}}.$$

Since $\text{III}(E_{/K}^d)_{2^\infty}^{G_{K/\mathbb{Q}}} \simeq \text{III}(E_{/\mathbb{Q}}^d)_{2^\infty}$ by Proposition 7.6, formula (13) for $p = 2$ follows.

Case II. $p \neq 2$.

Writing $\text{III}(E_{/K}^d)_{\bar{p}^\infty}$ for $(1 - c)\text{III}(E_{/K}^d)_{p^\infty}$, we have the decomposition

$$\text{III}(E_{/K}^d)_{p^\infty} = \text{III}(E_{/K}^d)_{p^\infty}^{G_{K/\mathbb{Q}}} \oplus \text{III}(E_{/K}^d)_{\bar{p}^\infty}.$$

Now the “minus component” $\text{III}(E_{/K}^d)_{\bar{p}^\infty}$ may be identified with $\text{III}(E_{/K}^{-7d})_{p^\infty}^{G_{K/\mathbb{Q}}}$ (for if χ denotes the non-trivial character of $G_{K/\mathbb{Q}}$, then E^{-7d} is the twist of E^d by χ and there is a K -isomorphism $\varphi : E^{-7d} \rightarrow E^d$ such that $\varphi^\sigma = \chi(\sigma)\varphi$ for all $\sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$). Therefore, using Proposition 7.6,

$$\#\text{III}(E_{/K}^d)_{p^\infty} = \#\text{III}(E_{/\mathbb{Q}}^d)_{p^\infty} \cdot \#\text{III}(E_{/\mathbb{Q}}^{-7d})_{p^\infty}.$$

Finally, the \mathbb{Q} -isogenous curves E^d and E^{-7d} have the same real period (Lemma 7.1(i)), the same Tamagawa product (Lemma 7.2) and the same number of \mathbb{Q} -rational points (Lemma 7.3(ii)). Thus by the result of Cassels referred to above

$$\#\text{III}(E_{/\mathbb{Q}}^{-7d})_{p^\infty} = \#\text{III}(E_{/\mathbb{Q}}^d)_{p^\infty}$$

which completes the proof of formula (13), and of conjecture (11).

8. THE EXCEPTIONAL CURVES

Recall that the curve E is called exceptional if E has bad reduction at \mathfrak{p} and good reduction at all other primes. In this section we will check formula (3) of the Introduction for these curves.

As before, write $K_\infty = K(E_{p^\infty})$ and $\mathcal{G} = \text{Gal}(K_\infty/K)$. Keep the rest of the notation from §§2–6 as well.

Lemma 8.1. (i) $K_\infty = K(\mathfrak{p}^\infty)$.
 (ii) $H^i(\mathcal{G}, E_{p^\infty}) = 0$ for all $i \geq 1$.

Proof. Since E has good reduction away from \mathfrak{p} , the extension K_∞/K is unramified outside of \mathfrak{p} . This gives (i). Statement (ii) follows from (i), noting that $\mathcal{G} = \text{Gal}(K(\mathfrak{p}^\infty)/K) \simeq \mathbb{Z}_2$. □

Proposition 8.2. *We have $\text{III}_{p^\infty} = 0$.*

Proof (notation as in §3). Part (ii) of the above lemma shows that the restriction homomorphism $H^1(K, E_{p^\infty}) \rightarrow H^1(K_\infty, E_{p^\infty})$ maps S (which is isomorphic to III_{p^∞}) injectively into $S_\infty(\mathfrak{p})$ (cf. the proof of Proposition 3.8). Now Lemmas 3.6, 8.1(i) and a result analogous to Lemma 5.7(i) (with K_∞^+ replaced by $K(\mathfrak{p}^\infty)$) show that $S_\infty(\mathfrak{p}) = 0$, which completes the proof. □

Recall that if $a, b \in K^\times$ then $a \sim b$ means that a/b is a unit at \mathfrak{p} .

Proposition 8.3. *If $L(\bar{\psi}, 1) \neq 0$, then*

$$L(\bar{\psi}, 1)/\Omega \sim (\#E(K)_{p^\infty})^{-1}.$$

Proof (notation as in §5). Setting $\mathfrak{f}' = \bar{\mathfrak{p}}^2$ in the discussion that precedes the statement of Proposition 5.3, we obtain an injective map $i = j : V_\infty \rightarrow \mathbf{D}[[\mathcal{G}]]$, where $V_\infty = \mathbf{N}_{K(\bar{\mathfrak{p}}^2 p^\infty)/K(\mathfrak{p}^\infty)}(\mathcal{U}(\bar{\mathfrak{p}}^2))$. Then results analogous to Proposition 5.3(ii) (with $(1 + \tau)V_\infty$ and Γ replaced by V_∞ and \mathcal{G} , respectively) and Lemmas 5.5 to 5.7 (with K_n^+ replaced by $K(\mathfrak{p}^n)$ for every n) hold true. It follows that $m(1)$ is a unit of $\mathbf{D}[[\mathcal{G}]]$, whence Theorem II.4.12 of [8] gives

$$(\kappa(\gamma_0) - 1)L(\bar{\psi}, 1)/\Omega \sim 1,$$

where γ_0 is a topological generator of \mathcal{G} and κ is the character giving the action of \mathcal{G} on E_{p^∞} . As $\kappa(\gamma_0) - 1 \sim \#E(K)_{p^\infty}$ by definition of κ , the proof is complete. □

We can now verify formula (3) for the exceptional curves. Using Propositions 8.2 and 8.3 and the analogue of Lemma 2.1(ii) for $\bar{\mathfrak{p}}$, we have

$$L(\bar{\psi}, 1)/\Omega \sim (\#E(K)_{p^\infty})^{-1} = 2 \cdot (\#E(K)_{2^\infty})^{-1} = 2^b \cdot (\#E(K)_{2^\infty})^{-1} \# \text{III}_{p^\infty},$$

as desired.

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