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HAMILTONIAN TORUS ACTIONS ON SYMPLECTIC ORBIFOLDS AND TORIC VARIETIES

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Abstract. In the first part of the paper, we build a foundation for further work on Hamiltonian actions on symplectic orbifolds. Most importantly, we prove the orbifold versions of the abelian connectedness and convexity theorems.

In the second half, we prove that compact symplectic orbifolds with completely integrable torus actions are classified by convex simple rational polytopes with a positive integer attached to each open facet and that all such orbifolds are algebraic toric varieties.

1. Introduction

In this paper we study Hamiltonian torus actions on symplectic orbifolds, with an emphasis on completely integrable actions.

The category of Hamiltonian group actions on symplectic manifolds is not closed under symplectic reduction; generically, the reduced space is an orbifold. In contrast, the symplectic reduction of an orbifold is generically an orbifold. Symplectic reduction is a powerful technique which has been used successfully in such diverse areas as Hamiltonian systems and representation theory.

Therefore we need to understand symplectic orbifolds even if we only want to understand symplectic manifolds. For example, a proof of the Guillemin-Sternberg conjecture that quantization commutes with reduction naturally encounters orbifolds [DGMW], [M]. In the same spirit, the proof of the non-abelian convexity theorem for manifolds can be reduced to an abelian convexity theorem for orbifolds [LMTW]. Orbifolds also arise in the study of resonances in Hamiltonian systems.

In the first half of the paper, we build a foundation for further work on Hamiltonian actions on symplectic orbifolds. For example, we classify the neighborhoods of isotropic orbits, and we discuss the extension of Bott-Morse theory to orbifolds. Most importantly, we prove the following analogues of the abelian connectedness and convexity theorems (see [A] and [GS1]).

Theorem. *Let a torus T act on a compact connected symplectic orbifold (M, ω), with a moment map $\phi : M \to \mathfrak{t}^*$. For every $a \in \mathfrak{t}^*$, the fiber $\phi^{-1}(a)$ is connected.*

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Theorem. *Let a torus T act on a compact connected symplectic orbifold (M, ω), with a moment map $\phi : M \to \mathfrak{t}^*$. The image of the moment map $\phi(M) \subset \mathfrak{t}^*$ is a rational convex polytope. In particular it is the convex hull of the image of the points in M fixed by T,*

$$\phi(M) = \textit{convex hull } (\phi(M^T)).$$

In the second half of the paper, we consider the special case of completely integrable torus actions, and prove the following theorem:

Theorem. *Compact symplectic toric orbifolds are classified by convex rational simple polytopes with a positive integer attached to each open facet.*

This theorem generalizes a theorem of Delzant [D] to the case of orbifolds. He proved that symplectic toric *manifolds* are classified by the image of their moment maps, that is, by a certain class of rational polytopes. It is easy to see that additional information is necessary for orbifolds:

Example 1.1. Given positive integers n and m, there exists a symplectic orbifold M which is topologically a two sphere, but which looks locally like $\mathbb{C}/(\mathbb{Z}/n\mathbb{Z})$ and $\mathbb{C}/(\mathbb{Z}/m\mathbb{Z})$ near its north and south poles, respectively. The circle action which rotates M around its north-south axis is Hamiltonian. Although the image of the moment map is a line interval for all m and n, these orbifolds are not isomorphic.

To state the above theorem precisely. we must define a few terms.

Definition 1.2. A *symplectic toric orbifold* is a quadruple (M, ω, T, ϕ), where ω is a symplectic form on a connected orbifold M and $\phi : M \to \mathfrak{t}^*$ is a moment map for an effective Hamiltonian action of a torus T on M such that $\dim T = \frac{1}{2} \dim M$.[1] Two symplectic toric orbifolds are *isomorphic* if they are equivariantly symplectomorphic (implicitly the torus is fixed in the definition of isomorphism).

Definition 1.3. Let x be a point in an orbifold M, and let $(\tilde{U}, \Gamma, \varphi)$ be a uniformizing chart for a neighborhood U of x (see [S] or section 2). Then the *(orbifold) structure* group of x is the isotropy group of $\tilde{x} \in \tilde{U}$, where $\varphi(\tilde{x}) = x$. This group is well defined as an abstract group.

Definition 1.4. Let $\mathfrak{t}$ be a vector space with a lattice ℓ; let $\mathfrak{t}^*$ denote the dual vector space. A convex polytope $\Delta \subset \mathfrak{t}^*$ is *rational* if

$$\Delta = \bigcap_{i=1}^{N} \{\alpha \in \mathfrak{t}^* \mid \langle \alpha, y_i \rangle \geq \eta_i\}$$

for some $y_i \in \ell$ and $\eta_i \in \mathbb{R}$. A *(closed) facet* is a face of Δ of codimension one in Δ. An *open facet* is the relative interior of a facet. An n dimensional polytope is *simple* if exactly n facets meet at every vertex. For this paper, we shall adopt the convenient but non-standard abbreviation that a *labeled polytope* in $\mathfrak{t}^*$ is a convex rational simple polytope Δ such that $\dim \Delta = \dim \mathfrak{t}$, plus a positive integer attached to each open facet. Two labeled polytopes are *isomorphic* if one can be mapped to the other by a translation and the corresponding open facets have the same integer labels.

[1] This is not the definition used in algebraic geometry. However we will show in Section 9 that every compact symplectic toric orbifold can be given the structure of an algebraic toric variety.

Theorem 1.5. *1. To every compact symplectic toric orbifold (M, ω, T, ϕ) there naturally corresponds a labeled polytope: The image of the moment map $\phi(M)$ is a rational simple polytope. For every open facet $\mathring{F}$ of $\phi(M)$ there exists a positive integer $n_{\mathring{F}}$ such that the structure group of every $x \in \phi^{-1}(\mathring{F})$ is $\mathbb{Z}/n_{\mathring{F}}\mathbb{Z}$.*

2. Two compact symplectic toric orbifolds are isomorphic if and only if their associated labeled polytopes are isomorphic.

3. Every labeled polytope arises from some compact symplectic toric orbifold.

This theorem is proved in three parts, each of which corresponds to one of the claims above: Theorem 6.4, Theorem 7.4, and Theorem 8.1.

Finally, in Section 9 we consider Kähler structures on symplectic toric orbifolds and show that every symplectic toric orbifold is an algebraic toric variety.

Definition 1.6. Let $\Delta \subset \mathfrak{t}^*$ be a convex polytope such that $\dim(\Delta) = \dim(\mathfrak{t})$. Given a face F of Δ, the *dual cone* to F is the set

$$C_F = \{\alpha \in \mathfrak{t} \mid \langle \alpha, \beta - \beta' \rangle \leq 0 \text{ for all } \beta \in F \text{ and } \beta' \in \Delta\}.$$

The *fan* of Δ is the set of cones dual to the faces of Δ.

Theorem 1.7. *1. Every compact symplectic toric orbifold admits an invariant complex structure which is compatible with its symplectic form.*

2. Two symplectic toric orbifolds with compatible complex structures are equivariantly biholomorphic exactly if their image polytopes have the same fans.

3. Every symplectic orbifold has the structure of an algebraic toric variety with the fan dual to the image polytope.

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PART 1. HAMILTONIAN TORUS ACTIONS ON SYMPLECTIC ORBIFOLDS

2. Group actions on orbifolds

In this section, we recall definitions related to actions of groups on orbifolds and describe some properties of actions of compact groups. The main result is the slice theorem. The presentation is largely self-contained and borrows heavily from a paper of Haefliger and Salem [HS].

We begin by defining orbifolds and related differential geometric notions. For more details, see Satake [S].

An *orbifold* M is a Hausdorff topological space $|M|$, plus an *atlas* $\mathcal{U}$ of *uniformizing charts* $(\tilde{U}, \Gamma, \varphi)$, where $\tilde{U}$ is an open subset of $\mathbb{R}^n$, Γ is a finite group which acts linearly on $\tilde{U}$ so that the set of points where the action is not free has codimension at least two, and $\varphi : \tilde{U} \to |M|$ induces a homeomorphism from $\tilde{U}/\Gamma$ to $U \subset |M|$.

Just as for manifolds, these sets U must cover $|M|$ and are subject to compatibility conditions:

1. For every $x \in |M|$ there exists $(\tilde{U}, \Gamma, \varphi) \in \mathcal{U}$ such that $x \in \varphi(\tilde{U})$.
2. For every $(\tilde{U}, \Gamma, \varphi)$ and $(\tilde{U}', \Gamma', \varphi')$ in $\mathcal{U}$ and every $x \in \varphi(\tilde{U})$ there exists $(\tilde{U}'', \Gamma'', \varphi'') \in \mathcal{U}$ such that $x \in \varphi''(\tilde{U}'') \subset \varphi(\tilde{U}) \cap \varphi'(\tilde{U}')$.
3. For every $(\tilde{U}, \Gamma, \varphi)$ and $(\tilde{U}', \Gamma', \varphi')$ in $\mathcal{U}$ such that $\varphi(\tilde{U}) \subset \varphi'(\tilde{U}')$ there exists an *injection* from $(\tilde{U}, \Gamma, \varphi)$ to $(\tilde{U}', \Gamma', \varphi')$—that is, a smooth open embedding $\lambda : \tilde{U} \to \tilde{U}'$ such that $\phi' \circ \lambda = \phi$.

The assumption that the set of points in a chart $\tilde{U}$ where the action is not free has codimension at least 2 in $\tilde{U}$ implies that the injections λ are *equivariant* (see [S]).

Additionally, there is a notion of when two atlases of charts are equivalent. First, every atlas is *directly equivalent* to any subatlas. More generally, we take the smallest equivalence relationship so that every pair of directly equivalent atlases are equivalent.

Let $\mathcal{U}$ and $\mathcal{V}$ be orbifold atlases on spaces $|M|$ and $|N|$, respectively. A map $f : M \to N$ is the following collection of objects: for each $\tilde{U} \in \mathcal{U}$, an element $f(\tilde{U}) \in \mathcal{V}$ and a smooth map $f_{\tilde{U}} : \tilde{U} \to \tilde{V}$, and for every injection $\lambda : \tilde{U} \to \tilde{U}'$ a map $f(\lambda) : f(\tilde{U}) \to f(\tilde{U}')$ such that $f(\lambda) \circ f_{\tilde{U}} = f_{\tilde{U}'} \circ \lambda$.

A vector field ξ on M is a Γ-invariant vector field $\tilde{\xi}$ on each uniformizing chart $(\tilde{U}, \Gamma, \phi)$; of course, these must agree on overlaps. More precisely, if λ is an injection from $(\tilde{U}, \Gamma, \varphi)$ to $(\tilde{U}', \Gamma', \varphi')$, then $\lambda_*(\tilde{\xi}') = \tilde{\xi}$. Similar definitions apply to differential forms, metrics, etc.

Let x be a point in an orbifold M, and let $(\tilde{U}, \Gamma, \varphi)$ be a uniformizing chart with $x \in \tilde{U}/\Gamma$. The *(orbifold) structure group* of x is the isotropy group Γ_x of $\tilde{x} \in \tilde{U}$, where $\varphi(\tilde{x}) = x$. The group Γ_x is well defined as an abstract group. The tangent space to $\tilde{x}$ in $\tilde{U}$, considered as a representation of Γ_x, is called the *uniformized tangent space at* x, and denoted by $\tilde{T}_x M$. The quotient $\tilde{T}_x M/\Gamma_x$ is $T_x M$, the fiber of the tangent bundle of M at x.

Remark 2.1. (*Orbifolds versus manifolds*) Throughout this paper, the reader will find many proofs which simply claim that the the proof for orbifolds is strictly "analogous" to the proof for manifolds. By this, we mean that because the usual proof (or the particular proof cited) is functorial, it will also work for orbifolds. Because each local uniformizing chart is itself a manifold, we can apply the manifold proof to construct the desired object on it. If the construction is natural, these local objects will form a global object on the orbifold. Sometimes the construction depends on an additional structure, but is natural once that structure is chosen. In this case, we choose that structure on the orbifold, and then apply the above reasoning.

Let us illustrate this general philosophy in the case of the tubular neighborhood theorem, which states that there exists a neighborhood of the zero section of the normal bundle of a suborbifold $X \subset M$ which is diffeomorphic to a neighborhood of X in M. First, we choose a metric on M. Let's examine the naturality condition explicitly in this case. Let $\tilde{U}$ and $\tilde{U}'$ be manifolds with metrics and submanifolds $\tilde{X}$ and $\tilde{X}'$ respectively. Let $N(\tilde{X})$ and $N(\tilde{X}')$ denote the normal bundles of $\tilde{X}$ and $\tilde{X}'$. Let $\lambda : \tilde{U} \to \tilde{U}'$ be an open isometric embedding such that $\lambda(\tilde{X}) = \lambda(\tilde{U}) \cap \tilde{X}'$. The embedding λ induces a map $\lambda_* : N(\tilde{X}) \mapsto N(\tilde{X}')$. Let $\psi : N(\tilde{X}) \to \tilde{U}$ and

$\psi' : N(\tilde{X}') \to \tilde{U}'$ be the diffeomorphisms constructed in the proof of the tubular neighborhood theorem for manifolds. The construction of ψ is natural in the sense that the following diagram commutes:

$$\begin{array}{ccc} N(\widetilde{X}) & \xrightarrow{\psi} & \tilde{U} \\ {\scriptstyle \lambda_*}\downarrow & & \downarrow{\scriptstyle \lambda} \\ N(\tilde{X}') & \xrightarrow{\psi'} & \tilde{U}' \end{array}$$

Therefore, these maps ψ form a diffeomorphism of orbifolds from a neighborhood of the zero section of the normal bundle of the suborbifold to a neighborhood of the suborbifold.

Most authors prefer natural constructions and make scrupulously clear which choices are necessary. Therefore, for the most part, we have not found it necessary to repeat what has been done well elsewhere.

This does not mean, of course, that orbifolds are identical to manifolds. On the contrary, it is clear that orbifolds have more local information. This is why for orbifolds Morse polynomials must be defined slightly differently. It is also why our polytopes have to have extra labels. A good rule of thumb is that the difference between manifolds and orbifolds already arises on the level of vector spaces and representations of finite groups.

Definition 2.2. Let G be a Lie group. A *smooth action* a of G on an orbifold M is a smooth orbifold map $a : G \times M \to M$ satisfying the usual group laws, that is, for all $g_1, g_2 \in G$ and $x \in M$

$$a(g_1, a(g_2, x)) = a(g_1 g_2, x) \quad \text{and} \quad a(1_G, x) = x,$$

where 1_G is the identity element of G, and "=" means "are equivalent as maps of orbifolds."

Definition 2.2 implies that the action a induces a continuous action $|a|$ of G on the underlying topological space $|M|$. In particular, for every $g_0 \in G$ and $x_0 \in M$ there are neighborhoods W of g_0 in G, U of x_0 in M, and U' of $a(g_0, x_0)$ in M, charts $(\tilde{U}, \Gamma, \varphi)$ and $(\tilde{U}', \Gamma', \varphi')$ and a smooth map $\tilde{a} : W \times \tilde{U} \to \tilde{U}'$ such that $\varphi'(\tilde{a}(g, \tilde{x})) = |a|(g, \varphi(x))$ for all $(g, \tilde{x}) \in W \times \tilde{U}$. Note that $\tilde{a}$ is not unique, it is defined up to composition with elements of the orbifold structure groups Γ of x_0 and Γ' of $g_0 \cdot x_0$.

If $g_0 = 1_G$, the identity of G, then we may assume $\tilde{U} \subset \tilde{U}'$, and we can choose $\tilde{a}$ such that $\tilde{a}(1_G, x) = x$. Then $\tilde{a}$ induces a local action of G on $\tilde{U}$.

An action of a Lie group G on an orbifold M induces an infinitesimal action of the Lie algebra $\mathfrak{g}$ of G on M. For a vector $\xi \in \mathfrak{g}$, denote the corresponding induced vector field by ξ_M. In particular, for any chart $(\tilde{U}, \Gamma, \varphi)$ there exists a Γ invariant vector field $\xi_{\tilde{U}}$ on $\tilde{U}$, and such vector fields satisfy compatibility conditions.

If a point x is *fixed* by the action of G and G is compact, then the local action of G on the uniformizing chart $\tilde{U}$ generates an action of $\tilde{G}$ on a subset $\tilde{V} \subset \tilde{U}$, where $\tilde{G}$ is a cover of the identity component of G. Note that the actions of $\tilde{G}$ and Γ on $\tilde{V}$ *commute*; otherwise a group action would not induce the corresponding infinitesimal actions of the Lie algebra.

More generally one can show that for a fixed point x with structure group Γ there exist a uniformizing chart $(\tilde{U}, \Gamma, \varphi)$ for a neighborhood U of x, an exact sequence

of groups

$$1 \to \Gamma \to \hat{G} \xrightarrow{\pi} G \to 1,$$

and an action of $\hat{G}$ on $\tilde{U}$ such that the following diagram commutes:

$$\begin{array}{ccc} \hat{G} \times \tilde{U} & \longrightarrow & \tilde{U} \\ \downarrow & & \downarrow \\ G \times U & \longrightarrow & U \end{array}$$

The extension $\hat{G}$ of G depends on x and, in particular, is not globally defined. The homomorphism $\pi : \hat{G} \to G$ induces an isomorphism of the Lie algebras $\varpi : \hat{\mathfrak{g}} \to \mathfrak{g}$, and for any $\xi \in \hat{\mathfrak{g}}$ we have $\xi_{\tilde{U}} = (\varpi(\xi))_{\tilde{U}}$.

The simplest examples of group actions on orbifolds are linear actions of groups on vector orbi-spaces. A *vector orbi-space* is a quotient of the form V/Γ, where V is a vector space and Γ is a finite subgroup of $\mathrm{GL}(V)$. We define

$$\mathrm{GL}(V/\Gamma) := N(\Gamma)/\Gamma,$$

where $N(\Gamma)$ is the normalizer of Γ in $\mathrm{GL}(V)$. The group $\mathrm{GL}(V/\Gamma)$ does act on the orbifold V/Γ in the sense of Definition 2.2. We define a *representation* $\rho : H \to \mathrm{GL}(V/\Gamma)$ of a group H on the vector orbi-space V/Γ to be a group homomorphism $\rho : H \to N(\Gamma)/\Gamma$. A representation of H on V/Γ defines an action of H on the orbifold V/Γ. For a more detailed discussion of representations, please see Section 3.

Let G be a compact Lie group acting on an orbifold M. Let x be a point in M, let Γ be its orbifold structure group, and let G_x be its stabilizer. Because Γ commutes with the local action of G_x on a chart $\tilde{U}$, the uniformized tangent space $\tilde{T}_x(G \cdot x) \subset \tilde{T}_x M$ is fixed by Γ. Thus, there is a natural representation of the isotropy group G_x of x on the vector orbi-space W/Γ, where $W = \tilde{T}_x M/\tilde{T}_x(G \cdot x)$ (we may also identify W with the orthogonal complement to $\tilde{T}_x(G \cdot x)$ in $\tilde{T}_x M$ with respect to some invariant metric). Because of Proposition 2.3 below we will refer to the vector orbi-space W/Γ as the *(differential) slice* for the action of G at x, and to the representation $G_x \to \mathrm{GL}(W/\Gamma)$ as the *(differential) slice representation.*

Proposition 2.3. (***Slice theorem***) *Suppose that a compact Lie group G acts on an orbifold M and $G \cdot x$ is an orbit of G. A G invariant neighborhood of the orbit is equivariantly diffeomorphic to a neighborhood of the zero section in the associated orbi-bundle $G \times_{G_x} W/\Gamma$, where G_x is the isotropy group of x with respect to the action of G, Γ is the orbifold structure group of x, and $W = \tilde{T}_x M/\tilde{T}_x(G \cdot x)$.*

Proof. This is strictly analogous to the slice theorem for actions on manifolds, and follows from the fact that metrics can be averaged over compact Lie groups (cf. Remark 2.1). □

Remark 2.4. As in the case of manifolds, the compactness of the group G is not necessary for the slice theorem. It is enough to require that the induced action on the underling topological space is proper.

For an action of a connected group G on an orbifold M, it follows from the existence of slices that the fixed point set M^G is a suborbifold.[2] Therefore, the

[2]Strictly speaking, M^G might have codimension 1 strata and so would not be an orbifold in the sense of this paper. However, for symplectic actions on symplectic orbifolds, the codimension of M^G is at least two.

decomposition of M into infinitesimal orbit types is a stratification into suborbifolds. On the other hand, the fixed point set M^G need not be a suborbifold if the group G is not connected.

Example 2.5. Let $\Gamma = \mathbb{Z}/(2\mathbb{Z})$ act on $\mathbb{C}^2$ by sending (x, y) to $(-x, -y)$ Let $G = \mathbb{Z}/(2\mathbb{Z})$ act on $\mathbb{C}^2/\Gamma$ by sending $[x, y]$ to $[x, -y]$. Then the fixed point set M^G is $\{[x, 0]\} \cup \{[0, y]\} = \mathbb{C}/\Gamma \cup \mathbb{C}/\Gamma$.

Consequently the decomposition of an orbifold according to the orbit type is not a stratification. Fortunately the following lemma holds.

Lemma 2.6. *If G is a compact Lie group acting on a connected orbifold M, then there exists an open dense subset of M consisting of points with the same orbit type.*

Proof. We first decompose the orbifold into the open dense set of smooth points M_{smooth} and the set of singular points. Since we assume that all the singularities have codimension 2 or greater, M_{smooth} is connected. A smooth group action preserves this decomposition. Since G is compact, the action of G on M_{smooth} has a principal orbit type (see for example Theorem 4.27 in [K]). The set of points of this orbit type is open and dense in M_{smooth}, hence open and dense in M. □

Corollary 2.7. *If a torus T acts effectively on a connected orbifold M, then the action of T is free on a dense open subset of M.*

3. Symplectic local normal forms

In this section, we write down normal forms for the neighborhoods of isotropic orbits of a compact Lie group G which acts on a symplectic manifold (M, ω) in a Hamiltonian fashion; that is, we classify such neighborhoods up to G equivariant symplectomorphisms. We also point out consequences of these normal forms.

The definitions of symplectic manifolds, symplectic group actions, moment maps, and Hamiltonian actions carry over verbatim to the category of orbifolds. To wit, a *symplectic orbifold* is an orbifold M with a closed non-degenerate 2-form ω. A group G acts *symplectically* on (M, ω) if the action preserves ω. A moment map $\phi : M \to \mathfrak{g}^*$ for this action is a G equivariant map such that

$$\iota(\xi_M)\,\omega = d\langle \xi, \phi \rangle, \text{ for all } \xi \in \mathfrak{g}.$$

If there is a moment map, we say that G acts on (M, ω) in a *Hamiltonian fashion.*

The simplest symplectic orbifold is a *symplectic vector orbi-space* V/Γ, where V is a symplectic vector space and Γ is a finite subgroup of the symplectic group $\mathrm{Sp}(V)$.[3] Two symplectic vector orbi-spaces are isomorphic if there exists a linear symplectic isomorphism $\beta : V \to V'$ such that $\beta\Gamma\beta^{-1} = \Gamma'$. In this case we write $\beta : V/\Gamma \xrightarrow{\simeq} V'/\Gamma'$.

Let $\mathrm{Sp}(V/\Gamma)$ denote the group of symplectic isomorphisms of V/Γ; it is the group $N(\Gamma)/\Gamma$, where $N(\Gamma)$ is the normalizer of Γ in $\mathrm{Sp}(V)$. A *symplectic representation* of a group H on a symplectic vector orbi-space V/Γ is a group homomorphism $\rho : H \to \mathrm{Sp}(V/\Gamma)$. Two symplectic representations $\rho : H \to \mathrm{Sp}(V/\Gamma)$ and $\rho' : H \to \mathrm{Sp}(V'/\Gamma')$ are isomorphic if there exists an isomorphism $\beta : V/\Gamma \to V'/\Gamma'$ such that $\rho = \beta\rho'\beta^{-1}$. In particular, two symplectic representations $\rho : H \to \mathrm{Sp}(V/\Gamma)$ and $\rho' : H \to \mathrm{Sp}(V/\Gamma)$ are isomorphic if there exists $\beta \in \mathrm{Sp}(V/\Gamma)$ such that $\rho = \beta\rho'\beta^{-1}$.

[3]The following discussion also holds, *mutatis mutandis,* for the general linear group, the orthogonal group, etc.

Lemma 3.1. *Let $\rho : H \to \mathrm{Sp}(V/\Gamma)$ be a symplectic representation of a group H on a symplectic vector orbi-space V/Γ, and let $N(\Gamma)$ denote the normalizer of Γ in $\mathrm{Sp}(V)$. The representation ρ and the short exact sequence $1 \to \Gamma \to N(\Gamma) \to \mathrm{Sp}(V/\Gamma) \to 1$ give rise to the* pull-back extension $\pi : \hat{H} \to H$ *and the* (symplectic) pull-back representation $\hat{\rho} : \hat{H} \to N(\Gamma) \subset \mathrm{Sp}(V)$*, so that Γ is naturally a subgroup of $\hat{H}$, and the following diagram is exact and commutes:*

$$
\begin{tikzcd}
1 \arrow[r] & \Gamma \arrow[r] \arrow[d, equal] & \hat{H} \arrow[r, "\pi"] \arrow[d, "\hat{\rho}"] & H \arrow[r] \arrow[d, "\rho"] & 1 \\
1 \arrow[r] & \Gamma \arrow[r] & N(\Gamma) \arrow[r] & \mathrm{Sp}(V/\Gamma) \arrow[r] & 1
\end{tikzcd}
\tag{1}
$$

If ρ is faithful then $\hat{\rho}$ is also faithful.

Conversely, given a symplectic representation $\hat{\rho} : \hat{H} \to \mathrm{Sp}(V)$ of a group $\hat{H}$ on a symplectic vector space V and a finite normal subgroup Γ of $\hat{H}$ such that $\hat{\rho}$ is faithful on Γ, there exists a symplectic orbi-representation $\rho : H \to \mathrm{Sp}(V/\Gamma)$ of the quotient $H = \hat{H}/\Gamma$ making the diagram (1) *commute.*

Proof. Pull-backs exist in the category of groups. □

Lemma 3.2. *Let $\rho : H \to \mathrm{Sp}(V/\Gamma)$ be a symplectic representation of a group H on a symplectic vector orbi-space $(V/\Gamma, \omega)$. The action of H on V/Γ is Hamiltonian with a moment map $\phi_{V/\Gamma} : V/\Gamma \to \mathfrak{h}^*$ given by the formula*

$$
\langle \xi, \phi_{V/\Gamma}(v) \rangle = \frac{1}{2}\omega(\xi \cdot v, v) \quad \text{for all } \xi \in \hat{\mathfrak{h}} \text{ and } v \in V/\Gamma, \tag{2}
$$

where $\xi \cdot v$ is the value at v of the vector field on V/Γ induced by the infinitesimal action of $\xi \in \mathfrak{h}$.

The diagram

$$
\begin{tikzcd}
V \arrow[r] \arrow[d, "\hat{\phi}_V"'] & V/\Gamma \arrow[d, "\phi_{V/\Gamma}"] \\
\hat{\mathfrak{h}}^* & \mathfrak{h}^* \arrow[l, "\varpi^*"']
\end{tikzcd}
$$

commutes, where $\hat{\phi}_V$ is the moment map for the action on V of the pull-back extension $\pi : \hat{H} \to H$, and $\varpi : \hat{\mathfrak{h}} \to \mathfrak{h}$ is the isomorphism of Lie algebras induced by the homomorphism $\pi : \hat{H} \to H$.

Proof. It is easy to see that equation (2) defines a moment map. To show that the diagram commutes, it is enough to show that the moment map $\hat{\phi}_V : V \to \hat{\mathfrak{h}}^*$ for the action of the pull-back extension $\hat{H}$ on V is Γ invariant. But Γ commutes with the identity component of $\hat{H}$. □

Although the following lemma is "well known", we give a proof in Appendix A since a published proof does not seem to be readily available.

Lemma 3.3. *There is a bijective correspondence between isomorphism classes of $2n$ dimensional symplectic representations of a torus H and unordered n tuples of elements (possibly with repetition) of the weight lattice $\ell^* \subset \mathfrak{h}^*$ of H.*

Let (V, ω) be a $2n$ dimensional symplectic vector space. Let $\rho : H \to \mathrm{Sp}(V, \omega)$ be a symplectic representation with weights $(\beta_1, \dots, \beta_n)$. There exist a decomposition $(V, \omega) = \bigoplus_i (V_i, \omega_i)$ into invariant mutually perpendicular 2-dimensional symplectic

subspaces and an invariant norm $|\cdot|$ *compatible with the symplectic form* $\omega = \bigoplus \omega_i$ *so that the representation of* H *on* (V_i, ω_i) *has weight* β_i *and the moment map* $\phi_\rho : V \to \mathfrak{h}^*$ *is given by*

$$\phi_\rho(v_1, \dots, v_n) = \sum |v_i|^2 \beta_i \quad \text{for all } v = (v_1, \dots, v_n) \in \bigoplus_i V_i. \tag{3}$$

Corollary 3.4. *Let* $\rho : H \to \mathrm{Sp}(V/\Gamma)$ *be a symplectic representation of a* torus H *on a symplectic vector orbi-space* $(V/\Gamma, \omega)$*. The image of the corresponding moment map* $\phi_{V/\Gamma}(V/\Gamma)$ *is a polyhedral cone in* $\mathfrak{h}^*$ *which is rational with respect to the lattice* $\ell \subset \mathfrak{h}$*, where* ℓ *is the kernel of the exponential map of* H*, a.k.a. the lattice of circle subgroups of* H*.*

Proof. This follows from Lemma 3.2 and equation (3) in Lemma 3.3. □

Linear symplectic actions of groups on symplectic vector orbi-spaces are not only relatively easy to understand; they lie at the heart of every nonlinear Hamiltonian action. Given a compact Lie group G which acts on a symplectic orbifold (M, ω) in a Hamiltonian fashion, we define the *symplectic slice* at a point $x \in M$ as follows. The 2-form ω induces a non-degenerate antisymmetric bilinear form on $\tilde{T}_x M$. Let $\tilde{T}(G \cdot x)^\omega$ be the symplectic perpendicular to the tangent space of $G \cdot x$ with respect to this form. The quotient

$$V = \tilde{T}(G \cdot x)^\omega / (\tilde{T}(G \cdot x) \cap \tilde{T}(G \cdot x)^\omega)$$

is naturally a symplectic vector space, The structure group Γ of x acts symplectically on $\tilde{T}_x M$ and acts trivially on $\tilde{T}(G \cdot x)$; therefore, Γ acts symplectically on V. The *symplectic slice* at x is the symplectic vector orbi-space V/Γ. The linear action of G_x on $\tilde{T}_x M/\Gamma$ is symplectic and preserves $\tilde{T}(G \cdot x)$. Therefore, it induces a symplectic representation of G_x on V/Γ, the *(symplectic) slice representation*.

As in the case of manifolds, the *differential* slice at x is isomorphic, as a G_x representation, to the product

$$\mathfrak{g}_x^\circ \times V/\Gamma,$$

where $\mathfrak{g}_x^\circ$ denotes the annihilator of $\mathfrak{g}_x$ in $\mathfrak{g}^*$. Thus, by Proposition 2.3, a neighborhood of the orbit $G \cdot x$ in (M, ω) is equivariantly diffeomorphic to a neighborhood of the zero section in the associated orbi-bundle

$$Y = G \times_{G_x} (\mathfrak{g}_x^\circ \times V/\Gamma).$$

Lemma 3.5. *Let* G *be a compact Lie group. Let* $G \cdot x$ *be an* isotropic *orbit in a Hamiltonian* G *orbifold* (M, ω) *and let* $G_x \to \mathrm{Sp}(V/\Gamma)$ *be the symplectic slice representation at* x*. For every* G_x *equivariant projection* $A : \mathfrak{g} \to \mathfrak{g}_x$*, there is a* G *invariant symplectic form* ω_Y *on the orbifold* $Y = G \times_{G_x} (\mathfrak{g}_x^\circ \times V/\Gamma)$ *such that*

1. *a neighborhood of* $G \cdot x$ *in* M *is equivariantly symplectomorphic to a neighborhood of the zero section in* Y*, and*
2. *the action of* G *on* (Y, ω_Y) *is Hamiltonian with a moment map* $\Phi_Y : Y \to \mathfrak{g}^*$ *given by*

$$\Phi_Y([g, \eta, v]) = Ad^\dagger(g)(\eta + A^* \phi_{V/\Gamma}(v)),$$

where $Ad^\dagger$ *is the coadjoint action,* $A^* : \mathfrak{g}_x^* \to \mathfrak{g}^*$ *is dual to* A*,* $\mathfrak{g}_x^\circ$ *is the annihilator of* $\mathfrak{g}_x$ *in* $\mathfrak{g}^*$*, and* $\phi_{V/\Gamma} : V/\Gamma \to \mathfrak{g}_x^*$ *is the moment map for the slice representation of* G_x*.*

Proof. The construction is standard in the case of manifolds (cf. [GS1]); we adapt it for orbifolds. Let $\hat{G}_x$ be the pull-back extension of the isotropy group G_x (cf. Lemma 3.1). The group G_x acts on G by $g_x \cdot g = gg_x^{-1}$; this lifts to a symplectic action on the cotangent bundle T^*G. The corresponding diagonal action of $\hat{G}_x$ on $T^*G \times V$ is Hamiltonian. The projection $A : \mathfrak{g} \to \mathfrak{g}_x$ defines a left G-invariant connection 1-form on the principal G_x bundle $G \to G/G_x$, and thereby identifies Y with the reduced space at zero $(T^*G \times V)_0$, thus giving Y a symplectic structure. The G moment map on $T^*G \times V$ descends to a moment map for the action of G on Y, giving the formula in (2). The proof that the neighborhoods are equivariantly symplectomorphic reduces to a form of the equivariant relative Darboux theorem; it is identical to the proof in the case of manifolds (see Remark 2.1). □

Remark 3.6. Symplectic slice representations classify neighborhoods of orbits in the following sense. Let a compact Lie group G act on symplectic orbifolds (M, ω) and (M', ω') in a Hamiltonian fashion with moment maps ϕ and ϕ' respectively. Let $G \cdot x \subset M$ and $G \cdot x' \subset M'$ be isotropic orbits.

Clearly, if there exist neighborhoods U of $G \cdot x$ and U' of $G \cdot x'$ and a G equivariant symplectomorphism $\psi : U \to U'$ such that $\psi(x) = x'$, then the stabilizer of x and x' is the same group H, and the slice representations at x and x' are isomorphic.

Conversely, if the stabilizer of x and x' is the same group H, and the symplectic slice representations at x and x' are isomorphic, then it follows from Lemma 3.5 that there exist neighborhoods U of $G \cdot x$ and U' of $G \cdot x'$ and a G equivariant symplectomorphism $\psi : U \to U'$ such that $\psi(x) = x'$ and $\phi' \circ \psi = \phi + const$.

Remark 3.7. Suppose again that a group G acts in a Hamiltonian fashion on a symplectic orbifold (M, ω) with moment map $\phi : M \to \mathfrak{g}^*$. Suppose further that the group G is a *torus* . Then the coadjoint action of G is trivial and every orbit is isotropic. It follows from Lemma 3.5 that, given an orbit $G \cdot x \subset M$, there exist an invariant neighborhood U of the orbit in M, a neighborhood $\mathcal{W}$ of the zero section in the associated bundle $G \times_{G_x} (\mathfrak{g}_x^\circ \times V/\Gamma)$ (where G_x is the isotropy group of x, $\mathfrak{g}_x^\circ$ is the annihilator of its Lie algebra in $\mathfrak{g}^*$ and V/Γ is the symplectic slice at x) and an equivariant diffeomorphism $\psi : \mathcal{W} \to U$ sending the zero section to $G \cdot x$ such that

$$(\phi \circ \psi)([g, \eta, v]) = \alpha + \eta + A^*(\phi_{V/\Gamma}(v)), \tag{4}$$

where $\alpha = \phi(x)$ and $A^* : \mathfrak{g}_x^* \hookrightarrow \mathfrak{g}^*$ is an inclusion with $\mathfrak{g}^* = \mathfrak{g}_x^\circ \oplus A^*(\mathfrak{g}_x^*)$. Consequently

$$\phi(U) = (\alpha + W_1) \times (C \cap W_2), \tag{5}$$

where $W_1 \subset \mathfrak{g}_x^\circ$ is a neighborhood of 0 and $W_2 \subset A^*(\mathfrak{g}_x^*)$ is a neighborhood of the vertex of a rational polyhedral cone C ($C = \phi_{V/\Gamma}(V/\Gamma)$ is a rational polyhedral cone by Corollary 3.4).

The following result is a consequence of Lemma 3.5 above.

Corollary 3.8. *If a subgroup $H \subset G$ is connected, then M^H, the set of points which are fixed by H, is a symplectic suborbifold.*

Lemma 3.9. *Let a compact group G act in a Hamiltonian fashion on a symplectic orbifold (M, ω) with moment map $\phi : M \to \mathfrak{g}^*$. For a regular value $\alpha \in \mathfrak{g}^*$ of ϕ which is fixed by the coadjoint action, the* reduced space *of M at α, $M_\alpha = \phi^{-1}(\alpha)/G$, is a symplectic orbifold.*

Proof. Consider $x \in \phi^{-1}(\alpha)$. Because α is regular, the stabilizer G_x of x is finite. Since α is fixed by the adjoint action, $G \cdot x$ is an isotropic orbit. Let $G_x \to \mathrm{Sp}(V/\Gamma)$ be the symplectic slice representation at x. By Lemma 3.5, there is a G invariant symplectic form on the orbifold $Y = G \times_{G_x} (\mathfrak{g}^* \times V/\Gamma)$ such that a neighborhood U of $G \cdot x$ in M is equivariantly symplectomorphic to a neighborhood of the zero section in Y, and the moment map $\Phi_Y : Y \to \mathfrak{g}^*$ is given by $\Phi_Y([g, \eta, v]) = \alpha + Ad^\dagger(g)(\eta)$, where $Ad^\dagger$ is the coadjoint action.

It is easy to see that $\phi^{-1}(\alpha) \cap U$ is isomorphic to $G \times_{G_x} V/\Gamma$. Therefore, a neighborhood of $[x]$ in M_α is isomorphic to $V/\hat{G}_x$, where $\hat{G}_x$ is the extension of G_x by Γ. □

Remark 3.10.
1. We believe that the assumptions that the group G is compact and that α is fixed by the coadjoint action are not necessary. It should be enough to assume that the action of G on the underlying topological space $|M|$ is proper and that the coadjoint orbit through α is locally closed.
2. If additionally we drop the assumption that α is a regular value of the moment map, then the quotient $\phi^{-1}(G \cdot \alpha)/G$ should be a symplectic stratified space in the sense of [SL].

A proof of these two assertions would take us too far afield, so we only note that we expect the argument in [BL] to carry over to the case of orbifolds.

4. Morse theory

In this section, we extend Morse theory to orbifolds.[4] We need Morse theory for the following result, which we will prove in the first part of this section.

Lemma 4.1. *Let M be a connected compact orbifold, and let $f : M \to \mathbb{R}$ be a Bott-Morse function such that no critical suborbifold has index* 1 *or* $\dim(M) - 1$. *The orbifold $M_{(a,b)} = f^{-1}(a, b)$ is connected for all $a, b \in \mathbb{R}$.*

We will use this result in the next section to prove that the fibers of a torus moment map are connected, and that the image of a compact symplectic orbifold under a torus moment map is a convex polytope. In the second part of the section we discuss the notion of Morse polynomials for functions on orbifolds and prove the Morse inequalities.

Most of the basic definitions needed for Morse theory on orbifolds are strictly analogous to their manifold counterparts. Let $f : M \to \mathbb{R}$ be a smooth function on an orbifold M. A critical point x of f is *non-degenerate* if the Hessian $H(f)_x$ of f is non-degenerate. More generally, a critical suborbifold $F \subset M$ is non-degenerate if for every point $x \in F$, the null space of the Hessian $H(f)_x$ is precisely the tangent space to F. In this case, the Hessian restricts to a non-degenerate quadratic form H on the normal bundle E of F in M.

A smooth function $f : M \to \mathbb{R}$ is *Bott-Morse* if the set of critical points is the disjoint union of non-degenerate suborbifolds, and is *Morse* if each of these suborbifolds is an (isolated) point.

Lemma 4.2. *Let M be a compact orbifold. Let $f : M \to \mathbb{R}$ be a smooth function. Choose $a < b \in \mathbb{R}$ such that $[a, b]$ contains no critical values. The orbifolds $M_a^- := f^{-1}(-\infty, a)$ and $M_b^- := f^{-1}(-\infty, b)$ are diffeomorphic, and so are the orbifolds $f^{-1}(a)$ and $f^{-1}(b)$.*

[4]Since orbifolds are stratified spaces, for Morse functions with isolated fixed points this is a special case of Morse theory on stratified spaces [GM].

Proof. The usual proof still applies, i.e., the diffeomorphism is given by flows of the (renormalized) gradient of f with respect to a Riemannian metric. □

For critical points, the situation is only slightly more complicated.

Lemma 4.3. (**Morse Lemma**) *Let F be a non-degenerate critical suborbifold of a smooth function f on an orbifold M. Let H be the restriction of the Hessian to the normal bundle E of F. There exists a homeomorphism ψ from a neighborhood of the zero section of E to a neighborhood of F such that*

$$H = f \circ \psi.$$

Proof. The proof of the Morse lemma in [MW] can be generalized to vector bundles by carrying out the construction fiber by fiber. Moreover, the construction is functorial once a metric has been chosen on M. (See Remark 2.1.)

Alternatively, one can apply the same reasoning to Palais' proof of the Morse lemma (cf. [La]). In this case, the map ψ will be a diffeomorphism. □

Let F be an non-degenerate critical suborbifold of a smooth function f on a compact orbifold M. Let H be the restriction of the Hessian to the normal bundle E of F. The bundle E splits as a direct sum of vector orbi-bundles E^- and E^+ corresponding to the negative and positive spectrum of H. (By Remark 2.1, this splitting exists because such a splitting exists and is natural in the manifold case, once a metric is chosen.) The *index* of F is the rank of E^-.

We need only the following homological consequence of the above lemmas.

Lemma 4.4. *Let M be a compact orbifold, and let $f : M \to \mathbb{R}$ be a Bott-Morse function. Choose an interval $[a, b] \subset \mathbb{R}$ which contains a unique critical value c. Let F be the critical suborbifold such that $f(F) = c$. Let E^- be the negative normal bundle, let $D = D_F$ denote a disc bundle of E^- and let $S = S_F$ denote the corresponding sphere bundle. The relative cohomology $H_*(M_b^-, M_a^-) = H_*(D_F, S_F)$. Moreover, the boundary map from $H_q(M_b^-, M_a^-)$ to $H_{q-1}(M_b^-)$ in the long exact sequence of relative homology is the composition of the isomorphism of $H_*(M_b^-, M_a^-)$ and $H_*(D_F, S_F)$, the boundary map from $H_q(D_F, S_F)$ to $H_{q-1}(S_F)$, and the map on homology induced by the "inclusion" map j from S_F to M_a^-. That is, the following diagram commutes:*

$$\begin{tikzcd}
H_q(M_b^-, M_a^-) \arrow[r, "\sim"] \arrow[d, "\partial"'] & H_q(D_F, S_F) \arrow[d, "\partial"] \\
H_{q-1}(M_a^-) & H_{q-1}(S_F) \arrow[l, "j_*"']
\end{tikzcd}$$

Proof. Again, the manifold proof (see for example [C]) can be adapted. The orbifold M_b^- has the homotopy type of the space obtained by attaching the disk orbi-bundle D_F to M_a by a map from S_F to $f^{-1}(a)$. The result then follows from excision. □

We now prove Lemma 4.1 with a sequence of lemmas.

Lemma 4.5. *Let F be an orbifold, let $\pi : E \to F$ be a λ dimensional real vector orbi-bundle, and let D and S be the corresponding disk and sphere orbi-bundles with respect to some metric. If $\lambda = 0$, then $H_0(S) = 0$, so $H_0(D, S) \neq 0$. Moreover, if $\lambda > 1$, then $H_1(D, S) = 0$. In either case, the boundary map $H_1(D, S) \to H_0(S)$ is trivial.*

Proof. If $\lambda = 0$ then S is empty, so the result is trivial.

Suppose that $\lambda > 1$; we wish to show that $H_1(D, S) = 0$. By the long exact sequence in relative homology it is enough to show that the maps $H_0(S) \to H_0(D)$ and $H_1(S) \to H_1(D)$ induced by inclusion are injective and surjective, respectively. But this follows from two facts: the fibers of $\pi : S \to F$ are path connected, and any path in the base F can be lifted to a path in the sphere bundle S. □

Lemma 4.6. *Let M be a connected compact orbifold, and let $f : M \to \mathbb{R}$ be a Bott-Morse function with no critical suborbifold of index 1. Then*

1. *$M_a^- := \{m \in M : f(m) < a\}$ is connected for all $a \in \mathbb{R}$, and*
2. *if $M_a^- \neq \emptyset$, then $H_1(M_a^-) \to H_1(M)$ is a surjection.*

Proof. Let $F \subset M$ be a critical suborbifold of f of index λ. Let D_F and S_F be the disk and sphere bundles of the negative orbi-bundle over F. Let $a = f(F)$, and let $\epsilon > 0$ be small. Assume, for simplicity, that no other critical suborbifold maps to a.

By Lemma 4.5, the map $H_1(D_F, S_F) \to H_0(S_F)$ is trivial. Therefore, by Lemma 4.4, the map $H_1(M_{a+\epsilon}^-, M_{a-\epsilon}^-) \to H_0(M_{a-\epsilon}^-)$ is also trivial. Thus by the long exact sequence in relative homology, the following sequence is exact:

$$0 \to H_0(M_{a-\epsilon}^-) \to H_0(M_{a+\epsilon}^-) \to H_0(M_{a+\epsilon}^-, M_{a-\epsilon}^-) \to 0.$$

Therefore, $\dim H_0(M_{a+\epsilon}^-) \geq \dim H_0(M_{a-\epsilon}^-)$. Since M is connected, this proves that M_a^- is connected for all $a \in \mathbb{R}$.

If $\lambda = 0$, then $\dim H_0(M_{a+\epsilon}^-)$ is strictly greater than $\dim H_0(M_{a-\epsilon}^-)$. Therefore, since M is connected, the minimum is the unique critical value of index 0. For any other critical value a, $H_1(M_{a+\epsilon}^-, M_{a-\epsilon}^-) = 0$, by Lemma 4.5 and Lemma 4.4. Thus, by the long exact sequence in relative homology, the map $H_1(M_{a-\epsilon}^-) \to H_1(M_{a+\epsilon}^-)$ is a surjection. □

Proof of Lemma 4.1. We may assume that a and b are regular values and that $M_{(a,b)} := \{m \in M \mid a < f(m) < b\}$ is not empty. By Lemma 4.6, $H_1(M_a^-) \oplus H_1(M_b^+) \to H_1(M)$ is a surjection, where $M_b^+ := f^{-1}(b, \infty)$. Therefore, by Mayer-Vietoris, the following sequence is exact:

$$0 \to H_0(M_{(a,b)}) \to H_0(M_a^-) \oplus H_0(M_b^+) \to H_0(M) \to 0.$$

By Lemma 4.6, M_a^- and M_b^+ are connected. Therefore $M_{(a,b)}$ is connected. □

4.1. **Morse polynomials.** We conclude the section with a few words about Morse polynomials for orbifolds. These observations are not used in the rest of the paper.

Let M be a compact orbifold. The *Poincaré polynomial* $\mathcal{P}(x)$ is defined by

$$\mathcal{P}(x) = \sum_{i=0}^{\infty} \dim H^i(M) x^i.$$

Let $f : M \to \mathbb{R}$ be a Bott-Morse function. For each critical set F, let E_F^- be the vector orbi-bundle corresponding to the negative spectrum of the Hessian of f along F. Let D_F denote a disc bundle of E_F^- with respect to any metric, and let S_F denote the corresponding sphere bundle. The *Morse polynomial* $\mathcal{M}(x)$ is defined by

$$\mathcal{M}(x) = \sum_F \sum_{i=0}^{\infty} \dim H^i(D_F, S_F) x^i,$$

where the sum is taken over all critical orbifolds F.

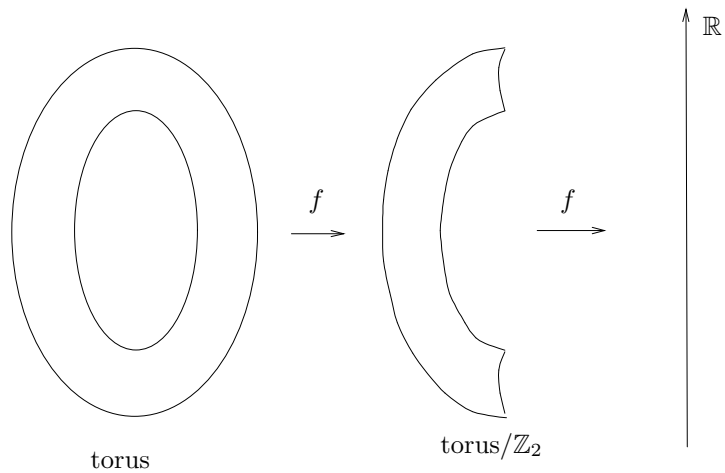


FIGURE 1

Theorem 4.7. *Let M be a compact orbifold, and let $f : M \to \mathbb{R}$ be a Bott-Morse function. If $\mathcal{P}$ is the Poincaré polynomial of M, and $\mathcal{M}$ is the Morse polynomial of f, then*

$$\mathcal{M}(x) - \mathcal{P}(x) = (1 + x)Q(x),$$

where Q is a polynomial with nonnegative coefficients.

Proof. Just as for manifolds, this follows immediately from Lemma 4.4 and a spectral sequence argument. □

However, there is one very important distinction between Morse theory on manifolds and on orbifolds. For simplicity consider the case of isolated critical points. For a manifold, $\dim H^i(D, S)$ is easy to compute; over any field, it is one if $\dim D = i$, and zero otherwise. Over finite fields, $\dim H^i(D/\Gamma, S/\Gamma)$ may be much more complicated. However, it is easy to see that $H_i(D/\Gamma, S/\Gamma; \mathbb{R}) = 0$ if $i \neq \dim D$; whereas if $i = \dim D$, then $H_i(D/\Gamma, S/\Gamma; \mathbb{R}) = \mathbb{R}$ if Γ preserves the orientation of D, and is trivial otherwise. Therefore, the ith coefficient of the Morse polynomial is the number of points x of index i such that the the orbifold structure group of x preserves the orientation of the negative eigenspace of the Hessian.

Corollary 4.8. *The number of critical points with index i is greater than or equal to the dimension of $H_i(M)$.*

Example 4.9. Let M be a torus, stood on end (see Figure 1). Let $f : M \to \mathbb{R}$ be the height function. Let $\Gamma = \mathbb{Z}/(2\mathbb{Z})$ act on M by rotating it 180 degrees around the vertical axis. Although there are four critical points, $\mathcal{M}(x) = 1 + 2x + x^2$.

5. Connectedness and convexity

In this section, we prove an analogue of the Atiyah connectedness theorem and the Atiyah-Guillemin-Sternberg convexity theorem for Hamiltonian torus actions

on symplectic orbifolds [GS1] [A]. That is, we prove that the fibers of the moment map are connected, and that the image of the moment map is a rational convex polytope. Our proofs are similar to Atiyah's proofs (op. cit.). The fibers are connected because the components of the moment map are Bott-Morse functions with even indices, and convexity is an consequence of connectedness. The precise statements of these theorems follow.

Theorem 5.1. *Let a torus T act on a compact connected symplectic orbifold (M, ω), with a moment map $\phi : M \to \mathfrak{t}^*$. For every $a \in \mathfrak{t}^*$, the fiber $\phi^{-1}(a)$ is connected.*

Theorem 5.2. *Let a torus T act on a compact connected symplectic orbifold (M, ω), with a moment map $\phi : M \to \mathfrak{t}^*$. The image of the moment map $\phi(M) \subset \mathfrak{t}^*$ is a rational convex polytope. In particular it is the convex hull of the image of the points in M fixed by T,*

$$\phi(M) = \textit{convex hull } (\phi(M^T)).$$

To prove these theorems, we will use the following lemma.

Lemma 5.3. *Let a compact Lie group G act on a compact symplectic orbifold (M, ω) with moment map $\phi : M \to \mathfrak{g}^*$. For every $\xi \in \mathfrak{g}$ the ξth component of the moment map $\phi^\xi := \xi \circ \phi$ is Bott-Morse, and the index of every critical orbifold is even.*

Proof. This is a generalization of Theorem 5.3 in [GS1] and of Lemma 2.2 in [A] to the case of orbifolds. The proof is the same, except one must use the orbifold version of the equivariant Darboux theorem (cf. Remark 2.1 and Lemma 3.5, which specializes to the equivariant Darboux theorem when the orbit is a point). □

Remark 5.4. If a component of the moment map ϕ^ξ has isolated critical points, the orbifold isotropy group of the critical points preserves the symplectic form, and hence the orientation, on the negative eigenspace of the Hessian. Therefore, these maps are perfect Morse functions, i.e., the ith coefficient of the Morse polynomial equals the ith coefficient of the Poincaré polynomial, $\mathcal{M}_i = \dim H_i(M)$.

Proof of Theorem 5.1. We will prove that the preimage of any ball is connected by induction on the dimension of T. Because the moment map ϕ is continuous and proper, this implies that the fibers of ϕ are connected.

Consider first the case of $\dim T = 1$. By Lemma 5.3, moment maps for circle actions are Bott-Morse functions of even index. By Lemma 4.1, the preimage of any ball is connected.

Suppose that T is a k dimensional torus, $k > 1$, and let B be a closed ball in $\mathfrak{t}^*$. Let $\ell \subset \mathfrak{t}$ denote the lattice of circle subgroups of T. For every $0 \neq \xi \in \ell$ the map $\phi^\xi \equiv \xi \circ \phi$ is a moment map for the action of the circle $S_\xi := \{\exp t\xi : t \in \mathbb{R}\}$. Let $\mathbb{R}_\xi$ denote the set of regular values of ϕ^ξ. For every $a \in \mathbb{R}_\xi$ the reduced space $M_{a,\xi} := (\xi \circ \phi)^{-1}(a)/S_\xi$ is a symplectic orbifold (reduction lemma). The $k-1$ dimensional torus $H := T/S_\xi$ acts on $M_{a,\xi}$ in a Hamiltonian fashion.

The affine hyperplane $\{\eta \in \mathfrak{t}^* : \xi(\eta) = a\}$ is naturally isomorphic to the dual of the Lie algebra of H. This isomorphism identifies the restriction of ϕ to $(\phi^\xi)^{-1}(a)$ with the pull-back of an H-moment map ϕ^H by the orbit map $\pi : (\phi^\xi)^{-1}(a) \to M_{a,\xi}$. By the inductive assumption the preimages of balls under ϕ^H are connected. Therefore, $\phi^{-1}(B \cap \{\eta : \xi(\eta) = a\}) = \pi^{-1}((\phi^H)^{-1}(B))$ is connected.

Now the set

$$U = \bigcup_{\xi \in \ell} \bigcup_{a \in \mathbb{R}_\xi} B \cap \{\eta : \xi(\eta) = a\}$$

is connected and dense in the ball B, and its preimage $\phi^{-1}(U)$ is connected. Therefore the closure $\overline{\phi^{-1}(U)}$ in M is connected. By Lemma 3.5, $\overline{\phi^{-1}(U)} = \phi^{-1}(\overline{U})$. Hence $\phi^{-1}(B) = \phi^{-1}(\overline{U})$ is connected. □

Proof of Theorem 5.2. Without loss of generality the action of T is effective, and hence, by Corollary 2.7, free on a dense subset. Consequently the interior of $\phi(M)$ is nonempty.

To prove that $\phi(M)$ is convex it suffices to show that the intersection $\mathcal{L} \cap \phi(M)$ is connected for any rational affine line $\mathcal{L} \subset \mathfrak{t}^*$, i.e., any line of the form $\mathcal{L} = \mathbb{R}v + a$ for some a in $\mathfrak{t}^*$ and v in the weight lattice ℓ^* of T. Define $\mathfrak{h} := \ker v$. Let i^* be the dual of the inclusion $i : \mathfrak{h} \to \mathfrak{t}$, and let $\alpha = i^*(a)$. The map $\phi^H := i^* \circ \phi : M \to \mathfrak{h}^*$ is the moment map for a subtorus $H = \exp(\mathfrak{h})$ of T. The fibers $(\phi^H)^{-1}(\alpha)$ are connected by Theorem 5.1. On the other hand,

$$(\phi^H)^{-1}(\alpha) = \phi^{-1}((i^*)^{-1}(\alpha)) = \phi^{-1}(\phi(M) \cap (a + \mathbb{R}v)).$$

Next, we show that $\phi(M)$ is the convex hull of $\phi(M^T)$). By Minkowski's theorem, since the set $\phi(M)$ is compact and convex, it is the convex hull of its extreme points. Recall that a point α in the convex set A is *extreme* for A if it *cannot* be written in the form $\alpha = \lambda\beta + (1-\lambda)\gamma$ for any $\beta, \gamma \in A$ and $\lambda \in (0,1)$. It follows from Lemma 3.5 and Remark 3.7 that for any point m in a Hamiltonian T orbifold M, the image $\phi(M)$ contains an open ball in the affine plane $\phi(m) + \mathfrak{t}_m^\circ$, where $\mathfrak{t}_m^\circ$ is the annihilator of the isotropy Lie algebra of m in $\mathfrak{t}^*$. Therefore the preimage of extreme points of $\phi(M)$ consists entirely of fixed points.

To show that $\phi(M)$ is a convex polytope it suffices to show that $\phi(M^T)$ is finite. This follows from that facts that M^T is closed, M is compact, and ϕ is locally constant on M^T. Therefore, we can write $\phi(M)$ as

$$\phi(M) = \bigcap_{i=1}^{N} \{\alpha \in \mathfrak{t}^* \mid \langle \alpha, \xi_i \rangle \leq \eta_i \text{ for all } i\},$$

for some $\eta_i \in \mathbb{R}$ and $\xi_i \in \mathfrak{t}$, where N is the number of facets.

To prove that $\phi(M)$ is rational we need to show that for each ξ_i, the subgroup $H = H_i \subset T$, which is the closure of $\{\exp t\xi_i : t \in \mathbb{R}\}$, is a circle. The global maximum of the function ϕ^{ξ_i} is η_i. Therefore the points in $(\phi^{\xi_i})^{-1}(\eta_i)$ are fixed by H. On the other hand, for a generic m in $(\phi^{\xi_i})^{-1}(\eta_i)$, the image $d\phi(T_m M) \subset \mathfrak{t}^*$ has codimension one, so the stabilizer of m is one dimensional. □

PART 2. TORIC VARIETIES

6. Local models for symplectic toric orbifolds

In this section, we prove the first claim in our main theorem by showing that a labeled polytope can be naturally associated to each symplectic toric orbifold. Additionally, we prove that if two symplectic toric orbifolds have the same labeled polytope associated to them, then they are locally isomorphic.

We begin with linear actions.

Lemma 6.1. *Let $\rho : H \to \mathrm{Sp}(V/\Gamma)$ be a faithful symplectic representation of a compact abelian Lie group H on a symplectic vector orbi-space such that $\dim V = 2\dim H$. The pull-back extension $\pi : \hat{H} \to H$ (cf. Lemma 3.1) is a torus.*

A fortiori, H is a torus.

Proof. Since the orbi-representation $\rho : H \to \mathrm{Sp}(V/\Gamma)$ is faithful, the pull-back representation $\hat{\rho} : \hat{H} \to \mathrm{Sp}(V)$ is also faithful. Let $\tilde{H}$ denote the identity component of $\hat{H}$. The subgroup $\hat{\rho}(\tilde{H})$ is an n dimensional torus in $\mathrm{Sp}(V)$, where $n = \frac{1}{2}\dim V$. Every maximal compact subgroup of $\mathrm{Sp}(V)$ is isomorphic to $U(n)$. Every maximal torus of $U(n)$ is n dimensional. Therefore, since the only elements of $U(n)$ which commute with a maximal torus are the elements of that torus, we will be done once we show that every $a \in \hat{H}$ commutes with $\tilde{H}$.

Given $a \in \hat{H}$, define a continuous homomorphism $f_a : \hat{H} \to \hat{H}$ by $f(h) = aha^{-1}$. Notice that $f_a(e) = e$, where e is the identity element of $\hat{H}$. Moreover, since H is abelian, $\pi(f_a(h)) = \pi(a)\pi(h)\pi(a)^{-1} = \pi(h)$. Therefore, for all $h \in \tilde{H}$, $f_a(h) = h$, i.e., a commutes with $\tilde{H}$. □

Let $\mathfrak{h}$ be a vector space with a lattice $\ell \subset \mathfrak{h}$. A vector $g \in \ell$ is *primitive* if there is no positive integer $n > 1$ such that $\frac{1}{n}g \in \ell$. A *rational cone* in $\mathfrak{h}^*$ is a set C such that

$$C = \{\eta \in \mathfrak{h}^* \mid \langle f_i, \eta\rangle \geq 0\},$$

where $f_i \in \ell$ for all i. We may assume the normals f_i are primitive. A cone is *strictly convex* if it contains no nontrivial subspace. A cone is *simplicial* if it spans $\mathfrak{h}^*$ and has $\dim \mathfrak{h}$ facets.

Lemma 6.2. *1. Let $\rho : H \to \mathrm{Sp}(V/\Gamma)$ be a faithful symplectic representation of a torus H on a symplectic vector orbi-space such that $\dim V = 2\dim H$, and let $\phi_\rho : V/\Gamma \to \mathfrak{h}^*$ be the corresponding moment map. The image $\phi_\rho(V/\Gamma)$ is a rational strictly convex simplicial cone (relative to the lattice of circle subgroups $\ell \subset \mathfrak{h}$). For every open facet $\mathring{F}$ of this cone there exists a positive integer $m_{\mathring{F}}$ such that $\mathbb{Z}/m_{\mathring{F}}\mathbb{Z}$ is the orbifold structure group of every point in $\phi_\rho^{-1}(\mathring{F})$, the preimage of $\mathring{F}$.*

Conversely, consider a strictly convex simplicial rational cone C in $\mathfrak{h}^*$ and a set of positive integers $\{m_{\mathring{F}}\}$ indexed by the open facets of C. There exists a unique faithful symplectic representation $\rho : H \to \mathrm{Sp}(V/\Gamma)$ such that $\dim V = 2\dim H$, $\phi_\rho(V/\Gamma) = C$, and for every open facet $\mathring{F}$ of C the orbifold structure group of every point $x \in M$ such that $\phi(x) \in \mathring{F}$ is $\mathbb{Z}/m_{\mathring{F}}\mathbb{Z}$.

2. The preimage $\phi_\rho^{-1}(0)$ is a single point.

3. The structure group $\Gamma = \ell/\hat{\ell}$, where $\ell \subset \mathfrak{h}$ is the lattice of circle subgroups of H, and $\hat{\ell}$ is the lattice generated by $\{m_{\mathring{F}} \cdot \eta_{\mathring{F}}\}_{\mathring{F} \in \mathcal{F}(C)}$, where $\mathcal{F}(C)$ is the set of open facets of C and $\eta_{\mathring{F}}$ is the primitive outward normal of an open facet $\mathring{F} \in \mathcal{F}(C)$.

Proof. We first show how an orbi-representation gives rise to a labeled cone.

Let $\pi : \hat{H} \to H$ be the the pull-back extension, let $\hat{\rho}$ be the pullback representation of $\hat{H}$ on V, and let $\phi_{\hat{\rho}} : V \to \hat{\mathfrak{h}}$ be the associated moment map. By Lemma 6.1, $\hat{H}$ is a torus. By Lemma 3.3, the vector space V can be split into the direct sum of symplectic, invariant two dimensional subspaces, $V = \bigoplus V_i$. Let $\hat{\rho}_i : \hat{H} \to V_i$ denote the ith subrepresentation, and let $f_i \in \hat{\ell}^*$ denote the corresponding weight.

Since the representation $\hat{\rho}$ is faithful, the vectors $\{f_i\}$ form a basis of the weight lattice. Let $\{e_i\}$ denote the dual basis of the lattice $\hat{\ell}$. By equation (3) in Lemma 3.3

the image $\phi_{\hat{\rho}}(V)$ is the strictly convex simplicial rational cone

$$\phi_{\hat{\rho}}(V) = \{\eta \in \hat{\mathfrak{h}}^* \mid \langle e_i, \eta\rangle \geq 0\}.$$

By Lemma 3.2, the diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d, "\hat{\phi}_V"'] & V/\Gamma \arrow[d, "\phi_{V/\Gamma}"] \\ \hat{\mathfrak{h}}^* & \mathfrak{h}^* \arrow[l, "\varpi^*"] \end{tikzcd}$$

commutes, where $\varpi : \hat{\mathfrak{h}} \to \mathfrak{h}$ is the isomorphism of Lie algebras induced by $\pi : \hat{H} \to H$, and ϖ^* is its transpose. Therefore, the image of ϕ_ρ is the rational simplicial cone

$$\phi_\rho(V/\Gamma) = \{\eta \in \mathfrak{h}^* \mid \langle \varpi(e_i), \eta\rangle \geq 0\},$$

the open facets $\mathring{F}_j$ of this cone are given by

$$\mathring{F}_j = \{\eta \in \mathfrak{h}^* \mid \langle \varpi(e_i), \eta\rangle > 0 \text{ for } i \neq j,\ \langle \varpi(e_j), \eta\rangle = 0\},$$

and their preimages are

$$\phi_\rho^{-1}(\mathring{F}_j) = \{[(v_1, \dots, v_n)] \in (\bigoplus V_i)/\Gamma \mid v_j = 0,\ v_i \neq 0 \text{ for } i \neq j\}.$$

Consequently for any $[v] \in \phi_\rho^{-1}(\mathring{F}_j)$ the stabilizer of $v \in V$ in $\hat{H}$ is $\bigcap_{i\neq j} \ker(\hat{\rho}_i)$, the circle subgroup with Lie algebra $\mathbb{R}e_j$. Therefore, the stabilizer of v in Γ is $\mathbb{Z}/m_j\mathbb{Z}$, where m_j is the integer such that the primitive element of the intersection of the ray $\mathbb{R}_{\geq 0}\varpi(e_j)$ with the lattice ℓ is $\frac{1}{m_j}\varpi(e_j)$.

Conversely, suppose we are given a rational strictly convex simplicial cone C in $\mathfrak{h}^*$, and a collection of positive integers $\{m_{\mathring{F}}\}$ indexed by the set $\mathcal{F}(C)$ of open facets of C. For every open facet $\mathring{F} \in \mathcal{F}(C)$ there exists a primitive vector $g_{\mathring{F}} \in \ell$ such that

$$C = \{\eta \in \mathfrak{h}^* \mid \langle g_{\mathring{F}}, \eta\rangle \geq 0, \text{ for all } \mathring{F} \in \mathcal{F}(C)\}.$$

Because C is simplicial, the set $\{e_{\mathring{F}} = m_{\mathring{F}} g_{\mathring{F}}\}_{\mathring{F}\in\mathcal{F}(C)}$ is a basis of a sublattice $\hat{\ell}$ of ℓ. Let $\{f_{\mathring{F}} = e^*_{\mathring{F}}\}$ denote the dual basis of $\hat{\ell}^*$. Let $\hat{H}$ be the torus $\mathfrak{h}/\hat{\ell}$. By Lemma 3.3 there exists a unique symplectic representation $\hat{\rho} : \hat{H} \to \mathrm{Sp}(V)$ with weights $\{f_{\mathring{F}}\}$. Clearly $\dim V = 2\dim \hat{H}$. Since $\{f_{\mathring{F}}\}$ is a basis, the representation $\hat{\rho}$ is faithful.

Let $\Gamma = \ell/\hat{\ell}$. There is a short exact sequence $1 \to \Gamma \to \hat{H} \xrightarrow{\pi} H \to 1$. Thus $\hat{\rho}$ defines an orbi-representation $\rho : H \to \mathrm{Sp}(V/\Gamma)$. We leave it for the reader to show that the image of the corresponding moment map ϕ_ρ is the cone C and that the orbifold structure groups of points in the preimages of open facets are cyclic groups of the correct orders. □

Lemma 6.3. *Let (M, ω, T, ϕ) be a compact symplectic toric orbifold. For a point $a \in \phi(M)$ let H be the isotropy group of any $x \in \phi^{-1}(a)$, and let $H \to \mathrm{Sp}(V/\Gamma)$ be the symplectic slice representation at x. Then $\dim V = 2\dim T$, the action of H on V/Γ is effective, and consequently H is a torus.*

Choose a subtorus K of T complementary to H. There exist a neighborhood $W \subset \mathfrak{t}^*$ of a and a T-equivariant symplectic embedding

$$\psi : \phi^{-1}(W) \to T^*K \times V/\Gamma$$

which maps the fiber $\phi^{-1}(a)$ *to* $K \times \{0\}$ *(the action of* $T = K \times H$ *on the right-hand side is the product action). Therefore*

$$W \cap \phi(M) = W \cap (a + \mathfrak{h}^\circ \times \phi_{V/\Gamma}(V/\Gamma)) \tag{6}$$

(where we have identified $\mathfrak{t}^* \simeq \mathfrak{k}^* \times \mathfrak{h}^* \simeq \mathfrak{h}^\circ \times \mathfrak{h}^*$*).*

Proof. By the discussion preceding Lemma 3.5, the differential slice at x is the product representation $\mathfrak{h}^\circ \times V/\Gamma$. Therefore, by the slice theorem (Proposition 2.3), a neighborhood of the orbit $T \cdot x$ is diffeomorphic to a neighborhood of the zero section in the associated bundle $T \times_H (\mathfrak{h}^\circ \times V/\Gamma)$. Since $\dim M = 2 \dim T$ and the action of T on M is effective, $\dim V/\Gamma = 2 \dim H$ and the action of H on V/Γ is effective. Therefore, by Lemma 6.1 H is a torus.

Consequently, $T \times_H (\mathfrak{h}^\circ \times V/\Gamma) = (K \times H) \times_H (\mathfrak{h}^\circ \times V/\Gamma) = T^*K \times V/\Gamma$. Note that the product $T^*K \times V/\Gamma$ has a natural product symplectic form, that the zero section of T^*K is isotropic in the product and that the symplectic slice at a point of the zero section is V/Γ. By Remark 3.6 there exist an invariant neighborhood U of the orbit $T \cdot x$ in M and an equivariant open symplectic embedding $\psi : U \to T^*K \times V/\Gamma$ such that $\psi(T \cdot x)$ is the zero section of T^*K.

Note that the moment map for the action of T on $T^*K \times V/\Gamma$ is the product of the maps $\phi_K : T^*K \to \mathfrak{k}^*$ and $\phi_{V/\Gamma} : V/\Gamma \to \mathfrak{h}^*$. Hence $\phi|_U = (\phi_K \times \phi_{V/\Gamma}) \circ \psi + a$, where we suppress the identification $\mathfrak{t}^* = \mathfrak{k}^* \times \mathfrak{h}^*$.

By Lemma 6.2, $\phi_{V/\Gamma}^{-1}(0)$ is a single point. Consequently, $\phi^{-1}(a) \bigcap U = T \cdot x$. Since $\phi^{-1}(a)$ is connected by Theorem 5.1 and since the orbit $T \cdot x$ is closed, it follows that $T \cdot x = \phi^{-1}(a)$. Since ϕ is proper, there exists a neighborhood W of a such that $\phi^{-1}(W) \subset U$. □

Theorem 6.4. *Let* (M, ω, T, ϕ) *be a compact symplectic toric orbifold. The image* $\phi(M)$ *is a* simple *rational polytope. Moreover, for every open facet* $\mathring{F}$ *of* $\phi(M)$*, there exists an integer* $m_{\mathring{F}}$ *such that the orbifold structure group of every* x *in* $\phi^{-1}(\mathring{F})$ *is* $\mathbb{Z}/m_{\mathring{F}}\mathbb{Z}$*. Thus, there is a labeled polytope* $(\phi(M), \{m_{\mathring{F}}\})$ *naturally associated to* (M, ω, T, ϕ)*.*

Proof. Given $x \in M$, let H be the isotropy group of x, let $H \to \mathrm{Sp}(V/\Gamma)$ be the symplectic slice representation, and let $\phi_{V/\Gamma} : V/\Gamma \to \mathfrak{h}^*$ be the associated moment map.

By Theorem 5.2, the image $\phi(M)$ is a rational convex polytope. By Lemma 6.3, for sufficiently small neighborhoods W of $a = \phi(x)$, we have $W \cap \phi(M) = W \cap (a + \mathfrak{h}^\circ \times \phi_{V/\Gamma}(V/\Gamma))$. By Lemma 6.2, $\phi_{V/\Gamma}(V/\Gamma)$ is a rational simplicial cone. Therefore, $\phi(M)$ is a rational simple polytope.

It follows from (6) and the fact that the cone $\phi_{V/\Gamma}(V/\Gamma)$ is strictly convex that $\phi(x)$ lies in a facet $\mathring{F}$ of $\phi(M)$ if and only if the isotropy group H of x is a circle. Consequently by part 3 of Lemma 6.2 the orbifold structure group Γ of x is cyclic.

By Lemma 6.3 the orbifold structure group is the same for every $x' \in \phi^{-1}(W \cap \mathring{F})$. Since $\mathring{F}$ is connected, the result follows. □

Proposition 6.5. *Let* $(M_i, \omega_i, T_i, \phi_i)$*,* $i = 1, 2$*, be two compact symplectic toric orbifolds. Suppose that both orbifolds have the same labeled polytope* $(\Delta, \{m_{\mathring{F}}\})$ *associated to them. Then for any point* $a \in \Delta$ *there exist a neighborhood* W *of* a *in* $\mathfrak{t}^* \supset \Delta$ *and an equivariant symplectomorphism* $\psi : {\phi_1}^{-1}(W) \to \phi_2^{-1}(W)$ *such that* $\phi_2 \circ \psi|_{{\phi_1}^{-1}(W)} = \phi_1|_{{\phi_1}^{-1}(W)}$*.*

Proof. Let $\mathring{F}$ be the open face of Δ containing a. By Lemma 6.3 the linear subspace of $\mathfrak{t}^*$ spanned by $\mathring{F}$ is the annihilator $\mathfrak{h}^\circ$ of the Lie algebra of a subtorus H of T. Moreover, H is the isotropy subgroup of any point in ${\phi_i}^{-1}(a)$ $(i = 1, 2)$. By Lemma 6.3 there exists an open neighborhood W_i of a in $\mathfrak{t}^*$ such that ${\phi_i}^{-1}(W_i)$ is equivariantly symplectomorphic to a neighborhood of $T/H \times \{0\}$ in $T^*(T/H) \times V_i/\Gamma_i$, where V_i/Γ_i is the symplectic slice representation of H at any point in ${\phi_i}^{-1}(a)$. Thus it is suffices to show that the two slice representations are the same, for we can then take $W = W_1 \cap W_2$.

By Lemma 6.3, the polytope Δ near a looks like $\mathfrak{h}^\circ \times \phi_{V_i/\Gamma_i}(V_i/\Gamma_i)$. Hence the cones $\phi_{V_i/\Gamma_i}(V_i/\Gamma_i)$ and the integer labels associated to the representations of H by Lemma 6.2 are the same. Hence the symplectic slice representations are identical, and we are done. $\square$

By combining the proof above with Lemma 6.2 we get a slightly stronger result.

Lemma 6.6. *Let (M, ω, T, ϕ) be a symplectic toric orbifold. The isotropy group and orbifold structure group of every point $x \in M$ can be read from the associated labeled polytope as follows:*

Let $\mathcal{F}(x)$ be the set of open facets of $\phi(M)$ whose closure contains $\phi(x)$. For each $\mathring{F} \in \mathcal{F}(x)$, let $\eta_{\mathring{F}} \in \mathfrak{t}$ denote the primitive outward normal, and let $m_{\mathring{F}}$ be the associated positive integer.

The isotropy group of x is the subtorus $H \subset T$ defined by the condition that its Lie algebra $\mathfrak{h}$ is the linear span of the normals $\eta_{\mathring{F}}$, for $\mathring{F} \in \mathcal{F}(x)$.

The orbifold structure group Γ is isomorphic to $\ell/\hat{\ell}$, where $\ell \subset \mathfrak{h}$ is the lattice of circle subgroups of H, and $\hat{\ell}$ is the lattice generated by $\{m_{\mathring{F}} \cdot \eta_{\mathring{F}}\}_{\mathring{F} \in \mathcal{F}(x)}$.

Remark 6.7. A useful consequence of Lemma 6.3 is the fact that for a compact symplectic toric orbifold (M, ω, T, ϕ), all the points in the interior of the polytope $\phi(M)$ are regular values of the moment map ϕ. More generally, for an open face $\mathring{F}$ of $\phi(M)$ of dimension k, the preimage $\phi^{-1}(\mathring{F})$ is a symplectic suborbifold of M of dimension $2k$ and $\phi|_{\phi^{-1}(\mathring{F})} \to \mathring{F}$ is a principal torus bundle with k-dimensional fibers.

In the example below we construct a 4-dimensional compact symplectic toric orbifold. We explicitly find the labeled polytope associated by Theorem 6.4. We then check the recipe given in Lemma 6.6 for reading the orbifold structure groups of every point from the labeled polytope.

Example 6.8. The map $A : \mathbb{Z}^3 \to \mathbb{Z}^2$ defined by the matrix

$$\begin{pmatrix} 3 & 0 & -2 \\ 0 & 5 & -1 \end{pmatrix}$$

induces a surjective group homomorphism $\bar{A}$ from the torus $\mathbb{T}^3 := \mathbb{R}^3/\mathbb{Z}^3$ to the torus $T := \mathbb{R}^2/\mathbb{Z}^2$. Let K be the kernel of $\bar{A}$. It is not hard to see that, under the identification of $\mathbb{T}^3$ with $U(1)^3$,

$$K = \{(\lambda^{10}, \lambda^3, \lambda^{15}) : \lambda \in U(1)\}.$$

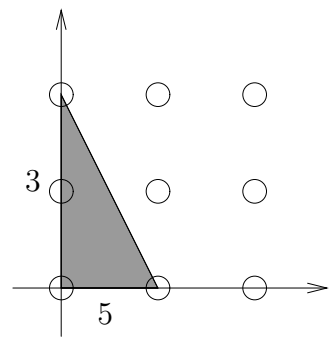
The product action of $U(1)^3$ on $\mathbb{C}^3$ is Hamiltonian with a moment map $\phi : \mathbb{C}^3 \to (\mathbb{R}^3)^*$ given by $\phi(z) = (|z_1|^2, |z_2|^2, |z_3|^2)$. The corresponding moment map ϕ_K for the action K on $\mathbb{C}^3$ is

$$\phi_K(z) = 10|z_1|^2 + 3|z_2|^2 + 15|z_3|^2.$$

Note that every non-zero value of ϕ_K is regular. In particular, $M := \phi_K^{-1}(30)/K$ is a symplectic orbifold. The Hamiltonian action of $\mathbb{T}^3$ on $\mathbb{C}^3$ induces a Hamiltonian action of $T \simeq \mathbb{T}^3/K$ on M, and the K-invariant map $\phi|_{\phi_K^{-1}(30)}$ descends to a map $\bar{\phi} : M \to (\mathbb{R}^3)^*$. The image $\bar{\phi}(M)$ is the intersection of the affine hyperplane $\alpha + \mathfrak{k}^\circ$ with $\phi(\mathbb{C}^3)$, where α is a point in $\phi(\phi_K^{-1}(30))$ and $\mathfrak{k}^\circ$ is the annihilator in $(\mathbb{R}^3)^*$ of the Lie algebra $\mathfrak{k}$ of K. More concretely, $\bar{\phi}(M) = \{(x_1, x_2, x_3) \in (\mathbb{R}^3)^* : x_i \geq 0 \text{ and } 10x_1 + 3x_2 + 15x_3 = 30\}$.

Since $\mathfrak{k}^\circ = Lie(\mathbb{T}^3/K)^*$ and $\mathbb{T}^3/K$ is isomorphic to T via the map induced by $\bar{A}$, the map $(\mathbb{R}^2)^* \to (\mathbb{R}^3)^*$ given by $y \mapsto A^T y + (0,0,2)$ identifies $Lie(T)^* = (\mathbb{R}^2)^*$ with the hyperplane $\mathfrak{k}^\circ + (0,0,2)$. Composing its inverse with $\bar{\phi}$ gives us a moment map $\Phi : M \to (\mathbb{R}^2)^*$ for the action of T on M.

Tracing through the identifications, we see that $\Phi(M)$ is the triangle with the vertices $(1,0)$, $(0,2)$, and $(0,0)$, as in the following picture:



Moreover, the coordinate hyperplanes map to the edges of the triangle $\Phi(M)$:

$$\begin{aligned}
\Phi((\phi_K^{-1}(30) \cap \{0\} \times \mathbb{C} \times \mathbb{C})/K) &= \{(0, 2t) : 0 \leq t \leq 1\}, \\
\Phi((\phi_K^{-1}(30) \cap \mathbb{C} \times \{0\} \times \mathbb{C})/K) &= \{(t, 0) : 0 \leq t \leq 1\}, \\
\Phi((\phi_K^{-1}(30) \cap \mathbb{C} \times \mathbb{C} \times \{0\})/K) &= \{(t, 2(1-t)) : 0 \leq t \leq 1\}.
\end{aligned}$$

For a point $z \in \{0\} \times \mathbb{C}^\times \times \mathbb{C}^\times$ the K-isotropy group is

$$K_z = \{\lambda \in U(1) : \lambda^3 = 1 \text{ and } \lambda^{15} = 1\} = \mathbb{Z}/3\mathbb{Z}.$$

For a point $z \in \mathbb{C}^\times \times \{0\} \times \mathbb{C}^\times$ the K-isotropy group is

$$K_z = \{\lambda \in U(1) : \lambda^{10} = 1 \text{ and } \lambda^{15} = 1\} = \mathbb{Z}/5\mathbb{Z}.$$

For a point $z \in \mathbb{C}^\times \times \mathbb{C}^\times \times \{0\}$ the K-isotropy group is

$$K_z = \{\lambda \in U(1) : \lambda^{10} = 1 \text{ and } \lambda^3 = 1\} = \{1\}.$$

Therefore the vertical edge of $\Phi(M)$ should be labeled "3," the horizontal edge "5," and the hypotenuse "1" (or not labeled).

The vertex $(1,0)$ corresponds to the point $a = (\sqrt{3}, 0, 0) \in \mathbb{C}^3$. The isotropy group $U(1)^3_a$ of a in $U(1)^3$ is $1 \times U(1) \times U(1)$. The isotropy group K_a of a in K is $\{(1, \lambda^3, \lambda^{15} : \lambda \in U(1) \text{ and } \lambda^1 0 = 1\}$. Therefore the corresponding point $[a] \in M$ is fixed by T and has orbifold structure group $\mathbb{Z}/10\mathbb{Z}$.

The vertex $(1,0)$ is contained in the closure of two open facets. The primitive outward normals of these facets are $(0,-1)$ and $(2,1)$; the integer labels are 5. These vectors span $\mathbb{R}^2$. Thus, by Lemma 6.6 $[a]$ should be a fixed point of T. This agrees with the computation above. Similarly, the lattice $\hat{\ell}$ is generated by $(0,-5)$ and $(2,1)$. It is easy to check that $\mathbb{Z}^2/\hat{\ell}$ is generated by $(1,0)$, and that this element

has order 10. Thus, Lemma 6.6 correctly predicts that orbifold structure group of $[a]$ is $\mathbb{Z}/10\mathbb{Z}$

Similar computations show that the orbifold structure group of the preimage of the vertex $(0,0)$ is $\mathbb{Z}/15\mathbb{Z}$ and that the orbifold structure group of the preimage of the vertex $(0,2)$ is $\mathbb{Z}/3\mathbb{Z}$. Both points are fixed by T.

7. From local to global

In this section, which is joint work with Chris Woodward, we show that two compact symplectic toric orbifolds which have isomorphic associated labeled polytopes are themselves isomorphic. First, we have already shown that they are *locally* isomorphic. By extending Proposition 2.4 in [HS] to the symplectic category, we show that these local isomorphisms can be glued together to construct a global isomorphism.

Two compact symplectic toric orbifolds (M,ω,T,ϕ) and (M',ω',T,ϕ') with $\phi(M)=\phi'(M')=\Delta$ are said to be *locally isomorphic over* Δ if every point in Δ has a neighborhood U such that ${\phi'}^{-1}(U)$ and $\phi^{-1}(U)$ are isomorphic as symplectic toric orbifolds.

Definition 7.1. Let (M,ω,T,ϕ) be a compact symplectic toric orbifold; let $\Delta=\phi(M)$. Define a sheaf $\mathcal{S}$ over Δ as follows: for each open set $U\subset\Delta$, $\mathcal{S}(U)$ is the set of isomorphisms of $\phi^{-1}(U)$.

Lemma 7.2. *Let (M,ω,T,ϕ) be a compact symplectic toric orbifold. Define a sheaf $\mathcal{S}$ over $\Delta=\phi(M)$ as in Definition 7.1. The cohomology group $H^1(\Delta,\mathcal{S})$ classifies (up to an isomorphism) compact symplectic toric orbifolds (M',ω',T,ϕ') such that $\phi'(M')=\Delta$ and M' is locally isomorphic to M over Δ.*

Proof. Let $\mathcal{U}=\{U_i\}_{i\in I}$ be a covering of Δ such that there is an isomorphism $h_i:\phi^{-1}(U_i)\to{\phi'}^{-1}(U_i)$ for each $i\in I$. Define $f_{ij}:\phi^{-1}(U_i\cap U_j)\to\phi^{-1}(U_i\cap U_j)$ by $f_{ij}=h_i^{-1}\circ h_j$. The set $\{f_{ij}\}$ is a closed cochain in $C^1(\mathcal{U},\mathcal{S})$. Moreover, the cohomology class of this cocycle is independent of the choices of the isomorphisms h_i.

Conversely, given a cocycle $\{f_{ij}\}\in C^1(\mathcal{U},\mathcal{S})$, we can construct a compact symplectic toric orbifold with moment polytope Δ by taking the disjoint union of the $\phi^{-1}(U_i)$'s and gluing $\phi^{-1}(U_i)$ and $\phi^{-1}(U_j)$ together using the isomorphisms f_{ij}. $\square$

Proposition 7.3. *Let (M,ω,T,ϕ) be a compact symplectic toric orbifold. Define a sheaf $\mathcal{S}$ over $\Delta=\phi(M)$ as in Definition 7.1. The sheaf $\mathcal{S}$ is abelian and the cohomology group $H^i(\Delta,\mathcal{S})$ is 0 for all $i>0$.*

Proof. Let $\underline{\ell\times\mathbb{R}}$ denote the sheaf of locally constant functions with values in $\ell\times\mathbb{R}$, where $\ell\subset\mathfrak{t}$ is the lattice of circle subgroups. Since Δ is contractible, $H^i(\Delta,\underline{\ell\times\mathbb{R}})=0$ for all $i>0$.

Define a sheaf $\mathcal{C}^\infty$ over Δ as follows: for each open set $U\subset\Delta$, $\mathcal{C}^\infty(U)$ is the set of smooth T invariant functions on $\phi^{-1}(U)$. We may think of elements of $\mathcal{C}^\infty(U)$ as continuous functions on U which pull back to smooth functions on $\phi^{-1}(U)$. [5]

Since $\mathcal{C}^\infty$ is a fine sheaf, $H^i(\Delta,C^\infty)=0$ for all $i>0$.

[5] One can show, using Lemma 6.3 and a theorem of G.W. Schwarz [Sch], and that for every T invariant smooth function f on $\phi^{-1}(U)$ there exists a smooth function $\tilde{f}$ on $\mathfrak{t}^*$ with $f=\tilde{f}\circ\phi$ on $\phi^{-1}(U)$.

Therefore, to prove that $\mathcal{S}$ is abelian and that $H^i(\mathcal{S}, \Delta) = 0$ for $i > 0$, it suffices to construct the following sequence of sheaves, and to show that it is exact:

$$0 \to \underline{\ell \times \mathbb{R}} \xrightarrow{j} \mathcal{C}^\infty \xrightarrow{\Lambda} \mathcal{S} \to 0.$$

Define $j : \underline{\ell \times \mathbb{R}} \to \mathcal{C}^\infty$ as follows: given $(\xi, c) \in \ell \times \mathbb{R}$ and a point $x \in M$, let $j(\xi, c)(x) = c + \langle \xi, \phi(x) \rangle$.

Next, we construct the map $\Lambda : C^\infty \to \mathcal{S}$. Let $U \subset \Delta$ be an open set and let $f : \phi^{-1}(U) \to \mathbb{R}$ be a smooth T invariant function. The flow of the Hamiltonian vector field Ξ_f of f on $\phi^{-1}(U)$ is T equivariant and preserves the moment map ϕ. Define $\Lambda(f)$ to be the time one flow of Ξ_f. We now show that the sequence of sheaves is exact.

The map j is clearly injective.

Recall that for every vector $\xi \in \mathfrak{t}$ there exists a vector field ξ_M on M induced by the action of T, that ξ_M is the Hamiltonian vector field of the function $\langle \xi, \phi(x) \rangle$. The time t flow of ξ_M is given by

$$x \mapsto e^{t\xi} \cdot x, \tag{7}$$

where $\xi \mapsto e^\xi$ is the exponential map from $\mathfrak{t}$ to T. Therefore $\operatorname{im} j \subset \ker \Lambda$.

To show that $\operatorname{im} j \supset \ker \Lambda$ we argue as follows. Without loss of generality we may assume that the subset U of Δ is the intersection of Δ with a ball in $\mathfrak{t}^*$. Let U_0 be the intersection of this ball with the interior $\mathring{\Delta}$ of the polytope Δ. Then both U and U_0 are convex, hence contractible. It follows from Remark 6.7 that $\phi^{-1}(U_0)$ is open and dense in $\phi^{-1}(U)$ and $\phi : \phi^{-1}(U_0) \to U_0$ is a principal T bundle.

Let $f : \phi^{-1}(U) \to \mathbb{R}$ be a T invariant function with $\Lambda(f) = id$. We want to show that $df = d\langle \xi, \phi \rangle$ on $\phi^{-1}(U)$ for some $\xi \in \ell$. It is enough to show that this equality holds on $\phi^{-1}(U_0)$.

Since f is T invariant and since $\phi : \phi^{-1}(U_0) \to U_0$ is a principal T bundle, there exists $h \in C^\infty(U_0)$ with $f = h \circ \phi$. Hence the Hamiltonian vector field Ξ_f of f at the points $x \in \phi^{-1}(U_0)$ is given by

$$\Xi_f(x) = (dh(\phi(x)))_M(x)$$

(since $dh(\phi(x)) \in T^*_{\phi(x)}\mathfrak{t}^* = \mathfrak{t}$, the expression $(dh(\phi(x)))_M$ makes sense).

Equation (7) implies that if $X : \phi^{-1}(U) \to \mathfrak{t}$ is a T invariant function and Y is a vector field on $\phi^{-1}(U)$ defined by $Y(x) = (X(x))_M(x)$, then the time t flow ψ_t of Y is given by

$$\psi_t : x \mapsto e^{tX(x)} \cdot x. \tag{8}$$

Consequently the time one flow $\Lambda(f)$ of Ξ_f is given by

$$\Lambda(f) : x \mapsto e^{dh(\phi(x))} \cdot x, \qquad x \in \phi^{-1}(U_0).$$

Since by assumption $\Lambda(f)(x) = x$ for all $x \in \phi^{-1}(U_0)$, we have $dh(u) \in \ell$ for all $u \in U_0$. Since U_0 is connected and since ℓ is discrete, the continuous function $dh : U_0 \to \ell$ is constant. Thus $df = d\langle \xi, \phi \rangle$ for some $\xi \in \ell$ and all $x \in \phi^{-1}(U_0)$.

The final step is to show that Λ is surjective. If the ball used in the definition of the set U is small enough, then by Lemma 6.3 the set $\phi^{-1}(U)$ is a tubular neighborhood of some T orbit in $\phi^{-1}(U)$. Let ψ be an isomorphism of $\phi^{-1}(U)$. We must show that there exists a T invariant function on $\phi^{-1}(U)$ whose time one flow is the map ψ.

Since ψ is an isomorphism, it is, a fortiori, a T-equivariant diffeomorphism of $\phi^{-1}(U)$ which preserves orbits. Therefore, by Theorem 3.1 in [HS], there exists a smooth T invariant map $\sigma : \phi^{-1}(U) \to T$ such that $\psi(x) = \sigma(x) \cdot x$. Since U is contractible and σ is T invariant, there exists a smooth map $X : \phi^{-1}(U) \to \mathfrak{t}$ such that $e^{X(x)} = \sigma(x)$. As before, define a vector field Y on $\phi^{-1}(U)$ by $Y(x) = (X(x))_M(x)$. By equation (8) the time one flow of Y is $x \mapsto e^{X(x)} \cdot x = \sigma(x) \cdot x = \psi(x)$. Thus it is enough to show that Y is a Hamiltonian vector field, i.e., that the contraction $\iota(Y)\omega$ is exact.

Just as for a free action of a compact Lie group on a manifold, we can, following Koszul [Ko], define on the orbifold M a *complex of basic forms*. Namely, a form $\alpha \in \Omega(M)$ is *basic* if α is T invariant and if for any vector $\xi \in \mathfrak{t}$, we have $\iota(\xi_M)\alpha = 0$. Similarly we can define basic forms on any open T invariant subset of M, such as $\phi^{-1}(U)$. We observe that basic forms have two properties.

1. A basic form α is preserved by any T equivariant map $\psi : M \to M$ which induces the identity map on the orbit space M/T, that is, $\psi^*\alpha = \alpha$. This is true because it is a closed condition, which holds on the open dense smooth subset of the orbifold M where the action is free.
2. The integral of a basic k form over a k cycle which lies entirely in a T orbit is zero. It follows that the cohomology of the complex of basic forms on a tubular neighborhood of an orbit is trivial. In other words, a closed basic form is exact on a tubular neighborhood of an orbit.

We now argue that the contraction $\iota(Y)\omega$ is a closed basic form on the tubular neighborhood $\phi^{-1}(U)$. Then, by property (2) above, there exists a basic zero form f such that $\iota(Y)\omega = df$.

Since Y and ω are T invariant, $\iota(Y)\omega$ is T invariant. Since the T orbits are isotropic in M, and since Y is tangent to T orbits, $\iota(\xi_M)\iota(Y)\omega = 0$ for any $\xi \in \mathfrak{t}$. The Lie derivative $L_Y\omega = d\iota(Y)\omega$ is also basic, since basic forms are a subcomplex. Consequently, since the time t flow ψ_t of Y induces the identity map on the orbit space, we have by property (1) above that $\psi_t^* L_Y\omega = L_Y\omega$. Since $\psi_1 = \psi$ and since ψ is symplectic, we have

$$\begin{aligned} 0 = \psi_1^*\omega - \omega &= \int_0^1 \frac{d}{dt}\psi_t^*\omega\, dt \\ &= \int_0^1 \psi_t^* L_Y\omega\, dt \\ &= \int_0^1 L_Y\omega\, dt = L_Y\omega = d\iota(Y)\omega. \end{aligned}$$

This proves there exists a T invariant function f whose time 1 flow is the isomorphism ψ. Hence Λ is surjective. □

Theorem 7.4. *Two compact symplectic toric orbifolds which have isomorphic associated labeled polytopes are themselves isomorphic.*

Proof. Without loss of generality, we may assume that the labeled polytopes associated to two orbifolds M and M' are equal. By Lemma 6.5, M and M' are locally isomorphic. By Proposition 7.3, $H^1(\Delta, \mathcal{S}) = 0$. By Lemma 7.2, this implies that M and M' are isomorphic as symplectic toric orbifolds. □

Remark 7.5. Given any labeled polytope Δ, one can construct local models for the symplectic toric orbifold associated to Δ. Since we've shown that $H^2(\mathcal{S}, \Delta) = 0$, an argument in [HS] similar to Lemma 7.2 shows that there exists a symplectic toric orbifold which corresponds to the given labeled polytope. In section 8, we give a more explicit construction.

8. Existence

Given any labeled polytope, we construct a compact symplectic toric orbifold such that its associated labeled polytope is the one which we began with. This construction is a slight variation of Delzant's construction [D]. A concrete example of the construction was worked out in Example 6.8.

Theorem 8.1. *Let T be a torus. Let $\mathfrak{t}$ denote its Lie algebra, and let $\ell \subset \mathfrak{t}$ denote the lattice of circle subgroups. Given a simple rational polytope $\Delta \subset \mathfrak{t}^*$ and a positive integer $m_{\mathring{F}}$ attached to each open facet $\mathring{F}$ of Δ, there exists a compact symplectic toric orbifold (M, ω, T, ϕ) such that $\phi(M) = \Delta$ and the orbifold structure group of a point in M which maps to a open facet $\mathring{F}$ is $\mathbb{Z}/m_{\mathring{F}}\mathbb{Z}$.*

Moreover, (M, ω, T, ϕ) is a symplectic reduction of $\mathbb{C}^N$ by an abelian subgroup of $\mathrm{SU}(N)$.

Proof. The polytope Δ can be written uniquely as

$$\Delta = \bigcap_{i=1}^{N} \{\beta \in \mathfrak{t}^* \mid \langle \beta, m_i y_i \rangle \leq \eta_i\},$$

where N is the number of facets, the vector $y_i \in \ell$ is the primitive normal to the ith facet, m_i is the integer attached to the ith open facet, and $\eta = (\eta_1, \cdots, \eta_N) \in (\mathbb{R}^N)^*$.

Define a linear projection $\varpi : \mathbb{R}^N \to \mathfrak{t}$ by $\varpi(e_i) = m_i y_i$, where $\{e_i\}$ is the standard basis for $\mathbb{R}^N$. This defines a short exact sequence and its dual:

$$0 \longrightarrow \mathfrak{k} \xrightarrow{j} \mathbb{R}^N \xrightarrow{\varpi} \mathfrak{t} \longrightarrow 0$$

and

$$0 \longrightarrow \mathfrak{t}^* \xrightarrow{\varpi^*} (\mathbb{R}^N)^* \xrightarrow{j^*} \mathfrak{k}^* \longrightarrow 0.$$

Let K denote the kernel of the map from $\mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$ to $T = \mathfrak{t}/\ell$ which is induced by ϖ. The kernel is given by

$$K = \{[\alpha] \in \mathbb{R}^N/\mathbb{Z}^N \mid \sum_{i=1}^{N} \alpha_i m_i y_i \in \ell\}.$$

The Lie algebra of K is $\mathfrak{k}$, the kernel of ϖ.

Consider $\mathbb{C}^N$ with the standard symplectic form $\sum \sqrt{-1} dz_j \wedge d\bar{z}_j$. The standard $\mathbb{T}^N$-action on $\mathbb{C}^N$ has moment map

$$\phi_{\mathbb{T}^N}(z_1, \ldots, z_N) = \sum_{i=1}^{N} |z_i|^2 e_i^* = (|z_1|^2, \ldots, |z_N|^2),$$

where $\{e_i^*\}$ is the basis dual to $\{e_i\}$. Clearly, $\langle \phi_{\mathbb{T}^N}(z), e_i \rangle \geq 0$ for all i, and $\langle \phi_{\mathbb{T}^N}(z), e_i \rangle = 0$ exactly if $z_i = 0$. Since K is a subgroup of $\mathbb{T}^N$, K acts on $\mathbb{C}^N$ with moment map $\phi_K = j^* \circ \phi_{\mathbb{T}^N}$.

The stabilizer of a point $z \in \mathbb{C}^N$ in $\mathbb{T}^N$ is

$$\mathbb{T}^N_z = \{[\alpha] \in \mathbb{R}^N/\mathbb{Z}^N \mid e^{2\pi\sqrt{-1}\alpha_i} = 1 \text{ for all } i \text{ with } z_i \neq 0\}.$$

Since the stabilizer of $z \in \mathbb{C}^N$ in K is $\mathbb{T}^N_z \cap K$,

$$K_z = \{[\alpha] \in \mathbb{R}^N/\mathbb{Z}^N \mid e^{2\pi\sqrt{-1}\alpha_i} = 1 \text{ for all } i \text{ with } z_i \neq 0 \text{ and } \sum_{i=1}^N \alpha_i m_i y_i \in \ell\}.$$

Define an affine embedding $\iota_\eta : \mathfrak{t}^* \to (\mathbb{R}^N)^*$ by $\iota_\eta(\beta) = \varpi^*(\beta) - \eta$. Note that

$$\begin{aligned} \iota_\eta(\Delta) &= \{\xi \in (\mathbb{R}^N)^* \mid \xi \in \varpi^*(\mathfrak{t}^*) - \eta \text{ and } \xi_i \geq 0 \text{ for all } i\} \\ &= \{\xi \in (\mathbb{R}^N)^* \mid \xi \in (j^*)^{-1}(j^*(-\eta)) \text{ and } \xi_i \geq 0 \text{ for all } i\}. \end{aligned}$$

Moreover,

$$\iota_\eta(\Delta \cap \{\beta \in \mathfrak{t}^* \mid \langle \beta, m_j y_j \rangle = \eta_j\}) = \iota_\eta(\Delta) \cap \{\xi \in (\mathbb{R}^N)^* \mid \xi_j = 0\}.$$

For every $z \in \mathbb{C}^N$ such that $\phi_K(z) = j^*(-\eta)$, $\phi_{\mathbb{T}^N}(z)$ is in $\iota_\eta(\mathfrak{t}^*)$. Since Δ is simple, for every point $\beta \in \Delta$ the set

$$\{y_i \mid \langle \beta, m_i y_i \rangle = \eta_i\}$$

is linearly independent. Consequently the set

$$\{y_i \mid z_i = 0\} = \{y_i \mid \phi_{\mathbb{T}^N}(z)_i = 0\}$$

is linearly independent. Hence the isotropy group K_z is discrete. Therefore, $j^*(-\eta)$ is a regular value of ϕ_K, and the reduced space $M = \phi_K^{-1}(j^*(-\eta))/K$ is a symplectic toric orbifold.

Since the action of $\mathbb{T}^N$ on $\mathbb{C}^N$ commutes with the action of K, it induces a Hamiltonian action of $\mathbb{T}^N$ on M. Moreover, the moment map $\phi_{\mathbb{T}^N}$ descends to a moment map $\tilde{\phi} : M \to (\mathbb{R}^N)^*$, and $\tilde{\phi}(M) = \iota_\eta(\Delta)$. In fact the action of $\mathbb{T}^N$ on M descends to a Hamiltonian action of $T = \mathbb{T}^N/K$, and $\phi_T = (\iota_\eta|_\Delta)^{-1} \circ \tilde{\phi}$ is a corresponding moment map.

We claim that the action of T on M is effective. It suffices to show that there exists a point $z \in \phi_K^{-1}(j^*(-\eta))$ so that its isotropy group in $\mathbb{T}^N$ is trivial. Such a point exists because the isotropy group in $\mathbb{T}^N$ of any point $z \in \phi_{\mathbb{T}^N}^{-1}(\{\xi \in (\mathbb{R}^N)^* \mid \xi_i > 0 \text{ for all } i\})$ is trivial, the embedding ι_η maps the interior of the polytope Δ into the set $\{\xi \in (\mathbb{R}^N)^* \mid \xi > 0 \text{ for all } i\})$ and Δ has non-empty interior.

It remains to show that the orbifold structure group of a point $[z]$ in M mapping to the interior of the facet cut out by the hyperplane $\{\beta \in \mathfrak{t}^* \mid \langle \beta, m_j y_j \rangle = \eta_j\}$ is $\mathbb{Z}/m_j\mathbb{Z}$. But $[z]$ lies in the interior of this facet if and only if $\phi_{\mathbb{T}^N}(z) \in \{\xi \in (\mathbb{R}^N)^* \mid \xi_j = 0 \text{ and } \xi_i \neq 0 \text{ for } i \neq j\}$. For such a point z the isotropy group K_z is $\mathbb{Z}/m_j\mathbb{Z}$. □

9. Compatible complex structures

In this section, we show that every compact symplectic toric orbifold possesses a invariant complex structure compatible with the symplectic form. Moreover, suppose two compact symplectic toric orbifolds (M, ω, T, ϕ) and (M', ω', T, ϕ') are given invariant complex structure which are compatible with their symplectic forms. They are equivariantly biholomorphic exactly if the polytopes $\phi(M)$ and $\phi'(M')$ have the same fan.

Theorem 9.1. *Every compact symplectic toric orbifold possesses an invariant complex structure which is compatible with its symplectic form, i.e., every such orbifold has an invariant Kähler structure.*

Proof. Let (M, ω, T, ϕ) be a compact symplectic toric orbifold.

By Theorem 6.4, $\Delta = \phi(M)$ is a simple rational polytope, and for each open facet $\mathring{F}$ of Δ, there exists a positive integer $m_{\mathring{F}}$ such that $\mathbb{Z}/m_{\mathring{F}}\mathbb{Z}$ is the orbifold structure group of every point in M which maps to $\mathring{F}$.

By Theorem 8.1, there exists a compact symplectic toric orbifold (M', ω', T, ϕ') which is a symplectic reduction of $\mathbb{C}^N$ by an abelian subgroup of $\mathrm{SU}(N)$, such that $\phi'(M) = \Delta$ and such that for every open facet $\mathring{F}$ of Δ, $\mathbb{Z}/m_{\mathring{F}}\mathbb{Z}$ is the orbifold structure group of every point in M' which maps to $\mathring{F}$. Since M' is the reduction of a Kähler manifold by a group which preserves its Kähler structure, by Theorem 3.5 in [GS2] M' possesses an equivariant Kähler structure which is compatible with its symplectic form.

By Theorem 7.4, M and M' are equivariantly symplectomorphic; therefore, M inherits an an invariant Kähler structure which is compatible with its symplectic form. □

Not only do all compact symplectic toric orbifolds admit compatible complex structures, they are, in fact, algebraic varieties.

Lemma 9.2. *Let (M, ω, T, ϕ) be a compact symplectic toric orbifold and let J be a T-invariant complex structure on M which is compatible with the symplectic form ω. Then M has the structure of an algebraic toric variety with the fan equal to the fan defined by the polytope $\phi(M)$.*

Remark 9.3. If the class of the Kähler form ω in $H^2(M)$ is rational, then by the Kodaira-Baily embedding theorem [Ba], M is a projective algebraic variety. The projective embedding provided by the Kodaira-Baily theorem is equivariant with respect to the action of the torus T, and so M is a projective toric variety. The class $[\omega]$ is rational if and only if the edges of the polytope $\phi(M)$ are rational vectors relative to the weight lattice of T. Therefore, in order to prove that *all* toric orbifolds are algebraic, we argue differently.

An immediate consequence of Lemma 9.2 is the following theorem.

Theorem 9.4. *Let two compact symplectic toric orbifolds*

$$(M, \omega, T, \phi) \quad \textit{and} \quad (M', \omega', T, \phi')$$

be given invariant complex structures which are compatible with their symplectic forms. The orbifolds are equivariantly biholomorphic exactly if the fans defined by their polytopes are equal.

Remark 9.5. This theorem shows, in particular, that if a compact symplectic toric orbifold admits two different compatible complex structures, they are equivariantly biholomorphic. In contrast, the Kähler structure is not unique. For instance, there are many S^1 invariant Kähler structures on S^2.

Remark 9.6. There are several reasons for the difference of the classification of symplectic toric orbifolds and of algebraic toric varieties.

The first reason is that some toric varieties do not admit any symplectic form. These correspond to fans which do not come from a polytope. The second reason is

that changing the cohomology class of the symplectic form corresponds to changing the length of the edges of the polytope. This information is lost in the algebraic category. Finally, the integers attached to the faces are lost. It is easy to see why. Give $\mathbb{C}$ the standard Kähler structure, and let $\mathbb{Z}/m\mathbb{Z}$ act on $\mathbb{C}$. The orbifolds $\mathbb{C}$ and $\mathbb{C}/(\mathbb{Z}/m\mathbb{Z})$ are not diffeomorphic, but they are $\mathbb{C}^\times$ equivariantly biholomorphic.

Proof of Lemma 9.2. Since the complex structure J is T-invariant, the action of T on M extends to the action of the complexification $T_\mathbb{C}$ of T. The action of $T_\mathbb{C}$ on M has a dense open orbit. Denote it by $T_\mathbb{C} \cdot m$.

The action of $T_\mathbb{C}$ on M can be linearized near fixed points. That is, if $x \in M$ is fixed by T, there exist a $T_\mathbb{C}$-invariant neighborhood V of 0 in T_xM, a $T_\mathbb{C}$ invariant neighborhood of x in M, and a biholomorphic map $f : Y \to U$ which is $T_\mathbb{C}$-equivariant. There are several ways to see that the linearization exists. For example, the linearization proof for group actions on Kähler manifolds in [LS] is natural and so, by Remark 2.1, translates into a proof in the orbifold case. Alternatively we can appeal to the holomorphic slice theorem in [HL], which holds for Kähler *spaces*, hence in particular, for orbifolds.

Since the action of T on M and hence on T_xM is faithful and since $\dim T = \frac{1}{2}\dim M$, the neighborhood V is all of T_xM and T_xM is a toric variety. The linearization map embeds this variety into M. The fan of this variety consists of a single simplicial cone together with its faces. Moreover, this is the cone dual to the image of the moment map corresponding to the linear action of T on the tangent space T_xM.

If y is another fixed point and $h : T_yM \to M$ another linearization, then both images $f(T_xM)$ and $h(T_yM)$ must contain the dense open orbit $T_\mathbb{C} \cdot m$ of $T_\mathbb{C}$. It is not hard to see that the transition map from $f^{-1}(T_\mathbb{C} \cdot m)$ to $h^{-1}(T_\mathbb{C} \cdot m)$ is rational and, in fact, is the same as the map defined by the intersection of the corresponding fans. The lemma now follows. □

Appendix A. Symplectic weights

In this section we show that weights for symplectic representations of tori are well defined by proving the following lemma.

Lemma A.1. *There is a bijective correspondence between isomorphism classes of $2n$ dimensional symplectic representations of a torus H and unordered n-tuples of elements (possibly with repetition) of the weight lattice $\ell^* \subset \mathfrak{h}^*$ of H.*

Let (V, ω) be a $2n$ dimensional symplectic vector space. Let $\rho : H \to \mathrm{Sp}(V, \omega)$ be a symplectic representation with weights $(\beta_1, \dots, \beta_n)$. There exist a decomposition $(V, \omega) = \bigoplus_i (V_i, \omega_i)$ into invariant mutually perpendicular 2-dimensional symplectic subspaces and an invariant norm $|\cdot|$ compatible with the symplectic form $\omega = \bigoplus \omega_i$ so that the representation of H on (V_i, ω_i) has weight β_i and the moment map $\phi_\rho : V \to \mathfrak{h}^*$ is given by

$$\phi_\rho(v_1, \dots, v_n) = \sum |v_i|^2 \beta_i \quad \text{for all } v = (v_1, \dots, v_n) \in \bigoplus_i V_i. \tag{9}$$

Proof. Since symplectic representations do not have naturally defined complex structures, we define a *character* of a torus H to be a homomorphism into an *oriented* circle. A *weight* is the differential of a character.

A symplectic representation $\rho : H \to \mathrm{Sp}(V, \omega)$ on a symplectic 2-plane (V, ω) has a well defined character. Assuming that ρ is non-trivial, the image $\rho(H)$ is

a compact abelian subgroup of $\mathrm{Sp}(V,\omega)$, and hence is a circle. Circle subgroups of $\mathrm{Sp}(V,\omega)$ are naturally oriented by the symplectic form ω via their orbits in V. Additionally, any two circle subgroups of $\mathrm{Sp}(V,\omega)$ are conjugate by an element of $\mathrm{Sp}(V,\omega)$, and conjugation preserves the induced orientations. We conclude that a conjugacy class of symplectic representations of a torus on a symplectic 2-plane has a well-defined character (and hence a well-defined weight) which determines this conjugacy class uniquely.

Therefore, we may assume that the plane is $\mathbb{C}$ with symplectic form $\sqrt{-1}dz\wedge d\bar{z}$ and that the action of a torus H is given by $(a,z)\mapsto e^{i\beta(\log a)}z$, where β is a weight of H. In agreement with formula (9), the moment map for this action is given by $z\mapsto |z|^2\beta$ and the image of the plane is a ray $\mathbb{R}_{\geq 0}\beta$ through the weight β.

By contrast, for the underlying real representation the weight is defined only up to a sign. For example, conjugation by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ sends the matrix

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

to

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Since a maximal compact subgroup of $\mathrm{Sp}(\mathbb{R}^{2n})\simeq \mathrm{Sp}(V,\omega)$ is isomorphic to $U(n)$, there exists on V an H invariant complex structure J which is compatible with the symplectic form, i.e., $\omega(J\cdot,J\cdot)=\omega(\cdot,\cdot)$ and $g(\cdot,\cdot)=\omega(\cdot,,J\cdot)$ is positive definite. Then V decomposes as a direct sum of mutually orthogonal invariant complex lines, $(V,\omega)=\bigoplus_{i=1}^n(V_i,\omega_i)$. These V_i's are mutually symplectically orthogonal invariant 2 planes. The moment map is given by formula (9), where $(\beta_1,\dots,\beta_n)$ are the corresponding weights.

Although this decomposition is not natural, we will show that if a symplectic representation (V,ω) of a torus H has two different decompositions, $(V,\omega)=\bigoplus_{i=1}^n(V_i,\omega_i)=\bigoplus_{i=1}^n(V_i',\omega_i')$ with weights $(\beta_1,\dots,\beta_n)$ and $(\beta_1',\dots,\beta_n')$, then the two n-tuples of weights are the same up to a permutation. For a weight $\alpha\in\ell^*$ of the torus H the isotypical subspace $W_\alpha=\bigoplus_{\beta_i=\pm\alpha}V_i$ is canonically defined for the underlying real representation. Therefore, if a weight β_i occurs in the first decomposition of (V,ω), then the subspace $W_{\beta_i}=\bigoplus_{\beta_j'=\pm\beta_i}V_j'$ is nonempty. The image of W_{β_i} under the moment map is the Minkowski sum of the rays through the weights β_j' such that $\beta_j'=\pm\beta_i$. On the other hand, neither the vector space W_{β_i} nor its image under the moment map depends on the decomposition of V into symplectic planes. Since the image contains the ray through β_i, there must exist β_j' with $\beta_i=\beta_j'$. We can then split off the 2 plane corresponding to β_i and repeat the above argument. □

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