SPHERICAL CLASSES AND THE ALGEBRAIC TRANSFER

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ABSTRACT. We study a weak form of the classical conjecture which predicts that there are no spherical classes in $Q_0S^0$ except the elements of Hopf invariant one and those of Kervaire invariant one. The weak conjecture is obtained by restricting the Hurewicz homomorphism to the homotopy classes which are detected by the algebraic transfer.

Let $P_k = \mathbb{F}_2[x_1, \ldots, x_k]$ with $|x_i| = 1$. The general linear group $GL_k = GL(k, \mathbb{F}_2)$ and the (mod 2) Steenrod algebra $\mathcal{A}$ act on $P_k$ in the usual manner. We prove that the weak conjecture is equivalent to the following one: The canonical homomorphism $j_k^*: \mathbb{F}_2 \otimes \mathcal{A}(P_k^G) \rightarrow \mathbb{F}_2 \otimes \mathcal{A}(P_k^G)$ induced by the identity map on $P_k$ is zero in positive dimensions for $k > 2$. In other words, every Dickson invariant (i.e. element of $P_k^G$) of positive dimension belongs to $A^+ \cdot P_k$ for $k > 2$, where $A^+$ denotes the augmentation ideal of $A$. This conjecture is proved for $k = 3$ in two different ways. One of these two ways is to study the squaring operation $Sq^0$ on $P(\mathbb{F}_2 \otimes P_k^G)$, the range of $j_k^*$, and to show it commuting through $j_k^*$ with Kameko’s $Sq^0$ on $P(\mathbb{F}_2 \otimes P_k^G)$, the domain of $j_k^*$. We compute explicitly the action of $Sq^0$ on $P(\mathbb{F}_2 \otimes P_k^G)$ for $k \leq 4$.

1. INTRODUCTION

The paper deals with the spherical classes in $Q_0S^0$, i.e. the elements belonging to the image of the Hurewicz homomorphism

$$H : \pi_*(S^0) \cong \pi_*(Q_0S^0) \rightarrow H_*(Q_0S^0).$$

Here and throughout the paper, the coefficient ring for homology and cohomology is always $\mathbb{F}_2$, the field of 2 elements.

We are interested in the following classical conjecture.

Conjecture 1.1. (conjecture on spherical classes). There are no spherical classes in $Q_0S^0$, except the elements of Hopf invariant one and those of Kervaire invariant one.

(See Curtis [9] and Wellington [21] for a discussion.)

Let $V_k$ be an elementary abelian 2-group of rank $k$. It is also viewed as a $k$-dimensional vector space over $\mathbb{F}_2$. So, the general linear group $GL_k = GL(k, \mathbb{F}_2)$
acts on $V_k$ and therefore on $H^*(BV_k)$ in the usual way. Let $D_k$ be the Dickson algebra of $k$ variables, i.e. the algebra of invariants

$$D_k := H^*(BV_k)^{GL_k} \cong \mathbb{F}_2[x_1, \ldots, x_k]^{GL_k},$$

where $P_k = \mathbb{F}_2[x_1, \ldots, x_k]$ is the polynomial algebra on $k$ generators $x_1, \ldots, x_k$, each of dimension 1. As the action of the (mod 2) Steenrod algebra (weak conjecture on spherical classes).

Conjecture 1.1. $\varphi_k = 0$ in any positive stem $i$ for $k > 2$.

It is well known that the Ext group has intensively been studied, but remains very mysterious. In order to avoid the shortage of our knowledge of the Ext group, we want to restrict $\varphi_k$ to a certain subgroup of Ext which (1) is large enough and worthwhile to pursue and (2) could be handled more easily than the Ext itself. To this end, we combine the above data with Singer's algebraic transfer.

Singer defined in [20] the algebraic transfer

$$Tr_k : \mathbb{F}_2 \otimes_{GL_k} PH_1(BV_k) \to Ext_{A}^{k,k+i}(\mathbb{F}_2, \mathbb{F}_2),$$

where $PH_*(BV_k)$ denotes the submodule consisting of all $A$-annihilated elements in $H_*(BV_k)$. It is shown to be an isomorphism for $k \leq 2$ by Singer [20] and for $k = 3$ by Boardman [4]. Singer also proved that it is an isomorphism for $k = 4$ in a range of internal degrees. But he showed it is not an isomorphism for $k = 5$. However, he conjectures that $Tr_k$ is a monomorphism for any $k$.

Our main idea is to study the restriction of $\varphi_k$ to the image of $Tr_k$.

Conjecture 1.3. (weak conjecture on spherical classes).

$$\varphi_k \cdot Tr_k : \mathbb{F}_2 \otimes_{GL_k} PH_*(BV_k) \to P(\mathbb{F}_2 \otimes_{GL_k} H_*(BV_k)) := (\mathbb{F}_2 \otimes D_k)^*$$

is zero in positive dimensions for $k > 2$. 
In other words, there are no spherical classes in $Q_0 S^0$, except the elements of Hopf invariant one and those of Kervaire invariant one, which can be detected by the algebraic transfer.

A natural question is: How can one express $\varphi_k \cdot Tr_k$ in the framework of invariant theory alone, and without using the mysterious Ext group?

Let $j_k : \mathbb{F}_2 \otimes (P_k^{GL})_{\mathcal{A}} \to (\mathbb{F}_2 \otimes P_k)_{\mathcal{A}}^{GL}$ be the natural homomorphism induced by the identity map on $P_k$. We have

**Proposition 3.1.** $j_k$ are equipped with canonical coalgebra structures. We get

$$j_k : \mathbb{F}_2 \otimes (P_k^{GL})_{\mathcal{A}} \to (\mathbb{F}_2 \otimes P_k)_{\mathcal{A}}^{GL}$$

is determined by Hu’ng–Peterson [18] for $k$. Furthermore, $j_1$ is isomorphic to $j_2$ mono.

**Proposition 4.2.** Let $j_k$ be the natural homomorphism induced by $P_k^{GL}$ is computed for $k = 3$ by Kameko [11], Alghamdi–Crabb–Hubbuck [3] and Boardman [4]. On the other hand, $\mathbb{F}_2 \otimes (P_k^{GL})_{\mathcal{A}}$ is determined by Hu’ng–Peterson [18] for $k = 3$ and 4.

Let $\mathbb{F}_2 \otimes (P_k^{GL})_{\mathcal{A}} := \bigoplus_{k \geq 0, A} \mathbb{F}_2 \otimes (P_k^{GL})_{\mathcal{A}}$ and $(\mathbb{F}_2 \otimes P_k)^{GL} := \bigoplus_{k \geq 0, A} (\mathbb{F}_2 \otimes P_k)_{\mathcal{A}}^{GL}$. They are equipped with canonical coalgebra structures. We get

**Proposition 3.1.** $j = \bigoplus j_k : \mathbb{F}_2 \otimes (P_k^{GL})_{\mathcal{A}} \to (\mathbb{F}_2 \otimes P_k)^{GL}$ is a homomorphism of coalgebras.

Let $\text{Sq}^0 : PH_*(BV_k) \to PH_*(BV_k)$ be Kameko’s squaring operation that commutes with the Steenrod operation $\text{Ext}^H_{\mathcal{A}} (\mathbb{F}_2, \mathbb{F}_2) \to \text{Ext}^H_{\mathcal{A}} (\mathbb{F}_2, \mathbb{F}_2)$ through the algebraic transfer $Tr_k$ (see [11], [3], [4], [17]). Note that $\text{Sq}^0$ is completely different from the identity map. We prove

**Proposition 4.2.** There exists a homomorphism

$$\text{Sq}^0 : P(\mathbb{F}_2 \otimes H_*(BV_k))^{GL_k} \to P(\mathbb{F}_2 \otimes H_*(BV_k))^{GL_k}$$

which commutes with Kameko’s $\text{Sq}^0$ through the homomorphism $j_k$.

These two propositions lead us to two different proofs of the following theorem.
Theorem 3.2. \( j_k^* = 0 \) in positive dimensions for \( k = 3 \). In other words, there is no spherical class in \( Q_0 S^0 \) which is detected by the triple algebraic transfer.

We compute explicitly the action of \( Sq^0 \) on \( P(\mathbb{F}_2 \otimes H_*(BV_k)) \) for \( k = 3 \) and 4 in Propositions 5.2 and 5.4.

The paper contains six sections and is organized as follows.

Section 2 is to prove Theorem 2.1. In Section 3, we assemble the \( j_k \) for \( k \geq 0 \) to get a homomorphism of coalgebras \( j_k = \bigoplus j_k \). By means of this property of \( j_k \) we give there a proof of Theorem 3.2. Section 4 deals with the existence of the squaring operation \( Sq^0 \) on \( P(\mathbb{F}_2 \otimes H_*(BV_k)) \) that leads us to an alternative proof for Theorem 3.2. This proof helps to explain the problem. In Section 5, we compute explicitly the action of \( Sq^0 \) on \( P(\mathbb{F}_2 \otimes H_*(BV_k)) \) for \( k \leq 4 \). Finally, in Section 6 we state a conjecture on the Dickson algebra that concerns spherical classes.

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2. Expressing \( \varphi_k \cdot Tr_k \) in the framework of invariant theory

First, let us recall how to define the homomorphism \( j_k \).

We have the commutative diagram

\[
\begin{array}{ccc}
P_k^{GL_k} & \xrightarrow{\subset} & P_k \\
\mathbb{F}_2 \otimes (P_k^{GL_k}) & \xrightarrow{p} & \mathbb{F}_2 \otimes P_k \\
\mathbb{F}_2 \otimes P_k & \xrightarrow{j_k} & \mathbb{F}_2 \otimes P_k \\
\end{array}
\]

where the vertical arrows are the canonical projections, and \( j_k \) is induced by the inclusion \( P_k^{GL_k} \subset P_k \). Obviously, \( p(P_k^{GL_k}) \subset (\mathbb{F}_2 \otimes P_k)^{GL_k} \). So, \( j_k \) factors through \( (\mathbb{F}_2 \otimes P_k)^{GL_k} \) to give rise to

\[
j_k : \mathbb{F}_2 \otimes (P_k^{GL_k}) \xrightarrow{\sim} (\mathbb{F}_2 \otimes P_k)^{GL_k},
\]

\[
j_k(1 \otimes Y) = 1 \otimes Y,
\]

for any polynomial \( Y \in D_k = P_k^{GL_k} \).

The goal of this section is to prove the following theorem.

Theorem 2.1. \( j_k = Tr_k^* \cdot \varphi_k^* \).

Now we prepare some data in order to prove the theorem at the end of this section.
First we sketch Lannes–Zarati’s work [13] on the derived functors of the destabilization. Let \( \mathcal{D} \) be the destabilization functor, which sends an \( \mathcal{A} \)-module \( M \) to the unstable \( \mathcal{A} \)-module \( \mathcal{D}(M) = M/B(M) \), where \( B(M) \) is the submodule of \( M \) generated by all \( S^i u \) with \( u \in M \), \( i > |u| \).

\( \mathcal{D} \) is a right exact functor. Let \( \mathcal{D}_k \) be its \( k \)-th derived functor for \( k \geq 0 \).

Suppose \( M_1, M_2 \) are \( \mathcal{A} \)-modules. Lannes and Zarati defined in [13, §2] a homomorphism

\[
\cap : \text{Ext}_A^r(M_1, M_2) \otimes \mathcal{D}_s(M_1) \to \mathcal{D}_{s-r}(M_2),
\]

\((f, z) \mapsto f \cap z\),

as follows.

Let \( F_s(M_i) \) be a free resolution of \( M_i \), \( i = 1, 2 \). A class \( f \in \text{Ext}_A^r(M_1, M_2) \) can be represented by a chain map \( F : F_s(M_1) \to F_{s-r}(M_2) \) of homological degree \(-r\). We write \( f = [F] \). If \( z = [Z] \) is represented by \( Z \in F_s(M_1) \), then by definition \( f \cap z = [F(Z)] \).

Let \( M \) be an \( \mathcal{A} \)-module. We set \( r = s = k \), \( M_1 = \Sigma^{-k} M \), \( M_2 = P_k \otimes M \), where

as before \( P_k = \mathbb{F}_2[x_1, \ldots , x_k] \), and get the homomorphism

\[
\cap : \text{Ext}_A^k(\Sigma^{-k} M, P_k \otimes M) \otimes \mathcal{D}_k(\Sigma^{-k} M) \to P_k \otimes M.
\]

Now we need to define the Singer element \( e_k(M) \in \text{Ext}_A^k(\Sigma^{-k} M, P_k \otimes M) \) (see Singer [20, p. 498]). Let \( \tilde{P}_1 \) be the submodule of \( \mathbb{F}_2[x, x^{-1}] \) spanned by all powers \( x^i \) with \( i \geq -1 \), where \( |x| = 1 \).

The \( \mathcal{A} \)-module structure on \( \mathbb{F}_2[x, x^{-1}] \) extends that of \( P_1 = \mathbb{F}_2[x] \) (see Adams [2], Wilkerson [22]). The inclusion \( P_1 \subset \tilde{P}_1 \) gives rise to a short exact sequence of \( \mathcal{A} \)-modules:

\[
0 \to P_1 \to \tilde{P}_1 \to \Sigma^{-1} \mathbb{F}_2 \to 0.
\]

Denote by \( e_1 \) the corresponding element in \( \text{Ext}_A^1(\Sigma^{-1} \mathbb{F}_2, P_1) \).

**Definition 2.2.** (Singer [20]).

(i) \( e_k = e_1 \otimes \cdots \otimes e_1 \in \text{Ext}_A^k(\Sigma^{-k} \mathbb{F}_2, P_k) \), \( k \) times.

(ii) \( e_k(M) = e_k \otimes M \in \text{Ext}_A^k(\Sigma^{-k} M, P_k \otimes M) \), for \( M \) an \( \mathcal{A} \)-module.

Here we also denote by \( M \) the identity map of \( M \).

The cap-product with \( e_k(M) \) gives rise to the homomorphism

\[
e_k(M) : \mathcal{D}_k(\Sigma^{-k} M) \to \mathcal{D}_0(P_k \otimes M) \equiv P_k \otimes M,
\]

\( e_k(M) (z) = e_k(M) \cap z \).

As \( \mathbb{F}_2 \) is an unstable \( \mathcal{A} \)-module, the following theorem is a special case (but would be the most important case) of the main result in [13].

**Theorem 2.3** (Lannes–Zarati [13]). Let \( \mathcal{D}_k \subset P_k \) be the Dickson algebra of \( k \) variables. Then \( e_k(\Sigma \mathbb{F}_2) : \mathcal{D}_k(\Sigma^{-1-k} \mathbb{F}_2) \to \Sigma D_k \) is an isomorphism of internal degree 0.

Next, we explain in detail the definition of the Lannes–Zarati homomorphism

\[
\varphi_k : \text{Ext}_A^k(\Sigma^{-k} \mathbb{F}_2, \mathbb{F}_2) \to (\mathbb{F}_2 \otimes \mathcal{D}_k)_A^*,
\]

which is compatible with the Hurewicz map (see [12], [13]).

Let \( N \) be an \( \mathcal{A} \)-module. By definition of the functor \( \mathcal{D} \), we have a natural homomorphism: \( \mathcal{D}(N) \to \mathbb{F}_2 \otimes N \). Suppose \( F_*(N) \) is a free resolution of \( N \). Then the above natural homomorphism induces a commutative diagram.
\[
\cdots \rightarrow DF_k(N) \rightarrow DF_{k-1}(N) \rightarrow \cdots \\
\downarrow i_k \quad \quad \quad \quad \quad \downarrow i_{k-1} \\
\cdots \rightarrow \mathbb{F}_2 \otimes F_k(N) \rightarrow \mathbb{F}_2 \otimes F_{k-1}(N) \rightarrow \cdots .
\]

Here the horizontal arrows are induced from the differential in \(F_\ast(N)\), and
\[
i_k[Z] = [1 \otimes Z]
\]
for \(Z \in F_k(N)\). Passing to homology, we get a homomorphism
\[
i_k : \mathbb{F}_2 \otimes D_k(N) \rightarrow \text{Tor}_A^k(\mathbb{F}_2, N),
\]
\[
1 \otimes [Z] \mapsto [1 \otimes Z]_A.
\]

Taking \(N = \Sigma^{1-k} \mathbb{F}_2\), we obtain a homomorphism
\[
i_k : \mathbb{F}_2 \otimes D_k(\Sigma^{1-k} \mathbb{F}_2) \rightarrow \text{Tor}_A^k(\mathbb{F}_2, \Sigma^{1-k} \mathbb{F}_2).
\]

Note that the suspension \(\Sigma : \mathbb{F}_2 \otimes D_k \rightarrow \mathbb{F}_2 \otimes \Sigma D_k\) and the desuspension
\[
\Sigma^{-1} : \text{Tor}_A^k(\mathbb{F}_2, \Sigma^{1-k} \mathbb{F}_2) \rightarrow \text{Tor}_A^k(\mathbb{F}_2, \Sigma^{-k} \mathbb{F}_2)
\]
are isomorphisms of internal degree 1 and \((-1)\), respectively. This leads us to

**Definition 2.4.** (Lannes–Zarati [13]). The homomorphism \(\varphi_k\) of internal degree 0 is the dual of
\[
\varphi_k^* = \Sigma^{-1} i_k \left( 1 \otimes e_k^{-1}(\Sigma \mathbb{F}_2) \right) \Sigma : \mathbb{F}_2 \otimes D_k \rightarrow \text{Tor}_A^k(\mathbb{F}_2, \Sigma^{-k} \mathbb{F}_2).
\]

Now we recall the definition of the algebraic transfer. Consider the cap-product
\[
\text{Ext}_A^r(\Sigma^{-k} \mathbb{F}_2, P_k) \otimes \text{Tor}_s^A(\mathbb{F}_2, \Sigma^{-k} \mathbb{F}_2) \rightarrow \text{Tor}_s^A(\mathbb{F}_2, P_k),
\]
\[
(e, z) \mapsto e \cap z.
\]

Taking \(r = s = k\) and \(e = e_k\) as in Definition 2.2, we obtain the homomorphism
\[
\text{Tr}^k_\ast : \text{Tor}_A^k(\mathbb{F}_2, \Sigma^{-k} \mathbb{F}_2) \rightarrow \text{Tor}_0^A(\mathbb{F}_2, P_k) \equiv \mathbb{F}_2 \otimes P_k,
\]
\[
\text{Tr}^k_\ast \otimes [Z]_A \equiv [1 \otimes E(Z)]_A \equiv 1 \otimes \mathbb{F}(Z),
\]
for \(Z \in F_k(\Sigma^{-k} \mathbb{F}_2)\), where \(e_k = [E]\) is represented by a chain map \(E : F_\ast(\Sigma^{-k} \mathbb{F}_2) \rightarrow F_{-k}(P_k)\).

Singer proved in [20] that \(e_k\) is \(GL_k\)-invariant, hence \(\text{Im}(\text{Tr}^k_\ast) \subset (\mathbb{F}_2 \otimes P_k)^{GL_k}\).

This gives rise to a homomorphism, which is also denoted by \(\text{Tr}^k_\ast\),
\[
\text{Tr}^k_\ast : \text{Tor}_A^k(\mathbb{F}_2, \Sigma^{-k} \mathbb{F}_2) \rightarrow (\mathbb{F}_2 \otimes P_k)^{GL_k}.
\]

**Definition 2.5.** (Singer [20]). The \(k\)-th algebraic transfer \(\text{Tr}_k : \mathbb{F}_2 \otimes PH_\ast(BV_k)
\rightarrow \text{Ext}_A^{k+k\ast}(\mathbb{F}_2, \mathbb{F}_2)\) is the homomorphism dual to \(\text{Tr}^k_\ast\).

We have finished the preparation of the needed data.
Proof of Theorem 2.1. Note that the usual isomorphism

$$\text{Ext}^k_A(\Sigma^{-k}F_2, P_h) \cong \text{Ext}^k_A(\Sigma^{1-k}F_2, \Sigma P_h)$$

sends $e_k(F_2)$ to $e_k(\Sigma F_2) = e_k(F_2) \otimes \Sigma F_2$. Moreover, if $e_k(F_2) = [E]$ is represented by a chain map $E : F_* (\Sigma^{-k}F_2) \to F_{*+k}(P_h)$ then $e_k(\Sigma F_2) = [E_{\Sigma}]$ is represented by the induced chain map $E_{\Sigma} : F_* (\Sigma^{1-k}F_2) \to F_{*+k} (\Sigma P_h)$, which is defined by $E_{\Sigma} = \Sigma E\Sigma^{-1}$.

By Theorem 2.3, $e_k(\Sigma F_2)$ is an isomorphism. So, for any $Y \in D_k$, there exists a representative of $e_k^{-1}(\Sigma Y)$, which is denoted by $E^{-1}\Sigma Y \in F_k(\Sigma^{1-k}F_2)$, such that $E_{\Sigma}(E^{-1}\Sigma Y) = \Sigma Y$.

The cap-product with $e_k(\Sigma F_2) = [E]$ induces the homomorphism

$$\tilde{Tr}^* : \text{Tor}_A(F_2, \Sigma^{1-k}F_2) \to \text{Tor}_0^A(\Sigma F_2, \Sigma P_h) \equiv F_2 \otimes \Sigma P_h, \quad \tilde{Tr}^*[1 \otimes Z] = e_k(\Sigma F_2) \cap [Z] = [1 \otimes E\Sigma(Z)] = 1 \otimes E\Sigma(Z),$$

for $Z \in F_k(\Sigma^{1-k}F_2)$. It is easy to check that $Tr^* = \Sigma^{-1}\tilde{Tr}^* \Sigma$. Moreover, set

$$\tilde{\varphi}_k = \Sigma \varphi_k^{-1} = i_k \left( 1 \otimes e_k^{-1}(\Sigma F_2) \right) : F_2 \otimes \Sigma D_k \to \text{Tor}_A^2(F_2, \Sigma^{1-k}F_2).$$

Obviously, $Tr^* \cdot \tilde{\varphi}_k = \Sigma^{-1}\tilde{Tr}^* \cdot \tilde{\varphi}_k \Sigma$. Now, for any $Y \in D_k$, we have

$$Tr^*_k \cdot \varphi^*_k(1 \otimes Y) = \Sigma^{-1}\tilde{Tr}^* \cdot \tilde{\varphi}_k \Sigma(1 \otimes Y)$$
$$= \Sigma^{-1}\tilde{Tr}^* \cdot \tilde{\varphi}_k(1 \otimes \Sigma Y)$$
$$= \Sigma^{-1}\tilde{Tr}^* \left[ 1 \otimes E\Sigma^{-1}\Sigma Y \right]$$
$$= \Sigma^{-1} \left( 1 \otimes E\Sigma(E^{-1}\Sigma Y) \right)$$
$$= 1 \otimes \Sigma^{-1}(\Sigma Y)$$
$$= 1 \otimes Y.$$

By definition of $j_k$, we also have $j_k(1 \otimes Y) = 1 \otimes Y$. The theorem is proved.

3. The homomorphism of coalgebras $j = \bigoplus j_k$

The canonical isomorphism $V_k \cong V_\ell \times V_m$, for $k = \ell + m$, induces the usual inclusion $GL_k \supset GL_\ell \times GL_m$ and the usual diagonal $\Delta : P_k \to P_\ell \otimes P_m$. Therefore, it induces two homomorphisms

$$\Delta_D : F_2 \otimes (P_k^{GL_k}) \to \left( F_2 \otimes (P_\ell^{GL_\ell}) \right) \otimes \left( F_2 \otimes (P_m^{GL_m}) \right),$$
$$\Delta_P : (F_2 \otimes P_k)^{GL_k} \to (F_2 \otimes P_\ell)^{GL_\ell} \otimes (F_2 \otimes P_m)^{GL_m}.$$

Here and in what follows, $\otimes$ means the tensor product over $F_2$, except when otherwise specified.
Set
\[ \mathbb{F}_2 \otimes D = \mathbb{F}_2 \otimes (P^{GL}_A) := \bigoplus_{k \geq 0} \mathbb{F}_2 \otimes (P^{GL_k}_A), \]
\[ (\mathbb{F}_2 \otimes P)^{GL}_A := \bigoplus_{k \geq 0} (\mathbb{F}_2 \otimes P_k^{GL_k})_A. \]

It is easy to see that \( \mathbb{F}_2 \otimes (P^{GL}) \) and \( (\mathbb{F}_2 \otimes P)^{GL}_A \) are endowed with the structure of a cocommutative coalgebra by \( \Sigma_D \) and \( \Sigma_P \), respectively. The coalgebra structure of \( (\mathbb{F}_2 \otimes P)^{GL}_A \) was first given by Singer [20].

**Proposition 3.1.** \( \psi = \bigoplus j_k : \mathbb{F}_2 \otimes (P^{GL}_A) \rightarrow (\mathbb{F}_2 \otimes P)^{GL}_A \) is a homomorphism of coalgebras.

**Proof.** This follows immediately from the commutative diagram
\[
\begin{array}{ccc}
\mathbb{F}_2 \otimes D_k & \xrightarrow{j_k} & (\mathbb{F}_2 \otimes P_k)^{GL}_A \\
\Sigma_D & & \downarrow \Sigma_P \\
(\mathbb{F}_2 \otimes D_k) \otimes (\mathbb{F}_2 \otimes D_m) & \xrightarrow{j_k \otimes j_m} & (\mathbb{F}_2 \otimes P_k)^{GL}_A \otimes (\mathbb{F}_2 \otimes P_m)^{GL}_A.
\end{array}
\]

**Remark.** According to Singer [20], \( \text{Tr}^* = \bigoplus \psi_k^* \) is a homomorphism of coalgebras. One can see that \( \varphi^* = \bigoplus \varphi_k^* \) is also a homomorphism of coalgebras. Then, so is \( j = \text{Tr}^* \cdot \varphi^* \). This is an alternative proof for Proposition 3.1.

Now let
\[
\mathbb{F}_2 \otimes PH_*(BV)^{GL} := \bigoplus_{k \geq 0} \left( \mathbb{F}_2 \otimes PH_*(BV_k) \right)^{GL} \cong \bigoplus_{k \geq 0} \left( \mathbb{F}_2 \otimes P_k^{GL_k} \right)^*,
\]
\[
P(\mathbb{F}_2 \otimes H_*(BV))^{GL} := \bigoplus_{k \geq 0} P(\mathbb{F}_2 \otimes H_*(BV_k))^{GL} \cong \bigoplus_{k \geq 0} \left( \mathbb{F}_2 \otimes (P_k^{GL_k}) \right)^*.
\]

Passing to the dual, we obtain the homomorphism of algebras
\[ j^*: \mathbb{F}_2 \otimes PH_*(BV)^{GL} \rightarrow P(\mathbb{F}_2 \otimes H_*(BV))^{GL}. \]

As an application of \( j^* \), we give here a proof for Conjecture 1.4 with \( k = 3 \).

**Theorem 3.2.** \( j_3 : \mathbb{F}_2 \otimes (P_3^{GL})^{GL_3}_A \rightarrow (\mathbb{F}_2 \otimes P_3)^{GL_3}_A \) is zero in positive dimensions.

**Proof.** We equivalently show that
\[ j^*_3 : \mathbb{F}_2 \otimes PH_*(BV_3) \rightarrow P(\mathbb{F}_2 \otimes H_*(BV_3))^{GL_3} \]
is a trivial homomorphism in positive dimensions.

\( \mathbb{F}_2 \otimes PH_*(BV_3) \) is described by Kameko [11], Alghamdi–Crabb–Hubbuck [3] and Boardman [4] as follows. \( \mathbb{F}_2 \otimes PH_*(BV_1) \) has a basis consisting of \( h_r, r \geq 0 \), where \( h_r \) is of dimension \( 2^r - 1 \) and is sent by the isomorphism \( \text{Tr}_1 \) to the Adams element, denoted also by \( h_r \), in \( \text{Ext}^{1,2}_A(\mathbb{F}_2, \mathbb{F}_2) \). According to [11], [3],
[4], $F_2 \otimes PH_*(BV_3)$ has a basis consisting of some products of the form $h_r h_s h_t$, where $r, s, t$ are non-negative integers (but not all such appear), and some elements $c_i (i \geq 0)$ with $\dim(c_i) = 2^{i+3} + 2^{i+1} + 2^i - 3$.

We will show in Lemma 3.3 that any decomposable element in $P(F_2 \otimes H_*(BV_3))$ is zero. Then, since $j^*$ is a homomorphism of algebras, $j^*_A$ sends any element of the form $h_r h_s h_t$ to zero.

On the other hand, by Hu’ng–Peterson [18], $F_2 \otimes D_3$ is concentrated in the dimensions $2^{r+2} - 4 (s \geq 0)$ and $2^{r+2} + 2^{s+1} - 3 (r > s > 0)$. Obviously, these dimensions are different from $\dim(c_i)$ for any $i$. Then $j^*_A$ also sends $c_i$ to zero.  

To complete the proof of the theorem, we need to show the following lemma.

**Lemma 3.3.** Let $\overline{D}_k = F_2 \otimes D_k$. Then the diagonal

$$\overline{\Delta}_D : \overline{D}_3 \to \overline{D}_1 \otimes \overline{D}_2 \otimes \overline{D}_1$$

is zero in positive dimensions.

**Proof.** Let us recall some informations on the Dickson algebra $D_k$. Dickson proved in [10] that $D_k \cong \mathbb{F}_2[Q_{k-1}, Q_{k-2}, \ldots, Q_0]$, a polynomial algebra on $k$ generators, with $|Q_s| = 2^k - 2^s$. Note that $Q_s$ depends on $k$, and when necessary, will be denoted $Q_{k,s}$. An inductive definition of $Q_{k,s}$ is given by

**Definition 3.4.**

$$Q_{k,s} = Q_{k-1,s-1}^2 + v_k \cdot Q_{k-1,s},$$

where, by convention, $Q_{k,k} = 1$, $Q_{k,s} = 0$ for $s < 0$ and

$$v_k = \prod_{\lambda_i \in \mathbb{F}_2} (\lambda_1 x_1 + \cdots + \lambda_{k-1} x_{k-1} + x_k).$$

Dickson showed in [10] that

$$v_k = \sum_{s=0}^{k-1} Q_{k-1,s} x_k^{2^s}.$$

Now we turn back to the lemma.

Since $\overline{\Delta}_D$ is symmetric, we need only to show that the diagonal

$$\overline{\Delta} : \overline{D}_3 \to \overline{D}_2 \otimes \overline{D}_1$$

is zero in positive dimensions.

For abbreviation, we denote $x_1, x_2, x_3$ by $x, y, z$, respectively, $Q_i = Q_{3,i}(x, y, z)$ for $i = 0, 1, 2$, $q_i = Q_{2,i}(x, y)$ for $i = 0, 1$. As is well known, $F_2 \otimes D_1$ has the basis

$\{z^{2^s-1} | s \geq 0\}$, and $F_2 \otimes D_2$ has the basis $\{q_1^{2^s-1} | s \geq 0\}$. By Hu’ng–Peterson [18], $F_2 \otimes D_3$ has the basis

$\{Q_0^{2^s-1} | s \geq 0\}, Q_2^{2^s-2^s-1} Q_1^{2^s-1} Q_0 (r > s > 0)\}.$

For $k \leq 3$, every monomial in $Q_0, \ldots, Q_{k-1}$ which does not belong to the given basis is zero in $F_2 \otimes D_k$. Note that the analogous statement is not true for $k \geq 4$ (see [18]).
Using the above inductive definitions of $Q_{k,s}$ and $v_k$, we get
\[ Q_0 = q_0^2 z + q_0 q_1 z^2 + q_0 z^4, \]
or
\[ \Delta(Q_0) = q_0^2 \otimes z + q_0 q_1 \otimes z^2 + q_0 \otimes z^4. \]
This implies easily that every term in $\Delta(Q_2^{s-2} - Q_1^{s-1} Q_0)$ is divisible by $q_0$, so it equals zero in $F_2 \otimes D_2$ as shown above. In other words,
\[ \Delta(Q_2^{s-2} - Q_1^{s-1} Q_0) = 0. \]
Similarly,
\[ Q_2 = q_1^2 + v_3 = q_1^2 + q_0 z + q_1 z^2 + z^4, \]
or
\[ \Delta(Q_2) = q_1^2 \otimes 1 + q_0 \otimes z + q_1 \otimes z^2 + 1 \otimes z^4, \]
\[ \Delta(Q_2^{s-1}) = (q_1^2 \otimes 1 + q_0 \otimes z + q_1 \otimes z^2 + 1 \otimes z^4)^{2s-1}. \]
By the same argument as above, we need only to consider terms in $\Delta(Q_2^{s-1})$ which are not divisible by $q_0$. Such a term is some product of powers of $q_1^2 \otimes 1$, $q_1 \otimes z^2$, $1 \otimes z^4$. If it contains a positive power of $z$ then this power is even and it equals zero in $F_2 \otimes D_1$. Otherwise, it should be $q_1^{2(2s-1)} \otimes 1$. Obviously, $q_1^{2(2s-1)}$ equals zero in $F_2 \otimes D_2$. So, $\Delta(Q_2^{s-1}) = 0$ for $s > 0$.

In summary, $\Delta = 0$ in positive dimensions. The lemma is proved. Then, so is Theorem 3.2.

As $Tr_3$ is an isomorphism (see Boardman [4]), we have an immediate consequence.

**Corollary 3.5.** $\varphi_3 : Ext^{3,3+i}_{A}(F_2, F_2) \rightarrow (F_2 \otimes D_3)^*_A$ is zero in every positive stem $i$.

4. The squaring operation: the existence

Liulevicius was perhaps the first person who noted in [14] that there are squaring operations $Sq^i : Ext^{k,i}_{A}(F_2, F_2) \rightarrow Ext^{k+i,2t}_{A}(F_2, F_2)$, which share most of the properties with $Sq^i$ on cohomology of spaces. In particular, $Sq^i(\alpha) = 0$ if $i > k$, $Sq^k(\alpha) = \alpha^2$ for $\alpha \in Ext^k_{A}(F_2, F_2)$, and the Cartan formula holds for the $Sq^i$’s. However, $Sq^0$ is not the identity. In fact,
\[ Sq^0 : Ext^{k,t}_{A}(F_2, F_2) \rightarrow Ext^{k,2t}_{A}(F_2, F_2), \]
\[ [b_1] \ldots [b_k] \mapsto [b_1^2] \ldots [b_k^2], \]
in terms of the cobar resolution (see May [16]).

Recall that $H_*(BV_k)$ is a divided power algebra
\[ H_*(BV_k) = \Gamma(a_1, \ldots, a_k) \]
generated by $a_1, \ldots, a_k$, each of degree 1, where $a_i$ is dual to $x_i \in H^1(BV_k)$. Here and in what follows, the duality is taken with respect to the basis of $H^*(BV_k)$ consisting of all monomials in $x_1, \ldots, x_k$. 
Let \( \gamma_t \) be the \( t \)-th divided power in \( H_\ast(BV_k) \) and for any \( a \in H_\ast(BV_k) \) let \( a^{(t)} = \gamma_t(a) \). So \( a^{(t)} \) is the element dual to \( x_t^1 \). One has

\[
a^{(2^r)}_t a^{(2^r)}_t = 0,
\]
and

\[
a^{(t)}_t = a^{(2^r)}_1 \cdots a^{(2^r)}_{t-1}
\]
if \( t = 2^{r_1} + \cdots + 2^{r_m} \), \( 0 \leq r_1 < \cdots < r_m \).

In [11] Kameko defined a \( GL_k \)-homomorphism

\[
Sq^0 : PH_\ast(BV_k) \rightarrow PH_\ast(BV_k),
\]

\[
a^{(i_1)}_1 \cdots a^{(i_k)}_k \mapsto a^{(2i_1+1)}_1 \cdots a^{(2i_k+1)}_k,
\]

where \( a^{(i_1)}_1 \cdots a^{(i_k)}_k \) is dual to \( x^{i_1}_1 \cdots x^{i_k}_k \). (See also [3].)

Crabb and Hubbuck gave in [8] a definition of \( Sq^0 \) that does not depend on the chosen basis of \( H_\ast(BV_k) \) as follows. The element \( a(V_k) = a_1 \cdots a_k \) is nothing but the image of the generator of \( \Lambda^k(V_k) \) under the (skew) symmetrization map

\[
\Lambda^k(V_k) \rightarrow H_k(BV_k) = \Gamma_k(V_k) = (V_k \otimes \cdots \otimes V_k)_{\Sigma_k}.
\]

Let \( F : H^\ast(BV_k) \rightarrow H^\ast(BV_k) \) be the Frobenius homomorphism defined by \( F(x) = x^2 \) for any \( x \), and let \( c : H_\ast(BV_k) \rightarrow H_\ast(BV_k) \) be the degree-halving dual homomorphism. It is obviously a surjective ring homomorphism. Then \( Sq^0 \) can be defined by

\[
Sq^0(c(y)) = a(V_k)y.
\]

Since \( y \in \ker c \) if and only if \( a(V_k)y = 0 \), \( Sq^0 \) is a monomorphism of \( GL_k \)-modules. Further, it is easy to see that \( cSq^{2i+1} = 0 \), \( cSq^{2i} = Sq^0 c \). So \( Sq^0 \) maps \( PH_\ast(BV_k) \) to itself.

Using a result of Carlisle and Wood [6] on the boundedness conjecture, Crabb and Hubbuck also noted in [8] that for any \( d \), there exists \( t_0 \) such that

\[
Sq^0 : PH_{2^d (2^{d-1}) k}(BV_k) \rightarrow PH_{2^{d+1} (2^{d+1} - 1) k}(BV_k)
\]
is an isomorphism for every \( t \geq t_0 \).

Kameko’s \( Sq^0 \) is shown to commute with \( Sq^0 \) on \( Ext^k_A(\mathbb{F}_2, \mathbb{F}_2) \) through the algebraic transfer \( TR_k \) by Boardman [4] for \( k = 3 \) and by Minami [17] for general \( k \).

One denotes also by \( Sq^0 \) the operation

\[
Sq^0 : \mathbb{F}_2 \otimes_{GL_k} PH_\ast(BV_k) \rightarrow \mathbb{F}_2 \otimes_{GL_k} PH_\ast(BV_k)
\]
induced by Kameko’s \( Sq^0 \). It preserves the product. Further, for \( k = 3 \), it satisfies

\[
Sq^0(h_t h_s h_t) = h_{r+1} h_{s+1} h_{t+1}, \quad Sq^0(c_i) = c_{i+1}
\]
(see Boardman [4]).

**Lemma 4.1.** \( Sq^0_{2^r+1} Sq^0 = 0 \), \( Sq^0_{2^r} Sq^0 = Sq^0_{2^r} Sq^0 \).

**Proof.** We need a formal notation. Namely, for \( a \in H_1(BV_k) \), set \( (a^{(t)})[2] = a^{(2t)} \). In general, \( (a^{(t)})[2] \neq \gamma_2(\gamma_t(a)) = (2^{t-1})a^{(2t)} \) (see Cartan [7]).
We start with a simple remark. Let \( x \in H^1(BV_k) \); then \( Sq^r(x^*) = (x^*)^{x^r} \). Let \( a \) denote the dual element of \( x \). Then, by dualizing,
\[
Sq^r_s(a^{(t)}) = \binom{t - r}{r} a^{(t-r)}.
\]

As a consequence, \( Sq^{2r+1}_s(a^{(2t+1)}) = 0 \) and
\[
Sq^{2r}_s(a^{(2t)}) = \binom{2t - 2r}{2r} a^{(2t - 2r)} = \binom{t - r}{r} (a^{(t-r)})^2 = (Sq^{r}_s(a^{(t)}))^2.
\]

Let \( \alpha = a^{(i_1)} \cdots a^{(i_k)} \). By the Cartan formula, we have
\[
Sq^*_s Sq^0(\alpha) = Sq^*_s(a^{(2i_1+1)}_1 \cdots a^{(2i_k+1)}_k)
= \sum_{r_1 + \cdots + r_k = r} Sq^{r_1}_s(a^{(2i_1+1)}_1) \cdots Sq^{r_k}_s(a^{(2i_k+1)}_k).
\]

The term corresponding to \((r_1, \ldots, r_k)\) equals 0 if at least one of \(r_1, \ldots, r_k\) is odd. Hence \( Sq^{2r+1}_s Sq^0(\alpha) = 0 \). Furthermore,
\[
Sq^*_s Sq^0(\alpha) = \sum_{r_1 + \cdots + r_k = r} Sq^{2r_1}_s(a^{(2i_1+1)}_1) \cdots Sq^{2r_k}_s(a^{(2i_k+1)}_k)
= \sum_{r_1 + \cdots + r_k = r} \left\{ Sq^{r_1}_s(a^{(2i_1)}_1) \cdots Sq^{r_k}_s(a^{(2i_k)}_k) \right\} a_1 \cdots a_k
= \left\{ Sq^{r_1}_s(a^{(i_1)}_1) \right\} \cdots \left\{ Sq^{r_k}_s(a^{(i_k)}_k) \right\} [2] a_1 \cdots a_k.
\]

The lemma is proved.

**Proposition 4.2.** For every positive integer \( k \), there exists a homomorphism
\[
Sq^0 : P(F_2 \otimes H_*(BV_k)) \to P(F_2 \otimes H_*(BV_k))
\]
that sends an element of degree \( n \) to an element of degree \( 2n + k \) and makes the following diagram commutative:

\[
\begin{array}{ccc}
F_2 \otimes PH_*(BV_k) & \xrightarrow{j_k^*} & P(F_2 \otimes H_*(BV_k)) \\
| & & |
\downarrow{Sq^0} & & \downarrow{Sq^0}
F_2 \otimes PH_*(BV_k) & \xrightarrow{j_k^*} & P(F_2 \otimes H_*(BV_k)).
\end{array}
\]

**Proof.** Since \( Sq^0 : H_*(BV_k) \to H_*(BV_k) \) is a \( GL_k \)-homomorphism, we can define \( Sq^0_{GL_k} = 1 \otimes Sq^0 \) and get a commutative diagram
where the horizontal arrows are the canonical projections.

Next, we show that $Sq^0 D$ sends the primitive part to itself. In other words, suppose $\alpha \in H_*(BV_k)$ satisfies

$$Sq^r(1 \otimes GL_k \alpha) = 1 \otimes GL_k Sq^r \alpha = 0$$

for any $r > 0$; we want to show that

$$Sq^r(Sq^0(1 \otimes \alpha)) = 0$$

for any $r > 0$. By definition of $Sq^0$ and Lemma 4.1, we have for every $r > 0$

$$Sq^r(Sq^0(1 \otimes \alpha)) = 1 \otimes GL_k Sq^r Sq^0(\alpha)$$

$$= \begin{cases} 1 \otimes GL_k Sq^0(Sq^r(\alpha)), & r \text{ even,} \\
0, & r \text{ odd,} \end{cases}$$

$$= \begin{cases} 0, & r \text{ odd,} \\
1 \otimes GL_k Sq^r Sq^0(\alpha), & r \text{ even,} \end{cases}$$

$$= 0.$$

Therefore, the above commutative diagram gives rise to a commutative diagram

$$\begin{array}{ccc}
PH_*(BV_k) & \xrightarrow{j_k} & P(\mathbb{F}_2 \otimes H_*(BV_k)) \\
\downarrow Sq^0 & & \downarrow Sq^0_D \\
PH_*(BV_k) & \xrightarrow{j_k} & P(\mathbb{F}_2 \otimes H_*(BV_k)).
\end{array}$$

By definition of $j_k$, the homomorphism $\tilde{j_k}$ factors through $F_2 \otimes PH_*(BV_k)$ and the previous diagram induces the commutative diagram stated in the proposition, in which $Sq^0_D$ is re-denoted by $Sq^0$ for short. The proposition is proved.

As an application of Proposition 4.2, we give an alternative proof of Theorem 3.2. By Kameko [11], Alghamdi et al. [3] and Boardman [4], $F_2 \otimes PH_*(BV_k)$ has a basis consisting of some products of the form $h_r h_s h_t$ and certain elements $c_i$ ($i \geq 0$) with $Sq^0(c_i) = c_{i+1}$ for any $i \geq 0$.

By Lemma 3.3, $j_3$ vanishes on any product $h_r h_s h_t$. Making use of Proposition 4.2, one has

$$j_3^*(c_i) = j_3^*(Sq^0)^i(c_0) = (Sq^0)^i(j_3^*(c_0)).$$
One needs only to show that $j_{c_0}^3 = 0$. Recall that $\dim(c_0) = 8$. The only element of dimension 8 in $D_3$ is $Q_2^8$. Obviously, $Q_2^8 = Sq^4Q_2$. So $P(F_2 \otimes H_*(BV_3))_8 = (F_2 \otimes D_3)_{b}^8 = 0$. Therefore, $j_{c_0}^3 = 0$. Theorem 3.2 is proved.

5. The squaring operation: an explicit formula for $k \leq 4$

Let $d_{(i_1, \ldots, i_{a})}$ be the dual element of $Q_{k-1}^{i_1} \cdots Q_{0}^{i_{a}} \in D_k$, where the duality is taken with respect to the basis of $D_k$ consisting of all monomials in the Dickson invariants $Q_{k-1}, \ldots, Q_{0}$.

It is well-known that

\begin{align*}
P(F_2 \otimes H_*(BV_1))_{GL_1} &= \text{Span} \{d_{(2^s-1)} | s \geq 0\}, \\
P(F_2 \otimes H_*(BV_2))_{GL_2} &= \text{Span} \{d_{(2^s-1,0)} | s \geq 0\}.
\end{align*}

By means of the definition of $Sq^0$ one can easily show that

\begin{align*}
Sq^0(d_{(2^s-1)}) &= d_{(2^{s+1}-1)}, \\
Sq^0(d_{(2^s-1,0)}) &= d_{(2^{s+1}-1,0)}.
\end{align*}

In this section we compute $Sq^0$ explicitly on $P(F_2 \otimes H_*(BV_3))$ for $k = 3$ and 4.

**Theorem 5.1** (Hu’ng–Peterson [18]). $PD^3_{GL} := P(F_2 \otimes H_*(BV_3))$ has a basis consisting of

\begin{align*}
d_{(2^s-1,0,0)}, & \quad s \geq 0, \\
d_{(2^s-2^r+1,2^r-1,1)}, & \quad r > s > 0.
\end{align*}

They are of dimensions $2^{s+2} - 4$ and $2^{r+2} + 2^{s+1} - 3$, respectively.

**Remark.** It is easy to check that $PD^3_{GL}$ has at most one non-zero element of any dimension.

**Proposition 5.2.** $Sq^0 : PD^3_{GL} \rightarrow PD^3_{GL}$ is given by

\begin{align*}
Sq^0(d_{(2^s-1,0,0)}) &= 0, \\
Sq^0(d_{(2^s-2^r+1,2^r-1,1)}) &= d_{(2^{s+1}-2^{r+1}-1,2^{r+1}-1,1)}.
\end{align*}

**Proof.** For brevity, we denote $x_1, x_2, x_3$ by $x, y, z$ and $a_1, a_2, a_3$ by $a, b, c$, respectively.

The first part of the proposition is an immediate consequence of dimensional information. To prove the second part we start by recalling that, from Definition 3.4, we have

\[Q_{3,0} = Q_{0} = x^4y^2z^1 + \text{(symmetrized)}\].

Suppose $m, n$ are non-negative integers. Let $x^\alpha y^\beta z^\gamma$ be the biggest monomial in $Q_2^n Q_1^m$ with respect to the lexicographic order on $(\alpha, \beta, \gamma)$. We claim that $x^\alpha y^\beta z^\gamma$ appears exactly one time in $Q_2^n Q_1^m Q_{0}$, or equivalently

\[(a) \quad Q_2^n Q_1^m Q_{0} = x^\alpha y^\beta z^{\gamma+1} + \text{(other terms)}\].
Indeed, suppose to the contrary that it appears more than once in \(Q_2^nQ_1^nQ_0\). That means there exists a monomial \(x^\alpha y^\beta z^\gamma\) in \(Q_2^nQ_1^n\), which is different from \(x^\alpha y^\beta z^\gamma\), and a permutation \(\sigma\) on the set \(\{4, 2, 1\}\) such that

\[ x^\alpha y^\beta z^\gamma = x^\alpha + \sigma(4) y^\beta + \sigma(2) z^\gamma + \sigma(1). \]

Since \(\alpha + 4 = \alpha + \sigma(4)\) and \(4 \geq \sigma(4)\), this implies \(\alpha \leq \alpha\). Combining this with the fact that \((\alpha, \beta, \gamma)\) is the biggest monomial in \(Q_2^nQ_1^n\) with respect to the lexicographic order on \((\alpha, \beta, \gamma)\), one gets \(\alpha = \alpha\) and \(\sigma(4) = 4\). Similarly, \(\beta = \beta\), \(\gamma = \gamma\) and \(\sigma\) is the identity permutation. This contradiction proves (a), or equivalently

\[ \text{proof of (a)} \]

(b) \(d_{(m,n,1)} = 1 \otimes a^{(\alpha+4)} b^{(\beta+2)} c^{(\gamma+1)} + (\text{other terms})\).

Here and throughout the proof, \(\otimes\) means the tensor product over \(GL_3\).

By definition of the squaring operation

(c) \(Sq^0(1 \otimes a^{(\alpha+4)} b^{(\beta+2)} c^{(\gamma+1)}) = 1 \otimes a^{(2\alpha+9)} b^{(2\beta+5)} c^{(2\gamma+3)}\).

Now a direct computation using Definition 3.4 shows that

\[ Q_2Q_1Q_0 = x^{12}y^4z + x^{10}y^6z + x^{10}y^5z^2 + x^{10}y^4z^3 + x^9y^6z^2 + x^9y^5z^3 + x^8y^6z^3 + x^8y^5z^4 + \text{symmetrized}. \]

Note that \(x^9y^5z^3\) and its symmetrized terms are the only terms of the form \(x^\text{odd} y^\text{odd} z^\text{odd}\) in \(Q_2Q_1Q_0\). On the other hand,

\[ Q_2^{2m}Q_1^{2n} = x^{2\alpha}y^{2\beta}z^{2\gamma} + \text{other terms}, \]

where \(x^{2\alpha}y^{2\beta}z^{2\gamma}\) is the biggest monomial in this polynomial with respect to the lexicographic order on \((2\alpha, 2\beta, 2\gamma)\). Focusing on monomials of the form \(x^\text{odd} y^\text{odd} z^\text{odd}\) and using the same argument as in the proof of (a), we have

\[ Q_2^{2m+1}Q_1^{2n+1}Q_0 = x^{2\alpha+9}y^{2\beta+5}z^{2\gamma+3} + \text{other terms}. \]

This is equivalent to

(d) \(1 \otimes a^{(2\alpha+9)} b^{(2\beta+5)} c^{(2\gamma+3)} = d_{(2m+1,2n+1,1)} + \text{other terms}\).

Combining (b), (c) and (d), we get

\[ Sq^0(d_{(m,n,1)}) = d_{(2m+1,2n+1,1)} + \text{other terms}. \]

Applying this for \((m,n,1) = (2^r-2^s-1, 2^s-1, 1)\), we obtain

\[ Sq^0(d_{(2^r-2^s-1,2^s-1,1)}) = d_{(2^s+1-2^r+1,1,2^r+2s+1)} + \text{other terms}. \]

In addition, \(Sq^0\) maps \(PD_3^*\) to itself (by Proposition 4.2) and \(PD_3^*\) consists of at most one non-zero element of any dimension. So the proposition is proved.

**Theorem 5.3** (Hu’ng–Peterson [18]). \(PD_1^* := P(\mathbb{Z}_2 \otimes H_*(BV_4))\) has a basis consisting of

\[
\begin{align*}
   d_{(2^s-1,0,0,0)} & , & s & \geq 0 , \\
   d_{(2^s-2^r-1,2^r-1,0,0)} & , & r & > s > 0 , \\
   d_{(2^s-2^r-1,2^r-1,2^r-1,2^r-1,2^r)} & , & t & > r > s > 1 , \\
   d_{(2^s-2^r-1,2^r-1,2^r-1,2^r-1,2^r-1)} & , & r & > s + 1 > 2 .
\end{align*}
\]

They are of dimensions \(2^{s+3} - 8, 2^{r+3} + 2^{s+2} - 6, 2^{t+3} + 2^{r+2} + 2^{s+1} - 4\) and \(2^{r+3} + 2^{s+1} - 4\), respectively.

**Remark.** \(PD_1^*\), as well as \(PD_3^*\), has at most one non-zero element of any dimension.
Proposition 5.4. \( Sq^0 : PD_4^* \to PD_4^* \) is given by

\[
\begin{align*}
Sq^0(d_{(2^r-1,0,0,0)}) &= Sq^0(d_{(2^r-2^r-1,2^r-1,1,0)}) = 0, \\
Sq^0(d_{(2^r-2^r-1,2^r-2^r-1,2^r-1,2)}) &= d_{(2^{r+1}-2^{r+1}-1,2^{r+1}-1,2^{r+1}-2^{r+1}-1,2)}, \\
Sq^0(d_{(2^r-2^{r+1}-2^r-1,2^r-1,2^{r+1}-1,2)}) &= d_{(2^{r+1}-2^{r+1}-2^r-1,2^{r+1}-1,2^{r+1}-1,2^{r+1}-1,2)}.
\end{align*}
\]

Proof. We denote \( x_1, x_2, x_3, x_4 \) by \( x, y, z, t \) and \( a_1, a_2, a_3, a_4 \) by \( a, b, c, d \), respectively, for brevity.

The first part of the proposition is an immediate consequence of dimensional information.

We claim that \( Q_0 = x^8 y^4 z^2 t + \) (symmetrized). It can be checked by a routine computation using Definition 3.4. Here we give an alternative argument. Indeed, the Dickson algebra \( D_4 \cong F_2[Q_3, Q_2, Q_1, Q_0] \) has exactly one non-zero element of dimension 15. To check the equality we need only to show that the right hand side is \( GL_4 \)-invariant. Recall that \( GL_4 \) is generated by the symmetric group \( \Sigma_4 \) and the transformation \( x \mapsto x + y, y \mapsto y, z \mapsto z, t \mapsto t \). So, it suffices to check that the right hand side is invariant under this transformation. We leave it to the reader.

Suppose \( m, n, p, q \) are non-negative integers with \( q > 0 \). Let \( x^a y^b z^c t^d \) be the biggest monomial in \( Q^m_3 Q^n_2 Q^p_1 Q^q_0^{-1} \) with respect to the lexicographic order on \( (\alpha, \beta, \gamma, \delta) \). By the same argument as in the proof of Proposition 5.2 we have

\[
Q^m_3 Q^n_2 Q^p_1 Q^q_0 = x^{\alpha+8} y^{\beta+4} z^{\gamma+2} t^{d+1} + \) (other terms).
\]

In other words,

(a) \[d_{(m,n,p,q)} = 1 \otimes a^{(\alpha+8)} b^{(\beta+4)} c^{(\gamma+2)} d^{(d+1)} + \) (other terms).

Here and throughout this proof, \( \otimes \) denotes the tensor product over \( GL_4 \).

By definition of the squaring operation

(b) \[ Sq^0(1 \otimes a^{(\alpha+8)} b^{(\beta+4)} c^{(\gamma+2)} d^{(d+1)}) = 1 \otimes a^{(2\alpha+17)} b^{(2\beta+9)} c^{(2\gamma+5)} d^{(2d+3)}.
\]

Using the same method that we used to compute \( Q_0 \) above, we can show that

\[
Q_3 Q_2 = \sum_{s_1 + s_2 + s_3 + s_4 = 20, s_i = 0 \text{ or a power of 2}} x^{s_1} y^{s_2} z^{s_3} t^{s_4},
\]

\[
Q_1 = \sum_{s_1 + s_2 + s_3 + s_4 = 14, s_i = 0 \text{ or a power of 2}} x^{s_1} y^{s_2} z^{s_3} t^{s_4}.
\]

In particular, we have

\[
Q_3 Q_2 = (x^{16} y^{2} z t + \text{symmetrized}) + \) (other terms),
\]

\[
Q_1 = (x^{8} y^{4} z t + \text{symmetrized}) + \) (other terms).
\]

Here, in both cases, any other term is of the form \( x^{\text{even}} y^{\text{even}} z^{\text{even}} t^{\text{even}} \). So

\[
Q_3 Q_2 Q_1 = (x^{17} y^{9} z^{5} t^{3} + \text{symmetrized}) + \) (other terms),
\]

where \( x^{17} y^{9} z^{5} t^{3} \) and its symmetrized terms are the only terms of the form \( x^{\text{odd}} y^{\text{odd}} z^{\text{odd}} t^{\text{odd}} \) in \( Q_3 Q_2 Q_1 \). On the other hand,

\[
Q_3^2 Q_2^2 Q_1^2 P_0^2 Q_0^{2q-2} = x^{2a} y^{2b} z^{2n} t^{2s} + \) (other terms),
\]

where \( x^{2a} y^{2b} z^{2n} t^{2s} \) is the biggest monomial in the polynomial with respect to the lexicographic order on \( (2a, 2b, 2n, 2s) \). Again, we focus on monomials of the form
and use the same argument as in the proof of Proposition 5.2 to get
\[ Q_3^{2m+1}Q_2^{2n+1}Q_1^{2p+1}Q_0^{2q-2} = x^{2\alpha+17}y^{2\beta+9}z^{2\gamma+5}t^{2\delta+3} + \text{(other terms)}, \]
or equivalently
\[ (c) \quad 1 \otimes a^{(2\alpha+17)}b^{(2\beta+9)}c^{(2\gamma+5)}d^{(2\delta+3)} = d^{(2m+1,2n+1,2p+1,2q-2)} + \text{(other terms)}. \]
Combining (a), (b) and (c), we get
\[ Sq^0((m,n,p,q)) = d^{(2m+1,2n+1,2p+1,2q-2)} + \text{(other terms)}. \]
Apply this for \((m, n, p, q) = (2t - 2r - 1, 2r - 2s - 1, 2s - 2q - 1, 2q - 1, 2t - 1, 2r - 1, 2s - 1, 2q - 1, 2t - 1, 2r - 1, 2s - 1, 2q - 1, 2t - 1, 2r - 1, 2s - 1, 2q - 1, 2t - 1, 2r - 1, 2s - 1, 2q - 1, 2t - 1, 2r - 1, 2s - 1, 2q - 1, 2t - 1, 2r - 1, 2s - 1, 2q - 1, 2t - 1, 2r - 1, 2s - 1, 2q - 1, 2t - 1, 2r - 1, 2s - 1, 2q - 1, 2t - 1, 2r - 1, 2s - 1, 2q - 1, 2t - 1, 2r - 1, 2s - 1, 2q - 1, 2t - 1, 2r - 1, 2s - 1, 2q).\]
Combining the resulting formulas and the facts that \( Sq^0 \) maps \( PD_*^1 \) to itself (by Proposition 4.2) and that \( PD_*^1 \) has at most one non-zero element of any dimension, we obtain the last two formulas of the proposition.

6. Final remark

Recall that \( \Omega_k := F \otimes D_k \). Let
\[ \Delta_D : \Omega_k \rightarrow \bigoplus_{\ell+m=k} \Omega_\ell \otimes \Omega_m \]
be the diagonal defined at the beginning of Section 3.

**Conjecture 6.1.** (Hu’ng–Peterson [19]). The diagonal \( \Delta_D \) is zero in positive dimensions for any \( k > 2 \).

This conjecture is proved in Lemma 3.3 for \( k = 3 \) and has been proved for \( 2 < k < 10 \) in [19]. It implies that \( j^* \) (respectively, \( \varphi_k \)) vanishes on the decomposable elements in \( F \otimes PH_*(BV_k) \) with respect to the product given by Singer [20] and discussed in Section 3 (respectively, in \( Ext^*_A(F_2, F_2) \) with respect to the cup product) for \( 2 < k < 10 \).

**Note added in proof.** Conjecture 6.1 has been established by F. Peterson and the author in the final version of [19].

**References**


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